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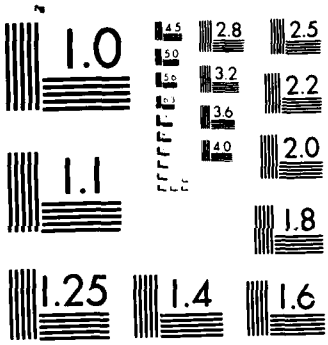

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Final Report
for

Grant No. DAAK11-82-K-0009

on

COMPUTATIONAL FLUID DYNAMICS OF LIQUID FILLED SPINNING SHELLS

for the period
5-25-82 to 5-25-83

Submitted to

U.S. Army Ballistic Research Laboratory/ARRADCOM
Aberdeen Proving Ground
Maryland 21005

by the

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A fully implicit, three-dimensional finite difference simulation of the incompressible Navier-Stokes equations has been cast using upwind operators. A non-inertial coordinate system was used to simulate the fluid motion in a precessing and spinning cylinder. The continuity equation was modified to include artificial compressibility. The code is capable of time accurate solutions, and it could be used to track time-dependent flows.

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1.0 INTRODUCTION

The Army is currently developing liquid payloads for spin-stabilized projectiles. A liquid filler can destabilize the flight of an aeroballistically well designed shell. The instability mechanism is a resonance between the coning frequency of the projectile and a natural vibration frequency of the spinning liquid. Only simplified models have been used in the past to study this liquid-induced, projectile instability. These include linear models and axisymmetric (independent of circumferential or azimuthal position within the fluid) finite-difference solutions to the governing equations, the Navier-Stokes equations [Refs. 1, 2, 3]. The linear models consist of closed form analytic solutions and/or numerical solutions. The primary deficiency with the available models is that the projectile yaw must be quite small. In fact, experiments to verify these modes indicate serious nonlinear effects at yaw amplitudes well below those common to projectiles [Refs. 4, 5]. Hence, large yaw effects must be incorporated into the modeling capabilities. Also, many of the linear models do not properly treat the boundary conditions at the liquid/solid interfaces. Corrections are required within these models, but the corrections are valid only for high Reynolds numbers. The axisymmetric Navier-Stokes code cannot model the three-dimensional disturbances of the liquid. The natural frequencies of the liquid are truly three-dimensional oscillations and most flow problems cannot be treated by an axisymmetric code. A full three-dimensional solution to the Navier-Stokes equations is required. Viscous effects must be maintained properly in all coordinate directions to properly model the rotating fluid. The code must also have a high Reynolds number capability. This requires the use of a computational algorithm which

is more complex than a standard finite-difference solution.

A numerical simulation capability developed specifically for the analysis of fluid flow in liquid-filled projectiles and satisfying the above requirements is described in Ref. 6. In that effort, the unsteady, incompressible Navier-Stokes equations for laminar flow are solved for the primitive variables without recourse to linearization or simplification of the equations of motion. The finite difference approximations to the governing equations are by choice either first or second-order time accurate and are second-order accurate in the axial, radial and azimuthal directions (with an option for fourth-order accuracy in the azimuthal direction). The method allows imposition of arbitrary body motions including spin and precession and the corresponding boundary conditions are easily and directly prescribed. The finite difference equations are solved using an implicit, approximate factorization procedure that permits the choice of reasonably large time steps and avoids limitations based on the magnitude of Reynolds number. Thus, the numerical simulation methodology considered provides a complete, and flexible framework for the computational analysis of fluid behavior in liquid filled projectiles.

This report describes progress made in the computattional fluid dynamics of fluid filled spinning shells beyond the capability developed in Ref. 6. One drawback of approximately factorized methods is the error due to approximate factorization and how it affects the accuracy of the numerical solution. This error has been eliminated in the current effort. Rotating frames of reference have been utilized resulting in simpler codes for simple coning, etc. Simple coning motion has been studied along with the case of a spinning container with a precessing lid. A new code using

the axisymmetric stream function formulation has also been developed.

In the next section, the derivation of implicit upwind schemes without approximate factorization errors is outlined. This methodology has been used in all the codes developed in the current effort. The third section deals with the development of the axisymmetric code based on the stream function formulation. The development of unfactorized implicit schemes for the primitive variable formulation is outlined in the fourth section. The fifth section deals with the governing equations in a rotating frame of reference. The sixth section covers the application of the method to simple coning and the seventh section describes the application to a spinning shell with a precessing lid. Concluding remarks are offered in the next section and a list of references rounds up the report.

2.0 IMPLICIT UPWIND SCHEMES WITHOUT APPROXIMATE FACTORIZATION

The unsteady Navier-Stokes equations can be divided into two sets of terms accounting in turn for the inviscid and viscous behaviour of the fluid. Along with the time-derivative terms, the inviscid terms constitute a hyperbolic system of equations dominated by wave phenomena described by the theory of characteristics. Implicit finite difference schemes for the unsteady Euler and Navier-Stokes equations have for the most part used central space differencing and have relied upon the techniques of approximate factorization or fractional steps to handle multidimensions [Refs. 7, 8]. Even researchers using various upwind schemes (split-flux method or Harten's scheme) [Refs. 9, 10], have resorted only to these conventional approximate ways of splitting the multidimensional operators into one-dimensional operators (which are then solved efficiently using block-tridiagonal elimination). A plus-minus splitting scheme [Ref. 11] has also been attempted for the split-flux scheme leading to a different but once again approximately factored implicit scheme. In contrast to the above, an implicit upwind algorithm is used in the current effort. This method is devoid of errors of any kind of approximate factorization. The advantages of using such an algorithm for the Navier-Stokes equations for incompressible flow will be explained in later sections. A brief outline of the fundamentals underlying the new approach is presented in this section. For a more complete description along with application to the compressible Euler equations, Ref. 12 may be consulted.

2.1 Theoretical Framework

It will be shown here that the new algorithm owes its existence to the beneficial properties of upwind schemes for hyperbolic systems of equations.

It cannot be constructed for central differencing schemes. (These remarks pertain to the hyperbolic part of the Navier-Stokes equations only. The viscous terms are approximated only with central difference formulae.) At the crux of the new method is the observation that upwind schemes result in a diagonally dominant system of finite difference equations governing the change of dependent variables between two marching (time) steps. Such a system can then be solved without factorization by using relaxation iterations between the two time levels. The Gauss-Seidel iteration technique lends itself naturally in this context as will soon be evident. Any other iteration technique will work if it is applicable to a diagonally dominant system of equations (thus pointwise Jacobi iterations will also suffice). The choice of iteration scheme will also be guided by the internal architecture of the computer used (parallel, pipeline, sequential processors, etc.) in as much as certain relaxation algoritms lend themselves better to certain types of processing to achieve faster computational speed.

2.2 Implicit Schemes

We are concerned here with hyperbolic systems of equations (including the unsteady Euler equations) in multi-dimensions. With the addition of centrally differenced viscous terms, the methodology is easily extended to the Navier-Stokes equations. Let the space variables be x and y and the time variable t. Only two spatial dimensions are considered here but all discussion extends in a straight forward manner to three dimensions (or to one, although this is not so interesting). We will be considering a quasi-linear system of equations of the type

$$q_t + A(q)q_x + B(q)q_y = 0 \tag{2.1}$$

where q is an m-vector and A, B are m by m matrices. For simple presentation of the underlying ideas, we will also be considering in more detail the linear wave equation given by

$$u_t + a\,u_x + b\,u_y = 0 \quad . \tag{2.2}$$

There can be several representations and versions of implicit schemes. A simple implicit scheme may be constructed for the equations given above by employing a backward discretization of the time derivative. Thus, the time discretized version of Eq. 2.1 is

$$\frac{q^{n+1}-q^n}{\Delta t} + A^{n+1}(q)q_x^{n+1} + B^{n+1}(q)q_y^{n+1} = 0 \quad . \tag{2.3}$$

The two time levels have been indicated by n and $n+1$. Except for the case of linear systems, these discretizations obviously result in nonlinear equations to solve for q^{n+1} . The spatial discretizations will be discussed soon.

By linearizing in some manner the Eqs. 2.3, linear implicit prediction equations can be constructed for q^{n+1} . For example, we can consider

$$\frac{q^{n+1}-q^n}{\Delta t} + A^n q_x^{n+1} + B^n q_y^{n+1} = 0 \quad . \tag{2.4}$$

2.3 A Simple Upwind Scheme

Let us discuss the two-dimensional wave equation. We shall see how a simple upwind scheme for it results in a diagonally dominant system to solve. Then we shall discuss the appropriateness of the Gauss-Seidel relaxation procedure.

Assuming a and b are positive, backward spatial derivatives can be used for u_x and u_y to result in the upwind scheme

$$\frac{u_{j,k}^{n+1}-u_{j,k}^{n}}{\Delta t} + a\,\frac{u_{j,k}^{n+1}-u_{j-1,k}^{n+1}}{\Delta x} + b\,\frac{u_{j,k}^{n+1}-u_{j,k-1}^{n+1}}{\Delta y} = 0 \quad . \tag{2.5}$$

This scheme is first-order accurate in space and time. By virtue of the linearity of the governing equation (and the discretization), Eq. 2.5 is a linear prediction equation for u^{n+1} .

2.4 Diagonal Dominance

Rewriting Eq. 2.5, we obtain

$$\left(\frac{1}{\Delta t} + \frac{a}{\Delta x} + \frac{b}{\Delta y}\right)u_{j,k}^{n+1} - \frac{a}{\Delta x}\,u_{j-1,k}^{n+1} - \frac{b}{\Delta y}\,u_{j,k-1}^{n+1} = \frac{1}{\Delta t}\,u_{j,k}^{n} \tag{2.6}$$

$$\text{for} \quad j = 1, \ldots, J$$

$$k = 1, \ldots, K$$

It is clear that if this is cast in matrix notation and boundary values of $u_{0,k}^{n+1}$ and $u_{j,0}^{n+1}$ are known, the system is diagonally dominant by rows and columns. In other words, if the elements of the matrix are $d_{j,k}$

$$|d_{j,j}| > \sum_{k \neq j} |d_{j,k}| \quad \text{for} \quad j=1,\ldots,J \tag{2.7a}$$

$$|d_{k,k}| > \sum_{j \neq k} |d_{j,k}| \quad \text{for} \quad k=1,\ldots,K \tag{2.7b}$$

For the benefit of a beginner and for use later on in this report, the full matrix is laid out in Table 1 for J=3 and K=3. It is clear that for large J and K, it would be computationally very expensive to solve the matrix equation by direct elimination. (However, in the special case of upwind differencing we are considering for the case a and b positive, the matrix is lower triangular and can be efficiently solved by direct elimination; the above remark about computational expense is a general remark for non-lower-triangular and non-upper-triangular systems of difference equations). Thus, iteration schemes may be sought for to solve the system of equations. Because of diagonal dominance, either point-Jacobi or Gauss-Seidel iteration procedures may be used to iteratively compute $u_{j,k}^{n+1}$ along with many other methods valid for diagonally dominant systems.

2.5 Jacobi iteration

Let $v_{j,k}^{i}$ denote successive approximations to $u_{j,k}^{n+1}$. The Jacobi (also known as pointwise or simultaneous) iteration scheme for Eq. 2.6 may be written as

$$v_{j,k}^{i} = \frac{1}{p}\left(\frac{u_{j,k}^{n}}{\Delta t} + q\, v_{j-1,k}^{i-1} + r\, v_{j,k-1}^{i-1}\right), \qquad i=1,\ldots \tag{2.8a}$$

with

$$v_{j,k}^{0} = u_{j,k}^{n} \tag{2.8b}$$

TABLE 1

Matrix form of implicit upwind scheme for wave equation

$$
\begin{bmatrix}
p & & & & & & & & \\
-q & p & & & & & & & \\
 & -q & p & & & & & & \\
-r & & & p & & & & & \\
 & -r & & -q & p & & & & \\
 & & -r & & -q & p & & & \\
 & & & -r & & & p & & \\
 & & & & -r & & -q & p & \\
 & & & & & -r & & -q & p
\end{bmatrix}
\begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix}
=
\begin{bmatrix} qu_{0,1} + ru_{1,0} \\ r_{2,0} \\ r_{3,0} \\ qu_{0,2} \\ \\ \\ qu_{0,3} \\ \\ \\ \end{bmatrix}
$$

$p = 1/(\Delta t) + a/(\Delta x) + b/(\Delta y)$, $q = q/(\Delta x)$, $r = b/(\Delta y)$

While convergence is guaranteed, it can be very slow for pointwise iteration.

2.6 Gauss-Seidel Iteration

The Gauss-Seidel procedure may be written as

$$v^i_{j,k} = \frac{1}{p}\left(\frac{u_{j,k}}{\Delta t} + q\, v^i_{j-1,k} + r\, v^i_{j,k-1}\right) \tag{2.9}$$

It is assumed here that the notation of Eq. 2.9 implies that $v^i_{j-1,k}$ and $v^i_{j,k-1}$ will be computed before $v^i_{j,k}$ and thus Eq. 2.9 is an explicit formula for $v^i_{j,k}$. Note also that $v^i_{0,k}$ and $v^i_{j,0}$ are known from boundary conditions. It is thus clear that in one full sweep (or fell swoop) of all the grid points, $u^{n+1}_{j,k}$ is known (i.e. $v^{\text{first iteration}}_{j,k} = u^{n+1}_{j,k}$). To make the procedure even clearer, the calculations may proceed in the following order of grid points (corresponding to the example grid of Table 1):

1,1 to 2,1 to 3,1 to or simultaneously with 1,2 to 1,3

to 2,2 to 2,3 to or simultaneously with 3,2

to 3,3 . (The "simultaneously with" operations can be exploited for parallel processing). It is thus clear that the Gauss-Seidel method is actually more natural for the wave equation than for a linear elliptic equation.

2.7 Notes on Direction of Sweep

However, it is also clear that the direction of sweep is very important to this one step convergence. Other directions of sweep implied for example by

$$v^{i}_{j,k} = \frac{1}{p}\left(\frac{u^{n}_{j,k}}{\Delta t} + q\, v^{i}_{j-1,k} + r\, v^{i-1}_{j,k-1}\right) \tag{2.10}$$

(backward k sweep , forward j sweep)

or

$$v^{i}_{j,k} = \frac{1}{p}\left(\frac{u^{n}_{j,k}}{\Delta t} + q\, v^{i-1}_{j-1,k} + r\, v^{i}_{j,k-1}\right) \tag{2.11}$$

(backward j sweep , forward k sweep)

will lead to slow convergence. The preferred Gauss-Seidel iteration, Eq. 2.9 is actually identical to a direct elimination method of solution of the matrix system of equations recognizing the fact that the matrix is of lower diagonal form and very sparse.

2.8 Notes on Signal Propagation Directions

The constants a and b of the scalar wave equation (Eq. 2.2) signify signal propagation directions by their signs and signal propagation speeds by their magnitudes. When a and b are both positive, signals travel to the right (along the positive coordinate direction) in both x and y. If a or b had been negative, it would have been appropriate to use forward differences for the space derivative(s) which the negative constant(s) multiplied. In that case, the directions of sweep in the Gauss-Seidel procedure must be changed. The boundary conditions must also be specified along different boundaries than those natural for the case when a and b are positive.

2.9 The Variable Coefficient Linear Wave Equation

Thus far, we have considered a and b to be of the same sign throughout the field. In such situations, one of the four different combinations

of forward or backward sweeps in j or k will result in one step convergence of the Gauss-Seidel procedure. Let us consider a and b to be not constant, but

$$a = a(x,y,t) \quad , \quad b = b(x,y,t) \tag{2.12}$$

A generalized upwind scheme for Eq. 2.2 with variable coefficients may be written in such a manner that the signs of a and b are automatically monitored and corresponding one-sided derivatives assigned. One representation can be

$$\begin{aligned} \frac{u^{n+1}-u^n}{\Delta t} &+ \left(\frac{a+|a|}{2}\right)\left(\frac{u^{n+1}_{j,k}-u^{n+1}_{j-1,k}}{\Delta x}\right) - \left(\frac{a-|a|}{2}\right)\left(\frac{u^{n+1}_{j,k}-u^{n+1}_{j+1,k}}{\Delta x}\right) \\ &+ \left(\frac{b+|b|}{2}\right)\left(\frac{u^{n+1}_{j,k}-u^{n+1}_{j,k-1}}{\Delta y}\right) - \left(\frac{b-|b|}{2}\right)\frac{u^{n+1}_{j,k}-u^{n+1}_{j,k+1}}{\Delta y} = 0 \ . \end{aligned} \tag{2.13}$$

2.10 Cycling Forward and Backward Sweeps

The following sequence of two Gauss-Seidel (forward and backward) sweeps is expected to be efficient for this general case.

$$\begin{aligned} \left(\frac{1}{\Delta t} + \frac{|a|}{\Delta x} + \frac{|b|}{\Delta y}\right) v^{i}_{j,k} &= \frac{u^{n}_{j,k}}{\Delta t} \\ + \left(\frac{a+|a|}{2\,\Delta x}\right) v^{i1}_{j-1,k} - \left(\frac{a-|a|}{2\,\Delta x}\right) v^{i2}_{j+1,k} &+ \left(\frac{b+|b|}{2\,\Delta y}\right) v^{i1}_{j,k-1} - \left(\frac{b-|b|}{2\,\Delta y}\right) v^{i2}_{j,k+1} \end{aligned} \tag{2.14}$$

$i1 = i$ and $i2 = i-1$ for odd i, $\quad i1 = i-1$ and $i2 = i$ for even i.

When a and b are of the same sign (+ or -) throughout the field, one

cycle (a forward and a backward sweep) is enough for convergence. In the general case, when a and/or b change sign in the field, one cycle will not be enough, but this procedure leads to a quite rapid convergence.

2.11 Second-Order Accuracy

We have so far discussed only algorithms that are first-order accurate in time and space. We now cover second-order accuracy in each.

Second-order time accuracy is very easily achieved by simply replacing the two-point backward temporal differencing by a three-point backward formula in time which for equal time steps can be written as

$$q_t \simeq \frac{1.5\, q^{n+1} - 2\, q^{n} + 0.5\, q^{n-1}}{\Delta t} \tag{2.15}$$

A modified formula may easily be obtained for varying time steps.

Second-order spatial accuracy is also achieved as easily as time accuracy. In practice, all one must do is to use second-order upwind discretizations on the right hand side of the equation that describes the relaxation algorithm before the step of writing the Gauss-Seidel sweeps. For example, the x-derivative term may be discretized as

$$q_x \simeq \frac{1.5\, q_j - 2\, q_{j-1} + 0.5\, q_{j-2}}{\Delta x} \tag{2.16}$$

for backward difference approximations needed with positive propagation. These second-order discretizations may have to be suitably modified or limited to result in a TVD scheme [Ref. 12]. Numerical experiments show that even without limiting, rapid convergence of the relaxation iterations is possible for the problems under consideration.

3.0 AXISYMMETRIC STREAM FUNCTION FORMULATION

The Navier-Stokes equations fit very well into the framework developed in the previous section. As a first application, we consider here the incompressible flow of a fluid contained in a spinning shell. For simplicity of treatment here, let the container be a body of revolution spinning about its axis. If z were the axial direction and r and θ the polar coordinates in the plane perpendicular to the axis, the governing equations may be expressed in terms of an axisymmetric stream function ψ, a vorticity ζ and a circulation γ (see Ref. 3 from which the governing equations are adapted here) in the r, θ, z coordinates as

$$\nabla^2\psi - \frac{\psi_r}{r} = r\,\zeta \tag{3.1a}$$

$$\zeta_t + u\,\zeta_r + w\,\zeta_z - u\,\frac{\zeta}{r} - 2\,\frac{\gamma}{r^3}\,\gamma_z = \frac{1}{Re}\left[\nabla^2\zeta + \frac{\zeta_r}{r} - \frac{\zeta}{r^2}\right] \tag{3.1b}$$

$$\gamma_t + u\,\gamma_r + w\,\gamma_z = \frac{1}{Re}\left[\nabla^2\gamma - \frac{\gamma_r}{r}\right] \quad . \tag{3.1c}$$

Here,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} \quad , \quad Re = \frac{\Omega\,a^2}{\nu} = \text{Reynolds number} \quad ,$$

$$\gamma = r\,v \;,\quad u = (\psi_z)/r \;,\quad w = -(\psi_r)/r \tag{3.1d}$$

where Ω is the spin rate and v is the azimuthal velocity.

All variables above are non-dimensional. The non-dimensionalization used is given below (once again taken from Ref. 3).

$$r = R/a \;,\quad u = U/(\Omega a) \;,\quad v = V/(\Omega a) \;,\quad w = W/(\Omega a) \;,\quad z = Z/a$$
$$\psi = \Psi/(\Omega a^3) \;,\quad \gamma = \Gamma/(\Omega a^2) \;,\quad \zeta = T/\Omega \;,\quad t = \Omega T \tag{3.2}$$

The dimensional quantities have been shown above as capital letters. The normalizing length scale a is usually chosen to be the cylinder radius.

The first equation (Eq. 3.1a) is an elliptic Poisson equation for the stream function. The other two (Eq. 3.1b and Eq. 3.1c) are parabolic vorticity transport equations. Without the viscous terms, these latter equations are hyperbolic. Note that these two equations for ζ and γ involve ther terms u and w which are derivatives of the stream-function, as only multipliers of derivatives of ζ and γ. Also, the first equation involves no derivatives of ζ or γ (in fact no γ even occurs there). Under these circumstances, it is conventional procedure to first update vorticity and circulation from the n-th time level to level $n+1$. Then, the updated vorticity values are used on the right hand side of the equation for stream function to update the value of ψ.

3.1 Numerical Method

In our procedure, along with this implementation, we use Gauss-Seidel iterations for all three equations after the transport terms of the equations for vorticity and circulation are approximated using one-sided spatial derivatives depending on the sign of the transport velocity multiplying them. The elliptic part (the viscous terms) of Eqs. 3.1b and 3.1c fit snugly into the structure of Gauss-Seidel iterations and Eq. 3.1a is nothing but elliptic.

We now write down the discretizations in detail for Eqs. 3.1. Let the coordinates r and z be indexed with the subscripts j and l respectively. Thus, j and l also denote computational coordinates with grid points located at integer values of j and l. The metrics

are then given by $\partial r/\partial j$, $\partial z/\partial l$, $\partial^2 r/\partial j^2$, $\partial^2 z/\partial l^2$, etc. The following transformations are then valid for any quantity f :

$$f_r = \frac{\partial f}{\partial j} \Big/ \frac{\partial r}{\partial j} \tag{3.3a}$$

$$\frac{\partial^2 f}{\partial r^2} = \left(\frac{\partial^2 f}{\partial j^2} - \left(\frac{\partial^2 r}{\partial j^2}\Big/\frac{\partial r}{\partial j}\right)\frac{\partial f}{\partial j}\right)\Big/\left(\frac{\partial r}{\partial j}\right)^2 \tag{3.3b}$$

$$f_z = \frac{\partial f}{\partial j} \Big/ \frac{\partial r}{\partial j} \tag{3.4a}$$

$$\frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial^2 f}{\partial l^2} \quad \left(\frac{\partial^2 z}{\partial l^2}\Big/\frac{\partial z}{\partial l}\right)\frac{\partial f}{\partial l}\right)\Big/\left(\frac{\partial z}{\partial l}\right)^2 \tag{3.4b}$$

Central difference formulae for any quantity f are given by

$$\frac{\partial f}{\partial j} = (f_{j+1} - f_{j-1})/2 \tag{3.5a}$$

$$(\partial^2 f/\partial j^2) = f_{j+1} - 2\, f_j + f_{j-1} \tag{3.5b}$$

along the radial computational coordinate. Similar expressions are valid along the axial coordinate. Forward and backward difference formulae for first derivatives are given by

$$\frac{\partial f}{\partial j} \simeq \mp(1.5\, f_j - 2\, f_{j\pm1} + 0.5\, f_{j\pm2}) \tag{3.6}$$

Velocities u and w are computed from central difference formulae. Central differences are used for all derivative terms in Eq. 3.la and all derivative terms divided by the Reynolds number (i.e. all viscous derivative

terms). All convective terms in Eqs. 3.1b and 3.1c (those first derivative terms multiplied by convection velocities u and w) are upwind differenced. For example, if u were positive in a term such as $u f_r$, backward differencing is used for $\partial f/\partial j$, and when u is negative, forward differencing is used. Any other derivative terms that have not been mentioned thus far are centrally differenced. For further details on the philosophy of upwind differencing (in the context of the non-conservation law representation of the equations used here), the reader may look up Refs. 13 or 14.

Conventional Gauss-Seidel relaxation is used for the Poisson equation Eq. 3.1a. Each of the other two equations is treated as a scalar equation and the Gauss-Seidel procedure similar to Eq. 2.14 is used for these but with second-order accurate spatial upwind differencing.

The boundary conditions used are outlined below (paraphrased from Ref. 3).

Along the cylinder axis:

$$\psi(t,0,z) = \gamma(t,0,z) = \zeta(t,0,z) \equiv 0 \quad . \tag{3.7}$$

Along the cylinder wall:

$$\psi(t,1,z) = 0,\ \gamma(t,1,z) = 1,\ \zeta(t,1.z) = \psi_{rr}(t,1,z) \quad . \tag{3.8}$$

Along the cylinder end wall:

$$\psi(t,r,0) = 0,\ \gamma(t,r,0) = r^2,\ \zeta(t,r,0) = (\psi_{zz}(t,r,0))/r \quad . \tag{3.9}$$

Along the cylinder symmetry plane:

$$\psi(t,r,\alpha) = \zeta(t,r,\alpha) = 0,\quad \gamma_z(t,r,\alpha) = 0 \tag{3.10}$$

with α being the ratio of cylinder half height to its radius.

Initial conditions are given by

$$\psi(0,r,z) = 0, \quad \gamma(0,r,z) = (\Omega_i/\Omega)r^2 \tag{3.11}$$

where Ω is the final cylinder rotation rate (also used in the non-dimensionalization process) and Ω_i is the initial rotation rate (= 0 for spin-up from rest).

4.0 PRIMITIVE VARIABLE FORMULATION

The primitive variable formulation is considered here in r, θ, z (cylindrical) coordinates. The equations and method considered in this section will be used in Sections 6 and 7.

4.1 The Governing Equations

The governing equations are written in the inertial coordinate system as follows:

Momentum equations

$$\partial_t u + u\partial_r u + (v/r)\partial_\theta u + w\partial_z u - v^2/r + \partial_r p = \nu(\nabla^2 u - (2/r^2)\partial_\theta v - u/r^2)$$

$$\partial_t v + u\partial_r v + (v/r)\partial_\theta v + w\partial_z v + uv/r + (1/r)\partial_\theta p = \nu(\nabla^2 v + (2/r^2)\partial_\theta u - v/r^2)$$

$$\partial_t w + u\partial_r w + (v/r)\partial_\theta w + w\partial_z w + \partial_z p = \nu(\nabla^2 w \qquad) \tag{4.1a}$$

Continuity equation

$$\partial_r u + (1/r)\partial_\theta v + \partial_z w + u/r = 0 \tag{4.1b}$$

In the above,

$$\nabla^2 = \partial_r^2 + (1\ r)\partial_r + (1/r^2)\partial_\theta^2 + \partial_z^2 \tag{4.1c}$$

and u = radial velocity, r = radial coordinate,

v = azimuthal velocity, θ = azimuthal coordinate,

w = axial velocity, z = axial coordinate.

The same non-dimensionalizations employed in the previous section has been used in the above equations. The quantity ν is the reciprocal of the

Reynolds number.

The continuity equation has no time derivative. To facilitate construction of an implicit numerical method, the continuity equation is modified (see Refs. 6, 15, 16 for a similar treatment applied to approximately factored schemes) to be

$$\partial_t p + \beta(\partial_r u + (1/r)\partial_\theta v + \partial_z w + u/r) = \partial_t p^* \quad . \tag{4.2}$$

The quantities β and p^* and the numerical algorithm will be chosen to let

$$\partial_r u + (1/r)\partial_\theta v + \partial_z w + u/r \rightarrow 0 \quad . \tag{4.3}$$

Equations 4.1a and 4.2 may be combined to yield

$$\vec{Q}_t + \underline{A}\,\vec{Q}_r + (1/r)\,\underline{B}\,\vec{Q}_\theta + \underline{C}\,\vec{Q}_z + \vec{H}_i = \vec{H}_v + (0,0,0,\ {}_t p^*)^{\text{transpose}} \tag{4.4}$$

where

$\vec{Q} = (u, v, w, p)^{\text{transpose}}$, $\vec{H}_v$ = viscous terms, and $\vec{H}_i$ = inviscid terms that do not involve derivatives, and

$$\underline{A} = \begin{bmatrix} u & 0 & 0 & 1 \\ 0 & u & 0 & 0 \\ 0 & 0 & u & 0 \\ \beta & 0 & 0 & 0 \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} v & 0 & 0 & 0 \\ 0 & v & 0 & 1 \\ 0 & 0 & v & 0 \\ 0 & \beta & 0 & 0 \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & w & 1 \\ 0 & 0 & \beta & 0 \end{bmatrix} \quad . \tag{4.5}$$

4.2 The SCM Method

The Split Coefficient Matrix (SCM) method of discretization is applied to the hyperbolic part of the modified Navier-Stokes equations. A detailed description of the SCM method can be found in Refs. 13 and 14. The method applied to the equations under consideration here is sufficiently described now for completeness.

Each of the coefficient matrices A, B, C can be written as (in the following, we do not type an underscore for the matrices and we delete the arrow on top for the vectors)

$$A = R_A \Lambda_A L_A \quad , \quad B = R_B \Lambda_B L_B \quad , \quad C = R_C \Lambda_C L_C \quad . \tag{4.6}$$

Here, the subscripts A, B, and C for matrices R, Λ, and L are used to denote what coefficient matrix the latter correspond to. The matrix of right (or column) eigenvectors of a coefficient matrix has been denoted by R. That is, each column of R is a right eigenvector of the corresponding coefficient matrix. Similarly, each row of L is a left (or row) eigenvector of the coefficient matrix. The diagonal matrix of eigenvalues of the coefficient matrix has been denoted by Λ. Thus, for example,

$$A\, R_A = R_A\, \Lambda_A \quad , \quad L_A\, A = \Lambda_A\, L_A \quad . \tag{4.7}$$

In Eqs. 4.6, it has been tacitly assumed that the left and right eigenvectors have been suitably normalized such that

$$R\, L = L\, R = I \tag{4.8}$$

where I is the identity matrix. When the coefficient matrices belong

to a hyperbolic system of equations, the left and right eigenvector matrices are of full rank (i.e. within each of these matrices, each eigenvector is linearly independent of the others).

For the SCM method, the eigenvalues are split into those that are positive and those that are negative (and those that are zero can be in either group).

$$\Lambda^{+} = (\Lambda + |\Lambda|)/2 \quad , \quad \Lambda^{-} = (\Lambda - |\Lambda|)/2 \tag{4.9}$$

Now, the split coefficient matrices may be defined to be

$$A_{\pm} \ , \ B_{\pm} \ , \ C_{\pm} \ = \ R_{A,B,C} \, \Lambda^{\pm}_{A,B,C} \ L_{A,B,C} \tag{4.10}$$

Rewriting each coefficient matrix as the sum of its positive and negative parts, it is appropriate to use backward difference approximations for those first derivative terms that are multiplied by the positive part of the coefficient matrix and vice-versa. Thus,

$$A \, Q_r \ \simeq \ A_{+} \, Q_{rb} \ + \ A_{-} \, Q_{rf} \tag{4.11}$$

with subscripts b and f denoting backward and forward differences. Equation 4.11 may be rewritten as

$$A \, Q_r \ \simeq \ \frac{1}{2}(A+|A|) \, Q_{rb} \ + \ \frac{1}{2}(A-|A|) \, Q_{rf} \tag{4.12}$$

with $|A| \ = \ R_A \, |\Lambda_A| \, L_A$. Equation 4.12 may be rewritten in turn as

$$A \, Q_r \ \simeq \ A \, (Q_{rb}+Q_{rf})/2 \ + \ |A| \, (Q_{rb}-Q_{rf})/2 \tag{4.13}$$

Assuming j, k, and l to be the indices corresponding to r, θ, and z (as was done in the previous section), and using second-order accurate one-sided formulae, we obtain

$$
\begin{aligned}
Q_{rb} &\simeq (1.5\, Q_j - 2\, Q_{j-1} + 0.5\, Q_{j-2})/(\partial r/\partial j)_j \\
Q_{rf} &\simeq -(1.5\, Q_j - 2\, Q_{j+1} + 0.5\, Q_{j+2})/(\partial r/\partial j)_j
\end{aligned}
\tag{4.15}
$$

Substituting this into Equation 4.14, we obtain

$$
\begin{aligned}
A\, Q_r \simeq & \frac{1}{(\partial r/\partial j)} A_j (Q_{j+1} - Q_{j-1} - (Q_{j+2} - Q_{j-2})/4 \\
& + \frac{1}{(\partial r/\partial j)} |A_j| (Q_{j-2} - 4Q_{j-1} + 6Q_j - 4Q_{j+1} + Q_{j+2})/4
\end{aligned}
\tag{4.16}
$$

The first term on the right hand side is a central difference approximation (albeit an unusual one) to AQ_r. The second term is a fourth difference diffusion term that arises naturally in the SCM method and helps squash high frequency oscillations.

The various eigenvalues and matrices are now defined. For coefficient Matrix A:

$$
\lambda_{1A} = u\ , \quad \lambda_{2A} = u\ , \quad \lambda_{3A} = (u + (u^2 + 4\beta)^{1/2})/2\ , \quad \lambda_{4A} = (u - (u^2 + 4\beta)^{1/2})/2 \tag{4.17a}
$$

$$
L_A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \lambda_{3A} & 0 & 0 & 1 \\ \lambda_{4A} & 0 & 0 & 1 \end{bmatrix}, \quad R_A = \begin{bmatrix} 0 & 0 & \dfrac{1}{(u^2+4\beta)^{1/2}} & \dfrac{-1}{(u^2+4\beta)^{1/2}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-\lambda_{4A}}{(u^2+4\beta)^{1/2}} & \dfrac{\lambda_{3A}}{(u^2+4\beta)^{1/2}} \end{bmatrix} \tag{4.17b}
$$

The eigenvalues and eigenvector matrices for the coefficient matrices B and C are similarly derived.

4.3 Relaxation of the Discretized Equations

The SCM method given above is used to discretize the inviscid terms. Central difference approximations are used for the viscous terms. Three-point or two-point backward discretization of the time-derivative terms leads to second-order or first-order time accuracy as desired. The time-discretized implicit version of Eq. 4.4 can then be written as

$$(1+\phi/2)\left(\frac{Q^{n+1}-Q^{n}}{\Delta t}\right) - (\phi/2)\left(\frac{Q^{n}-Q^{n-1}}{\Delta t}\right) + (AQ_r+(1/r)BQ_\theta+CQ_z+H_i)^{n+1}$$
$$= H_v^{n+1} + (0,0,0,\partial_t p^*)^{\text{transpose}} \tag{4.20}$$

with $\phi = 0$ for first-order time accuracy and $\phi = 1$ for second-order time accuracy.

Writing q^i to be the i-th iterate approximation to Q^{n+1}, we can write the Gauss-Seidel method for Eq. 4.20, coupled with appropriate SCM (upwind) and central difference approximations, to be

$$\left[\frac{1}{\Delta t}\left(1+\frac{\phi}{2}\right)I+\frac{1}{\partial r/\partial j}|A|+\frac{1}{r}\frac{1}{\partial\theta/\partial k}|B|+\frac{1}{\partial z/\partial l}|C|-\left(\frac{\partial H_v}{\partial q_{j,k,l}}\right)+\left(\frac{\partial H_i}{\partial q_{j,k,l}}\right)\right](q^i-q^{i-1})$$

$$= (1+\phi/2)\frac{1}{\Delta t}(q^{i-1}-Q^n) - \frac{\phi}{2}\left(\frac{Q^n-Q^{n-1}}{\Delta t}\right) + (AQ_r + \frac{1}{r}BQ_\theta + CQ_z)^{i,i-1}_{SCM} \tag{4.21}$$

$$+ (H_i)^{i,i-1} - H_v^{i,i-1} + (0,0,0,p_t^*)^{\text{transpose}} = 0.$$

To implement the Gauss-Seidel sweep, all terms with the superscript

"i,i-1" are evaluated using all available i-th level values along with i-1 level values for those quantities for which i-th level values have not been computed in the sweep. Each cycle of sweeps comprises a forward sweep (increasing values of indices j, k and l) and a backward sweep (decreasing values of indices, starting from the maximum values of these indices).

The pseudo pressure transient term p_t^* in Eq. 4.21 is taken to be

$$p_t^* = (1+\frac{\phi}{2})(p^{i-1}-p^n)/(\Delta t) - (\phi/2)(p^n-p^{n-1})/(\Delta t) \tag{4.22}$$

Thus, when the relaxation iterations converge between two time steps, the two transient terms for pressure, namely (p^i-p^n) and $(p^{i-1}-p^n)$ will cancel each other out, leading to exact satisfaction of the continuity equation. However, perfect convergence of the relaxation iterations is not sought for every time step; approximate convergence of the subiterations will almost always suffice.

4.4 Advantages of the Unfactored Scheme

Approximately factorized schemes incur an error due to the factorization. Also, for the imcompressible Navier-Stokes equations, when the continuity equation in modified as in Eq. 4.2, for large values of β, the momentum equations are contaminated by this factorization error. Unfactored schemes avoid this error. Thus, larger values of β can be chosen. When $\beta \sim 1/(\Delta t)$ the exact continuity equation (Eq. 4.1b) is satisfied to first-order accuracy. And larger values of β lead to a greater degree of satisfaction of the continuity equation. The other advantage of unfactored implicit schemes is that, as a steady state is approached, larger and larger values of the time step may be taken.

5.0 ROTATING FRAMES OF REFERENCE

Thus far, we have been considering an inertial reference frame and the dependent variables were the inertial velocity components. It is often very convenient to work with a frame of reference attached to the spinning and precessing shell and also compute the velocity components in the rotating frame of reference. The governing equations are given below for a coordinate frame attached to a rotating shell. The treatment follows that of Greenspan [Ref. 17].

Let L, Ω^{-1}, U characterize reference length, time and velocity, respectively, of a particular motion. The position vector $\vec{r}$, time t, velocity vector $\vec{q}$, rotation vector $\vec{\Omega}$, and pressure can then be scaled respectively by L, Ω^{-1}, U, Ω and $\rho\Omega UL$. The vectors are in a Cartesian frame of reference fixed to the spinning and precessing shell. Then for

$$\vec{\Omega}(t) = \Omega(\hat{k} + \varepsilon\vec{\delta}(t)) \tag{5.1}$$

the governing equations may be written in non-dimensional form as

Momentum equations:

$$\partial_t\vec{q} + \varepsilon\vec{q}.\vec{\nabla}\vec{q} + 2(\hat{k} + \varepsilon\vec{\delta}(t)) \times \vec{q} = -\vec{\nabla}p + \vec{r} \times (d/dt)\vec{\delta} + E\nabla^2 q \tag{5.2a}$$

Continuity equation:

$$\vec{\nabla}.\vec{q} = 0 \tag{5.2b}$$

where the Ekman number is given by

$$E = \nu/(\Omega L^2) \tag{5.3a}$$

and the Rossby number is

$$\varepsilon = U/(\Omega L) \tag{5.3b}$$

The reduced pressure p appearing above includes along with the static pressure, any body forces (assumed to be conservative), and the centrifugal acceleration.

$$p = \text{static pressure} + \text{conservative body force} - (1/2)\rho(\vec{\Omega} \times \vec{r}).(\vec{\Omega} \times \vec{r}) \tag{5.4}$$

The z axis and consequently the $\hat{k}$ unit vector is assumed to be the axis of spin of the container. The precession $\vec{\delta}(t)$ is superposed on this basic spin.

The governing equations 5.1 and 5.2 have been written in a Cartesian coordinate system attached to the spinning shell. The z coordinate lies along the spin axis. The following remarks are useful to transform these equations into a set of equations written for a cylindrical coordinate system attached to the rotating shell and for cylindrical dependent variables in that system. The z axis is identical in both Cartesian and cylindrical systems. The other axes are linked by

$$x = r\cos\theta \quad , \quad y = r\sin\theta \tag{5.5}$$

The velocity components are linked by

$$\begin{aligned} u_{cartesian} &= u\cos\theta - v\sin\theta \\ v_{cartesian} &= u\sin\theta + v\cos\theta \\ w_{cartesian} &= w \end{aligned} \tag{5.6}$$

where the cylindrical velocity components have been denoted as usual by u, v and w.

The metrics transform according to

$$x_r = \cos\theta \quad , \quad x_\theta = -r\sin\theta$$
$$y_r = \sin\theta \quad , \quad y_\theta = r\cos\theta \tag{5.7}$$

and

$$r_x = \cos\theta \quad , \quad \theta_x = -(\sin\theta)/r$$
$$r_y = \sin\theta \quad , \quad \theta_y = (\cos\theta)/r \tag{5.8}$$

There are three scalar equations identifiable in the vector equation 5.2a, one each for each Cartesian velocity component. To go from the equations in 5.2a and 5.2b to a cylindrical variable formulation similar to Eq. 4.1a and Eq. 4.1b, we can take the following steps. First, in each of the equations in 5.2, transform the independent variables using

$$\partial/\partial x = \cos\theta\, \partial/\partial r - (1/r)\sin\theta\, \partial/\partial\theta$$
$$\partial/\partial y = \sin\theta\, \partial/\partial r + (1/r)\cos\theta\, \partial/\partial\theta \tag{5.9}$$
$$\partial^2/\partial x^2 + \partial^2/\partial y^2 = \partial^2/\partial r^2 + (1/r)\partial/\partial r + (1/r^2)\partial^2/\partial\theta^2$$

Now, we have four equations for the Cartesian velocity components but in the cylindrical independent variables. Into each of these equations, substitute Eqs. 5.6 to obtain four equations involving the cylindrical velocity components. The continuity equation and the equation for Cartesian w component of velocity will now be in the form of Eqs. 4.1. However, the other two equations will each involve linear combinations of u and v. To obtain mutually exclusive prediction equations for

u and v, recombine these two equations as follows: to obtain an equation for u, multiply the equation for $u_{cartesian}$ by $\cos\theta$, the equation for $v_{cartesian}$ by $\sin\theta$, and add; to obtain an equation for v, multiply the equation for $u_{cartesian}$ by $(-\sin\theta)$, the equation for $v_{cartesian}$ by $(\cos\theta)$ and add.

This procedure has been used for the sets of governing equations given in the next two sections. The difference between those sets and Eqs. 4.1 can be summarized as follows: all convective terms are now scaled by the Rossby number; new source terms corresponding to the Coriolis acceleration $2(\hat{k} + \varepsilon\vec{\delta}(t)) \times \vec{q}$ and corresponding to the acceleration of the rotating frame given by $\vec{r} \times d\vec{\delta}/dt$, are added. Thus, it is an easy matter to adapt the methodology developed for the primitive variable formulation in the last section for the inertial coordinate system to rotating frames of reference. The source terms have no spatial derivatives in them and are added in a very straight-forward manner. For computing efficiency, they must however be treated implicitly in the relaxation iterations. That is, the Jacobian matrix of these source terms must be included along with the other terms in the diagonal matrix of the left hand side of the operator in the implicit method.

6.0 CONING - SPIN AND PRECESSION

We now apply the governing equations developed in the last section for rotating frames of reference to the particular case of simple coning which is a combination of spin and precession. Let the cylindrical shell be spinning with spin rate Ω and precessing with precession rate ω. We define

$$T_{NU} = \omega / \Omega \tag{6.1}$$

from which we can define

$$\sigma = 1/ (1 + T_{NU}) \tag{6.2}$$

Let the axis be precessing with a yaw angle α . In a slight departure from the normalization procedure of the previous section, we define first the Rossby number to be

$$\varepsilon = \alpha \sigma \tag{6.3}$$

from which we can derive the normalizing velocity U to then be

$$U = \varepsilon \Omega L \tag{6.4}$$

where L is the usual length scale (typically taken to be the radius of the cylinder). All we have done here is to pick the Rossby number first and obtain the non-dimensionalizing velocity from it, rather than the other way around.

The perturbation $\vec{\delta}(t)$ is then given by [Ref. 17]

$$\vec{\delta} = -T_{NU}\,(\cos(\sigma t)\,\hat{i} + \sin(\sigma t)\,\hat{j}) \tag{6.5}$$

The source terms in Eq. 5.2a are thus given by

$$2(\hat{k} + \varepsilon\vec{\delta}(t)) \times \vec{q} =$$

$$\begin{aligned} & -(v_{cartesian} + \varepsilon T_{NU} \sin\sigma t)\,\hat{i} \\ 2 \quad & +(u_{cartesian} + \varepsilon T_{NU} \cos\sigma t)\,\hat{j} \\ & +\varepsilon T_{NU}(u_{cartesian} \sin\sigma t + v_{cartesian} \cos\sigma t)\,\hat{k} \end{aligned} \tag{6.6}$$

and

$$\vec{r} \times d\vec{\delta}/dt = \sigma T_{NU} \begin{matrix} z \cos \sigma t\,\hat{i} \\ +z \sin \sigma t\,\hat{j} \\ -(x \cos \sigma t + y \sin\sigma t)\hat{k} \end{matrix} \tag{6.7}$$

In the cylindrical rotating frame of reference, the source terms that must be added to the left hand side can be computed from the above.

Source term for radial momentum equation

$$= -2v - 2\varepsilon T_{NU}\sin(\sigma t-\theta) - z\sigma T_{NU}\cos(\sigma t-\theta) \tag{6.8}$$

Source term for azimuthal momentum equation

$$= +2u + 2\varepsilon T_{NU}\cos(\sigma t-\theta) - z\sigma T_{NU}\sin(\sigma t-\theta) \tag{6.9}$$

Source term for axial momentum equation

$$= 2\varepsilon T_{NU}[u \sin(\sigma t+\theta) + v \cos(\sigma t+\theta)] + \sigma T_{NU}[x \cos\sigma t + y \sin\sigma t] \tag{6.10}$$

The boundary conditions along all walls are given to be

$$u = v = w = 0 \tag{6.11}$$

in the rotating frame.

7.0 SPINNING SHELL WITH PRECESSING LID

In this section, we consider a spinning cylindrical shell with only the lid precessing at one end of the cylinder. Once again, α, Ω, ω are defined to be the precession angle, spin rate and precession rate, respectively (as before). However, for this case, we define

$$\sigma = 1 - T_{NU} \tag{7.1}$$

and we define the Rossby number to be

$$\varepsilon = \sin(\alpha) \quad . \tag{7.2}$$

The rotating frame of reference does not precess. Thus,

$$\vec{\delta}(t) = 0 \tag{7.3}$$

and the source term is simply given by $2\vec{k} \times \vec{q}$. This contributes -2v to the radial momentum equation and 2u to the azimuthal momentum equation.

The boundary conditions along the walls are given as follows:
Along the walls,

$$u = v = 0 \quad . \tag{7.4}$$

Along the cylinder wall and lower end wall,

$$w = 0 \quad . \tag{7.5}$$

Along the upper end wall,

$$w = r\,\sigma\,\sin(\sigma t+\theta_k) \tag{7.6}$$

where θ_k is a given phase angle.

8.0 CONCLUDING REMARKS

A new formulation is presented for constructing finite difference algorithms for the incompressible Navier-Stokes equations. The new method is based on constructing relaxation procedures for unfactored implicit upwind schemes for the hyperbolic part of the system of governing equations taken together with central differencing for the viscous terms. The new approach has been applied to the axisymmetric stream function formulation as well as for the primitive variable equations in three spatial dimensions. The formulation has been extended for rotating frames of reference. The two cases considered here are 1) coning motion, and 2) spinning shell with precessing lid. The computer programs developed in this study have been installed at the Ballistic Research Laboratory, Aberdeen Proving Ground, and are being used to obtain results by BRL personnel for many cases of interest. Those results will be incorporated into a BRL Technical Report at a future date.

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