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A THEORY OF CONSOLIDATION OF SOFT SEDIMENTS

-FINAL REPORT-

by

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1. Introduction

The objective of the research undertaken was to provide a theory of consolidation of fine-grained soils of sufficient generality to enable predictions to be made in engineering situations where the sediment is so soft that allowance must be made for changes in material properties during the progress of consolidation, where large strains and displacements must occur, account being taken of pore water flow in two or three dimensions.

The study proceeded in two Stages. The first Stage involved a critical examination of presently available theories of consolidation which were modified and extended to meet the objective of the research. The aim in Stage I was to develop mathematical equations governing the consolidation process based on physical assumptions which accord with the known behavior of soft sediments.

In Stage II numerical procedures and associated algorithms were considered based on the analysis which would permit prediction of the distribution and time variation within the medium of quantities of engineering interest, namely: the pore water pressure, the effective stresses and the void ratio.

2. Theories of Consolidation: A Brief Survey

The mechanism of the consolidation process in fine-grained soils and its expression in analytical form were given first by Terzaghi in 1923 (1). The theory was restricted to pore water flow in one (Cartesian) dimension but as was recently pointed out did take some account of large strains (2). It was aimed at providing a means for predicting the progress of settlement in uniformly loaded clay layers. Since that time the theory of one-dimensional consolidation has been greatly extended to take account of non-linear behavior, including the variation during consolidation of the coefficients of permeability and compressibility, and other effects associated with large strain and deformation (see, for



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example, ³, ⁴, ⁵) and this development was made possible by the intrinsic simplicity of one-dimensional deformation.

The problems of consolidation which face the civil engineer are rarely one-dimensional and a need soon developed for a more general formulation of the theory which would, for example, allow the settlement of structures to be predicted or sand drain installations to be rationally designed: both cases where soil deformation and pore water flow occur in three dimensions. Severe difficulties were soon encountered in this quest and it was found that only by introducing radical simplifying assumptions could any progress be made. This phase of development is associated with Terzaghi and Rendulic - both engineers - and with the mathematician Maurice Biot. The nature of their contributions reflect their difference in outlook and objective. Biot developed a rigorous and self-consistent theory (⁶) based on the assumption that the soil skeleton behaves as a porous perfectly elastic medium, although this is only a crude approximation to the known behavior of real soils. Terzaghi, on the other hand, tried to avoid commitment to particular constitutive relations between components of strain and effective stress and he succeeded, but only at the cost of implying that during consolidation (under constant loading) all components of total stress remain constant everywhere (^{7,8}). In reviewing these theories in 1943 (⁹) Terzaghi remarks "...the existing theoretical methods for dealing with two- and three-dimensional problems of the consolidation of clay under load are not yet ready for practical application". Owing to the complexity of the problem progress was slow, but by 1969 the consequences of these two differing approaches had been examined in a sufficient number of particular cases for some general conclusions to be drawn by Schiffman and co-workers (¹⁰).

In all these studies the assumption was invariably made, either tacitly or explicitly, that during consolidation the effective stress-strain relations are linear, the coefficient of permeability of the soil remains constant and that the strains and deformations of the soil are small. However, these factors can have an important influence on the

progress of consolidation, in particular upon the magnitude and changing pattern of pore water pressure distribution. More recently engineers have sought to remove those restrictions from the theory, but owing to the complexity of the general case have confined their attention to problems of one-dimensional compression and flow (see, for example, 4, 17, 18). Contemporaneously, Biot's work has been extended to large strains and deformations (19, 20, 21, 22), but the "soil" considered remains highly idealized.*

3. The Present Study

The typical problem with which we are concerned here is the consolidation under its own weight of a mound of soft saturated clay which has been placed beneath water. It has been assumed that the strains and deformations that may develop during consolidation are large: that is, account must be taken in the theory of the variation of compressibility and permeability during consolidation.

Attention has been restricted to two-dimensional pore-water flow and one-dimensional soil deformation. The details of the analyses are given in Appendixes A (Eulerian description) and B (Lagrangian description). Appendix C formulates the governing relationships for an axially symmetric geometry using a Lagrangian description. The most promising approach seems at present to lie with a Lagrangian description allied with the void ratio (e) as the dependent variable. The governing equations (B19, B26) which result are non-linear and must be solved numerically using an iterative procedure.

A numerical study of specific cases may reveal that strains and deformations are not unduly large, in which case alternative approximations to those adopted may prove to be more appropriate. This is likely to be helpful if the more general problem of successive

*We remark that a theory of large strain consolidation which regards the coefficient of permeability as a soil constant is a contribution exclusively to Applied Mechanics.

accretion of material on an existing mound is to be treated economically, although the present theory can, in fact, be used to cover this case.

4. Numerical Analysis: One-Dimensional Theory

Several numerical analysis techniques can and have been applied to the equation governing one-dimensional consolidation. These include an implicit finite difference technique ⁽²³⁾, and explicit finite difference technique ⁽²⁴⁾* and the method of lines ⁽²⁵⁾. The finite difference procedures approximate the spatial derivatives by centered differences and the time derivative by a forward difference. The method of lines is a classical method which reduces a partial differential equation to a system of ordinary differential equations by discretizing all but one of the independent variables. Given a partial differential equation of the form of equation (A22) and using a centered difference approximation for the spatial coordinate, one obtains a system of ordinary differential equations of the form

$$\frac{\partial u_i}{\partial t} = f(t, u_1, u_2, \dots, u_m); \quad i = 1, 2, \dots, m \quad (1)$$

where there are m spatial mesh points.

The system of ordinary differential equations is solved by an implicit technique. An Adams-Bashforth predictor is used in conjunction with an Adams-Moulton corrector. If the system of equations (1) is stiff, the predictor-corrector method is coupled with Newton's method to solve the nonlinear equations by iteration ⁽²⁶⁾.

A number of software packages exist for the purpose of solving one-dimensional nonlinear finite strain consolidation problems.

- A suite of packages have been developed by Bromwell Engineering, Inc. (now Bromwell and Carrier, Inc.) of Lakeland, Florida. These packages are based on the governing equation expressed in terms of the excess pore water pressure. They use an implicit

*We prefer the explicit technique since, unlike the implicit procedure, a stability criterion exists.

finite difference solution technique. These programs assume that the void ratio-permeability and void ratio-effective stress relationships are governed by power laws. They have been used in practice in the design and analysis of dredged fills and mine waste planning (23, 27, 28).

- A series of programs have been developed at the University of Colorado and at the Waterways Experiment Station. These programs have been based on the assumption that the void ratio-effective stress and void ratio-permeability relationships are exponential functions. These programs are based on a governing relationship where the void ratio is the dependent variable. An explicit finite difference procedure has been employed to solve the governing equation. The programs developed at the University of Colorado were developed primarily for research purposes (17). The second generation programs developed at the Waterways Experiment Station were designed primarily to assist in the planning of dredge fill operations (29, 30).
- The University of Colorado has developed a series of programs which are unrestricted with respect to the form of the void ratio-effective stress and void ratio-permeability relationships. These programs are based on the void ratio as the dependent variable in the governing equation. They use the method of lines as the numerical procedure of choice. They have been used for research purposes and to analyze mine waste and marine geotechnology problems (18, 31, 32, 33, 34, 35, 36, 37).
- A program developed for Ardaman and Associates, Orlando, Florida has been reported (38). Unfortunately, at the time of this writing, there are insufficient details available concerning the algorithms, numerical method and use of this software item.

5. Numerical Analysis: Two-Dimensional Theory

There are four numerical procedures which can be applied to the governing equation for two-dimensional nonlinear finite strain consolidation developed in the Appendices.

First, an explicit finite difference procedure can be developed. In this procedure the spatial derivatives can be expanded in terms of centered differences, at time t . The time derivative can be expanded as a forward difference. This will result in an explicit difference equation. The time marching procedure should be investigated for stability. However stability will be assured if each mesh size Δa and Δb and the time increment Δt meets the stability criterion for the two-dimensional problem at each time step.

The second procedure is implicit. Here the spatial derivatives are expanded in terms of centered differences at time $(t + \Delta t)$. This procedure will establish a set of simultaneous equations for which an alternating-direction implicit procedure may be feasible. However, the stability criterion for implicit formulations is not established.

The third suggested procedure is a variation of the method of lines described previously. Here a system of ordinary differential equations of the form (see Appendix D)

$$\frac{\partial e_{ij}}{\partial t} = f(t, e_{ij}); \quad \begin{array}{l} i = 1, 2, \dots, m \\ j = 1, 2, \dots, n \end{array} \quad (2)$$

is obtained. The one-dimensional results in m ordinary differential equations. This two-dimensional problem results in $m \times n$ ordinary differential equations. These equations can be solved by the same techniques as discussed for the one-dimensional formulation. This method is always stable.

A fourth possible method would apply a finite element discretization to the spatial variables along with an explicit time marching scheme. This approach has the disadvantage that a variational principle for this class of problems must be established a priori.

It is noted that the governing (B19) equation contains an integral term from equation (B25). This can be discretized by means of Simpson's rule.

The suggested solution algorithm is as follows:

- Given the boundary value problem in terms of the void ratio a solution is developed for $e(a,b,t)$.
- The vertical effective stress $\sigma'(a,b,t)$ is determined from the void ratio-effective stress relationship which is known a priori.
- The vertical total stress $\sigma(a,b,t)$ is determined from the bulk equilibrium relationship

$$\frac{\partial \sigma}{\partial a} + [n\rho_f + (1-n)\rho_s] \left[\frac{1+e}{1+e_o} \right] = 0 \quad (3)$$

- The static pore water pressure $u_o(a,b,t)$ is determined by integrating the fluid equilibrium relationship,

$$\frac{\partial u_o}{\partial a} + \frac{1+e}{1+e_o} \rho_f = 0 \quad (4)$$

(We note that this assumes a constant level of height of water above the base of the clay layer. The consideration of a variable free water surface is a trivial extension to this algorithm.)

- Each a station at each time of interest is converted to the appropriate ξ station by

$$\xi(a,b,t) = \int_0^a \frac{1+e(a',b,t)}{1+e(a',b,0)} da' \quad (5)$$

The development of a detailed step-by-step algorithm and the code to implement the algorithm are beyond the scope of the current contract.

5. References

- (¹) Terzaghi, K., (1923), "Die Berechnung der Durchlässigkeitsziffer des Tones aus dem Verlauf der Hydrodynamischen Spannungerscheinungen", Akademie der Wissenschaften in Wien, Sitzungsberichte, Mathematischnaturwissenschaftliche Klasse, Part IIa, 132, No. 3/4, pp. 125-138.
- (²) Znidarcic, D. and Schiffman, R.L., (1982), "On Terzaghi's Concept of Consolidation", *Geotechnique*, 32, pp. 387-389, Discussions and Reply in *Geotechnique* 34, pp. 130-134.
- (³) Schiffman, R.L. and Gibson, R.E., (1964), "Consolidation of Nonhomogeneous Clay Layers", *Journal of the Soil Mechanics and Foundations Division, ASCE*, 90, No. SM5, Proceedings Paper4043, pp. 1-30.
- (⁴) Gibson, R.E., England, G.L., and Hussey, M.J.L., (1967), "The Theory of One-Dimensional Consolidation of Saturated Clays, I. Finite Non-Linear Consolidation of Thin Homogeneous Layers", *Geotechnique*, 17, pp. 261-263.
- (⁵) Lee, K. and Sills, G.C., (1979), "A Moving Boundary Approach to Large Strain Consolidation", *Proceedings, Third International Conference on Numerical Methods in Geomechanics*, (edited by W. Wittke), A.A. Balkema, Rotterdam, The Netherlands, 1, pp. 163-173.
- (⁶) Biot, M.A., (1941a), "General Theory of Three-Dimensional Consolidation", *Journal of Applied Physics*, 12, pp. 155-164.
- (⁷) Gibson, R.E., Knight, K., and Taylor, P.W., (1963), "A Critical Experiment to Examine Theories of Three-Dimensional Consolidation", *Proceedings, European Conference on Soil Mechanics and Foundation Engineering in Wiesbaden*, 1, pp. 69-76.
- (⁸) Gibson, R.E. and Lumb, P., (1953), "Numerical Solution of Some Problems in the Consolidation of Clay", *Proceedings, Institution of Civil Engineers*, 2, Part 1, pp. 182-198.
- (⁹) Terzaghi, K., (1942), Theoretical Soil Mechanics, John Wiley and Sons, Inc., New York, New York.
- (¹⁰) Schiffman, R.L., Chen, A., T.-F., and Jordan, J.C., (1969), "An Analysis of Consolidation Theories", *Journal of the Soil Mechanics and Foundations Division, ASCE*, 95, No. SM1, Proceedings Paper 6370, pp. 285-312, Discussions, 96, No. SM1, pp. 331-336, Closure, 96, No. SM5, pp. 1793-1795.
- (¹¹) Crank, J. and Gupta, R.S., (1972a), "A Moving Boundary Problem Arising from the Diffusion of Oxygen in Absorbing Tissue", *Journal of the Institute of Mathematics and its Applications*, 10, pp. 19-33.

- (12) Crank, J. and Gupta, R.S., (1972b), "A Method for Solving Moving Boundary Problems in Heat Flow Using Cubic Splines or Polynomials", *Journal of the Institute of Mathematics and its Applications*, 10, pp. 296-304.
- (13) Casagrande, A., (1942), "Fifth Progress Report on Triaxial Shear Research", Harvard University, Cambridge, Massachusetts.
- (14) Casagrande, A., (1944), "Seventh Progress Report on Triaxial Shear Research", Harvard University, Cambridge, Massachusetts.
- (15) Rutledge, P.C., (1947), "Triaxial Shear Research", U.S. Waterways Experiment Station, Vicksburg, Mississippi.
- (16) Bishop, A.W., (1952), "The Stability of Earth Dams", Ph.D. Thesis, University of London.
- (17) Gibson, R.E., Schiffman, R.L., and Cargill, K.W., (1981), "The Theory of One-Dimensional Consolidation of Saturated Clays, II. Finite Non-Linear Consolidation of Thick Homogeneous Layers", *Canadian Geotechnical Journal*, 18, pp. 280-293.
- (18) Schiffman, R.L., Pane, V., and Gibson, R.E., (1984), "The Theory of One-Dimensional Consolidation of Saturated Clays, IV. An Overview of Nonlinear Finite Strain Sedimentation and Consolidation", *Sedimentation/Consolidation Models*, (Edited by R.N. Yong and F.C. Townsend), ASCE, pp. 1-29.
- (19) Small, J.C., Booker, J.R., and Davis, E.H., (1976), "Elasto-Plastic Consolidation of Soil", *International Journal of Solids and Structures*, 12, pp. 431-448.
- (20) Carter, J.P., Small, J.C., and Booker, J.R., (1977), "A Theory of Finite Elastic Consolidation", *International Journal of Solids and Structures*, 13, pp. 467-478.
- (21) Carter, J.P., Booker, J.R., and Small, J.C., (1979), "The Analysis of Finite Elasto-Plastic Consolidation", *International Journal for Numerical and Analytical Methods in Geomechanics*, 3, pp. 107-129.
- (22) Meijer, K.L., (1984), "Comparison of Finite and Infinitesimal Strain Consolidation by Numerical Experiments", *International Journal for Numerical and Analytical Methods in Geomechanics*, 8, pp. 531-548.
- (23) Somogyi, F., (1980), "Large Strain Consolidation of Fine-Grained Slurries", Presented at the Canadian Society for Civil Engineering 1980 Annual Conference, Winnipeg, Manitoba, Canada.

- (24) Pane, V., (1981), "One-Dimensional Finite Strain Consolidation", M.S. Thesis, Department of Civil Engineering, University of Colorado, Boulder, Colorado.
- (25) Pane, V., (1985), "Sedimentation and Consolidation of Clays", Ph.D. Thesis, Department of Civil Engineering, University of Colorado, Boulder, Colorado.
- (26) Shampine, L.F., Gordan, H.K., (1975), Computer Solution of Ordinary Differential Equations, W.H. Freeman and Company, San Francisco, California.
- (27) Bromwell Engineering, Inc., (1982), "Waste Clay Disposal in Mine Cuts", 2 Volumes, Bromwell Engineering, Inc., Lakeland, Florida, Report.
- (28) Carrier, W.D., III, Bromwell, L.G., and Somogyi, F., (1983), "Design Capacity of Slurried Mineral Waste Ponds", *Journal of Geotechnical Engineering, ASCE*, 109, No. 5, pp. 699-716.
- (29) Cargill, K.W., (1983), "Procedures for Prediction of Consolidation in Soft Fine-Grained Dredged Material", Dredging Operations Technical Support, U.S. Army Engineer Waterways Experiment Station, Technical Report D-83-1.
- (30) Cargill, K.W., (1984), "Prediction of Consolidation of Very Soft Soil", *Journal of Geotechnical Engineering, ASCE*, 110, No. 6, pp. 775-795.
- (31) Caldwell, J.A., Ferguson, K., Schiffman, R.L., and Van Zyl, D., (1984), "Application of Finite Strain Consolidation Theory for Engineering Design and Environmental Planning of Mine Tailings Impoundments", Sedimentation/Consolidation Models, (Edited by R.N. Yong and F.C. Townsend), ASCE, pp. 581-606.
- (32) Ferguson, K.A., Hutchison, I.P.G., and Schiffman, R.L., (1985a), "Water Balance Approach to Prediction of Seepage from Mine Tailings Impoundments, Part I. General Water Balance Approach and Some Typical Modeling Results", Seepage and Leakage from Dams and Impoundments, (Edited by R.L. Volpe and W.E. Kelly), ASCE, pp. 250-262.
- (33) Ferguson, K.A., Hutchison, I.P.G., and Schiffman, R.L., (1985b), "Water Balance Approach to Prediction of Seepage from Mine Tailings Impoundments, Part II. Theoretical Aspects of Water Balance Approach to Seepage Modeling and Detailed Core History Results", Seepage and Leakage from Dams and Impoundments, (Edited by R.L. Volpe and W.E. Kelly), ASCE, pp. 263-285.

- (³⁴) Scully, R.W., Schiffman, R.L., Olsen, H.W., and Ko, H.-Y., (1984), "Validation of Consolidation Properties of Phosphatic Clay at Very High Void Ratios", Sedimentation/Consolidation Models, (Edited by R.N. Yong and F.C. Townsend), ASCE, pp. 158-181.
- (³⁵) Schiffman, R.L. and Pane, V., (1984), "Nonlinear Finite Strain Consolidation of Soft Marine Sediments", Seabed Mechanics, (Edited by B. Denness), Graham & Trotman, London, United Kingdom, pp. 123-130.
- (³⁶) Schiffman, R.L. and Cargill, K.W., (1981), "Finite Strain Consolidation of Sedimenting Clay Deposits", Proceedings, Tenth International Conference on Soil Mechanics and Foundation Engineering, 1, pp. 239-242.
- (³⁷) Templeton, J.S., III, Murff, J.D., Goodwin, R.H., and Klejbuk, L.W., (1985), "Evaluating Soils and Hazards in the Mississippi Canyon", Proceedings, Seventeenth Annual Offshore Technology Conference, Paper OTC 4964, 3, pp. 63-72.
- (³⁸) Ardaman and Associates, Inc. (1983), "Evaluation of Phosphatic Clay and Reclamation Methods, Volume 6: Predictive Methodology for Evaluating Disposal Methods", Florida Institute of Phosphate Research, Publication No. 80-02-002.

APPENDIX A

In this Appendix we consider the case of pore-water flow and soil skeleton deformation restricted to two dimensions (the x, z -plane) and we seek to derive the equations governing the motion of these phases by adopting the so-called Eulerian scheme of description. We therefore consider the soil grains and water which at any time (t) reside within the rectangular element of sides (δx , δz) located at the point (x, z).

We denote by (w_x , w_z) the velocity components of the solid phase and by (v_x , v_z) those of the pore water. These components will be functions of the three independent variables (x, z, t).

1. Equations of Continuity

The volume porosity (volume of void space V_v per unit of bulk volume V) we denote by n , so that

$$n = V_v/V ,$$

while the area porosity (area of void space A_v per unit of bulk area A) we denote by n_a . If the number of grains occupying the rectangle is in some sense large, it can be shown that

$$n_a = n .$$

The rate of increase of weight of solids within (δx , δz) must equal the net rate of flow of weight across the four faces of the element and this leads us to the equation

$$\frac{\partial}{\partial t} [(1-n)\rho_s] + \frac{\partial}{\partial x} [w_x(1-n)\rho_s] + \frac{\partial}{\partial z} [w_z(1-n)\rho_s] = 0 \quad (A1)$$

where ρ_s is the unit weight of the solids.

Similarly, for the pore-water

$$\frac{\partial}{\partial t} [n\rho_f] + \frac{\partial}{\partial x} [v_x n\rho_f] + \frac{\partial}{\partial z} [v_z n\rho_f] = 0 \quad (A2)$$

where ρ_f is the unit weight of the fluid. These are the equations of continuity.

Since, in the range of stress with which we are concerned, the unit weights ρ_s , ρ_f can be regarded as constants, it follows from (A1) and (A2) by addition, that

$$\frac{\partial}{\partial x} [nv_x + (1-n)w_x] + \frac{\partial}{\partial z} [nv_z + (1-n)w_z] = 0 \quad (A3)$$

2. Flow Rule

We shall assume that the pore-water moves through the soil skeleton in accordance with Darcy's law, due account being taken of the fact that it is the relative velocity between the phases which induces a drag on the soil skeleton. Accordingly, we write

$$n(v_x - w_x) = -\frac{k_x}{\rho_f} \frac{\partial u}{\partial x} \quad (A4)$$

$$n(v_z - w_z) = -\frac{k_z}{\rho_f} \frac{\partial u}{\partial z} \quad (A5)$$

where (k_x , k_z) are the coefficients of permeability appropriate to the (x,z) directions and u is an excess pore water pressure which is defined in terms of the pore-water pressure (p) by

$$u = p + \rho_f z + \text{const.} \quad (A6)$$

where the positive direction of z is against gravity.

3. The Governing Equations

If we set (A4), (A5) in (A3) we find that

$$\frac{\partial}{\partial x} \left[k_x \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial z} \left[k_z \frac{\partial u}{\partial z} \right] = \rho_f \left[\frac{\partial w_x}{\partial x} + \frac{\partial w_z}{\partial z} \right] \quad (A7)$$

and it can be seen that if the soil skeleton is stationary ($w_x = 0$, $w_z = 0$), then a familiar equation governing the excess pore-water pressure is obtained, namely

$$\frac{\partial}{\partial x} \left[k_x \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial z} \left[k_z \frac{\partial u}{\partial z} \right] = 0 . \quad (A8)$$

It is worth noting that just after dumping a mound, $w_x = 0$ and $w_z = 0$ almost everywhere and so (A8) holds at this instant: this equation again holds when consolidation is complete.

Now, from (A1)

$$\frac{\partial w_x}{\partial x} + \frac{\partial w_z}{\partial z} = \frac{1}{(1-n)} \left[\frac{\partial n}{\partial t} + w_x \frac{\partial n}{\partial x} + w_z \frac{\partial n}{\partial z} \right] \quad (A9)$$

and eliminating the rate of soil skeleton dilatation between (A7) and (A9) we find

$$\frac{\partial}{\partial x} \left[\frac{k_x}{\rho_f} \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial z} \left[\frac{k_z}{\rho_f} \frac{\partial u}{\partial z} \right] + \frac{D}{Dt} [\ln(1-n)] = 0 \quad (A10)$$

where the operator

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + w_x \frac{\partial}{\partial x} + w_z \frac{\partial}{\partial z} .$$

In order to proceed further the relations connecting strain increments with the effective stresses and the effective stress increments must be assumed. These relations, together with the equations of equilibrium, would allow a complete theory to be developed. However, in view of the wide variety of soil types and their complicated response to load, a completely general approach is at present not feasible.

We shall therefore proceed in the spirit of Terzaghi and Rendulic's early work and seek plausible simplifications which will permit solutions to be obtained to problems which will be sufficiently exact for engineering purposes.

4. One-Dimensional Compression

We shall assume here that the development of a mound takes place in a number of stages each consisting of the sudden dumping of soil followed by a period of consolidation during which no further loading occurs.

During the consolidation it will be assumed that:

- (a) Lateral displacement of the soil can be ignored.

The mound will therefore consolidate with the development of vertical strains only but the pore water flow we shall not assume to be constrained in any way.

The porosity (or void ratio) at any point in the clay will depend on the initial porosity and state of effective stress there and also upon the subsequent development of the components of effective stress as consolidation proceeds. The detailed effective stress-strain behavior of soils is very complicated and to obtain an engineering solution to the problem we introduce the following simplifying assumptions.

- (b) The void ratio (e) of a soft saturated clay depends to a good approximation only upon the major principal effective stress σ_1' (the so-called "American Hypothesis"; see, for example, ref. ¹³⁻¹⁵).
- (c) The major principal stress is (almost everywhere) equal to the vertical stress σ_{zz} (see Ref. ¹⁶) in a mound-like structure. It follows from these two assumptions that

$$e = e(\sigma_1') = e(\sigma_{zz}') . \quad (A11)$$

For simplicity of exposition we restrict our remarks to the case where $k_x = k_z = k(\text{const})$; the more general problem can if required readily be treated later in detail. Equation (A10) can be written in the form:

$$\frac{k}{\rho_f} \nabla^2 u = \frac{1}{(1+e)} \frac{de}{d\sigma_1'} \frac{D\sigma_{zz}'}{Dt} \quad (A12)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + w_z \frac{\partial}{\partial z} \quad (A13)$$

Since

$$\begin{aligned} \sigma_{zz}' &= \sigma_{zz} - p \\ c_v \nabla^2 p &= \frac{Dp}{Dt} - \frac{D\sigma_{zz}'}{Dt} \end{aligned} \quad (A14)$$

where

$$c_v = - \frac{k(1+e)}{\rho_f} \frac{d\sigma_1'}{de}$$

which is, apart from the factor $(1+e)$ in place of $(1+e_0)$, Terzaghi's coefficient of one-dimensional consolidation.

5. One-Dimensional Consolidation

It is worth noting that in cases of strictly one-dimensional compression and pore water flow, the above assumptions (a), (b) and (c) are satisfied and equation (A14)

$$c_v \frac{\partial^2 p}{\partial z^2} = \frac{Dp}{Dt} - \frac{D\sigma_{zz}'}{Dt} \quad (A15)$$

is exact.

This equation governing the pore water pressure has a structure very similar to that encountered in Terzaghi's theory and indeed his equation can be recovered by replacing the differential operator D/Dt by $\partial/\partial t$. From (A13) this can be seen to be tantamount to ignoring, for example, $w_z \partial p/\partial z$ compared with $\partial p/\partial t$.

There is no a priori reason to suppose that this can be justified. Some indication of the error involved can be found from (A3) which reduces to

$$nv_z + (1-n)w_z = 0 \quad (A15)$$

when there exists a plane on which the condition $v_z = w_z = 0$ persists. (This is so in the oedometer test: the mid-plane with two-way drainage, or the base if there is no flow from it.) It then follows from (A16) and (A5) that

$$w_z = \frac{k}{\rho_f} \frac{\partial u}{\partial z}$$

and so, for example:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{k}{\rho_f} \left(\frac{\partial u}{\partial z} \right)^2 \quad (A17)$$

What then is the equation governing the pore-water pressure which is exact within the context of the assumptions made in the theory, when the Eulerian description is used?

We commence from the form of (A15) which allows for the variation of permeability, namely

$$\frac{1}{m_v} \frac{\partial u}{\partial z} \left[\frac{k}{\rho_f} \frac{\partial u}{\partial z} \right] = \frac{Dp}{Dt} - \frac{D\sigma_{zz}}{Dt} \quad (A18)$$

where, from (A6):

$$\frac{Dp}{Dt} = \frac{\partial u}{\partial t} + \frac{k}{\rho_f} \frac{\partial u}{\partial z} \left[\frac{\partial u}{\partial z} - \rho_f \right] \quad (A19)$$

Also,

$$\frac{D\sigma_{zz}}{Dt} = \frac{\partial \sigma_{zz}}{\partial t} + \frac{k}{\rho_f} \frac{\partial u}{\partial z} \frac{\partial \sigma_{zz}}{\partial z} \quad (A20)$$

but vertical equilibrium requires that

$$\frac{\partial \sigma_{zz}}{\partial z} = -n\rho_f - (1-n)\rho_s \quad (A21)$$

Setting (A21) in (A20) and subtracting the resulting equation from (A19), it is found that (A18) becomes

$$\frac{1}{m_v} \frac{\partial}{\partial z} \left[\frac{k}{\rho_f} \frac{\partial u}{\partial z} \right] - \frac{k}{\rho_f} \left(\frac{\partial u}{\partial z} \right)^2 - \frac{k(\rho_s - \rho_f)}{(1+e)\rho_f} \frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} - \frac{\partial \sigma_{zz}}{\partial t} . \quad (A22)$$

Using (A6) a similar equation can be derived for the pore-water pressure (p).

This equation is highly non-linear even in the "thin" layer case ($\rho_s = \rho_f$), and its use in problems of large displacement and strain is limited for the following reasons:

- (i) The parameters k and m_v are related to the void ratio e , but this in turn is not connected to u (or p) in a straightforward way.
- (ii) The vertical total stress σ_{zz} and its time derivative $\partial \sigma_{zz} / \partial t$ for a fixed value of z varies with time in a way which is not given as part of the data of the problem and must be discovered as part of the solution:
- (iii) The geometry of the mound boundary on which a condition of the type $u = 0$ persists is, again, not known ab initio and must be found during the solution process.

However, one distinct advantage of working in terms of u is that the initial conditions can be specified in terms of the initial state of total stress in the mound. This allows a more complete and exact description than can be achieved when the void ratio or porosity is used as the independent variable (see Appendix B).

A rather more promising governing equation emerges if the porosity (n) is taken as the dependent variable. Commencing from (A10) we can write

$$\frac{1}{\rho_f} \frac{\partial}{\partial z} \left[k \frac{\partial u}{\partial z} \right] = \frac{1}{(1-n)} \left[\frac{\partial n}{\partial t} + v_s \frac{\partial n}{\partial z} \right] . \quad (A23)$$

Now

$$\frac{\partial u}{\partial z} = \frac{\partial p}{\partial z} + \rho_f = \frac{\partial \sigma_{zz}}{\partial z} - \frac{\partial \sigma'_{zz}}{\partial z} + \rho_f \quad (\text{A24})$$

so that using (A21)

$$\frac{\partial u}{\partial z} = - (1-n)(\rho_s - \rho_f) - \frac{d\sigma'}{dn} \frac{\partial n}{\partial z} \quad (\text{A25})$$

Setting this expression into (A23) we find, after some algebra, that

$$\frac{\partial}{\partial z} \left[c_v \frac{\partial n}{\partial z} \right] = \frac{\partial n}{\partial t} + \left[\frac{\rho_s}{\rho_f} - 1 \right] \frac{d}{dn} \left[k(1-n)^2 \right] \frac{\partial n}{\partial z} \quad (\text{A26})$$

which is essentially equation (23) of (5). It is not open to objections (i) and (ii) above. The difficulty (iii) remains but it has been claimed that this can be overcome by using suitable numerical techniques*.

*Lee and Sills (Ref. 5) use a numerical technique originating from work by Crank and Gupta (11, 12) which effectively updates the position of the moving boundary as the solution proceeds. They take the case of a "thin" soil layer where the approximation $\rho_s = \rho_f$ can be justified on physical grounds, and compare their numerical solution with that given by Gibson, England and Hussey (Ref. 4). The agreement is apparently excellent, but is not wholly convincing as the coefficient c_F in ref. 4 is taken incorrectly as $c_F = c_v / (1+e)^2$ instead of $c_F = c_v (1+e_0)^2 / (1+e)^2$.

APPENDIX B

In this Appendix we consider the physical problem discussed in Appendix A, in particular the case of two-dimensional pore-water flow and one-dimensional compression in a submarine clay mound. However, here we use the Lagrangian scheme of description and examine the motion of an element of soil which contains the same solids throughout its history.

For reasons mentioned in Appendix A we shall not seek to derive an equation governing the pore-water pressure (or an excess pore-water pressure), but rather work in terms of the void ratio (e) or the porosity (n). The same physical assumptions used there will again be adopted here. The argument which ensues follows closely that used in ref. ⁴ and we shall not repeat the details here.

1. Equations of Continuity

The deformation of the soil skeleton is described by following the motion of an element of initial area ($\delta a \times \delta b$) lying at $t = 0$ at $(a,b)^*$ which during its subsequent motion always contains the same solids. The location of a material point initially at (a,b) is given subsequently by

$$\xi = \xi(a,b,t) \quad (B1)$$

$$\eta = \eta(a,b,t) \quad (B2)$$

where, by definition,

$$a = \xi(a,b,0) \quad (B3)$$

$$b = \eta(a,b,0) \quad (B4)$$

*The coordinate a points upward against gravity.

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*The coordinate a points upward against gravity.

Since we shall assume that the motion of the soil skeleton is confined to the a direction it follows that (B2) may be replaced by

$$\eta = b . \quad (B2)_{bis}$$

The equation of continuity of solids takes the simple form

$$\frac{\partial \xi}{\partial a} = \frac{1-n_0}{1-n} = \frac{1+e}{1+e_0} \quad (B5)$$

where

$$n_0 = n(a,b,0)$$

$$e_0 = e(a,b,0)$$

are, respectively, the initial ($t = 0$) porosity and void ratio at (a,b).

As the element translates and deforms during its subsequent motion, pore-water moves into and out from its four faces and the equation of continuity of this (incompressible) phase is found to be

$$\frac{\partial}{\partial a} [n(v_a - w_a)] + \frac{\partial}{\partial b} [nv_b \frac{\partial \xi}{\partial a}] + \frac{\partial}{\partial t} [n \frac{\partial \xi}{\partial a}] = 0 . \quad (B6)$$

2. Flow Rule

Darcy's Law takes the form*

$$n(v_a - w_a) = - \frac{k}{\rho_f} \frac{\partial u}{\partial \xi} \quad (B7)$$

$$n(v_b - w_b) = - \frac{k}{\rho_f} \frac{\partial u}{\partial \eta} \quad (B8)$$

with this description, but since

$$\frac{\partial u}{\partial a} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial a} , \quad (B9)$$

* We have taken $k_z = k_b = k$ here, but the extension to the anisotropic case is trivial.

$$\frac{\partial u}{\partial b} = \frac{\partial u}{\partial \eta} \quad (\text{B10})$$

and

$$w_b = 0 \quad (\text{B11})$$

it follows that

$$n(v_a - w_a) \frac{\partial \xi}{\partial a} = - \frac{k}{\rho_f} \frac{\partial u}{\partial a} \quad (\text{B12})$$

$$nv_b = - \frac{k}{\rho_f} \frac{\partial u}{\partial b} \quad (\text{B13})$$

The excess pore-water pressure (u) is related to the pore-water pressure (p) by

$$u = p + \rho_f \xi - \text{const.} \quad (\text{B14})$$

3. The Governing Equations

Setting (B12) and (B13) in (B6) we find

$$\frac{\partial}{\partial a} \left[\frac{k}{\rho_f} \frac{\partial u}{\partial a} \left(\frac{1+e_0}{1+e} \right) \right] + \frac{\partial}{\partial b} \left[\left(\frac{1+e}{1+e_0} \right) \frac{k}{\rho_f} \frac{\partial u}{\partial b} \right] = \frac{\partial}{\partial t} \left[\frac{e}{1+e_0} \right] \quad (\text{B15})$$

where (B5) has been used. Using now (B5) and (B14), equation (B15) can be expressed in terms of the pore-water pressure:

$$\frac{\partial}{\partial a} \left[\frac{k}{\rho_f} \frac{\partial p}{\partial a} \left(\frac{1+e_0}{1+e} \right) + k \right] + \frac{\partial}{\partial b} \left[\frac{k}{\rho_f} \frac{\partial p}{\partial b} \left(\frac{1+e}{1+e_0} \right) \right] = \frac{1}{(1+e_0)} \frac{\partial e}{\partial t} \quad (\text{B16})$$

Vertical equilibrium requires that

$$\frac{\partial \sigma}{\partial a} + [n\rho_f + (1-n)\rho_s] \frac{\partial \xi}{\partial a} = 0, \quad (\text{B17})$$

where σ is the vertical total stress, and this may be used in conjunction with

$$p = \sigma - \sigma' \quad (\text{B18})$$

which defines the vertical effective stress (σ'), to eliminate p from (B16).

The resulting equation governing the void ratio (e) is

$$\frac{\partial}{\partial a} \left[\frac{C_F}{1+e_0} \frac{\partial e}{\partial a} \right] + \frac{\partial}{\partial b} \left[C_F \frac{(1+e)^2}{(1+e_0)^3} \frac{\partial e}{\partial b} \right] - (G_S - 1) \frac{d}{de} \left[\frac{k}{1+e} \right] \frac{\partial e}{\partial a} + \frac{\partial}{\partial b} \left[\frac{k}{\rho_f} \left(\frac{1+e}{1+e_0} \right) \frac{\partial \sigma}{\partial b} \right] = \frac{1}{(1+e_0)} \frac{\partial e}{\partial t}$$

in the general case, which reduces when e_0 is a constant to

$$\frac{\partial}{\partial a} \left[C_F \frac{\partial e}{\partial a} \right] + \frac{\partial}{\partial b} \left[C_F \left(\frac{1+e}{1+e_0} \right)^2 \frac{\partial e}{\partial b} \right] - (G_S - 1) \frac{d}{de} \left[k \left(\frac{1+e_0}{1+e} \right) \right] \frac{\partial e}{\partial a} + \frac{\partial}{\partial b} \left[\frac{k}{\rho_f} (1+e) \frac{\partial \sigma}{\partial b} \right] = \frac{\partial e}{\partial t} \quad (B19)$$

where (as in Ref. ⁴) the coefficient of finite consolidation

$$C_F = - \frac{k}{\rho_f} \frac{(1+e_0)^2}{(1+e)} \frac{d\sigma'}{de} \quad (B20)$$

and

$$G_S = \rho_s / \rho_f \quad (B21)$$

4. The Total Stress Term

The last term on the left-hand side of (B19) is not known a priori and must be found and updated during the solution. In this Section we show how the term $\partial \sigma / \partial b$ can be found at any time from the current void ratio distribution $e(a,b,t)$.

Consider a vertical cylinder of material, of unit cross-sectional area, extending from the base of the mound to the water level above the mound and denote this height by H . It is evident that the weight of material (solids and water) within this cylinder would remain constant if no water flowed across the surfaces (i.e., strict one-dimensional consolidation) or was added ($H = \text{constant}$). Under these conditions $\partial \sigma / \partial b$ on the base depends on b alone. However, on the surface of

the mound $a = a_0(b)$ settlement increases the water pressure and hence the total vertical stress there. Thus at the top of the clay $\partial\sigma/\partial b$ depends both on b and t . At any other height a , or when the conditions of simple vertical flow of pore-water no longer obtain or H varies with time, the term $\partial\sigma/\partial b$ can be expected to depend upon (a,b,t) . We shall now determine this relation.

Integrating (B17) with respect to a , and using (B5), it is found that

$$\sigma = \rho_f \int_a^{a_0(b)} \frac{e}{1+e_0} da + \rho_s \int_a^{a_0(b)} \frac{da}{1+e_0} + \rho_f [H(t) - \xi(a_0(b), b, t)] \quad (B22)$$

where the last term represents the water pressure on the surface $a = a_0(b)$ of the mound, which equals the total stress σ in (B22) when $a = a_0(b)$.

From (B5):

$$\xi(a, b, t) = \int_0^a \frac{1+e(a, b, t)}{1+e_0(a, b)} da \quad (B23)$$

since $\xi(0, b, t) = 0$. Substituting (B23) into (B22) and rearranging, it is found that*

$$\sigma = \rho_f H - \rho_f \int_0^a \left(\frac{1+e}{1+e_0} \right) da + (\rho_s - \rho_f) \int_a^{a_0} \frac{da}{1+e_0} \quad (B24)$$

It follows from (B24) that

$$\frac{\partial\sigma}{\partial b} = -\rho_f \int_0^a \frac{\partial}{\partial b} \left(\frac{1+e}{1+e_0} \right) da + (\rho_s - \rho_f) \left[\frac{1}{1+e_0(a_0, b, t)} \frac{da_0}{db} + \int_a^{a_0} \frac{\partial}{\partial b} \left(\frac{1}{1+e_0} \right) da \right] \quad (B25)$$

*The first term represents the weight of water if it completely filled the cylinder from the base to the water surface. The second term is the weight of water if it filled the cylinder from the base to the height $\xi(a, b, t)$ at which σ is acting. The last term is the buoyant weight of solids above the plane on which σ acts.

which reduces in the case $e_0 = \text{const.}$ to

$$\frac{\partial \sigma}{\partial b} = \frac{(\rho_s - \rho_f)}{1+e_0} \frac{da_0}{db} - \frac{\rho_f}{1+e_0} \int_0^a \frac{\partial e}{\partial b} da, \quad (\text{B26})$$

the first term arising from the initial non-uniform surface profile of the mound, the second being a time-varying correction to this resulting from the settlement.

5. Some Typical Boundary Conditions

If a mass of saturated clay of more-or-less uniform void ratio e_0 is deposited in a short period of time beneath water and rests on (say) an impervious base, the void ratio distribution in the mound at subsequent times will, on the basis of the assumptions that have been made herein, be governed by (B19) with (B26) and subject to:

(a) the initial condition

$$e(a,b,0) = e_0 \text{ (constant);} \quad (\text{B27})$$

(b) the mound surface boundary condition

$$e(a_0(b),b,t) = e_0 \text{ (constant);} \quad (\text{B28})$$

(c) the impervious base condition

$$w_a(0,b,t) = 0 \quad (\text{B29})$$

$$v_a(0,b,t) = 0 \quad (\text{B30})$$

which result in the following condition on the void ratio:

$$\frac{\partial e}{\partial a} + \frac{(\rho_s - \rho_f)}{(1+e_0)} \frac{de}{d\sigma'} = 0 \quad (\text{B31})$$

on $a = 0$.

APPENDIX C

In this Appendix the equations governing the consolidation of a submarine clay mound under conditions of axial symmetry are derived in Lagrange coordinates. The work described in Appendix B was confined to plane flow of pore-water and would approximate the conditions encountered in long mounds of uniform cross-section away from the ends. The present treatment extends, therefore, the study to mounds of compact form which can be approximated by bodies within a surface of revolution about a vertical axis.

The physical assumptions and notation adopted in Appendix B will be retained here and the development below should be read in conjunction with that work upon which it relies heavily.

1. The Physical Equations

The initial coordinates of a point of the soil skeleton are (a,b) where a is taken as vertically upward against gravity and b is now a radial distance measured horizontally from the axis of symmetry of the mound.

The continuity equation of the solid phase retains the form

$$\frac{\partial \xi}{\partial a} = \frac{1+e}{1+e_0} \quad (C1)$$

but that of the pore water now becomes

$$\frac{\partial}{\partial a} [n(v_a - w_a)] + \frac{\partial}{\partial b} [nv_b \frac{\partial \xi}{\partial a}] + \frac{\partial}{\partial t} [n \frac{\partial \xi}{\partial a}] + \frac{nv_b}{b} \frac{\partial \xi}{\partial a} = 0 \quad (C2)$$

The flow of the pore water is still governed by the pair of equations

$$n(v_a - w_a) \frac{\partial a}{\partial a} = - \frac{k}{\rho_f} \frac{\partial u}{\partial a} \quad (C3)$$

$$nv_b = - \frac{k}{\rho_f} \frac{\partial u}{\partial b} \quad (C4)$$

Setting (C3) and (C4) in (C2) and using (C1), we find that

$$\frac{\partial}{\partial a} \left[\frac{k}{\rho_f} \frac{\partial u}{\partial a} \frac{1+e_o}{1+e} \right] + \frac{1}{b} \frac{\partial}{\partial b} \left[b \frac{k}{\rho_f} \frac{\partial u}{\partial b} \frac{1+e_o}{1+e} \right] - \frac{1}{1+e_o} \frac{\partial e}{\partial t} = 0 . \quad (C5)$$

The spatial gradients of the excess pore water pressure (u) are related to those of the pore water pressure (p) through the equations

$$\frac{\partial u}{\partial a} \left(\frac{1+e_o}{1+e} \right) = \frac{\partial p}{\partial a} \left(\frac{1+e_o}{1+e} \right) + \rho_f \quad (C6)$$

$$\frac{\partial u}{\partial b} = \frac{\partial p}{\partial b} \quad (C7)$$

which allow (C5) to be written in terms of the pore water pressure and the void ratio (e).

When the equation of vertical equilibrium, namely

$$\frac{\partial \sigma}{\partial a} + [n\rho_f + (1-n)\rho_s] \frac{\partial \xi}{\partial a} = 0 , \quad (C8)$$

is made use of, together with the definition of the vertical effective stress

$$\sigma' = \sigma - p , \quad (C9)$$

equation (C5) can be written in terms of the void ratio (e) above as

$$\begin{aligned} \frac{\partial}{\partial a} \left[\frac{C_F}{1+e_o} \frac{\partial e}{\partial a} \right] + \frac{1}{b} \frac{\partial}{\partial b} \left[b \frac{k}{\rho_f} \left(\frac{1+e_o}{1+e} \right) \frac{\partial \sigma}{\partial b} \right] + \frac{1}{b} \frac{\partial}{\partial b} \left[b \frac{(1+e)^2}{(1+e_o)^3} C_F \frac{\partial e}{\partial b} \right] \\ - (G_s - 1) \frac{d}{de} \left[\frac{k}{1+e} \right] \frac{\partial e}{\partial a} = \frac{1}{1+e_o} \frac{\partial e}{\partial t} \end{aligned} \quad (C10)$$

which is the axisymmetric form of equation (B19) and the governing equation we seek.

In all other respects the results arrived at for the case of plane pore water flow, and documented in Appendix B, hold mutatis mutandis for this case also.

APPENDIX D

In this Appendix we formulate the basic procedure for the method of lines. This is the method we suggest for the solution of the governing equations presented in the previous appendices.

We consider a governing equation of the form given by equation (B19). In a general form this equation can be written as

$$A(e) \frac{\partial^2 e}{\partial a^2} + B(e) \left(\frac{\partial e}{\partial a} \right)^2 + C(e) \frac{\partial^2 e}{\partial b^2} + D(e) \left(\frac{\partial e}{\partial b} \right)^2 + E(e) \frac{\partial e}{\partial a} = \frac{\partial e}{\partial t} \quad (D1)$$

The method of lines consists of a procedure in which all but one of the independent variables are discretized. This yields a coupled system of ordinary differential equations. For the equation given by (D1) we choose to discretize a into m equidistant mesh points, having coordinates (a_i ; $i = 1, 2, \dots, m$) and b into n equidistant mesh points, having coordinates (b_j ; $j = 1, 2, \dots, n$).

The partial derivatives are approximated at the mesh points by

$$\left(\frac{\partial e}{\partial a} \right)_{i,j} = \frac{e_{i+1,j} - e_{i-1,j}}{2(\Delta a)} \quad (D2)$$

$$\left(\frac{\partial^2 e}{\partial a^2} \right)_{i,j} = \frac{e_{i+1,j} - 2e_{i,j} + e_{i-1,j}}{(\Delta a)^2} \quad (D3)$$

with similar expressions at (i) and $(j-1)$, (i) and $(j+1)$ for the derivatives with respect to b .

The discretized system of governing equations now becomes

$$\begin{aligned} & A(e_{i,j}) \left(\frac{e_{i+1,j} - 2e_{i,j} + e_{i-1,j}}{(\Delta a)^2} \right) + B(e_{i,j}) \left(\frac{e_{i+1,j} - e_{i-1,j}}{2\Delta a} \right)^2 \\ & + C(e_{i,j}) \left(\frac{e_{i,j+1} - 2e_{i,j} + e_{i,j-1}}{(\Delta b)^2} \right) + D(e_{i,j}) \left(\frac{e_{i,j+1} - e_{i,j-1}}{2\Delta b} \right)^2 \\ & + E(e_{i,j}) \left(\frac{e_{i+1,j} - e_{i-1,j}}{2\Delta a} \right) = \frac{\partial e_{i,j}}{\partial t} \end{aligned} \quad (D4)$$

where $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. Thus we have a system of $m \times n$ ordinary differential equations.

1. Method of Solution

The system of ordinary differential equations (D4) is solved by an Adams-Bashforth predictor which is used in conjunction with an Adams-Moulton corrector (26).

For the purpose of developing the formulation of the method we concentrate on a single equation of the system given by equation (D4). We aim to obtain the solution to

$$\frac{de}{dt} = f(t, e(t)) \quad (D5)$$

in the closed interval $[0, t_{fin}]$, where $f(t, e(t))$ denotes the function representing the left hand side of the chosen equation from the system (D4). To accomplish our objective we divide the closed interval $[0, t_{fin}]$ into a set of time steps (t_0, t_1, \dots, t_p) such that $(t_0 = 0)$ and $(t_p = t_{fin})$. We obtain the solution at these time steps. The basic computational task relates to the advancement of the numerical solution to (t_{k+1}) after having computed the value of (e) at time (t_k) .

We note that

$$e(t_{k+1}) = e(t_k) + \int_{t_k}^{t_{k+1}} \frac{de}{dt} dt \quad (D6)$$

or from equation (D5)

$$e(t_{k+1}) = e(t_k) + \int_{t_k}^{t_{k+1}} f(t, e(t)) dt \quad (D7)$$

The Adams method approximates the solution by replacing $f(t, e(t))$ with a polynomial $P_{l,k}(t)$ of order (l) which interpolates the computed derivatives at the preceding points from the set of discrete time values and then by integrating the polynomial. Thus the polynomial $P_{l,k}(t)$ must satisfy the following condition

$$P_{l,k}(t_i) = f(t_i, e(t_i)); \quad i = 0, 1, 2, \dots, k \quad (D8)$$

The expression

$$e_{k+1} = e(t_k) + \int_{t_k}^{t_{k+1}} P_{l,k}(t) dt \quad (D9)$$

defines the (l^{th}) order predictor of the solution $e(t_{k+1})$ at time (t_{k+1}) .

By choosing a constant time step

$$t_k = t_{k-1} + h; \quad k = 1, 2, \dots, p \quad (D10)$$

the backward formulation of the predictor is given by

$$e_{k+1} = e(t_k) + h \sum_{i=1}^l \gamma_{i-1} \nabla^{i-1} f_k \quad (D11)$$

where

$$f_k = f(t_k, e(t_k)) \quad (D12)$$

$$\gamma_i = \frac{1}{i!h} \int_{t_k}^{t_{k+1}} \frac{(t-t_k)(t-t_{k-1}) \dots (t-t_{k-1-i})}{h^{i-1}} dt \quad (D13)$$

and the backward difference operator ∇^i is defined as

$$\nabla^i f_k = \nabla(\nabla^{i-1} f_k) = \nabla^{i-1} f_k - \nabla^{i-1} f_{k-1} \quad (D14)$$

where we note that

$$\nabla^0 f_k = f_k \quad (D15a)$$

$$\nabla^1 f_k = \nabla f_k = f_k - f_{k-1} \quad (D15b)$$

For purposes of efficiency with respect to the time to solve the system of differential equations the Adams method uses a variable time step (h_k) and a variable order of the predictor. The backward difference formulation is replaced by a modified divided difference formulation.

Given the prediction e_{k+1} as computed above we evaluate its derivative with respect to time \dot{e}_{k+1} by

$$\dot{e}_{k+1} = P_{l,k}(t_{k+1}) \approx f_{k+1} \quad (D16)$$

The corrector to solve for $e(t_{k+1})$ is formulated as

$$e(t_{k+1}) = e(t_k) + \int_{t_k}^{t_{k+1}} P_{l+1,k}(t) dt \quad (D17)$$

The polynomial $P_{l+1,k}(t)$ interpolates the same data as $P_{l,k}(t)$ with an additional point, namely

$$P_{l+1,k}(t_{k+1}) = f_{k+1} \quad (D18)$$

It now remains only to evaluate $f(t_{k+1}, e(t_{k+1}))$ for the next time step.

The algorithm described (26) uses low orders of the predictor polynomials (P). This maximizes the stability properties. The order of the predictor polynomial is accomplished separately from the time step selection.

The time step is determined as the largest value for which the local error meets a given input tolerance. The control of the local error assumes that the local solution is uniformly approximated over the closed interval $[t_k, t_{k+1}]$. The local error is controlled per unit time step; i.e. relative to the size of the step. Using this procedure the solution overshoots the last time (t_{fin}). Since the approximation is uniform over the whole interval, however, an interpolation provides the solution at the required value of time.

The procedure sets all the parameters for continuation of the predictor-corrector process. Thus a computation at a predetermined sequence of time is provided by this algorithm.

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