AN INEQUALITY CONCERNING THE DEVIATION BETWEEN
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AN INEQUALITY CONCERNING THE DEVIATION BETWEEN THEORETICAL AND EMPIRICAL DISTRIBUTIONS*

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August 1985
Technical Report No. 85-30

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*Part of the work of this author is sponsored by the Air Force Office of Scientific Research (AFSC) under contract F49620-85-C-0008. The United States Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation herein.

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1. INTRODUCTION

The result. Let \( x_1, \ldots, x_r \) be \( r \) points in \( \mathbb{R}^d \), and \( \mathcal{A} \) be a class of Borel sets in \( \mathbb{R}^d \). Denote by \( \Delta(x_1, \ldots, x_r) \) the number of distinct sets in \( \{ \{ x_1, \ldots, x_r \} \cap \mathcal{A} \mid \mathcal{A} \in \mathcal{A} \} \). Define

\[
\mathcal{A}^r(x) = \max_{x_1, \ldots, x_r \in \mathbb{R}^d} \Delta^r(x_1, \ldots, x_r).
\]

Vapnik and Chervonenkis (1971) showed that either \( \mathcal{A}^r(x) = 2^r \) for any positive integer \( r \) or \( \mathcal{A}^r(x) < r^{s+1} \), where \( s \) is the smallest \( k \) such that \( \mathcal{A}^k(x) \neq 2^k \). A class of sets \( \mathcal{A} \) for which the latter case holds will be called a V-C class with index \( s \).

Suppose that \( \mu \) is a probability measure on \( \mathbb{R}^d \). Let \( X_1, X_2, \ldots \) be a sequence of i.i.d. random vectors with common distribution \( \mu \), and \( \mu_n \) be the empirical distribution of \( X_1, \ldots, X_n \). Denote a "distance" between \( \mu_n \) and \( \mu \) by

\[
D_n(\mu, \mu) = \sup_{\mathcal{A} \in \mathcal{A}} |\mu_n(\mathcal{A}) - \mu(\mathcal{A})|.
\]

Throughout this paper we assume that \( D_n(\mu, \mu), \sup_{\mathcal{A} \in \mathcal{A}} |\mu_n(\mathcal{A}) - \mu_2(\mathcal{A})| \) and \( \sup_{\mathcal{A} \in \mathcal{A}} \mu_n(\mathcal{A}) \) are all random variables. We shall prove the following theorem.

**Theorem 1.** Let \( \mathcal{A} \) be a V-C class with index \( s \) such that

\[
\sup_{\mathcal{A} \in \mathcal{A}} \mu(\mathcal{A}) \leq \delta \leq 1/8. \tag{1}
\]

Then for any \( \epsilon > 0 \) we have

\[
P(D_n(\mu, \mu) > \epsilon) \leq 5(2n)^s \exp\left(-\frac{n\epsilon^2}{4(9\delta^2 + 4\epsilon)}\right) \tag{2}
\]

\[
+ 7(2n)^s \exp\left(-\frac{n\delta^2}{6s}\right) \tag{2}
\]

\[
+ 2^{2+s}n^{1+2s} \exp\left(-\frac{n\delta}{8}\right),
\]
provided $n \geq \max \left( \frac{12\sigma}{\varepsilon^2}, 68(1+s)(\log 2)/\delta \right)$.

The proof of (2) is based on an important inequality proved by Devroye and Wagner (1980).
2. HISTORICAL NOTES

A few remarks concerning this inequality are in order. In 1971, Vapnik and Chervonenkis proved that, for any \( \varepsilon > 0 \)

\[
P(D_n(A, \mu) > \varepsilon) \leq 4\exp(-ne^2/8) E^{A}(x_1, \ldots, x_{2n}).
\]

(3)

This inequality is quite general since no restrictions such as (1) are imposed. In using this inequality, an estimate of \( m^A(n) \) must be given, see, for example, Gaenssler and Stute (1979), Wenocur and Dudley (1981).

The weakness of (3) lies in the fact that, in many applications \( \varepsilon = \varepsilon_n \to 0 \) as \( n \to \infty \). In this case \( ne^2_n \) may not tend to \( 0 \) or tend to \( 0 \) very slowly. For this reason, the inequality proved by Devroye and Wagner (1980) is sometimes more useful. They proved that, if \( \sup_A \mu(A) \leq \delta \leq \frac{1}{4} \), then for any \( \varepsilon > 0 \)

\[
P(D_n(A, \mu) > \varepsilon) \leq 4m^A(2n)\exp(-ne^2/(64\delta + 4\varepsilon))
\]

(4)

\[
+ 2P(\sup_A \mu_{2n}(A) > 2\delta)
\]

for \( n \geq 8\delta/e^2 \). If we further have

\[
\sup_A \sup_{x,y \in A} ||x - y|| \leq \rho < \infty
\]

and

\[
\sup_{x \in \mathbb{R}^d} \mu(S(x, \rho)) \leq \delta \leq \frac{1}{4},
\]

(5)

here \( || \cdot || \) is the \( L_2 \) or \( L_\infty \) norm in \( \mathbb{R}^d \), and \( S(x, \rho) \) is the closed ball with radius \( \rho \).
centered at $x$, then

$$P(D_n(A,u) > \varepsilon) \leq 4m^A(2n)\exp(-n\varepsilon^2/(64\delta+4\varepsilon))$$

$$+ 4n \exp(-n\delta/10)$$

for $n \geq \max(1/\delta, 8\delta/\varepsilon^2)$.

This inequality is most useful when $A$ is the class of balls with the same diameter (norm $L_2$ or $L_\infty$). Otherwise $\delta$ may be much larger than $\text{Sup}_A u(A)$, and (6) gives no improvement over (3). Chen and Zhao (1984) made an essential improvement in the one-dimensional case:

Let $A$ be a class of intervals in $\mathbb{R}^1$, satisfying $\text{Sup}_{I \in A} u(I) \leq \delta < 1$. Then there exists positive absolute constants $C_0, C_1, \ldots, C_4$ such that for any $\varepsilon > 0$

$$P(\text{Sup}_{I \in A} |u_n(I) - u(I)| > \varepsilon)$$

$$\leq C_1 \varepsilon^{-1/6/n} \exp(-C_2 n\varepsilon^2/\delta) + C_3 \exp(-C_4 n\varepsilon),$$

provided $n/\log n > C_0/\varepsilon$.

The proof of (7) relies on a result concerning the strong approximation to Brownian bridge of the empirical process on $\mathbb{R}^1$. The argument fails in the general case $d > 1$. The inequality (2), to be proved in the next section, gives a satisfactory generalization to the case $d > 1$. 
3. PROOF OF THEOREM 1

Set

\[ \delta_j = 2^{-1} + 2^{-2} + \ldots + 2^{-j}, \quad j = 1, 2, \ldots, r, \]

where \( r \) will be chosen later. Then

\[ \delta < \delta_1 < \delta_2 < \ldots < \delta_r < 2\delta \leq \frac{1}{4} \]

When \( n \geq 12\delta/e^2 \) we have \( n \geq 8\delta_1/e^2 \). From (4), the definition of V-C class and the fact that

\[ \sup_A u(A) \leq \delta_1 \leq \frac{1}{4}, \]

it follows that

\[ P\{D_n(A, u) > \varepsilon\} \leq 4((2n)^S + 1) \exp\left(-ne^2/(64\delta_1 + 4\varepsilon)\right) \]

\[ + 2P\{\sup_A u(A) > 2\delta_1\} \]

\[ \leq 5(2n)^S \exp\left(-ne^2/(64\sqrt{2}\delta + 4\varepsilon)\right) + 2P\{D_n(A, u) > \delta_1\}, \]

provided \( n \geq 12\delta/e^2 \).

When \( \delta n \geq 68(1+s)\log 2 \), we have \( 2^{j-1} n \geq 8\delta_j/\delta_j-1 \) for \( j = 2, 3, \ldots, r \). As before, from (4) and \( \sup_A u(A) \leq \delta_2 \leq \frac{1}{4} \), it follows that

\[ P\{D_n(A, u) > \varepsilon\} \leq 5(2n)^S \exp\left(-ne^2/(91\delta + 4\varepsilon)\right) \]

\[ + (2.5X2.2n)^S \exp\left(-2n\delta_1^2/(64\delta_2 + 4\delta_1)\right) \]

\[ + 2^2 P\{D_{2^n}(A, u) > \delta_2\}, \]
provided \( n \geq \max(68(1+s)\log_2/\delta, 12\delta/e^2) \).

Using (4) and \( \sup_A \mu(A) \leq \delta_j \leq n \) repeatedly, we obtain

\[
P(D_n(A, \mu) > \epsilon) \leq 5(2n)^s \exp(-n\epsilon^2/(9\delta+4\epsilon))
\]
\[
+ \sum_{j=1}^{r-1} 2^j \cdot 5(2^{1+s}n)^s \exp(-2^j n \delta_j^2/(68\delta_j+1))
\]
\[
+ 2^r P(D_{2^r n} (A, \mu) > \delta_r) = J_{1,n} + J_{2,n} + J_{3,n},
\]
provided \( n \geq \max(68(1+s)\log_2/\delta, 12\delta/e^2) \).

It is easy to see that

\[
2^j n \delta_j^2/\delta_{j+1} \geq 2j\delta_j, \ j=1, \ldots, r-1.
\]

Hence it follows from (8), (9) and \( 2^{1+s} \leq e^{\delta n/68} \) that

\[
J_{2,n} \leq 5(2n)^s \sum_{j=1}^{r-1} 2^{(1+s)j} \cdot \exp(-2^j n \delta_j^2/(68\delta_j+1))
\]
\[
\leq 5(2n)^s \sum_{j=1}^{r-1} (2^{1+s}n)^j \exp(-2j\delta_j n/68)
\]
\[
\leq 5(2n)^s \sum_{j=1}^{r-1} \exp(-j\delta_j n/68)
\]
\[
= 5(2n)^s e^{-\delta n/68} (1-e^{-\delta n/68})^{-1}
\]
\[
\leq 5(2n)^s (1-2^{-(1+s)})^{-1} e^{-\delta n/68}
\]
\[
\leq 7(2n)^s \exp(-\delta n/68),
\]

where \( s \geq 1 \) is invoked.

When \( \delta n \geq 68(1+s)\log_2 \), we have \( 2^r n \delta_r \geq 2. \) By (3)
\[ J_{3,n} \leq 2^{r+1}((2^{r+1}n)^{s+1}) \exp(-2^{r}n\delta_r^2/8). \]  \hspace{1cm} (11)

Take \( r = r_n \) to be an integer such that \( n/2 < 2^r \leq n \). When \( \delta n \geq 68(1+s)\log 2 \), we have \( n^2\delta_r^2 \geq 2 \), \( n\delta \geq \sqrt{2} \) and \( n\delta_r^2 \geq 2\delta \). By (11) we have

\[ J_{3,n} \leq 2n((2n^2)^{s+1}) \exp(-n^2\delta_r^2/16) \]  \hspace{1cm} (12)

\[ \leq 4n(2n^2)^{s}\exp(-\delta n/8). \]

Formula (2) follows from (8), (10) and (12). The theorem is proved.
4. APPLICATIONS

Theorem 1 has some applications in strong convergence problems involving the uniform deviation between frequencies and probabilities of a class of events. As an example, we consider the nearest neighbor (NN) density estimates proposed by Loftsgarden and Quesenberry (1965). Suppose that \(X\) is a \(\mathbb{R}^d\)-valued random vectors with distribution \(\mu\) and unknown density function \(f\). The so-called NN estimate of \(f(x)\) has the form

\[
\hat{f}_n(x) = \frac{k}{\{n(2a_n(x))\}^d}, \quad x = (x(1), \ldots, x(d)) \in \mathbb{R}^d,
\]

where \(k = k_n \leq n\) is a positive integer chosen in advance, \(a_n(x)\) is the smallest \(a > 0\) such that the cube \([x-a, x+a] = \prod_{i=1}^d [x(i)-a, x(i)+a]\) contains at least \(k\) sample points. As an application of Theorem 1, we prove a theorem about the convergence rate of \(\sup_{x \in \mathbb{R}^d} |\hat{f}_n(x) - f(x)|\).

In the sequel, we use \(c, \alpha, c_1, c_2, \ldots\) for some positive constants independent of \(n\) and \(x\). For \(x = (x(1), \ldots, x(d)) \in \mathbb{R}^d\), \(y = (y(1), \ldots, y(d)) \in \mathbb{R}^d\), write \(f'(x)(y-x) = \sum_{i=1}^d \frac{\partial f}{\partial x(i)} (y(i)-x(i))\), and take \(||y - x|| = \max_{1 \leq i \leq d} |y(i) - x(i)|\).

We say that the density function \(f\) belongs to \(\lambda\)-class for some \(\lambda \in (0,1]\), if \(\lambda \in (0,1]\) and \(|f(y) - f(x)| \leq C ||y-x||^\lambda\) for any \(x, y \in \mathbb{R}^d\), or \(\lambda \in (1,2]\) and, \(f\) are bounded and

\[
|f(y) - f(x) - f'(x)(y-x)| \leq C ||y - x||^\lambda
\]

for any \(x, y \in \mathbb{R}^d\). We have

**Theorem 2.** Suppose that \(f\) belongs to \(\lambda\)-class for some \(\lambda \in (0,2]\). Take \(k = o(n)\) and

\[
k/n \geq \beta \left(\frac{\log n}{n}\right)^{(d+\lambda)/(d+3\lambda)}
\]

(14)
where \( \beta > 0 \) is any given constant. Then

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{(x)} (f_n(x) - f(x)) \leq C \ a.s. \tag{15}
\]

To prove this theorem, we need the following lemma. In the sequel, \( \nu_n \) denotes the empirical measure of \( X_1, \ldots, X_n \). Besides, a cube of the form \([x-a, x+a]\) is called a regular cube.

**Lemma 3.** Let \( A \) be a class of regular cubes satisfying the measurability conditions mentioned in paragraph 1 and the condition

\[
\sup_{A \in A} \nu(A) \leq k/n \leq 1/8.
\]

Take \( k = o(n) \) and

\[
k/n \geq \beta \left( \frac{\log n}{n} \right)^{1/(1+2r)},
\]

where \( r > 0 \) and \( \beta > 0 \) is any given constant. Then

\[
\limsup_{n \to \infty} \left( \frac{n}{k} \right)^{1+r} \sup_{A \in A} |\nu_n(A) - \nu(A)| \leq C_1 \ a.s.
\]

Notice that \( A \) is a V-C class, one can obtain Lemma 3 from Theorem 1 immediately. The proof is omitted.

**Proof of Theorem 2.** Take \( k = o(n) \) and

\[
k/n \geq \beta \left( \frac{\log n}{n} \right) (d+\lambda)/(d+3\lambda)
\]

Put

\[
\nu_n = \theta_1^{-1}(k/n)^{\lambda/(d+\lambda)}
\]

\[
q_n = \theta_2 \nu_n = \theta_1^{-1} \theta_2 (k/n)^{\lambda/(d+\lambda)}
\]
\[ B_n = \{ x : f(x) \geq V_n \} \]

where \( \theta_1, \theta_2 \in (0, 1) \) will be chosen later.

Let \( \mu(x, a) \) and \( \mu_n(x, a) \) be the probability measure and empirical measure of \([x-a, x+a]\) respectively. Put \( M = \max(\sup_x f(x), 1) \). We have

\[
P\left( \sup_{x \in B_n} |\hat{f}_n(x) - f(x)| > q_n \right) \leq I_n + J_n \tag{17}
\]

where

\[
I_n = P(U_{x \in B_n} \{ \hat{f}_n(x) > f(x) + q_n \}), \tag{18}
\]

\[
J_n = P(U_{x \in B_n} \{ \hat{f}_n(x) < f(x) - q_n \}).
\]

Thus

\[
I_n \leq P(U_{x \in B_n} \{ a_n(x) < b_n(x) \}), \tag{19}
\]

where

\[
2b_n(x) = \left( \frac{k}{nf(x)} \right) \left( 1 + q_n/f(x) \right)^{-1} \right)^{1/d}.
\]

Fix \( x \in B_n = \{ x : f(x) \geq V_n \} \). Take \( \theta_2 < 1/8 \), then \( q_n/f(x) \leq \theta_2 < 1/8 \). Noticing \( 1/(1+t) < 1 - 7t/8 \) for \( 0 \leq t < 1/8 \), we have

\[
2b_n(x) \leq \left( \frac{k}{nf(x)} \right) (1 - 7q_n/8f(x))^{1/d} \leq (k/nf(x))^{1/d}.
\]

It follows that

\[
\mu(x, b_n(x)) = \int_{x-b_n(x)}^{x+b_n(x)} f(t)dt
\]
\[ (2b_n(x))^{d+\lambda} = (2b_n(x))^d f(x) + C_2(2b_n(x))^{d+\lambda} \]

\[ \leq \frac{k}{n}(1 - \frac{7}{8} q_n/f(x))(1 + C_2(\frac{k}{n f(x)})^{\lambda/d} f(x)) \]

\[ \leq \frac{k}{n}(1 - \frac{7}{8} q_n/f(x) + C_2(\frac{k}{n f(x)})^{\lambda/d} f(x)). \]

Fix \( \theta_2 \), take \( \theta_1 \) small enough such that \( C_2 \theta_1^{(\lambda+d)/d} < \frac{3}{8} \theta_2 \), then \( C_2(\frac{k}{n f(x)})^{\lambda/d} \)

\[ \leq C_2 \theta_1^{\lambda/d}(k/n)^{\lambda/(\lambda+d)} < \frac{3}{8} \theta_1 \theta_2 (k/n)^{\lambda/(\lambda+d)} = \frac{3}{8} q_n. \]

It follows that

\[ \mu(x, b_n(x)) \leq \frac{k}{n}(1 - \frac{1}{2} q_n/f(x)) < k/n, \]

and

\[ \frac{k}{n} - \mu(x, b_n(x)) \geq k q_n/(2nM). \]

Hence, by (19) and Theorem 1, we have

\[ I_n \leq P\{\sup_{x \in B_n} (\mu_n(x, b_n(x)) - \mu(x, b_n(x)) \geq k q_n/(2nM)\} \]

\[ \leq C_5 n^\alpha(\exp(-\frac{n(kq_n^2/2nM)^2}{91k/n+2kq_n/nM}) + \exp(-k/68)) \]

where \( \alpha \) is a constant depending only on \( d \). In view of (14), we have for large \( n \)

\[ I_n \leq C_5 n^\alpha(\exp(-\theta_1^2 M^{-2}B_{1+2}\lambda/(\lambda+d) \log n/400) \]

\[ + \exp(-k/68)). \]

Take \( \theta_1 \) small enough, we have
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\[ \sum I_n < \infty. \quad (20) \]

In the same way, we can take \( \theta_1 \) and \( \theta_2 \) such that

\[ \sum J_n < \infty \quad (21) \]

By (17), (18), (20) and (21), we have

\[ \sum P\{q_n^{-1}\text{Sup}_{x \in B_n} |\hat{f}_n(x) - f(x)| > 1\} < \infty. \]

By Borel-Cantelli's lemma,

\[ \lim_{n \to \infty} \text{Sup}_{x \in B_n} q_n^{-1} \text{Sup}_{x \in B_n} |\hat{f}_n(x) - f(x)| \leq 1 \text{ a.s.} \quad (22) \]

Fix \( \theta_1, \theta_2 \), and take \( 2b_n = C_3(k/n)^{1/(d+\lambda)} \). Fix \( x \in B_n^c = \{x: f(x) < V_n\} \). With small \( C_3 \) we have

\[ \mu(x, b_n) = \int_{x-b_n}^{x+b_n} f(t) dt \]

\[ \leq (2b_n)^d f(x) + C_2(2b_n)^{d+\lambda} \]

\[ \leq \frac{k}{n}[\theta_1^{-1} C_3^d + C_2 C_3^{d+\lambda}] < k/2n < k/n. \]

Taking \( r = \lambda/(d+\lambda) \) in Lemma 3, we can assert with probability one that, for \( n \) large enough, the inequality

\[ \mu_n(x, b_n) \leq \mu(x, b_n) + 2C_1(k/n)^{(d+2\lambda)/(d+\lambda)} \]

\[ < k/2n + 2C_1(k/n)^{(d+2\lambda)/(d+\lambda)} < k/n \]

holds uniformly for \( x \in B_n^c \). By definition, for \( x \in B_n^c \),
$$a_n(x) \geq b_n = \frac{1}{2} C_3(k/n)^{1/(d+\lambda)}.$$  
$$\hat{f}_n(x) \geq C_4(k/n)^{\lambda/(d+\lambda)}$$

It follows that

$$\lim_{n \to \infty} \sup_{x \in B_n^C} \frac{(n/k)^{\lambda/(d+\lambda)} \sup_{x \in B_n} |\hat{f}_n(x) - f(x)|}{C_4 a.s.} \leq C_4 \quad (23)$$

Theorem 2 is proved in view of (22) and (23).
REFERENCES


An inequality concerning the deviation between theoretical and empirical distributions

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August 1985

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Empirical distribution, Uniform deviation, Probability inequality, Uniform strong convergence, Nearest neighbor density estimate.

In this paper the author established an inequality concerning the uniform deviation between theoretical and empirical distributions. An application in strong convergence of nearest neighbor density estimate is also discussed.