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MIXED MODE I AND II FULLY PLASTIC CRACK GROWTH FROM

SIMULATED WELD DEFECTS(U) MASSACHUSETTS INST OF TECH

CAMBRIDGE DEPT OF MECHANICAL ENGIN.  G A KARDOMATEAS

UNCLASSIFIED  23 OCT 95  N00014-82-K-0025

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MIXED MODE I AND II FULLY PLASTIC CRACK GROWTH FROM SIMULATED WELD DEFECTS

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Unlimited Distribution

23 October 1985


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Massachusetts Institute of Technology
Cambridge, MA 02139

Office of Naval Research
Code 432S

PhD thesis, September 1985
Frank A. McClintock, Supervisor
Department of Mechanical Engineering
Massachusetts Institute of Technology

Steels 1018, HY-80, HY-100, and A36; 5086-Hill Aluminum,
asymmetric, weld defect, shear band, mixed Mode I and II,
oblique, strain hardening, fully plastic, limit load, ductile
fracture, J-integral, crack initiation, crack tip displacement,
crack growth ductility, crack opening angle, tearing modulus,

In symmetrical specimens, the crack advances into the relatively
undamaged region between two plastic shear zones. A crack near
a weld or shoulder, loaded into the plastic range, may have only
a single shear band, along which the crack grows in pretrained
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symmetrical configurations. A crack ductility can be defined as
the minimum displacement per unit crack growth. A low crack
ductility requires higher stiffness of the surrounding structure,
19 (cont.) tearing modulus, fracture-stable design, tearing modulus, fracture-stable design, fractography, shear slip, experiments, theory

20 (cont.) for fracture-stable design. Tests of six alloys showed that, for the low hardening alloys, the crack ductility in the asymmetric case is less than a third that of the symmetric. In the higher hardening alloys the crack ductility in the asymmetric case is smaller by a factor of 1.2 at most. A noteworthy result is the presence of a Mode I opening component even with asymmetry, as is shown by the far field displacement vector being more than $45^\circ$ from the transverse direction. The crack direction is less than $45^\circ$, indicating the effect of triaxiality on cracking.

A macro mechanical model for crack advance by combined fracture and sliding off along two slip planes gives the independent physical parameters (cracking and two shear directions, relative amounts of cracking and shearing) in terms of the observable quantities of the macroscopic fracture (flank angles, flank lengths, back angle). This two slip-plane model admits a Mode I opening component and describes, based on an idealization of underlying physical mechanisms, the development of deformation in ductile crack growth for both the symmetric and asymmetric specimens. A finite element study of the asymmetric specimens gave a crack direction within two degrees and a far field displacement vector at initiation within three degrees of that experimentally found. Stress and strain fields indicate the presence of a Mode I component. Early growth, studied by successive removal of the most damaged element, resulted in crack growth rate for the lower hardening case about twice that of the higher hardening one. Finally, a logarithmic tensile singularity in the mean stress was found for rigid-plastic flow past a growing crack of finite angle with rigid flanks under combined shear and tension. The tensile singularity predicts yielding of the crack flanks.
MIXED MODE I AND II FULLY PLASTIC CRACK GROWTH FROM SIMULATED WELD DEFECTS

by

GEORGE A. KARDOMATEAS

Diploma, Ethnikon Metsovion Polytechnion, Greece (1981)
M.Sc., Massachusetts Institute of Technology (1982)

SUBMITTED TO THE DEPARTMENT OF MECHANICAL ENGINEERING IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 1985

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September 10, 1985

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Frank A. McClintock
Thesis Supervisor

Accepted by

Aia A. Sone
Chairman, Departmental Graduate Committee
Mixed Mode I and II Fully Plastic Crack Growth From Simulated Weld Defects

by

GEORGE A. KARDONATEAS

Submitted to the Department of Mechanical Engineering on September 10, 1986 in partial fulfillment of the requirements for the degree of Doctor of Philosophy

Abstract

In symmetric specimens the crack advances into the relatively undamaged region between two plastic shear zones. A crack near a weld or shoulder, loaded into the plastic range, may have only a single shear band, along which the crack grows into pretrained and damaged material with less ductility than the usual symmetrical configurations. A crack ductility can be defined as the minimum displacement per unit crack growth. A low crack ductility requires higher stiffness of the surrounding structure for fracture-stable design. Tests of six alloys showed that, for the low-hardening alloys, the crack ductility in the asymmetric case is less than a third that of the symmetric. In the higher hardening alloys the crack ductility in the asymmetric case is smaller by a factor of 1.2 at most. A noteworthy result is the presence of a Mode I opening component even with asymmetry, as is shown by the far field displacement vector being more than 45° from the transverse direction. The crack direction is less than 45°, indicating the effect of triaxiality on cracking.

A macro-mechanical model for crack advance by combined fracture and sliding off along two slip planes gives the independent physical parameters (cracking and two shear directions, relative amounts of cracking and shearing) in terms of the observable quantities of the macroscopic fracture (flank angles, flank lengths, back angle). This two slip plane model admits a Mode I opening component and describes, based on an idealization of underlying physical mechanisms, the development of deformation in ductile crack growth for both the symmetric and asymmetric specimens. A finite element study of the asymmetric specimens gave a crack direction within two degrees and a far field displacement vector at initiation within three degrees of that experimentally found. Stress and strain fields indicate the presence of a Mode I component. Early growth, studied by successive removal of the most damaged element, resulted in crack growth rate for the lower hardening case about twice that of the higher hardening one. Finally, a logarithmic tensile singularity in the mean normal stress was found for rigid-plastic flow past a growing crack of finite angle with rigid flanks under combined shear and tension. The tensile singularity predicts yielding of the crack flanks.

Thesis Supervisor: Frank A. McClintock
Title: Professor of Mechanical Engineering
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I would like to express my deep gratitude to Professor Frank McClintock for his advice and interest in my research and to the Office of Naval Research for sponsoring the project under Contract N0014-82K-0025.

Having finished my student years I would like to thank my earlier teachers from the National Technical University of Athens for providing me with an excellent scientific background.
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CHAPTER ONE

INTRODUCTION

In symmetric singly grooved unconstrained tensile specimens the crack advances into the relatively undamaged region between two symmetric shear zones. An asymmetry, introduced through a weld fillet or a harder, heat-affected zone or a shoulder on one side of the crack suppresses one of the two slip lines that would appear in a symmetrical specimen. This is likely to give asymmetric cracking along the remaining active slip line, with less ductility because the crack is advancing into prestrained and pre-damaged material. A reduced ductility requires higher stiffness of the surroundings for fracture-stable design.

Near the tip of the growing crack, strain hardening will cause the deformation field to fan out. For power law creep or deformation theory plasticity and a stationary crack, the asymptotic stress and strain distribution may be found from the extended by Shih [1] HRR [2,3] solutions for the general mixed mode I and II case. Notice, however, that such a superposition of stationary singularities does not take into account the hardening of the material left behind the growing crack. Indeed, the stress and strain fields near the tips of growing cracks in ductile materials are known to differ from the stress and strain state around stationary cracks in the same materials as is shown from asymptotic solutions [4,5,6] supplemented through finite element calculations [7,8].

McClintock and Slocum [9] developed a formulation for the accumulation of damage directly ahead of an asymmetric crack, based on strain increments adapted from Shih's [1] analysis. The crack was assumed to follow the center of the 45° shear band. It was found that the crack growth per unit displacement increases
approximately as the logarithm of the total crack advance per inclusion spacing $p$ and varies inversely as the critical fracture strain $\gamma_c$. Little effect of strain hardening on the growth rate was found.

The objective of the current study is to investigate through experimental, analytical and numerical work, the ductility of asymmetric, fully plastic, unconstrained configurations. First, the approximate pure Mode II incremental solution [9] is extended to admit a crack growing at an angle to the shear band. This deviation from the shear band is expected from the higher triaxiality. Far field displacement is assumed again to be parallel to the shear band. Next, tests results on symmetric and asymmetric specimens of six alloys are presented. A method for quantifying and representing the ductility is suggested. A macromechanical model of crack growth by combined fracture on one plane and sliding off along two others describes, for this idealization of the physical mechanisms, the ductile crack growth for both the asymmetric and symmetric specimens. To account for the effect of the finite width of the shear band and study the stress and strain fields at initiation, a finite element investigation is undertaken. Early growth is also studied by successive element removal. Finally, a stream function technique is used to investigate whether rigid-plastic strain hardening flow past a growing crack of finite angle with rigid flanks can be sustained. The last chapter contains an overview of the results and summarizes the conclusions. It also contains recommendations for further research.

REFERENCES


2. Hutchinson J.W. “Singular behaviour at the end of a tensile crack in a


CHAPTER TWO

DIRECTIONAL EFFECTS IN FULLY PLASTIC CRACK GROWTH NEAR A SHEAR BAND.

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<table>
<thead>
<tr>
<th>Symbol</th>
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<tr>
<td>$F_t$</td>
<td>hole growth ratio</td>
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<tr>
<td>$I_{1/n}(n, M_P)$</td>
<td>dimensionless parameter</td>
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<td>$J$</td>
<td>J-integral</td>
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<td>$k$</td>
<td>shear yield</td>
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<tr>
<td>$M_P$</td>
<td>mixity parameter (eq. 2)</td>
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<tr>
<td>$n$</td>
<td>strain hardening exponent</td>
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<tr>
<td>$T_j$</td>
<td>traction vector</td>
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<tr>
<td>$U^i$</td>
<td>far field displacement (along shear band)</td>
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<tr>
<td>$W$</td>
<td>work per unit volume</td>
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<td>$\sigma$</td>
<td>mean normal stress</td>
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<td>$\sigma_1$</td>
<td>flow stress at unit strain</td>
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<td>$\phi$</td>
<td>angle between crack and shear band</td>
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<td>$\eta$</td>
<td>damage</td>
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<td>$\Delta c$</td>
<td>increment of crack advance</td>
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<td>$\gamma$</td>
<td>principal shear strain</td>
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<td>$r$</td>
<td>principal shear strain</td>
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<td>$\rho$</td>
<td>mean inclusion spacing</td>
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<td>$\tilde{\sigma}_{ij}$</td>
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<tr>
<td>$\tilde{\varepsilon}_{ij}$</td>
<td>angular strain functions</td>
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<tr>
<td>$\tilde{u}_i$</td>
<td>angular displacement functions</td>
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ABSTRACT

Welds, shoulders, or other asymmetries may eliminate one of the shear bands of symmetrically cracked parts and thus give crack propagation through pre-damaged material, instead of through the relatively unstrained region between the two plastic shear zones of the symmetric case. Previous work is extended to include sites at several angles ahead of the crack. Far field displacement is assumed to take place parallel to the shear band. Strain increments are approximated from the mixed mode, power-law elastic solution for a stationary crack and used with a
fracture criterion for hole growth in shear bands to predict the direction and rate of crack growth. The crack is assumed to advance to advance in the direction that requires the minimum far field displacement to reach critical damage. For a shear band at 45° the crack progresses at 21°-30° from the transverse (depending on strain hardening), indicating the effect of higher triaxiality. The crack growth rate is about 6-15% higher than if the directional effects are neglected. Lower strain-hardening results in a 5% higher rate of crack advance per unit displacement, a higher fracture strain, and the final crack orientation being closer to the 45° shear band.

INTRODUCTION

Most fracture tests use symmetric specimens. The crack advances into relatively undamaged material between two shear bands. This will not happen if one of the bands is eliminated due to a weld bead, or a harder heat-affected zone, for example (Fig. 1). A fatigue crack or other defect near such an asymmetry will tend to advance along the remaining shear band through highly strained material. Lower ductility is thus expected. An example of lowered ductility in asymmetric flow is the formation of a shear lip at the end of an ordinary cup and cone fracture in a tensile test.

McClintock and Slocum [1] developed an approximate formulation for the accumulation of damage ahead of the crack in a power-law strain hardening material, by using the strain and displacement fields derived by Shih [2] for a stationary mixed-mode crack and the McClintock, Kaplan, and Berg [3] criterion for fracture by hole growth. It was assumed that the crack advances directly along the shear band. Preliminary experiments, however, have indicated that the crack actually advances at an angle from the shear band which reflects the effect of the
higher triaxiality on one side.

In the following we modify the Pure Mode II [1] solution by considering several sites around the crack tip. The far field displacement is again assumed to be parallel to the shear band and the strain increments to follow Shih's [2] mixed mode stationary crack fields. The accumulated damage from initiation and prior growth is calculated and the necessary far-field displacement for critical damage is found for each site. The crack is assumed to advance in the direction requiring the least far-field displacement.

ANALYSIS

1. Crack initiation. A nonlinear elastic solution for the small scale yielding of mixed Modes I and II stationary crack problems was developed by Shih [2]. The material was assumed to be power-law hardening according to the relation between equivalent stress and strain:

\[ \sigma = \sigma_1 \varepsilon^n, \]  

(1)

where \( \sigma_1 \) is the flow strength at unit strain and \( n \) is the strain hardening exponent. Shih [2] introduced a Mode I mixity parameter \( M_P \), defined in terms of the near field stresses by

\[ M_P = \frac{2}{\pi} \tan^{-1} \left( \lim_{r \to 0} \frac{\sigma_{\theta \theta}(r, \theta=0)}{\tau_{r \theta}(r, \theta=0)} \right). \]

(2)

The mixity parameter varies from 0 for pure Mode II to 1 for pure Mode I. McClintock [4] restated the dominant singularity governing the behavior of the stresses, strains and displacements (for large plastic strains) in terms of the J integral as
\[
\frac{\sigma_{ij}}{\sigma_1} = \left[ \frac{J}{\sigma_1 r I_{1/n}(n,M^p)} \right]^{n/(n+1)} \tilde{\sigma}_{ij}(\theta,M^p,n),
\]

\[
\epsilon_{ij} = \left[ \frac{J}{\sigma_1 r I_{1/n}(n,M^p)} \right]^{1/(n+1)} \tilde{\epsilon}_{ij}(\tilde{\epsilon},M^p,n),
\]

\[
u_i = \left[ \frac{J}{\sigma_1 r I_{1/n}(n,M^p)} \right]^{1/(n+1)} \tilde{\nu}_i(\theta,M^p,n).
\]

The J integral is defined (x_1 axis along crack) as

\[
J = \int W dx_2 - T_j \frac{\partial u_j}{\partial x_1} ds,
\]

where \(T_j\) is the traction vector, \(u_j\) is the displacement vector, and \(W\) is the work per unit volume. For a single shear band, \(J\) can be evaluated in terms of the shear strength \(k\), the far field relative displacement \(U\) and the angle \(\phi\) between crack and shear band (Fig. 2) by

\[
J = \frac{kU}{\cos \phi}.
\]

The dimensionless functions \(\tilde{\sigma}_{ij}(\theta,M^p,n)\), \(\tilde{\epsilon}_{ij}(\tilde{\epsilon},M^p,n)\) and \(I_{1/n}(n,M^p)\) have been numerically determined by Shih [2] for \(n=1/3\) and \(n=1/13\). The dimensionless functions \(\tilde{\nu}_i(\theta,M^p,n)\) are derived in the Appendix from the strain functions and are shown in Fig. 3 for \(n=1/13\) and \(M^p=0.5\).

Assume that currently the shear band forms an angle \(\phi\) with a recent average crack direction to be defined below. With the crack not advancing along the shear band, there is no longer pure Mode II. When the relative far-field displacement is assumed to be parallel to a single narrow shear band, as is valid in the non-hardening limit, and that direction is assumed to be the same as the local relative displacement across the flanks, the mixity parameter can be determined from the angular
functions ($\gamma$) of the displacement field relative to $\theta=-\pi$ since, from Fig. 2,

$$\tan \phi = \frac{u_x}{u_r} = \frac{\bar{u}_x(MP,n)}{\bar{u}_r(MP,n)}.$$ (5)

Fig. 4 shows the resulting variation of the mixity parameter with the angle $\phi$ for $n=1/13$. Thus, the angle $\phi$ determines the angular stress and strain functions and hence the local stress and strain for a given $J$ through (3). The angular functions turn out to affect the fracture criterion through the triaxiality and the shear strain, as follows. The mean normal stress for plane incompressible flow is

$$\sigma = \frac{\sigma_{rr} + \sigma_{\theta\theta}}{2}.$$ (6)

The triaxiality $\sigma/\tau$ used in the fracture criterion is given in terms of the dimensionless principal shear stress $\tilde{\tau}$, defined by

$$\tilde{\tau} = \left[ \frac{\tilde{\tau}_{\theta}^2 + \left(\frac{\sigma_{rr} - \sigma_{\theta\theta}}{2}\right)^2}{2} \right]^{1/2},$$ (7)

by

$$\frac{\sigma}{\tau} = \frac{\tilde{\sigma}_{rr} + \tilde{\sigma}_{\theta\theta}}{2\tilde{\tau}}.$$ (8)

The angular variation of this triaxiality, $\sigma/\tau$, is shown in Fig. 5 for $n=1/13$. Note that the triaxiality is highest for negative values of $\theta$ for all cases except pure Mode I. This is the primary reason for exploring the directional effects.

Similarly, the dimensionless principal shear strain $\tilde{\gamma}$ can be expressed for the plane incompressible case as:

$$\tilde{\gamma} = 2\left[ \frac{\tilde{\gamma}_{r\theta}^2 + \left(\frac{\tilde{\tau}_{rr} - \tilde{\tau}_{\theta\theta}}{2}\right)^2}{2} \right]^{1/2} = 2\sqrt{\tilde{\gamma}_{r\theta}^2 + \tilde{\gamma}_{rr}^2}.$$ (9)

Introducing (4) into the first of (3) and solving for the displacement gives the far-field displacement $U$ in terms of the principal shear strain $\gamma$ at any point in the near
The critical displacement for crack initiation occurs when the fracture strain is reached at the point \((p, \theta)\) where \(p\) is a fracture process zone size (e.g. the mean inclusion spacing). The fracture strain is found by using the fracture criterion of McClintock, Kaplan and Berg [3] by which it is postulated that fracture due to micro-void coalescence occurs when the “damage”, \(\eta\), reaches unity. The damage is expressed in terms of a hole growth ratio \(F_t\), the principal shear strain \(\gamma\) and the triaxiality \(\sigma/\tau\):

\[
\eta = \frac{1}{\ln F_t} \left[ \ln \sqrt{1+\gamma^2} + \frac{\gamma}{2(1-n)} \sinh \left( \frac{(1-n)\sigma}{\tau} \right) \right].
\] (11)

The above damage equation is associated with growth of cylindrical holes.

Alternatively one might use, for example, the eqs for growth of spherical holes in nonhardening material (Rice and Tracey, [5]).

A Newton-Raphson technique is used to solve (10) and (11) at \(r=p\) for the critical far-field displacements for initiation in a number of directions. The actual initial direction is that which minimizes the required displacement. Once the initiation displacement \(U_i\) for the critical strain at the point \((p, \theta_c)\) is known, the strain at all other points can be found by re-arranging (10):

\[
\gamma = \left( \frac{kU_i}{r\sigma_1 I(n,M^P) \cos \phi} \right)^{1/(n+1)} \gamma. \] (12)

2. Crack growth. After initial growth by \(\Delta c\), further growth requires reaching the critical damage at some new site \(p\) from the current crack tip. The damage at each site is that from crack initiation plus those for any following crack growth.
increments. The strain to bring the damage to unity is found by differentiating (11) to find the damage increments in terms of the strain increment and the strain itself:

\[ d\eta = \frac{1}{\ln F_t} \left[ \frac{\gamma}{1+\gamma^2} + \frac{1}{2(1-n)} \sinh\frac{(1-n)\sigma}{\tau} \right] d\gamma. \] (13)

In the absence of an incremental strain-hardening solution for a growing crack, we follow McClintock and Slocum [1] and approximate the strain increment in terms of the far-field displacement increment by differentiating and rearranging (10) (this is strictly valid only for non-linear elasticity):

\[ d\gamma = \frac{k\tilde{\gamma}}{(n+1)\sigma_1 I(n,M_P) r\cos\phi} \left(\frac{\gamma}{\tilde{\gamma}}\right)^n dU. \] (14)

The damage at any point in front of the growing crack is given by the sum of the damage due to crack initiation, as found from (12) and (11), and all of the damage increments from prior crack growth, as found from (13) and (14), with \( r \) taken to be the distance from the prior crack tip to the point in question, and \( \phi \) the prior angle between crack and shear band.

The necessary increment in damage for fracture is \( \delta\eta = 1-\eta. \) The corresponding strain increment can be found from (13):

\[ \delta\gamma = (1-\eta)\ln F_t / \left[ \frac{\gamma}{1+\gamma^2} + \frac{1}{2(1-n)} \sinh\frac{(1-n)\sigma}{\tau} \right]. \] (15)

The necessary increment in far-field displacement to cause this strain increment can then be found from (14):

\[ \delta U = (n+1) \frac{\sigma_1 I_{1/n}^1(n,M_P) \rho \cos\phi}{\tilde{\gamma} k} \left(\frac{\gamma}{\tilde{\gamma}}\right)^n \delta\gamma. \] (16)

The crack will advance in the direction \( \phi \) which requires the least far field displacement to reach critical damage, not necessarily toward the most severely
i.e. the angle between crack and shear band is smaller and the triaxiality is smaller.

(5) For the low strain hardening $n=1/13$, increasing $\phi_0$ gave smaller initiation displacements and strains and larger initial crack growth rates ($dc/du$).

(6) Strains and triaxialities during growth are relatively insensitive to the initial angle between crack and shear band.

(7) For both strain hardening exponents and all the angles $\phi_0$, the final average angle between crack and shear band $\phi_{avg}$ after growth by $c/\rho=100$ was between $23^0$ and $32^0$ from the shear band. The Mode I mixity corresponding to the final crack orientation was also within a correspondingly narrow range for each of the strain hardening exponents.

(8) These rigid-plastic results do not predict instability (infinite crack advance per unit far field displacement). Instability could, however, arise from the compliance of the surrounding structure.

SUMMARY OF CONCLUSIONS

A part containing a crack near a weld or a shoulder, loaded into the plastic range, can give an asymmetric shear band extending from the crack tip. The resulting crack propagation into previously damaged material gives less ductility than the typical symmetric case. A previous incremental solution for crack growth using Shih's asymptotic fields for a stationary crack in nonlinear elastic material is extended to account for the effect of triaxial stress in advancing a crack at an angle to the shear band. Far field displacement is assumed to take place along the shear band. Cracking is assumed to occur at the site around the crack tip that needs the least far field displacement for critical damage. For a $45^0$ shear band, it is found that the crack does not advance along the shear band but at an angle of about $21^0$ from the transverse under a higher triaxiality. The crack growth rate is higher by
about 6-15% (the larger increase with less hardening) than if the directional effects are neglected and the crack is assumed to progress along the shear band. A higher strain hardening decreases slightly (about 5%) the crack growth rate and the final angle from the transverse of the growing crack. Strains and triaxiality during growth are not sensitive to the initial angle between the crack and the shear band.

REFERENCES


TABLE 1

Comparison of numerical and approximate pure Mode II solutions

<table>
<thead>
<tr>
<th></th>
<th>Numerical</th>
<th>Approx. Mode II</th>
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<tr>
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<td>0°</td>
<td>45°</td>
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<tr>
<td>n</td>
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</tr>
<tr>
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<tr>
<td>Δc/Δu</td>
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Appendix - Displacement Functions

The displacement functions $u_j$ are determined from the strain functions for the plane strain considered here. The radial displacement $u_r$ may be found from the radial strain

$$
\varepsilon_{rr} = \frac{\partial u_r}{\partial r},
$$

so

$$
u_r = \int_0^r \varepsilon_{rr} dr + u_r(0, \theta).
$$

For zero rigid-body translation at $r=0$, $u_r(0, \theta)=0$. Eliminating $\varepsilon_{rr}$ with (3) and integrating gives

$$
u_r = \frac{J}{\sigma_1 n} \left[ \frac{n+1}{1/(n+1)} \right] \frac{\varepsilon_{rr}}{n}.
$$

Using the displacement equation (3) gives the radial displacement function $\tilde{u}_r$ relative to that at $\theta=-\pi$:

$$
\tilde{u}_r(\theta, MP, n) = \frac{n+1}{n} \left( \varepsilon_{rr}(\theta, MP, n) - \varepsilon_{rr}(-\pi, MP, n) \right).
$$

The tangential displacement function $u_\theta$ is determined from

$$
\varepsilon_{\theta\theta} = \frac{1}{r} u_r + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}
$$

as

$$
u_\theta = \int_0^\theta (r \varepsilon_{\theta\theta} - u_r) d\theta + f(r).
$$

Noting that $\varepsilon_{\theta\theta} = -\varepsilon_{rr}$ for plane strain incompressibility and using (3) with (20) gives the tangential displacement:

$$
u_\theta = 1 \left[ \frac{J}{\sigma_1 n} \right] \frac{2n+1}{n} \int_\pi^\theta \varepsilon_{rr} d\theta + f(r)/r.
$$
With respect to the displacement at \( \theta = -\pi \), \( f(r) = 0 \). By using (3) we can thus find the dimensionless tangential displacement function relative to the displacement at \( \theta = -\pi \), in terms of the dimensionless strain function \( \tilde{\varepsilon}_{rr} \):

\[
\tilde{u}_\phi(\theta, M^p, n) = \frac{2n+1}{n} \int_{-\pi}^{\theta} \tilde{\varepsilon}_{rr}(\theta, M^p, n) d\theta.
\]  

(24)

As an example, the displacement functions for \( n = 1/13 \), \( M^p = 0.50 \), determined numerically by (21), (20) are given in Fig. 3.
Fig. 1 Asymmetric crack from a defect near a weld. The symmetric case is shown above.
Fig. 2  J-integral parameters.
Fig 3  Angular variation of the radial and tangential near tip
dimensionless displacement functions for plane strain
with n=1/13 and k^p=0.5
Fig. 4  Mode I mixity parameter $M^p$ as a function of the crack-shear band angle $\phi$. 
Fig 5: Angular variation of the triaxiality $r$ for plane strain
with $n=1/3$ and mixity $M^p = 0, 0.5, 0.82, 1.0$
Fig. 6 Instrumental crack growth
Fig. 7 Growing crack path for an initial crack-shear band angle $\phi_0$.
Fig. 9 Mode I mixity $M^p$ vs. projected crack advance.
CHAPTER THREE
EXPERIMENTAL STUDY

TABLE OF SYMBOLS

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<tr>
<th>Symbol</th>
<th>Description</th>
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<td>$D_{\text{ext}}$</td>
<td>minimum extension rate (eq. 10)</td>
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<td>$T$</td>
<td>tearing modulus</td>
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<td>fracture strain</td>
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<td>$\theta_c$</td>
<td>crack direction from transverse.</td>
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ABSTRACT

Most fracture tests use symmetric specimens, with the crack advancing into the relatively undamaged region between two plastic shear zones. However, a crack near a weld or shoulder, loaded into the plastic range, may have only a single shear band, along which the crack grows into prestrained and damaged material with less
ductility than the usual symmetrical configurations. An experimental study on six alloys shows that while the crack initiation displacements are similar, the growth displacement is much less for the asymmetrical specimens, especially with less hardening. Indeed, for the low-hardening alloys ($n \approx 0.1$) the crack growth ductility, defined as the minimum displacement per unit ligament reduction, is less in the asymmetric case than the symmetric by a factor of three. In the higher hardening alloys the crack growth ductility is less in the asymmetric case by a factor of 1.2 at most. Triaxiality on one side of the asymmetric shear crack diverts it from $45^0$ to $38^0-41^0$ from the transverse direction, the larger angles with smaller strain hardening. In addition, the far field displacement vector is more axial than the $45^0$ line, at $51^0$ to $63^0$ from transverse, suggesting a Mode I component even with asymmetry.

INTRODUCTION

For fracture-stable structures it is important not only that fully plastic conditions be attained before fracture, but also that the load does not fall off too rapidly during crack growth. Flow fields such as Fig. 1, in which the far-field deformation consists of a single shear band, may arise in practice due to the constraint of weld material. These specimens may exhibit less ductility than the symmetric ones, because the crack is advancing into pre-strained and damaged material, rather than into the new material encountered by a crack advancing between two symmetrical shear bands. Being able to predict such increased crack growth can have useful applications in the design, inspection and maintenance of pressure vessels and ships.

Near the tip of the growing crack, strain hardening will cause the deformation
field to fan out. For power law creep or deformation theory plasticity, the stress and strain in the neighborhood of a stationary crack may be found from Shih's [1] mixed mode solutions. More realistically, a corresponding, fully-plastic, incremental plasticity solution should be obtained for a growing crack, taking into account the hardening of the material left behind the growing crack. McClintock and Slocum [2] developed a formulation for the accumulation of damage directly ahead of an asymmetric crack, based on strain increments adapted from Shih's [1] analysis for stationary cracks in a power law material. The crack was assumed to follow the center of the 45° shear band. It was found that the crack growth per unit displacement increases approximately as the logarithm of the total crack advance per inclusion spacing $\rho$ and varies inversely as the critical fracture strain $\gamma_c$. To correct for triaxial effects, several sites around the current crack tip were considered in chapter 2. The damage at each site due to crack initiation and prior growth was determined and then the necessary increment in far field displacement was found for each site. The crack was assumed to advance in the direction requiring the least displacement. This numerical investigation resulted in growth directions not along the 45° shear band but at a smaller angle from the transverse depending on the hardening and the initial crack-shear band angle, and lower ductility by 6-15% (larger decrease with less hardening) than with growth along the shear band.

A test with pure shear (Mode II) loading was carried out by Chant et al. [3] of high hardening carbon manganese steel (B.S. 1501-151-430A, Y.S.=329 MN/m$^2$, T.S. = 490 MN/m$^2$). Small specimens were subjected to both Mode II and Mode I testing but the ductility, measured by $dJ/da$, was practically the same although the microscopic features for the pure shear specimens are different than those observed in the Mode I specimens. The objective of the current study is to present experimental evidence on the ductility of asymmetric crack configurations.
EXPERIMENTAL PROCEDURES

Material. Tests were performed on six alloys with the mechanical properties listed in Table 1. True stress-true plastic strain curves (Figs. 2a,b) for these alloys were obtained using standard 6.35 mm. dia. specimens with 25.4 mm. gage length. It is convenient to represent a curve of equivalent stress $\bar{\sigma}$ vs. equivalent plastic strain $\bar{\varepsilon}$ by:

$$\bar{\sigma} = \sigma_1 (\varepsilon_0 + \bar{\varepsilon})^n,$$

where $\sigma_1$, $\varepsilon_0$, and $n$ are three constants that were determined from the flow strengths at yield point, $\varepsilon=0.125$ and $\varepsilon=0.250$, and are given in Table 2. The lower hardening alloys are the 1018 cold finished steel, HY-80 and HY-100 steels ($n\approx0.10$) and the higher hardening alloys are the 1018 normalized and A36 hot rolled steels ($n\approx0.24$). The 5086-H111 aluminum is between these two groups.

Test Method. From 12.7 mm. dia. round bars of each alloy, seven specimens were first machined as shown in Fig. 3a, with side grooves to ensure a straight fatigue pre-crack approximately 1.3 mm deep. For the four asymmetric specimens (Fig. 3b), further side grooves were machined at 40° from the transverse direction. This corresponded to the crack direction found in preliminary tests and served to reduce 3-dimensional effects. For the three symmetric specimens, since the crack grows by alternating shear at $\pm45^0$, orthogonal triangles were machined, as shown in Fig. 3c.

Stability of the tests turned out to be an important consideration due to the high crack growth rate expected in the asymmetric case. Thus short specimens, stiff adapters, and locknuts were used. The tensile tests were performed on an MTS 50
metric ton testing machine with resulting compliances of $2.3 \times 10^{-6}$, $4.8 \times 10^{-6}$, $1.08 \times 10^{-6}$ mm/N for the steel specimens, the adapters, and the machine respectively. The axial and transverse displacements across the notch were plotted continuously.

A typical plot of load vs. axial displacement is shown in Fig. 4; the breakthrough point is when the fracture first breaks through the back surface, with some shear lips remaining on the sides. The displacement during crack initiation and growth, $u_i$ and $u_g$, are found from the drawings of the crack path. The topographies of the fracture surfaces were thus subsequently plotted using a metallurgical microscope with a travelling stage. The horizontal and vertical coordinates of the travelling stage are recorded with two linear potentiometers; several points are obtained to give an impression of the surface profile of the broken specimens. A typical microscope plot, as in Fig. 5, consists of the $60^0$ notch, the fatigue crack (with some amount of deformation, $\overrightarrow{v_1}$-$\overrightarrow{v_2}$), an initiation zone which shows some blunting, and a growth zone. The initiation displacement is $\overrightarrow{v_1}$-$\overrightarrow{v_g}$. These quantities were also checked against the data from the load-extension curves. In addition, fracture profiles were used to obtain the angular quantities such as the crack opening angle, $\omega$, the lower and upper flank angles, $\theta_l$ and $\theta_u$, and the orientation of the total displacement vector, $\phi$, in the asymmetric case.

RESULTS

Initiation Displacement. Stable tests were obtained except for the lower hardening alloys, which were unstable for less than 20% of the falling part of the load-displacement curve. The results of the tests are summarized in Table 3. An idealized initiation displacement, $u_i^{1}/\theta_0$, can be defined as the extension between the initial elastic loading and the steepest unloading parts of the load-displacement curve.
at maximum load, normalized with the initial ligament $l_0$ (Fig. 4). This quantity, given in the first row of the table, is a convenient measure of initiation and can be compared with the initiation displacement $u_i$, measured from the fracture surface profiles after complete separation. The normalized crack initiation displacement $u_i/l_0$, does not appreciably differ between the asymmetric and symmetric configurations. It is, however, dependent on the strain hardening, being for the higher hardening alloys two to four times that of the lower hardening ones.

**Ductility.** For a measure of the crack growth resistance, the crack ductility, $D_g$, is defined as the minimum displacement, $du_c$, per unit projected ligament reduction, $dl$. Thinning of the ligament from the far side in fully plastic flow makes the reduction in ligament rather than crack advance more appropriate for describing load drop. The displacement $du_c$ is associated with the crack opening stretch and consists of the gauge displacement $du$ and the elastic unloading $du\text{unl}$ (Fig. 4):

$$du_c = du + du\text{unl}.$$  \hspace{1cm} (2)

The ligament reduction can be approximated from the relative load drop and thus we can define:

$$D_g = \left(\frac{du_c/l_0}{dP/P_{\text{max}}}\right)_{\text{min}} \approx \left(\frac{du_c}{dl}\right)_{\text{min}}.$$  \hspace{1cm} (3)

From Fig. 6, this is also related to the crack opening angle by

$$D_g \approx \frac{(\text{COA})}{\cos^2\theta_c},$$  \hspace{1cm} (4)

where $\theta_c$ is the crack orientation. In addition, the above defined quantity is the normalized compliance requirement for fracture-stable design:

Compliance of surrounding $< \frac{D_g l_0}{P_{\text{max}}}$.  \hspace{1cm} (5)
The crack ductility, given in the second row, is smaller for the asymmetric case by a factor ranging from 3.4 for the lower hardening HY100 to 1.1 for the higher hardening A36 hot rolled steel. Notice that in the lower hardening alloys 1018 CF, HY-80, HY-100 steel the factor by which the ductility is reduced is larger than three which shows also that these alloys have much larger stiffness requirements for stability. A comparison of $D_g$ among the alloys reveals that, in the asymmetric case, the crack growth rate in the lower hardening alloys is about 2 times larger than in the higher hardening alloys. In the symmetric case, on the contrary, the crack growth rate is practically insensitive to strain hardening.

The third row is a parameter analogous to the "tearing modulus" $T$ of Paris et al. [4] defined in terms of the yield or tensile strength $\sigma_0$, the modulus $E$ and the $J$-integral by:

$$T = \frac{E}{\sigma_0^2} \frac{dJ}{dc}.$$  \hspace{1cm}  (6)

To approximate the $J$-integral, consider the simple case of the far-field displacement taking place along a single shear band [2] and express it in terms of the shear strength, $k$, and the displacement along the band $u\sqrt{2}$,

$$J = ku\sqrt{2},$$  \hspace{1cm}  (7)

and thus define a parameter, $T^*$, analogous to the tearing modulus $T$, which allows comparing the ductility of alloys of different strength. In terms of the tensile strength $T.S. \approx k\sqrt{3}$, by:

$$T_{\text{asym}}^* = \frac{E/\sqrt{3}}{(T.S.)} D_g.$$  \hspace{1cm}  (8)

In the symmetric case the expression for the $J$-integral, $J=2ku$ [5] leads to an analogous to (8) expression.
\[ T_{\text{sym}}^* = \left( \frac{2E/\sqrt{3}}{\text{T.S.}} \right) D_g. \]  

\[ T_{\text{asym}}^* \] is about 3 times larger for the higher hardening alloys, as is shown in Table 3.

The load-displacement curve of Fig. 4 can be described in terms of the initial elastic compliance, the idealized initiation displacement \( u_i^I \), and the minimum gauge displacement per unit crack advance (steepest slope of the falling part), \( D_{\text{ext}} \), given by

\[ D_{\text{ext}} = \left( \frac{du/l_0}{dP/P_{\text{max}}} \right)_{\text{min}} \simeq \left( \frac{du}{dl} \right)_{\text{min}}. \]  

This definition includes the effect of the compliance in the shoulders and is thus smaller than \( D_g \). Results for 25 mm gauge length are given in the fourth row of Table 3.

**Growth Displacement.** The growth displacement until the fracture breaks through the back of the specimen, \( u_g \), can be found from the fracture surface profiles. The normalized displacements during crack growth, \( u_g/l_0 \), is more than 3 times larger in the symmetric than the asymmetric specimens for the lower hardening alloys but only about 18\% larger for the higher hardening A36 HR steel. It is also larger by about a factor of two in the higher hardening relative to that of the low hardening alloys.

The far-field displacement vector angle from the transverse in the asymmetric case, defined from the slope of the transverse-axial displacement curves, is found to be greater than 45° and larger initially in the lower hardening alloys. As the crack grows the displacement vector becomes less axial (Fig. 7a). The final orientation \( \theta_c \), measured after fracture from the microscope plots (Fig. 5) is between 53° and 63°.
larger for the higher hardening case. The fact that the axial component of the
displacement is larger than the transverse one suggests a Mode I mixity of the local
plastic flow.

Crack direction. In the symmetric specimens the crack runs within ten degrees
of horizontal except for shear lips near the ends of the cracks. In the lower
hardening alloys, even with the symmetric geometry, the fracture turned often into
the asymmetric mode, the fracture advancing close to the $45^\circ$ slip plane or, in some
cases, half of the cross section following the one and the other half following the
other slip plane. In the asymmetric specimens the crack progresses at an angle of
about $38^\circ-41^\circ$ from the transverse. This smaller than $45^\circ$ angle was expected from
the higher triaxiality. In the lower strain hardening alloys the crack grows closer to
the $45^\circ$ band, at $40^\circ-41^\circ$ from the transverse and in the higher hardening alloys at
$38^\circ-39^\circ$ (Table 3). Finally more blunting occurred with the higher hardening alloys
and in the symmetric case.

Load. To summarize the load performance, a parameter dealing with the
maximum load will be defined. The nominal load-carrying ability is simply the
tensile strength multiplied by the net area at the end of the fatigue crack (and
corrected by the plane strain factor). In terms of the initial ligament $l_0$, the width $w$, and the tensile strength T.S.,

$$P_{\text{nom}} = l_0 w (\text{T.S.}) (2/\sqrt{3}).$$

A load factor $F_L$ can be defined in terms of the actual maximum load $P_{\text{max}}$ as:

$$F_L = P_{\text{max}} / P_{\text{nom}}.$$
case, where the overall deformation is bigger, and in the higher hardening alloys.

The normalized load-normalized displacement (and transverse-axial displacement for the asymmetric case) curves, obtained for the lower hardening HY-100 steel and higher hardening A36 HR steel for both the asymmetric and symmetric case are shown in Figs. 7a, 7b, 8a and 8b. For the HY-100 steel, notice the sharp increase in the slope of the falling part of the load-displacement curve of the asymmetric case relative to that of the symmetric; this is not the case in the A36 HR steel. The microscope plots for these alloys are shown in Figs. 9a, 9b, 10a and 10b. In the HY-100 steel, notice the large reduction in the crack opening angle of the asymmetric case relative to that of the symmetric, whereas in the A36 steel the difference in the crack opening angle between the two geometries is not appreciable.

Size effects. To investigate size effects, tests were performed in 38.1 mm. dia. specimens of 5086-H111 aluminum and the results were compared with those from the 12.7 mm. specimens. Table 4 summarizes the results Comparing with the data given in Table 3 for the smaller 5086-H111 specimens, we conclude that the ductility and the normalized growth displacement is only 4% smaller for the larger specimens and the load factor is slightly larger. Notice that the size effects that were predicted in [2] are associated with a transient behavior (increasing crack advance per unit far field displacement).

Marking the crack front. In the large 38.1 mm dia. 5086-H111 aluminum specimens, the crack front was marked by imposing unloading-loading cycles at selected points during crack advance. The spacing of these fatigue marks was measured with a stereo microscope at about 50x. The corresponding displacements were then obtained from the load-displacement curves. In this manner, points on the
The c-u curve can be accurately determined. Figure 11a and 11b show the load-displacement curves and in Figure 12 the corresponding crack growth-displacement data.

**Comparing with theoretical formulations.** An approximate formulation for the accumulation of damage ahead of an asymmetric crack, based on strain increments following a power law relationship was presented by McClintock and Slocum [2]. The crack was assumed to progress along the 45° shear band (pure Mode II) with the far field displacement along the shear band. The initiation displacement was expressed by:

\[ u_i = \frac{\sigma_f}{k} I_{1/n} \rho \left( \frac{\gamma_c}{2\epsilon} \right)^{n+1}, \quad (13) \]

where \( \gamma_c \) is the fracture strain, \( k \) is the shear strength, \( \epsilon = 0.88 \) for the assumed pure Mode II and \( I_{1/n} = 0.72-0.83 \) for \( n = 0.1-0.2 \). The initiation displacement is thus of the order of the inclusion spacing (0.010 mm), much smaller than the one found experimentally. This discrepancy is due to the blunting that occurs during crack initiation. For a quasi-steady growth, the crack advance per unit displacement was practically insensitive to the strain hardening exponent \( n \) and was found:

\[ \frac{d(u/u_i)}{d(c/\rho)} = \frac{n+1}{\ln[(c-c_i)/\rho + \exp(n+1)]}. \quad (14) \]

The above formula, for a mean inclusion spacing \( \rho = 0.01 \) mm and growth by the ligament length of \( l_0 = 2.54 \) mm, gives \( du/\rho \approx 0.200 \), which is closer to the test data for the higher hardening alloys. Equation (14) underestimates the crack growth rate in the lower hardening alloys by a factor of two. For the size effects, predicted in [2], use the mean inclusion spacing of about 10 microns and find the ratio of the crack growth rate for the large 38.1 mm specimens (initial ligament \( l_0 = 7.62 \) mm) to that of the small 12.7 mm (initial ligament \( l_0 = 2.54 \) mm) ones as
Thus the resulting from the integration of stationary crack fields [2] increasing crack advance per unit displacement (associated with the strain distribution flattening out in front of the crack at a decreasing rate) leads to larger size effects than those experimentally observed. Notice, however, that a solution based on a superposition of stationary singularities does not take into account the hardening of the material left behind the growing crack. More realistically, a corresponding fully-plastic, incremental plasticity solution should be obtained for a growing mixed mode crack.

To study the directional effects, an incremental solution was developed in chapter 2. The far field displacement was again assumed to be along the shear band. At the initiation and at each growth step several sites around the current crack tip were considered and the crack was assumed to advance to the direction requiring the least far-field displacement to reach critical damage. The program predicted that a smaller strain-hardening coefficient would cause the crack to grow closer to the shear band and this was confirmed from the experimental results. It also gave 6-15% higher crack growth rates (the larger increase with less hardening) than the pure Mode II [2] solution and thus closer to the experimental findings. The effect of strain hardening was very small, although it was correctly found that a lower strain hardening increases the crack growth rate. The 45° shear band gave however a crack angle of 21° from the transverse. The experimentally found angle of approximately 40° from the transverse can be obtained by assuming a 65° shear band. Notice however that the displacement vector angles (Table 3) suggest that we cannot assume the far-field displacement taking place along a 45° shear band as this model did.
CONCLUSIONS

In asymmetrical configurations with only a single shear band, (which can occur with cracks near welds for example), the crack progresses into prestrained material instead of the new material between the two shear bands of the symmetric case. Experiments on six alloys have shown that the resulting reduction in ductility is primarily dependent on the strain hardening exponent. In the lower hardening alloys the crack ductility, defined as the minimum displacement per unit ligament reduction, in the asymmetric case is less that a third that of the symmetric one but in the higher hardening alloys the reduction is no more than 20%. The high crack growth rate of the asymmetric configuration leads also to correspondingly higher stiffness requirements for fracture-stable design. The initiation displacement is not much different and a fair amount of blunting was observed during initiation for both cases. The crack growth direction is $38^0 - 41^0$ from the transverse (instead of $45^0$) as expected from triaxiality, the higher angles with the smaller strain hardening. The displacement vector is at about $51^0 - 63^0$ from the transverse. Angles greater than $45^0$ suggest a Mode I component, even with asymmetry.
REFERENCES


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<td>1018 steel</td>
<td>586</td>
<td>600</td>
<td>157</td>
<td>49.3</td>
</tr>
<tr>
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<td></td>
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</tr>
<tr>
<td>0.15-0.20% C,</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>0.60-0.90% Mn</td>
<td></td>
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<tr>
<td>1018 steel</td>
<td>321</td>
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<td>61.7</td>
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<tr>
<td>normalized</td>
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<tr>
<td>at 1700°F</td>
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<tr>
<td>in argon flow</td>
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<tr>
<td>A36 steel</td>
<td>281</td>
<td>348</td>
<td>105</td>
<td>61.1</td>
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<tr>
<td>hot rolled</td>
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<tr>
<td>0.29% C max,</td>
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<tr>
<td>0.60-0.90% Mn</td>
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</tr>
<tr>
<td>HY80 steel</td>
<td>587</td>
<td>692</td>
<td>175</td>
<td>69.9</td>
</tr>
<tr>
<td>0.18% C, 2-3.25% Ni,</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>0.10-0.40% Mn,</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.15-0.35% Si</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HY100 steel</td>
<td>693</td>
<td>772</td>
<td>195</td>
<td>68.6</td>
</tr>
<tr>
<td>0.20% C, 2.25-3.50% Ni,</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>0.10-0.40% Mn,</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.15-0.35% Si</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5086-H111 aluminum</td>
<td>210</td>
<td>264</td>
<td>70</td>
<td>45.8</td>
</tr>
<tr>
<td>4% Mg, 0.4% Mn,</td>
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<td></td>
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</tr>
<tr>
<td>0.15% Cr</td>
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</tbody>
</table>
TABLE 2

Stress-strain equation parameters.

<table>
<thead>
<tr>
<th>Alloy</th>
<th>$\sigma_1$</th>
<th>$\epsilon_0$</th>
<th>n</th>
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<td>796</td>
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<tr>
<td>1018 steel</td>
<td>818</td>
<td>0.01718</td>
<td>0.23</td>
</tr>
<tr>
<td>normalized</td>
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<td></td>
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<tr>
<td>A36 steel</td>
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<td></td>
<td></td>
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<tr>
<td>HY80 steel</td>
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<td>0.00702</td>
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<td>1180</td>
<td>0.00488</td>
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<tr>
<td>5086-H111</td>
<td>589</td>
<td>0.00554</td>
<td>0.19</td>
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<tr>
<td>aluminum</td>
<td></td>
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</tr>
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</table>
TABLE 3 - TEST RESULTS (Ligament $l_0=2.54$ mm)

<table>
<thead>
<tr>
<th>Alloy</th>
<th>1018 CF</th>
<th>HY80</th>
<th>HY100</th>
<th>1018 norm.</th>
<th>A36 HR</th>
<th>5086-H111</th>
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</thead>
</table>

**INITIATION**

Idealized initiation displacement, $u_i/l_0$ (Fig. 4)

<table>
<thead>
<tr>
<th></th>
<th>Sym</th>
<th>Asym</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.072</td>
<td>0.108</td>
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<tr>
<td></td>
<td>0.073</td>
<td>0.110</td>
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</tbody>
</table>

**DUCTILITY MEASURES**

Crack Growth Ductility, $D_g$, eq. (3), $\sim(du_c/dt)_{\text{min}}$

<table>
<thead>
<tr>
<th></th>
<th>Sym</th>
<th>Asym</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.233</td>
<td>0.072</td>
</tr>
<tr>
<td></td>
<td>0.320</td>
<td>0.096</td>
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</tbody>
</table>

Modified Tearing Modulus, $T^*$, eqs. (8), (9), $\propto(E/T.S.)D_g$

<table>
<thead>
<tr>
<th></th>
<th>Sym</th>
<th>Asym</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>14.1</td>
<td>14.1</td>
</tr>
<tr>
<td></td>
<td>16.2</td>
<td>16.2</td>
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</tbody>
</table>

Min. extension rate, $D_{ext}$, eq. (10), $\sim(du/dt)_{\text{min}}$ (25mm gauge length)

<table>
<thead>
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<th></th>
<th>Sym</th>
<th>Asym</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.199</td>
<td>0.046</td>
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<tr>
<td></td>
<td>0.285</td>
<td>0.060</td>
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</tbody>
</table>

**DISPLACEMENTS from fracture profiles, Fig. 5**

Initiation Displ., $u_i/l_0$

<table>
<thead>
<tr>
<th></th>
<th>Sym</th>
<th>Asym</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.021</td>
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<tr>
<td></td>
<td>0.051</td>
<td>0.072</td>
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</tbody>
</table>

Growth Displ., $u_g/l_0$

<table>
<thead>
<tr>
<th></th>
<th>Sym</th>
<th>Asym</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.262</td>
<td>0.084</td>
</tr>
<tr>
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<td>0.362</td>
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Displacement vector angle, $\phi$

<table>
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<tr>
<th></th>
<th>Sym</th>
<th>Asym</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>51°</td>
<td>51°</td>
</tr>
<tr>
<td></td>
<td>55°</td>
<td>55°</td>
</tr>
</tbody>
</table>

Crack Direction, $\theta_c=(\theta_u+\theta_I)/2$ (Fig. 5)

<table>
<thead>
<tr>
<th></th>
<th>Sym</th>
<th>Asym</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>41°</td>
<td>41°</td>
</tr>
<tr>
<td></td>
<td>40°</td>
<td>40°</td>
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</table>

Load Factor, $F_L=P_{\text{max}}/l_0(T.S.)(2/\sqrt{3})$

<table>
<thead>
<tr>
<th></th>
<th>Sym</th>
<th>Asym</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.02</td>
<td>0.88</td>
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<tr>
<td></td>
<td>1.16</td>
<td>1.05</td>
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</table>

(continued)
### TABLE 4 - RESULTS FROM LARGE SPECIMENS (Ligament $l_0=7.62$ mm)

<table>
<thead>
<tr>
<th>Alloy</th>
<th>5086-H111 Aluminum</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>INITIATION</strong></td>
<td></td>
</tr>
<tr>
<td>Idealized initiation displacement, $u_i^L/l_0$</td>
<td></td>
</tr>
<tr>
<td>Sym</td>
<td>0.081</td>
</tr>
<tr>
<td>Asym</td>
<td>0.072</td>
</tr>
<tr>
<td><strong>DUCTILITY MEASURES</strong></td>
<td></td>
</tr>
<tr>
<td>Crack Ductility, $D_g$</td>
<td></td>
</tr>
<tr>
<td>Sym</td>
<td>0.165</td>
</tr>
<tr>
<td>Asym</td>
<td>0.105</td>
</tr>
<tr>
<td>Modified Tearing, Modulus, $T^*$</td>
<td></td>
</tr>
<tr>
<td>Sym</td>
<td>43.4</td>
</tr>
<tr>
<td>Asym</td>
<td>13.8</td>
</tr>
<tr>
<td>Min. extension rate, $D_{ext}$ for 25 mm gauge length</td>
<td></td>
</tr>
<tr>
<td>Sym</td>
<td>0.118</td>
</tr>
<tr>
<td>Asym</td>
<td>0.080</td>
</tr>
<tr>
<td><strong>DISPLACEMENTS from fracture profiles</strong></td>
<td></td>
</tr>
<tr>
<td>Initiation Displ., $u_i^L/l_0$</td>
<td></td>
</tr>
<tr>
<td>Sym</td>
<td>0.026</td>
</tr>
<tr>
<td>Asym</td>
<td>0.024</td>
</tr>
<tr>
<td>Growth Displ., $u_g/l_0$</td>
<td></td>
</tr>
<tr>
<td>Sym</td>
<td>0.280</td>
</tr>
<tr>
<td>Asym</td>
<td>0.133</td>
</tr>
<tr>
<td>Displacement vector angle, $\phi$</td>
<td></td>
</tr>
<tr>
<td>Sym</td>
<td>$(\approx 90^0)$</td>
</tr>
<tr>
<td>Asym</td>
<td>$57^0$</td>
</tr>
<tr>
<td>CRACK DIRECTION, $\theta_c$</td>
<td></td>
</tr>
<tr>
<td>Sym</td>
<td>$(\approx 0^0)$</td>
</tr>
<tr>
<td>Asym</td>
<td>$40^0$</td>
</tr>
<tr>
<td>LOAD FACTOR, $F_L$</td>
<td></td>
</tr>
<tr>
<td>Sym</td>
<td>1.21</td>
</tr>
<tr>
<td>Asym</td>
<td>1.18</td>
</tr>
</tbody>
</table>
Figure 1. Asymmetric and Symmetric shear from cracks.
Figure 2a. True Stress-True Plastic Strain for HY-80, HY-100 and A36 Hot Rolled Steel. Dashed lines indicate Bridgman necking correction from the semiempirical relations.
Figure 2b. True Stress-True Plastic Strain for 1018 Cold Finished, 1018 Normalized Steel and 5086-H111 Aluminum. Dashed lines indicate Bridgman necking correction from the semiempirical relations.
Figure 3b  Second Machining (after fatigue precracking) for the Asymmetric Specimen: \( a \) is the fatigue crack.
Figure 3c. Second Machining (after fatigue precracking) for the Symmetric Specimens.
Figure 4. Schematic of the load vs. axial gauge-point displacement curve. The displacements $u_g$ and $u_1$ are measured after fracture.
Figure 5. Schematic of the Fracture Surface Profile for the Asymmetric and Symmetric cases.
Figure 6. Deriving the relation between the crack opening angle and the crack growth rate.
Figure 7a. Test Data for the HY-100 Steel Asymmetric Specimens.
Figure 7b. Test Data for the HY-100 Steel Symmetric Specimens.
Figure 8a. Test Data for the A36 Hot Rolled Steel Asymmetric Specimens.
A36 HR Steel
Symmetric

Normalized Load, $P/P_{nom}$

Normalized Axial Displacement, $u_a/l_o$

Figure 8b. Test Data for the A36 Hot Rolled Steel Symmetric Specimens.
Figure 1a  Fracture Surface Profile for the HY-100 Steel Asymmetric Specimens
HY-100 Steel
Symmetric

Figure 9b Fracture Surface Profile for the HY-100 Steel Symmetric Specimens.
Figure 10a Fracture Surface Profile for the A36 Hot Rolled Steel Asymmetric Specimens
Figure 10b Fracture Surface Profile for the A36 Hot Rolled Steel Symmetric Specimens
Figure 11a. Test Data for the 38.1 mm dia. 5086-H111 asymmetric specimens showing the unloading-loading points for marking the crack front.
Figure 11b. Test Data for the 38.1 mm dia. 5086-H111 symmetric specimens showing the unloading-loading points for marking the crack front.
Figure 12. Crack advance-displacement data for the specimens of Figs 11a, 11b. The fatigue marks provided the crack positions.

Projected Crack Growth $C_x/l_x$ vs. Axial Displacement, $u_a/l_a$.
APPENDIX

FRACTOGRAPHIC OBSERVATIONS IN ASYMMETRIC AND SYMMETRIC FULLY PLASTIC SPECIMENS.

Observations of ductile fracture suggest that it results from a multi-step process initiated by the cracking of inclusions or the separation of inclusion-metal interfaces, followed by void growth and coalescence. The coalescence has been observed to occur on a plane of high shear stress, giving elongated dimples form, or on a plane normal to the direction of maximum tensile stress, giving equiaxed dimples [1]. Furthermore, fracture surfaces have been studied to identify and classify the characteristic surface markings that are produced by the deformation mechanisms [2].

Tests on symmetric and asymmetric specimens were performed on six alloys for which X-ray spectrography gave the predominant inclusions: 1018 cold finished steel with Si-bearing inclusions, 1018 steel normalized at 1700\(^0\), A36 hot rolled steel with MnS inclusions, HY80 steel with Al-bearing inclusions, HY100 steel with MnS inclusions and 5086-H111 aluminum with Fe-bearing inclusions. These alloys can be separated into the lower hardening ones (1018 cold finished, HY80 and HY100 steel) and the higher hardening ones (A36 hot rolled, 1018 normalized steel). In this work the microscopic features of the fracture surface for the two geometries are quantitatively compared.

In general, for a given crack tip opening displacement, the amount of crack extension can be separated into two components: a sliding off component and a fracture component. To quantify the ductility, as observed from the fractographs, an "apparent crack ductility", \( D_{AC} \), observed fractographically, can be defined as
the ratio of that part of the projected crack area exposed by pure plastic flow to the total projected area, including that exposed by fracture. For instance, with \( n \) parabolic dimple markings per unit area, each having tip radius \( r \), the apparent crack ductility may be found by assuming that the area \( \pi r^2 \) of each parabola opens up before arrival of the crack front, and the balance of the surface is formed by sliding off. Then \( D_{AC} = 1 - n \pi r^2 \). Due to the difficulty in measuring these quantities, only rough approximations for \( D_{AC} \) can be obtained. Table 1 shows these approximate findings (estimated from the lower flank fractographs, surface normal to the beam) for the lower hardening HY100 steel and the higher hardening 1018 normalized steel from the asymmetric and symmetric specimens which are more ductile. These results are another manifestation of the fact that higher hardening alloys are more ductile than the lower hardening ones in the asymmetric configuration but almost equally ductile in the symmetric one.

**TABLE 1 - Apparent Crack Ductility \( D_{AC} \)**

<table>
<thead>
<tr>
<th>Steel Type</th>
<th>Asymmetric</th>
<th>Symmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td>HY-100 steel</td>
<td>0.51</td>
<td>0.64</td>
</tr>
<tr>
<td>1018 normalized steel</td>
<td>0.68</td>
<td>0.67</td>
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</tbody>
</table>

Fig. 1 shows micrographs of the upper and lower flanks for 5086-H111 aluminum with different degrees of void formation and shearing. Fracture is more "shear type" in the lower flank, indicating a larger sliding off component in the crack extension. This suggests a macro-mechanical model for crack growth by combined void growth and sliding off, where the lower flank slides off along the upper slip plane and the upper flank slides off along the lower slip plane by a smaller amount. Thus the lower flank consists of a larger amount of "sheared" material
than the upper.

A qualitative understanding of the differences in ductility from the fractographs can be obtained by comparing in Fig. 2, the micrograph for the less ductile HY100 asymmetric specimen with the corresponding one for the more ductile higher hardening 1018 normalized steel (larger and less elongated voids).

To compare the symmetric and asymmetric cases, consider Fig. 3 which shows micrographs of the low-hardening 1018 cold finished asymmetric and symmetric specimens. This alloy shows a substantial reduction in ductility in the asymmetric configuration. In the asymmetric case the fracture is more the "shear type" with voids elongated in the direction of crack growth; in the symmetric case the fracture is more the "normal type" with equiaxed voids. In the high-hardening A36 hot rolled steel, with small differences in the ductility between the asymmetric and the symmetric cases, the corresponding micrographs (Fig. 4) are not much different: the fracture in the asymmetric case is almost as much the "normal type" as in the symmetric case.

"Zig-zagging" of the fracture surface is a characteristic of some symmetric specimens, where two slip planes are active and the crack grows by alternating shear. Fig. 5 shows this for the 5086-H111 aluminum. The wavy (zig-zag) region followed the fatigue precrack. In the end the fracture turned into a shear lip. Symmetric specimens in the lower hardening alloys often turned into asymmetric ones, following only one slip plane. In some instances, half of the specimen followed the $+45^\circ$ slip plane and half the $-45^\circ$ plane.

In conclusion, fractographic observation of deformation during crack extension
in the asymmetric specimens suggests a mechanism by fracture followed by a
different amount of sliding off in the two flanks. The usual symmetric case suggests
alternating shear and fracture and in some cases the macroscopic surface is
characterized by zig-zagging. Noteworthy features in the asymmetric specimens are
the "shear type" fracture, more evident in the lower hardening alloys and a larger
amount of sliding off in the lower flank. The symmetric specimens, with the larger
ductility, show in turn the "normal type" fracture with more equiaxed voids than
the corresponding asymmetric specimens.

REFERENCES

1. Hayden H.W. and Floreen S. "Observations of Localized Deformation

2. Beachem C.D. and Meyn D.A. "Fracture by Microscopic Plastic
59-88 (1968).
Figure 1 Fracture surface of 5086-H111 aluminum asymmetric specimen showing the difference between the two flanks
(a) Upper flank, (b) Lower flank with more shearing
Figure 2  Fracture surface of the upper flank of asymmetric specimens in
(a) HY-100 steel (lower hardening, less ductile)
(b) 1018 normalized steel (higher hardening, more ductile)
Figure 3  Fracture surface of 1018 cold finished steel (lower hardening alloy). (a) Asymmetric, (b) Symmetric    Distinctly more "shear type" fracture in the asymmetric, less ductile case
Figure 4 Fracture surface of A36 hot rolled steel (higher hardening alloy). (a) Asymmetric. (b) Symmetric. Without appreciable difference. Both cases have almost the same ductility.
Figure 5  Fracture surface of 5086-H111 aluminum symmetric specimen showing the macroscopic roughness of the specimen. After the fatigue crack a "wavy" region (shown better in (b)) and finally a shear lip at the end.
APPENDIX

ON FRACTURE CHARACTERIZATION

In this section the concepts that are commonly used in characterizing fracture are discussed.

Growth Resistance. Representing ductile crack propagation has been based on the introduction of $d(COD)/da$ [1,2,3] and the tearing modulus $T$ or $dJ/da$ concept [4]. In general, past work has implied that the results are all characterized by a common triaxiality (or are independent of it) whereas in fact both $d(COD)/da$ and $T$ should depend on triaxiality, since both cleavage and hole growth do. Analogous to the above measures of growth resistance are the previously defined crack ductility $D_g = du/c_0/ds$ or gauge displacement per unit reduction in ligament $D_{ext} = du/ds$, and the modified tearing modulus $T^*$ to include asymmetric cracks. Table 1 gives some values of $d(COD)/da$ and $T$ for some common tests. They can be compared with the much lower values of about 0.010 for $D_g$ and 0.060 for $D_{ext}$ and of about 15 for $T^*$ found in the asymmetric low hardening tests. Symmetric tests, on the other hand, show values of $D_g$ and $D_{ext}$ close to those in Table 1. It may also be that an asymmetry, introduced in the bending specimens could result in values of $d(COD)/da$ below those of the symmetric bending specimens.

J-controlled growth. In large-scale yielding the HRR singularity is embedded in a plastic zone that extends throughout the remaining ligament. J-control depends on material and crack geometry (McMeeking and Parks [7]). The finite element study of the asymmetric specimens showed that stress and strain fields are consistent with the HRR singularity at initiation. The COA (crack opening angle) concept has also been alternatively used to characterize growth. Shih [1] in his experimental study found that the COA appears to be constant over a larger range of growth than the
tearing modulus. Instead of $dJ/da$, Shih [1] suggested characterizing growth by COA. A condition in terms of a tearing modulus $T_\delta$ based on the COA ($=d\delta/da$):

$$T_\delta = \frac{d\delta E}{da \sigma_0} >> 1$$

In other words the COA, $d\delta/da$, must be large compared to the yield strain, $\sigma_0/E$. In similar fashion, the tearing modulus $T$ should be much larger than unity. How large $T$ or $T_\delta$ must be for a J-controlled or COA controlled growth is yet to be explored. COA's of more than $10^0-20^0$, were reported in the COA-controlled tests in [1].

**Stability.** Stability depends again on triaxiality and geometry. Paris et al. [5] developed instability relations for fully-plastic (nonhardening) conditions including some common test piece configurations. For example, in the double edge cracked strip in tension the imposed constraint leads to a critical value of $T$ for instability six times that in the center cracked strip in tension. In the expression for the 3-point bending case the remaining ligament size, $l$, comes into the instability criterion, so that if $l$ is small enough in the first place the situation remains stable throughout.

In conclusion, single-test characterization of crack propagation can apply only if crack extension occurs in a certain mode and configuration. Instead of a single parameter representation like $d$(COD)/$da$, a set of $d$(COD)/$da$, each referring to a certain mode and triaxiality, could conceivably describe adequately the material resistance in crack propagation. For instance, asymmetric (mixed mode I and II) fully plastic configurations in low hardening alloys have been found less ductile than the corresponding symmetric singly grooved unconstrained tensile specimens. Extended work could involve studying the effect of triaxiality by performing constrained asymmetric tests. For example, tensile testing on doubly-grooved specimens with the asymmetry introduced through varying notch angles and
positions; or wedge-splitting of a doubly grooved specimen; or ductile fracture under asymmetric bending with the asymmetry introduced not only by specimen geometry but also through shear loading.

REFERENCES


<table>
<thead>
<tr>
<th>Material</th>
<th>Test</th>
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<th>T</th>
<th>Ref.</th>
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<td>0.205</td>
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</tr>
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<td>tension 4T</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T.S. = 574 MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ASTM A471 rotor steel</td>
<td>3-point bend</td>
<td>36.1</td>
<td></td>
<td>[4]</td>
</tr>
<tr>
<td>Y.S. = 931 MN/m²</td>
<td>8 x 1 x 0.5 in</td>
<td></td>
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<tr>
<td>T.S. = 1022 MN/m²</td>
<td>Notched and Fatigue</td>
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<tr>
<td></td>
<td>precracked, a/w = 0.502</td>
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<tr>
<td>Free-cutting</td>
<td>3-point bend</td>
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<td>[3]</td>
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<tr>
<td>Mild steel</td>
<td>fatigue precracked</td>
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<tr>
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<td>3-point single</td>
<td>0.250</td>
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<td>[2]</td>
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<tr>
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<td>T.S. = 526 MN/m²</td>
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<td>Asymmetric and Symmetric tensile tests.</td>
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<tr>
<td>1018 CF steel</td>
<td>Asymmetric</td>
<td>0.072 (0.046)*</td>
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<tr>
<td>Y.S. = 586 MN/m²</td>
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<tr>
<td>T.S. = 612 MN/m²</td>
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<td>0.233 (0.199)</td>
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<tr>
<td>HY80 steel</td>
<td>Asymmetric</td>
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<tr>
<td>Y.S. = 587 MN/m²</td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>T.S. = 708 MN/m²</td>
<td>Symmetric</td>
<td>0.320 (0.285)</td>
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</tbody>
</table>

*Values in parentheses are based on gauge extension.
CHAPTER FOUR

SHEAR-BAND CHARACTERIZATION OF MIXED
MODE I AND II FULLY PLASTIC CRACK GROWTH

TABLE OF SYMBOLS

- \( f \): amount of fracture
- \( s_l \): amount of slip along lower slip plane
- \( s_u \): amount of slip along upper slip plane
- \( \theta_f \): fracture plane
- \( \theta_{sl} \): lower slip plane
- \( \theta_{su} \): upper slip plane
- \( \chi \): fracture parameter \((=f/s_u)\)
- \( \xi \): shearing parameter \((=s/s_u)\)
- \( l_i \): projected lower flank length
- \( l_u \): projected upper flank length
- \( l_0 \): initial ligament
- \( \theta_u \): upper flank angle from transverse
- \( \theta_l \): lower flank angle from transverse
- \( \omega \): crack opening angle
- \( u \): axial displacement
- \( \phi \): displacement vector angle from transverse
- \( \beta_u \): upper back angle
- \( \beta_l \): lower back angle
- \( \gamma_u \): strain for upper shearing
- \( \gamma_l \): strain for lower shearing

ABSTRACT

Asymmetric fully plastic specimens give higher crack growth rates and thus smaller deformation to fracture than the corresponding symmetric specimens. A macro-mechanical model of crack growth by combined fracture on one plane and sliding off along two others describes, for this idealization of the physical mechanisms, the ductile crack growth for the general mixed mode I, II case. The analysis allows determining the independent physical parameters (shear and cracking...
directions, relative amounts of cracking and shearing) in terms of the observable quantities of the macroscopic fracture (flank angles, flank lengths, back angle).

INTRODUCTION

In studying fracture there is a need for understanding and characterizing the deformation and crack growth in the fully plastic range for both the usual symmetric case and for the asymmetric case shown in Fig. 1. The asymmetric configuration may occur near welds due to the constraint of a heat-affected zone or due to some geometric asymmetry, such as near-by shoulders. These cracks exhibit less ductility than symmetric ones, because the crack is advancing into prestrained and damaged material rather than into the new material encountered by a crack advancing between two symmetric shear bands.

The nonhardening rigid plastic flow field of Fig. 1b consists of a single slip line at 45°. Strain hardening, however, causes the deformation field to fan out. It also leads to adding a Mode I component, as suggested by the direction of the far field displacement being more axial than 45°. To account for the presence of the Mode I component and the spreading out of the deformation in the more general asymmetric case we assume two slip planes at arbitrary angles.

Crack growth is a mixture of sliding off and fracture. In the general case it may be idealized by assuming cycles of first sliding off on the upper slip plane, then on the lower, and finally fracture on possibly a third (Fig. 2). The combination of cracking and sliding off gives the two new surfaces of the macro fracture. These define the crack opening angle and the crack direction. In the symmetric case the two slip planes and the fracture are symmetric. These ideas will now be developed.
quantitatively, giving a description of the mixed mode ductile crack growth based on an idealization of the underlying physical mechanisms. The single-band pure Mode II asymmetric and the pure Mode I symmetric behavior can be obtained as limiting cases.

ANALYSIS

Consider lower and upper slip planes at angles $\theta_{sl}$ and $\theta_{su}$ (Fig. 2). The upper crack flank is formed by sliding off along the lower slip plane through a distance $s_i$ at $\theta_{si}$ combined with fracture over a distance $f$ at an angle $\theta_f$. The lower flank is formed by sliding off along the upper slip plane at $\theta_{su}$ and the fracture $f$ at $\theta_f$. A cracking parameter $\chi = f/s_u$ and a shearing parameter $\xi = s_i/s_u$ can be defined. As independent physical variables consider the cracking and shearing parameters $\chi$ and $\xi$, the fracture angle $\theta_f$ and the slip angles $\theta_{si}$, $\theta_{su}$. The limiting case of Mode I, with two symmetric slip lines corresponds to $\theta_f=0$, $\theta_{su}=-\theta_{si}$, $s_i=s_u$ and the limiting Mode II case of slip on a single plane corresponds to $s_i=0$.

Observable quantities that allow solving for the above physical variables turn out to be the angles between the faces of the crack and the transverse direction $\theta_u$, $\theta_p$, the transverse components of the crack flank lengths after complete separation, normalized with the initial ligament, $l_u/l_0$, $l_p/l_0$, and the angle that the deformed upper back surface makes to the load axis, $\beta_u$ (Fig. 3). Other dependent variables of interest are the crack opening angle (COA), the total axial displacement per initial ligament $u_y/l_0$, the orientation of the displacement vector $\phi$, and the angle that the deformed lower back surface makes to the load axis $\beta_i$ (Fig. 3). These can be deduced from the analysis and observed from the tests, except for the lower back angle which is suppressed by the shoulder.
The orientations of the crack flanks $\theta_u$ and $\theta_l$ from the transverse direction can be found from Fig. 1:

$$
\theta_l = \tan^{-1} \frac{f \sin \theta_f + s_u \sin \theta_{su}}{f \cos \theta_f + s_u \cos \theta_{su}} = \tan^{-1} \frac{\chi \sin \theta_f + \sin \theta_{su}}{\chi \cos \theta_f + \cos \theta_{su}},
$$

(1)

$$
\theta_u = \tan^{-1} \frac{f \sin \theta_f + s_l \sin \theta_{sl}}{f \cos \theta_f + s_l \cos \theta_{sl}} = \tan^{-1} \frac{\chi \sin \theta_f + \xi \sin \theta_{sl}}{\chi \cos \theta_f + \xi \cos \theta_{sl}}.
$$

(2)

The crack opening angle is

$$
\text{COA} = \omega = \theta_l - \theta_u,
$$

(3)

Fig. 4 shows the crack opening angle as a function of $\chi$ and $\xi$ with the slip angle difference $\theta_{su} - \theta_{sl}$ as a parameter. An increasing cracking ratio $\chi = f/s_u$ or shearing ratio $\xi = s_l/s_u$ leads to a decreasing COA (notice that the COA is more sensitive to $\chi$ than $\xi$). A larger slip angle difference, $\theta_{su} - \theta_{sl}$, leads to a larger COA. Large slip angle differences represent spreading out of the deformation and can simulate the effect of a high strain hardening exponent which has been found experimentally to result in a bigger crack opening angle.

The original ligament thickness, $l_0$, projected onto the transverse direction, is reduced to zero by the cracking $f$ and by the sliding off $s_l$ and $s_u$ (Fig. 2):

$$
l_0 = f \cos \theta_f + s_u \cos \theta_{su} + s_l \cos \theta_{sl}.
$$

(4)

The corresponding axial extension is

$$
u_y = s_u \sin \theta_{su} - s_l \sin \theta_{sl}.
$$

(5)

From (4) and (5) the deformation ratio, defined as the total axial displacement per initial ligament, which is a measure of the ductility, is found to be
\[
\frac{u_y}{l_0} = \frac{\sin\theta_{su} - \xi \sin\theta_{sl}}{\chi \cos\theta_f + \cos\theta_{su} + \xi \cos\theta_{sl}} \tag{6}
\]

The deformation ratio behaves as the COA, being higher for a lower \(\chi\) or \(\xi\) and smaller for a larger slip angle difference \(\theta_{su} - \theta_{sl}\). Fig. 5 shows an example of variation of \(u_y/l_0\).

An expression for the flank lengths is desirable because the final projected flank lengths per initial ligament can be measured. The projected upper and lower flank lengths, \(l_u\) and \(l_l\), are given by

\[
l_u = l \cos\theta_f + s_l \cos\theta_{sl}. \tag{7}
\]

\[
l_l = l \cos\theta_f + s_u \cos\theta_{su}. \tag{8}
\]

Using (4) and substituting the expression for the cracking and shearing ratios \(\chi\) and \(\xi\) gives

\[
\frac{l_u}{l_0} = \frac{\chi \cos\theta_f + \xi \cos\theta_{sl}}{\chi \cos\theta_f + \cos\theta_{sl} + \xi \cos\theta_{su}}. \tag{9}
\]

\[
\frac{l_l}{l_0} = \frac{\chi \cos\theta_f + \cos\theta_{su}}{\chi \cos\theta_f + \xi \cos\theta_{sl} + \cos\theta_{su}}. \tag{10}
\]

The lower flank ratio \(l_l/l_0\) decreases with an increasing shearing parameter \(\xi\) in contrast to the upper flank ratio \(l_u/l_0\) which increases (Fig. 6). The amount by which the upper back surface is drawn in, projected along the transverse, is (Fig. 2)

\[
t_u = s_u \cos\theta_{su}. \tag{11}
\]

A "thinning" ratio for the upper surface, \(t_u/l_0\), can be defined in terms of the independent parameters by noting that
\[ t_u/l_0 = 1 - l_u/l_0. \] (12)

For the lower surface similarly,

\[ t_l/l_0 = 1 - l_l/l_0. \] (13)

Of interest is also the orientation of the displacement vector from the transverse, \( \phi \). In terms of the flank lengths and angles, it can be found from

\[ \phi = \tan^{-1} \frac{(l_u/l_0) \tan \theta_f - (l_l/l_0) \tan \theta_u}{(l_l/l_0) - (l_u/l_0)}. \] (14)

The back angle, defined as the angle that the deformed back surface makes to the load axis, can be observed macroscopically. For the upper slip line from Fig. 2

\[ \beta_u = \tan^{-1} \frac{ds_u \cos \theta_{su}}{[df \sin(\theta_{su} - \theta_f) + ds_l \sin(\theta_{su} - \theta_{sl})]/\cos \theta_{su} + ds_u \sin \theta_{su}}. \] (15)

Substituting the expressions for the cracking and shearing ratios we can write

\[ \beta_u = \tan^{-1} \frac{\cos \theta_{su}}{\chi (\cos \theta_f \tan \theta_{su} - \sin \theta_f) + \xi (\cos \theta_{sl} \tan \theta_{su} - \sin \theta_{sl}) + \sin \theta_{su}}. \] (16)

Similarly for the lower surface

\[ \beta_l = \tan^{-1} \frac{ds_l \cos \theta_{sl}}{[df \sin(\theta_{sl} - \theta_f) + ds_u \sin(\theta_{sl} - \theta_{su})]/\cos \theta_{sl} + ds_l \sin \theta_{sl}}, \] (17)

or

\[ \beta_l = \tan^{-1} \frac{\xi \cos \theta_{sl}}{\chi (\cos \theta_f \tan \theta_{sl} - \sin \theta_f) + (\cos \theta_{su} \tan \theta_{sl} - \sin \theta_{su}) + \xi \sin \theta_{sl}}. \] (18)

There are five independent macroscopically observable parameters: the flank angles \( \theta_u, \theta_f \), the projected flank lengths per initial ligament, \( l_u/l_0, l_l/l_0 \), and the back angle \( \beta_u \). Equations (1) for \( \theta_u \), (2) for \( \theta_f \), (9) for \( l_u/l_0 \), (10) for \( l_l/l_0 \), and (16) for \( \beta_u \) give the corresponding physical variables \( \chi, \xi, \theta_f, \theta_{sl}, \theta_{su} \), as described in the Appendix.
Finally, an expression for the shear strain can also be found. It can be expressed in terms of the slip and the normal separation between corresponding slip planes. For the upper shear band, $s_u$,

$$
\gamma_u = \frac{s_u}{f \sin(\theta_{su} - \theta_f) + s_l \sin(\theta_{su} - \theta_{sl})} = \\
\frac{1}{x \sin(\theta_{su} - \theta_f) + \xi \sin(\theta_{su} - \theta_{sl})}, \quad (19)
$$

Similarly, for the lower shearing, $s_l$,

$$
\gamma_l = \frac{s_l}{f \sin(\theta_{sl} - \theta_f) + s_u \sin(\theta_{sl} - \theta_{su})} = \\
\frac{\xi}{x \sin(\theta_{sl} - \theta_f) + \sin(\theta_{sl} - \theta_{su})}, \quad (20)
$$

APPLICATION TO TESTS

Tests were performed on 12.7 mm dia. round bars of six alloys: 1018 cold finished, 1018 normalized, A36 hot rolled, HY-80 and HY-100 steel and 5086-H111 aluminum in both the asymmetric and symmetric configurations. The alloys tested can be grouped into the lower hardening ones (1018 cold finished, HY-80 and HY-100) and the higher hardening ones (A36 hot rolled, 1018 normalized). The lower hardening alloys exhibited a significantly lower ductility in the asymmetric configuration than the symmetric; the higher hardening alloys showed only a small reduction. The profiles of the fracture surface and the deformed back surface were plotted with a travelling stage microscope to obtain the flank lengths, the flank angles, the back angle, the displacement to separation and the orientation of the displacement vector.

To apply the above model to the tests, the projected crack length ratios $l_u/l_0$, $l_l/l_0$ for the upper and lower flanks and the flank angles $\theta_u$, $\theta_l$ were measured from
the profiles of the fracture surface. In addition the back angle for the upper surface
$\beta_u$ was measured from the microscope plots of the back surface. The projected
length ratios depend on the strain hardening exponent, being smaller for a higher
strain hardening. These quantities were used in equations (1), (2), (9), (10), (16) to
yield the cracking parameter $\chi = f/s_u$, the shearing parameter $\xi = s/\tau$, the slip
angles $\theta_{sl}$ and $\theta_{su}$ and the fracture angle $\theta_r$. The axial displacement $u_y/l_0$ and the
orientation of the displacement vector $\phi$ can also be obtained and compared with the
test data. Results are shown in the tables at the end of this chapter. For the
asymmetric specimens the shearing ratio $\zeta$ is found to be about 0.5 indicating
shearing in lower flank about twice that in upper flank. SEM fractographs have
confirmed that the lower flank shows indeed more "shear type" fracture than the
upper one (chapter three). The slip angle difference $\theta_{sl} - \theta_{su}$ is a measure of the
spreading out of deformation and is found to be $4^0-6^0$ in the high hardening alloys as
opposed to $1^0-2^0$ in the low hardening ones. The cracking ratio $\chi$ is a measure of the
relative amount of fracture and sliding off and allows defining an comparing with the
"apparent crack ductility", $D_{AC}$, as the sliding off to total area. Thus,

In upper flank $D_{AC,u} = s/\tau (f+s) = 1/(\chi + 1)$

In lower flank $D_{AC,l} = s_u/\tau (f+s_u) = 1/(\chi + 1)$

A smaller cracking ratio means higher ductility. In the low hardening alloys the
cracking ratio $\chi$ is smaller in the symmetric case, whereas in the higher hardening
alloys it is about the same in both the symmetric and asymmetric configurations.
The cracking ratio $\chi$ in the higher hardening asymmetric specimens is also smaller
that of the low hardening asymmetric ones.

The Mode I symmetric case corresponds to the limit of $\theta_{sl} = \theta_{su}$, $\theta_r = 0$, and
\( \xi = s_i/s_u = 1 \). Results are also shown in the tables at the end of the chapter. The projected flank ratio \( l_u/l_0 = I/\ell_0 \) and the flank orientation \( \theta_l = -\theta_u \) were measured to give the cracking ratio \( \chi \) and the slip angle \( \theta_s \). As dependent variables, the displacement to separation \( u_y/l_0 \), and the back angle \( \beta_u = -\beta_i \) can be found. The displacement is more than twice that of the asymmetric case in the lower hardening HY-100 steel. Fig. 7 shows the variation of the deformation ratio \( u_y/l_0 \) and the crack opening angle \( \omega \) vs. the cracking parameter \( \chi \) with \( \theta_s = \theta_{su} = -\theta_{sl} \) as a parameter.

CONCLUSIONS

In asymmetrical, singly grooved, fully plastic tensile specimens the crack progresses into pre-strained material. This results in less ductility than in symmetrical specimens where the crack grows into new material between two shear bands. A macro-mechanical model for crack advance by sliding off along two slip planes and fracture in the asymmetric specimens gives the independent parameters (shear and cracking directions, relative amounts of cracking and shearing) in terms of the observable quantities of the macroscopic fracture (flank angles, flank lengths, back angle). This two slip plane model accounts for the presence of a Mode I component (far field displacement more axial than \( 45^0 \)) that was experimentally confirmed in the asymmetric case. Higher hardening alloys are found to exhibit more thinning of the ligament (hence smaller projected length ratios), a larger slip angle difference, indicating more fanning out of the deformation and a bigger sliding off component. The analysis, based on an idealization of underlying physical mechanisms, describes the deformation that leads to a larger crack opening angle and displacement to separation in the higher hardening asymmetric specimens relative to the lower hardening ones as well as the symmetric specimens relative to
the asymmetric ones.

REFERENCES


APPENDIX

The problem is to determine $\chi$, $\xi$, $\theta_f$, $\theta_{sl}$, $\theta_{su}$ from the observable quantities $\theta_p$, $\theta_u$, $l_f/l_0$, $l_u/l_0$, $\beta_u$. The relevant equations are summarized:

\[
\begin{align*}
\theta_i &= \tan^{-1} \frac{\chi \sin \theta_f + \sin \theta_{su}}{\chi \cos \theta_f + \cos \theta_{su}}, \\
\theta_u &= \tan^{-1} \frac{\chi \sin \theta_f + \xi \sin \theta_{sl}}{\chi \cos \theta_f + \xi \cos \theta_{sl}}, \\
l_f &= \frac{\chi \cos \theta_f + \cos \theta_{su}}{\chi \cos \theta_f + \xi \cos \theta_{sl}}, \\
l_u &= \frac{\chi \cos \theta_f + \xi \cos \theta_{sl} + \cos \theta_{su}}{\cos \theta_{su}}, \\
l_0 &= \frac{\chi \cos \theta_f + \xi \cos \theta_{sl} + \cos \theta_{su}}{\cos \theta_{su}}, \\
\beta_u &= \tan^{-1} \frac{\chi \cos \theta_f \tan \theta_{su} - \sin \theta_f + \xi \cos \theta_{sl} \tan \theta_{su} - \sin \theta_{sl} + \sin \theta_{su}}{\chi \cos \theta_f + \xi \cos \theta_{sl} + \cos \theta_{su}}.
\end{align*}
\]

For convenience define the upper thinning ratio from (24)

\[
t_u/l_0 = 1 - \frac{\cos \theta_{su}}{\chi \cos \theta_f + \xi \cos \theta_{sl} + \cos \theta_{su}}.
\]

Dividing Eqs (21)-(25) by $\cos \theta_{su}$ leaves them in terms of four parameters, (A, B, C, D defined in the following) plus $\tan \theta_{su}$ that can be solved from the observed variables.

The first is found by dividing (23) by (26):

\[
A \equiv \frac{\chi \cos \theta_f}{\cos \theta_{su}} = l_f/t_u - 1.
\]

Dividing (23) by (24) and introducing A from (27) gives

\[
B \equiv \frac{\xi \cos \theta_{sl}}{\cos \theta_{su}} = (l_u/l_f) (A+1) - A.
\]

Introducing (27) into (21) and rearranging,

\[
C \equiv \frac{\chi \sin \theta_f}{\cos \theta_{su}} = (\tan \theta_f)(A+1) - \tan \theta_{su}.
\]

Now introduce (27), (28), and (29) into (22)
\[
D \equiv \frac{\xi \sin \theta_{sl}}{\cos \theta_{su}} = (\tan \theta_u)(A+B) - C .
\] (30)

From (25) and the above definitions of A,B,C,D,

\[
\tan \beta_u = \frac{1}{(A+B+1) \tan \theta_u - (C+D)} .
\] (31)

From (30) C+D of (31) is given in terms of observed variables. Solve for \( \theta_{su} \)

\[
\tan \theta_{su} = \frac{1 + (A+B)\tan \beta_u \tan \theta_u}{(A+B+1) \tan \beta_u} .
\] (32)

Having found \( \theta_{su} \), find C from (29) and D from (30). Now find \( \theta_f \) from (27) and (29)

\[
\tan \theta_f = C/A ,
\] (33)

and \( \theta_{sl} \) from (28) and (30):

\[
\tan \theta_{sl} = D/B .
\] (34)

Then \( \chi \) is determined from (27) by using the already determined values of \( \theta_{su} \) in (32) and of \( \theta_f \) in (33)

\[
\chi = A \cos \theta_{su}/\cos \theta_f ,
\] (35)

and similarly \( \xi \) is found from (28) by using the values of \( \theta_{su} \) from (32) and \( \theta_{sl} \) from (34)

\[
\xi = B \cos \theta_{su}/\cos \theta_{sl} .
\] (36)
TABLE 1
Deformation of singly-grooved asymmetrical specimens

<table>
<thead>
<tr>
<th>Alloy</th>
<th>HY-100 steel (low hardening, n=0.10)</th>
<th>1018 normalized (high hardening, n=0.24)</th>
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<tr>
<td>Observations</td>
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<tr>
<td>Projected upper flank ratio, $l_u/l_0$</td>
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<td>0.750</td>
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<td>Projected lower flank ratio, $l_l/l_0$</td>
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<td>0.870</td>
</tr>
<tr>
<td>Upper flank angle, $\theta_u$</td>
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<td>36°</td>
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<tr>
<td>Lower flank angle, $\theta_l$</td>
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<td>Corresponding slip and fracture parameters</td>
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<tr>
<td>Slip angle $\theta_{sl}$</td>
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<td>Slip angle $\theta_{su}$</td>
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<td>Cracking angle $\theta_f$</td>
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<td>Cracking parameter $\chi$</td>
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<td>Dependent variables</td>
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<tr>
<td>Growth Displ. ratio, $u_y/l_0$</td>
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<td>0.115</td>
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<td>Growth displ. vector angle from transverse</td>
<td>56°</td>
<td>54°</td>
</tr>
</tbody>
</table>


**TABLE 2**

Deformation of singly-grooved symmetrical specimens

Assumed $\theta_s = \theta_{su} = -\theta_{sl}$, $\xi = s_u/s_l = 1$, $\theta_f = 0^\circ$.

<table>
<thead>
<tr>
<th>Alloy</th>
<th>HY-100 steel</th>
<th>1018 normalized</th>
</tr>
</thead>
</table>

**Observations**

- Projected flank ratio, $l_u/l_o = l/l_0$
  - HY-100: 0.780, 0.740

- Flank angle, $\theta_f = -\theta_u$
  - HY-100: 14$^\circ$, 12$^\circ$

**Corresponding parameters**

- Slip angle, $\theta_s$
  - HY-100: 41$^\circ$, 31$^\circ$

- Cracking parameter, $\chi$
  - HY-100: 1.907, 1.579

**Dependent variables**

- Growth Displ. ratio, $u_y/l_0$
  - HY-100: 0.390, 0.404, 0.315, 0.317

- Back angle, $\beta_u$
  - HY-100: 12$^\circ$, 13$^\circ$, 19$^\circ$, 15$^\circ$
MIXED MODE I AND II FULLY PLASTIC CRACK GROWTH FROM SIMULATED WELD DEFECTS(U) MASSACHUSETTS INST OF TECH CAMBRIDGE DEPT OF MECHANICAL ENGIN. G A KARDOMATEAS

UNCLASSIFIED 23 OCT 85 N00014-82-K-0025
Deformation of singly-grooved asymmetrical specimens

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### Observations

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<th>1018 CF steel</th>
<th>HY-80 Steel</th>
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<td>Projected upper flank</td>
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<td>0.960</td>
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<td>Lower flank angle. $\theta_l$</td>
<td>41°</td>
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<td>Upper back angle. $\beta_u$</td>
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### Corresponding slip and fracture parameters

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<th>1018 CF steel</th>
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<td>Slip angle $\theta_{sl}$</td>
<td>50°</td>
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<td>Slip angle $\theta_{su}$</td>
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<td>54°</td>
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<td>Growth Displ. ratio. $u_y/l_0$</td>
<td>0.088 0.084</td>
<td>0.120 0.115</td>
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<td>Growth displ. vector angle from transverse</td>
<td>51° 52°</td>
<td>56° 54°</td>
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Deformation of singly-grooved symmetrical specimens

Assumed $\theta_s = \theta_{su} = -\theta_{sl}$. $\xi = s_1 / s_u = 1$. $\theta_f = 0^\circ$.

Alloy: 1018 CF steel HY-80 steel

Observations

Projected flank ratio, $l_u/l_0 = l_1/l_0$

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<th>Projected flank ratio</th>
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<td>0.820</td>
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Flank angle, $\theta_l = -\theta_u$

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Corresponding parameters

Slip angle $\theta_s$

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Cracking parameter, $\chi$

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Dependent variables

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<th>HY-80 steel</th>
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<tr>
<td>Growth Displ. ratio, $u_y/l_0$</td>
<td>0.260</td>
<td>0.262</td>
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<tr>
<td>Back angle, $\beta_u$</td>
<td>12$^\circ$</td>
<td>12$^\circ$</td>
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Deformation of singly-grooved asymmetrical specimens

### Alloy
- A36 hot rolled steel
- 5086-H111 aluminum

#### Observations

<table>
<thead>
<tr>
<th>Description</th>
<th>A36</th>
<th>H111</th>
</tr>
</thead>
<tbody>
<tr>
<td>Projected upper flank ratio, $l_u/l_0$</td>
<td>0.770</td>
<td>0.810</td>
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<tr>
<td>Projected lower flank ratio, $l_1/l_0$</td>
<td>0.890</td>
<td>0.900</td>
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<tr>
<td>Upper flank angle, $\theta_u$</td>
<td>36°</td>
<td>41°</td>
</tr>
<tr>
<td>Lower flank angle, $\theta_1$</td>
<td>41°</td>
<td>39°</td>
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<tr>
<td>Upper back angle, $\beta_u$</td>
<td>13°</td>
<td>16°</td>
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#### Corresponding slip and fracture parameters

<table>
<thead>
<tr>
<th>Description</th>
<th>A36</th>
<th>H111</th>
</tr>
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<tbody>
<tr>
<td>Slip angle $\theta_{sl}$</td>
<td>53°</td>
<td>51°</td>
</tr>
<tr>
<td>Slip angle $\theta_{su}$</td>
<td>57°</td>
<td>53°</td>
</tr>
<tr>
<td>Cracking angle $\theta_l$</td>
<td>32°</td>
<td>37°</td>
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<tr>
<td>Cracking parameter $\chi$</td>
<td>1.834</td>
<td>2.821</td>
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<td>Shearing parameter $\xi$</td>
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#### Dependent variables

<table>
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<th>Description</th>
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<th>H111</th>
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<tr>
<td>Growth Displ ratio, $u_y/l_0$</td>
<td>0.214</td>
<td>0.216</td>
</tr>
<tr>
<td>Growth displ. vector angle from transverse</td>
<td>61°</td>
<td>60°</td>
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</table>
Deformation of singly-grooved symmetrical specimens

Assumed \( \theta_s = \theta_u = -\theta_f \), \( \xi = s_l/s_u = 1 \), \( \theta_f = 0 \).

Alloy: A36 hot rolled steel 5086-H111 aluminum

Observations

- Projected flank ratio, \( l_u/l_0 = l/l_0 \):
  - A36: 0.780 0.760
  - 5086-H111: 0.275 0.254

- Flank angle, \( \theta_f = \theta_u \):
  - A36: 10° 10°
  - 5086-H111: 2.158 1.892

Corresponding parameters

- Slip angle \( \theta_s \):
  - A36: 32° 29°
  - 5086-H111: 16° 15°

- Cracking parameter, \( \chi \):
  - A36: 2.158 1.892

- Dependent variables:
  - Growth Displ. ratio, \( u_y/l_0 \):
    - A36: deduced = 0.275 observed = 0.254
    - 5086-H111: deduced = 0.263 observed = 0.278

  - Back angle, \( \beta_u \):
    - A36: 16° 15°
    - 5086-H111: 19° 16°
Figure 1  Symmetric (a) and Asymmetric (b) shear from cracks
Figure 2. Development of deformation for the two-slip plane model.
Figure 3. Micro- and Macroscopic geometry
Figure 4. Crack opening angle as a function of the cracking parameter \( \chi \) for \( \xi = s_f/s_u = 0.5 \), and the shearing parameter \( \xi \) for \( \chi = f/s_u = 2.0 \). In both cases \( \theta_f = 30^\circ \) and \( \theta_s = 40^\circ \).
Figure 5. Deformation ratio as a function of the cracking parameter \( \chi \) for \( \xi=s_u/s_f=0.5 \), and the shearing parameter \( \xi \) for \( \chi=f/s_u=2.0 \). In both cases \( \theta_f=38^\circ \) and \( \theta_s=40^\circ \).
Figure 6. Projected lower and upper flank ratio as a function of the shearing parameter $\xi$ for $\chi=f/s_u=2.0$, $\theta_f=38^0$, $\theta_{sf}=40^0$ and $\theta_{su}=49^0$. 
Figure 7. Deformation ratio and crack opening angle as a function of the cracking parameter $X$ for the limiting Mode I symmetric case. $\theta = 0$, $\xi = s / s_u = 1$, $\theta_s = \theta_{su} = -\theta_{su}$. 
CHAPTER FIVE

FINITE ELEMENT INVESTIGATION OF PLANE STRAIN ASYMMETRIC FULLY PLASTIC SPECIMENS

TABLE OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>strain hardening exponent</td>
</tr>
<tr>
<td>σ₁</td>
<td>flow stress at unit strain</td>
</tr>
<tr>
<td>η</td>
<td>damage (eq. 3)</td>
</tr>
<tr>
<td>σ</td>
<td>mean normal stress</td>
</tr>
<tr>
<td>γ</td>
<td>principal shear strain</td>
</tr>
<tr>
<td>τ</td>
<td>principal shear stress</td>
</tr>
<tr>
<td>σₑq</td>
<td>equivalent stress</td>
</tr>
<tr>
<td>℅ₑq</td>
<td>equivalent strain</td>
</tr>
<tr>
<td>θₖ</td>
<td>critical orientation</td>
</tr>
<tr>
<td>ρ</td>
<td>mean inclusion spacing</td>
</tr>
<tr>
<td>φ</td>
<td>displacement vector angle from transverse</td>
</tr>
<tr>
<td>Uᵧ</td>
<td>axial component of far field displacement vector</td>
</tr>
<tr>
<td>Uₓ</td>
<td>transverse component of far field displacement vector</td>
</tr>
<tr>
<td>M⁵</td>
<td>mixity parameter (eq. 4)</td>
</tr>
<tr>
<td>uᵧ</td>
<td>axial component of relative crack tip displacement (=uᵧ+−uᵧ−)</td>
</tr>
<tr>
<td>uₓ</td>
<td>transverse component of relative crack tip displacement (=uₓ+−uₓ−)</td>
</tr>
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</table>

ABSTRACT

Crack initiation and early growth in asymmetric, fully plastic, plane strain configurations in power-law hardening materials is investigated numerically via the finite element method. In such asymmetric configurations a single shear band is present instead of the two shear bands of the symmetric case. Results for two strain hardening exponents, n=0.12 and n=0.24, indicate that cracking occurs at an angle of 39°-43° from the transverse, smaller than 45° due to the higher triaxiality. The direction of cracking is closer to 45° for lower strain hardening exponents and is
within $2^\circ$ of those experimentally found. The stress and strain field is consistent with the power law singularity of the HRR fields. The far field displacement vector is not along the shear band but at about $68^\circ-70^\circ$ from the transverse at initiation, indicating the presence of a Mode I component. Early growth, studied by successive removal of elements reaching unit damage, results in crack growth per unit displacement for the lower hardening case about twice that of the higher hardening one.

**INTRODUCTION**

Asymmetric plane strain specimens have been used to study crack growth along a single shear band. Such cases may occur when a weld fillet or a harder heat-affected zone on one side of the crack suppresses the other shear zone that would appear in a symmetric specimen. Based on Shih's extension to mixed mode [1] of the HRR [2,3] fields, McClintock and Slocum [4] developed an approximate formulation for the accumulation of damage directly ahead of an asymmetric crack. The crack was assumed to follow the center of a $45^\circ$ shear band and the far-field displacement was assumed to be parallel to the shear band. It was found that the initiation displacement was of the order of the fracture process zone size $\rho$. To study the directional effects, several sites around the current crack tip were considered in chapter two and the crack was assumed to advance to the direction requiring the least far field displacement for critical damage. The far field displacement vector was again assumed to be at $45^\circ$ from the transverse. Both these solutions found only little effect of strain hardening on the crack growth rate. However, tests have shown that the far field displacement vector is not at $45^\circ$ but more axial, at an angle of about $60^\circ$ from the transverse. In addition, a lower strain hardening exponent $n$ was found to increase the crack growth rate dramatically. Strain hardening causes the
deformation field to fan out. The effect of the finite width of the shear band can be captured with a finite element investigation. In the following the finite element method is used to study crack initiation and early growth in fully plastic plane strain asymmetric specimens.

TECHNIQUE

The finite element grid used is indicated in Fig. 1, with the details of the refined mesh for the first circle around the crack tip shown in Fig. 2. An increased element concentration near the 45° line is used to account for the high strain gradients there. Angular spacings of 3.75° for four sectors, 7.5° for two sectors, 15° for four sectors and 30° for nine sectors are used. Minimum radial size for the 3.75° elements is \( \rho = 0.01 \) mm, the approximate value for the mean inclusion spacing. The radial size ratio was \( s = 1.155 \) for the 3.75° sectors becoming \( s^2 \) for the 7.5° sectors, \( s^4 \) for the 15° and \( s^8 \) for the 30° sectors. The net ligament of the specimen is \( l_0 = 2.55 \) mm. 8-node plane strain isoparametric elements are used. The mesh consisted of a total of 207 elements with 722 nodes and 1444 degrees of freedom. The nodes at the bottom were on rollers with the center node pinned. An axial displacement with zero shear traction was applied at the nodes of the upper end. The analysis was carried on a Data General computer available at M.I.T. and the general purpose finite element code ABAQUS [5] was used.

The mesh was checked by comparing the theoretical strain distribution for the elastic and the low hardening \( n = 1/13 \) HRR [2,3] fields with the linear variation of strains within the 8-node elements. The radial variation in strain showed a maximum deviation of 15% from the elastic solution for the first element around the tip. For the HRR \( n = 1/13 \) solution the deviation was 33%, dropping to 5.6% for the
second element. The angular distribution in $\epsilon_{\theta}$ showed a maximum deviation of 8% from the HRR for the 30° sectors. In addition, a circular portion of the finite element mesh with 16 radially elements at the finest sectors was tested by imposing Mode I HRR displacement boundary conditions. The HRR singularity in $\epsilon_{\theta}$ was reproduced with no more than 5% deviation in all elements except the first one, where the maximum deviation was 14% at the first integration point.

The material is modeled as isotropic power-law hardening: the stress $\sigma$ is given in terms of the plastic strain $\varepsilon_P$, the flow stress at unit strain $\sigma_1$, a strain hardening exponent $n$, and a pre-strain $\varepsilon_0$ by

$$\sigma = \sigma_1 (\varepsilon_0 + \varepsilon_P)^n.$$  \hspace{1cm} (1)

Two cases were considered, $n=0.24$, $\sigma_1=826$ MN/m², yield strength $Y=333$ MN/m² and $n=0.12$, $\sigma_1=909$ MN/m², $Y=435$ MN/m².

The fracture criterion of McClintock, Kaplan and Berg [6] is used, by which it is postulated that fracture due to micro-void coalescence occurs when a quantity $\eta$, named “damage”, reaches a critical value of unity. The damage is expressed in terms of a hole growth ratio $F_t$, the principal shear strain $\gamma$, and the triaxiality (defined as the ratio of the mean normal stress $\sigma$ to principal shear strain $\gamma$). In terms of the equivalent stress and strain,

$$\tau = \frac{\sigma_{eq}}{\sqrt{3}} \quad \gamma = \frac{\varepsilon_{eq}}{\sqrt{3}}.$$  \hspace{1cm} (2)

The damage is:

$$\eta = \frac{1}{\ln F_t} \left[ \ln \sqrt{1+\gamma^2} + \frac{\gamma}{2(1-n)} \sinh \left( \frac{(1-n)\sigma}{\gamma} \right) \right].$$  \hspace{1cm} (3)

The critical displacement for growth initiation occurs when the damage becomes
unity at a point \((\rho, \theta_c)\) where \(\rho\) is the fracture process zone size and \(\theta_c\) is the critical orientation. To study the first steps of crack growth, successive elements were removed as they reached a damage of unity.

RESULTS AND DISCUSSION

The axial displacement, at the upper end, \(U_y\), was gradually increased and the damage from (3) was calculated at each site around the tip. Cracking occurs when the fracture criterion of \(\eta=1\) is first satisfied.

The initiation conditions (critical orientation from the transverse, \(\theta_c\), critical strain \(\gamma_c\), critical triaxiality \(\sigma/\tau\), far-field displacement \(u_f/\rho\)) are shown in Table 1. The crack tip initiation displacement is of the order of the mean inclusion spacing as was also found by McClintock and Slocum [4] and in chapter 2. The critical orientation of \(39-43^0\) from the transverse and the far field displacement vector orientation of about \(68-70^0\) at initiation can be compared with the values of \(38-41^0\) for the crack direction and \(58^0-69^0\) for the displacement vector at initiation from tests. The lower hardening \(n=0.12\) case results in fracture closer to the shear band, as found experimentally.

For a crack at \(\theta=0^0\), a Mode I mixity parameter \(M^p\) was introduced by Shih [1], defined in terms of the near field stresses by

\[
M^p = \frac{2}{\pi} \tan^{-1} \left| \lim_{\theta \to 0} \frac{\sigma_{r\theta}(r, \theta=0)}{\sigma_{rr}(r, \theta=0)} \right| \tag{4}
\]

The mixity parameter varies from 0 for pure Mode II to 1 for pure Mode I. This parameter can be referred to either the initial crack direction \(\theta=0^0\) or the final (critical) one \(\theta=\theta_c\), giving values as shown in Table 1. Notice that the above definition of the Mode I mixity is with respect to both shearing and crack advance at
\[ \theta = 0, \text{ for both the limiting cases } M_P = 0 \text{ or } 1. \] In the problem at hand, pure Mode I is crack advance along the line \( \theta = 0 \) (corresponding to the symmetric case) and pure Mode II would be relative deformation and crack advance along the 45\(^\circ\) shear band. Alternatively, for experiments and finite element studies, a definition of a Mode I mixity in terms of the displacement field is helpful:

\[ M_1^* = \frac{2}{\pi} \tan^{-1} \left( \lim_{r \to 0} \frac{u_{\theta}(r,\pi) - u_{\theta}(r,-\pi)}{u_{\pi}(r,\pi) - u_{\pi}(r,-\pi)} \right) \]  

(5)

Values for this parameter are also shown in Table 1. Notice that for the non-hardening rigid plastic pure Mode II limit with a single slip line at \( \theta = 45^\circ \), \( M_1^* = 0.5 \) for the crack at \( \theta = 0^\circ \) but \( M_1^* = 0 \) for the crack at \( \theta = 45^\circ \).

Fig. 3a shows the angular variation of the \( \sigma_{\theta\theta} \) stress component. The curve is consistent with the Shih's [1] curves and has a maximum near \( \theta = 65^\circ \). This compares with the case \( M_P = 1, n = 1/13 \) which has a maximum at an angle near 97\(^\circ\) and the \( M_P = 1, n = 1/3 \) case with a maximum near 88\(^\circ\) whereas the case of \( M_P = 0 \) has a maximum \( \sigma_{\theta\theta} \) at \( \theta = 0^\circ \). Furthermore, the maximum for \( n = 1/13, M_P = 0.82 \) is near 55\(^\circ\) and for \( n = 1/3, M_P = 0.79 \) is near 40\(^\circ\) [1]. The \( \theta \)-variation of \( \epsilon_{\theta\theta} \) is shown in Figs. 3b, 3c. Of the two peaks in \( \epsilon_{\theta\theta} \), the one for positive \( \theta \) is the dominant the other peak tending to vanish during growth when the Mode I mixity is reduced (Fig. 3d).

The radial variation of the equivalent strain for \( n = 0.12 \) (along the critical orientation) is shown in Fig. 4. The asymptotic solution for power-law hardening materials yield singularities in the stress and strain of the form \( r^{-n/(n+1)} \) and \( r^{-1/(n+1)} \), respectively. The agreement between the theoretical curve and the finite element results is within 5%.

Fig. 5 shows the axial displacement of the upper flank relative to the lower
flank at the initiation point. The components of the relative displacement of the crack tip $u_x(x=0, y=0)$ and $u_y(x=0, y=0)=\text{CTOD}$ are included in Table 1. A higher CTOD occurs in the higher hardening case. Figs. 6a and 6b show the angular variation of the near tip displacement field for the two hardening exponents $n=0.24$ and $n=0.12$, along with the nonhardening limit. The far-field displacement (displacement at the upper boundary) components $U_x$ and $U_y$ at the initiation point are also included in Table 1 together with the far field displacement vector orientation from the transverse, $\phi$. The value of $\phi=68^0$ to $70^0$, instead of $45^0$, indicates that we cannot consider the far field displacement taking place parallel to the shear band, as assumed by McClintock and Slocum [4]. We can observe that the displacement vector at initiation is more axial for the lower hardening case with larger $M_P$. The higher triaxiality for angles smaller than $45^0$ is the main reason for the cracking direction deviating towards the transverse. The triaxiality is smaller for $n=0.24$ because of the smaller Mode I mixity $M_P$.

Tests have shown that, in the asymmetric case, the lower hardening alloys exhibit a maximum crack growth rate more than twice that of the higher hardening alloys. The finite element mesh of Figure 1 was used to study the early growth. Crack was grown by successive removal of the most heavily damaged element. After initiation and removal of the most damaged element, the far field displacement is further increased until critical damage $\eta=1$ occurs in the next row of elements. At this point the next step of crack growth takes place by removing the critical element. After growth by four steps (1.9% of the ligament) it was found that the average displacement per unit projected crack advance $\Delta u/\Delta l$ is about 88% smaller for the lower hardening $n=0.12$ case than that of the higher hardening $n=0.24$ case (Table 1). Another noteworthy result is that the far field displacement vector $U$ becomes less axial as the crack grows. For the case $n=0.12$, at the end of the fourth
step, the angle of the displacement vector from the transverse is $\phi=67.6^0$ instead of
the initiation value of $\phi=69.5^0$. Decreasing $\phi$ angles with crack growth have been
experimentally observed (chapter three). During these steps the critical elements
were at the same angular sector and no appreciable acceleration of the crack was
observed.

CONCLUSIONS

A finite element investigation of fully plastic asymmetric specimens with a
single slip band, as might be encountered near a weld, has provided the stress, strain
and displacement fields around the tip. Results indicate the presence of a large
Mode I component with the far field displacement vector at initiation not along the
$45^0$ shear band but at an angle about $67^0$ from the transverse. The initiation
conditions were found by using the fracture criterion for hole growth by McClintock
Kaplan and Berg [6]. The critical direction was at $39-43^0$, less than $45^0$ from the
transverse, increasing for a lower strain hardening exponent. Displacement to crack
initiation is about twice the fracture process zone size. Stress and strain fields are
consistent with the solutions for the mixed mode extended HRR fields. Early
growth, studied by successive removal of the most damaged element, resulted in
crack growth rate for the lower hardening case about twice that of the higher
hardening one. The angle of the far-field displacement vector from the transverse
was found to be decreasing with crack growth.

REFERENCES

1. Shih C.F. "Small Scale Yielding Analysis of Mixed Mode Plane Strain
Crack Problems", Fracture Analysis, ASTM STP 560,


5. ABAQUS from Hibbitt, Karlsson and Sorensen, Inc.

## TABLE 1
Results of the finite element study

<table>
<thead>
<tr>
<th>Initiation Conditions</th>
<th>$n = 0.12$</th>
<th>$n = 0.24$</th>
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</thead>
<tbody>
<tr>
<td>Critical angle $\theta_c$</td>
<td>43.1°</td>
<td>39.4°</td>
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<tr>
<td>Far field displ. components $U_y/\rho$</td>
<td>2.1</td>
<td>1.8</td>
</tr>
<tr>
<td>$U_x/\rho$</td>
<td>0.782</td>
<td>0.737</td>
</tr>
<tr>
<td>Displacement vector-angle from transverse</td>
<td>$\phi$</td>
<td>69.5°</td>
</tr>
<tr>
<td>Displacements at crack tip $U_x(x=0, y=0)/\rho$</td>
<td>0.134</td>
<td>0.192</td>
</tr>
<tr>
<td>$U_y(x=0, y=0)/\rho$</td>
<td>0.518</td>
<td>0.564</td>
</tr>
<tr>
<td>Principal Shear Strain</td>
<td>$\gamma_c$</td>
<td>0.246</td>
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<tr>
<td>Triaxiality</td>
<td>$\sigma/\tau$</td>
<td>2.18</td>
</tr>
<tr>
<td>Mixity parameter</td>
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<td></td>
</tr>
<tr>
<td>Mode I Mixity $M_p(\text{Shih})$ defined by Shih (rel. to $\theta=0°$)</td>
<td>0.936</td>
<td>0.927</td>
</tr>
<tr>
<td>(based on stresses) (rel. to $\theta=\theta_c$)</td>
<td>0.717</td>
<td>0.710</td>
</tr>
<tr>
<td>Displ. based Mode I Mixity $M_I^*$</td>
<td>0.752</td>
<td>0.815</td>
</tr>
<tr>
<td>Early Growth</td>
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<tr>
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<td>0.143</td>
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CHAPTER SIX

ON THE FULLY PLASTIC FLOW PAST A GROWING ASYMMETRIC CRACK AND ITS RELATION TO MACHINING MECHANICS

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<td>ψ</td>
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<td>σ</td>
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<tr>
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<td>equivalent strain</td>
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<td>σ'</td>
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<tr>
<td>V_l</td>
<td>rigid body velocity at lower flank</td>
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<td>&quot;slip angle&quot; (eq. 49)</td>
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ABSTRACT

A tensile logarithmic singularity in the mean normal stress is found for steady flow of rigid-plastic, linearly strain-hardening material, with rigid material flowing past straight flanks. For cracks, this indicates that the flanks of the crack tend to deform. For the machining case it explains the tendency for precracking ahead of the tool which contributes to a built-up edge, or the formation of a discontinuous chip. Finally, an approximate analysis of the quasi-steady integral of the stationary crack solution shows a tendency of the crack flanks to form a cusp. The strains for a cusp field would be dominated by the elastic-plastic field which shows instead a
vertical tangent at the crack tip.

INTRODUCTION

Fully plastic flow before fracture is desirable even in structures containing cracks. Such ductility is reduced if plastic flow is limited to one shear band, for example, by a weld (Fig. 1). In such asymmetric mixed mode I and II configurations, the crack accelerates as it advances into pre-strained and damaged material. Further evidence for a lowered ductility in asymmetric cracking is the tendency to form a shear lip at the end of a cup and cone fracture in a tensile test. Non-hardening plasticity gives a shear band of infinitesimal thickness. Strain hardening, however, causes the deformation field to fan out, leaving a finite strain except possibly at the crack tip.

In orthogonal machining the geometry is similar, with the cutting tool progressing steadily below the plastic zone. Here, again due to strain hardening, Christopherson et al. [2] found that the plastic zone fans out over $10^\circ$-$30^\circ$, as opposed to the single plane required by the perfectly plastic solid.

STRESS SINGULARITY WITH RIGID FLANKS

Postulate a steady flow past a crack in rigid-plastic, linearly strain hardening material. The mechanics of the problem should determine whether or not the crack tip has a finite angle. Start by assuming a crack of finite angle $\omega$ and rigid body velocity of the material flowing along the flanks. To satisfy incompressibility assume a stream function $\psi$ in polar coordinates $r$ and $\theta$. Seek the form of the stream function in the immediate vicinity of the crack tip where the velocities should be
nonzero and finite. In a separable expansion of $\psi$ in the dominant term as $r \to 0$, $r^s F(\theta)$, the exponent $s$ must be unity since the velocities at the tip are nonzero and finite. Thus,

$$\psi = r F(\theta). \quad (1)$$

The corresponding velocities are

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = F'(\theta), \quad (2)$$

$$u_\theta = - \frac{\partial \psi}{\partial r} = -F(\theta). \quad (3)$$

The strain rates are

$$\dot{\epsilon}_r = \frac{\partial u_r}{\partial r} = 0 = -\dot{\epsilon}_\theta, \quad (4)$$

$$\gamma_{r\theta} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_r}{r} = \frac{F''(\theta) + F(\theta)}{r}. \quad (5)$$

Thus the only component of strain is shear. The equivalent strain rate is

$$\bar{\epsilon} = \sqrt{\frac{2}{3} \epsilon_r^2 + \epsilon_\theta^2 + 2 \left( \frac{\gamma_{r\theta}}{2} \right)^2} = \frac{F''(\theta) + F(\theta)}{\sqrt{3} r}. \quad (6)$$

The stress deviators $s_{ij}$ are found from the stress-strain relations and the equivalent stress $\bar{\sigma}$:

$$\epsilon_{ij} = -\frac{3 s_{ij}}{2 \bar{\sigma}}. \quad (7)$$

Since $\dot{\epsilon}_r = \dot{\epsilon}_\theta = 0$, from (4),

$$s_r = s_\theta = 0 \text{ and } s_{r\theta} = \bar{\sigma}/\sqrt{3}. \quad (8)$$

Assume the material is rigid-plastic, linearly strain hardening:

$$\bar{\sigma} = Y + H \bar{\epsilon}. \quad (9)$$
The accumulated equivalent strain \( \bar{\varepsilon} \) is calculated by integration along a streamline, where the time increment is expressed in terms of that required for an element to traverse an increment of angle:

\[
\bar{\varepsilon} = \int_{t_0}^{t} \bar{\varepsilon} dt = \int_{u_0}^{u} \bar{\varepsilon} dt = - \int_{0}^{\theta} \frac{F'' + F}{\sqrt{3} F} d\theta.
\]  

(10)

Thus the equivalent strain is independent of radius. The same holds for the equivalent stress \( \sigma \), by (9), and also for the shear stress, by (8),

\[
s_{r\theta} = s_{r\theta}(\theta).
\]  

(11)

Now turn to the equilibrium equations. In terms of the mean normal stress \( \sigma \),

\[
\frac{\partial \sigma}{\partial r} + \frac{1}{r} \frac{\partial s_{r\theta}}{\partial \theta} + \frac{1}{r} \frac{s_{r\theta} - s_{\theta\theta}}{r} = 0.
\]  

(12)

\[
\frac{\partial s_{r\theta}}{\partial r} + \frac{1}{r} \frac{1}{\partial \sigma} + \frac{1}{r} \frac{\partial s_{\theta\theta}}{\partial \theta} - \frac{2s_{r\theta}}{r} = 0.
\]  

(13)

Introducing (8) to eliminate \( s_r, s_\theta \), (11) to eliminate \( \partial s_{r\theta}/\partial r \), and cross-differentiation to eliminate \( \sigma \) leads to:

\[
d^2 s_{r\theta}/d\theta^2 = 0, \text{ from which } ds_{r\theta}/d\theta = \text{const.} = s_{r\theta\theta}.
\]  

(14)

Now, (12) simplifies with (8), and can be integrated

\[
\frac{\partial \sigma}{\partial r} + \frac{1}{r} \frac{ds_{r\theta}}{d\theta} = 0, \quad \sigma = -s_{r\theta\theta} \ln(r/R) + C(\theta).
\]  

(15)

To find \( C(\theta) \), differentiate (13) with respect to \( \theta \), and again note \( \partial s_{r\theta}/\partial r = 0 \) from (11) and \( s_\theta = 0 \) from (8). Equating the result to the second partial of (15) gives

\[
\frac{\partial^2 \sigma}{\partial \theta^2} = -2 \frac{ds_{r\theta}}{d\theta} = \frac{d^2C(\theta)}{d\theta^2}; \quad C(\theta) = -s_{r\theta\theta} \theta^2 + C_1 \theta + C_2.
\]  

(16)

Define \( \sigma(R,0) \) as the mean normal stress at \( \theta = 0 \) and a convenient radius R. Equation (15) then becomes:
\[ \sigma(r, \theta) - \sigma(R, 0) = -s_{r\theta, \theta}(\ln(r/R) + \theta^2) + C_1 \theta. \]  

(17)

Thus, the assumption of rigid flanks subtending a finite angle would require that the mean normal stress at the crack tip \((r \to 0)\) have a logarithmic singularity. Let us now complete the study of the field specified by the stream function (1) by applying the boundary conditions and deriving the streamlines.

Two possible flow fields are consistent with the constant rate of shear stress from (14) and the hardening of the material (increase in equivalent stress \(\sigma\) from (8) as it flows along the streamline). The first field, shown in Fig. 2a, is for \(s_{r\theta, \theta} > 0\). From (17) this model gives a tensile logarithmic singularity in the mean normal stress as \(r \to 0\), and thus the field will be called "tensile". The second field, shown in Fig. 2b, is for \(s_{r\theta, \theta} < 0\). Here the singularity in the mean normal stress is compressive and, accordingly, this field will be called "compressive". A compressive singularity, however, would require strains of order unity or more for fracture. Since such large strains are not actually observed [1], the "compressive" field is not plausible for the growing crack.

Two other conceivable fields can be excluded. A single band being split by the crack (Fig. 2c) would have shear stresses of the same sign, but increasing in magnitude both above and below the line of advance due to increasing strains along a streamline. This change in sign of \(s_{r\theta, \theta}\) would give tensile and compressive singularities adjacent to each other, and a discontinuity in normal stress. If the shear in a band being split by the crack were to change sign, on the other hand, there would be an intermediate region below yield, and the band would separate into two, corresponding to those of Figs. 2a,b. In the limit, the Mode I field would be approached.
Thus only the "tensile" field of Fig. 2a remains. From (8) and (9) for positive shearing,

\[
\frac{d\tau_{r\theta}}{d\theta} = \frac{1}{\sqrt{3}} \frac{d\sigma}{d\theta} = \frac{1}{H} \frac{d\varepsilon}{d\theta}.
\]  
(18)

(18) and (14) give, with \(s_{r\theta,\theta} = \text{const.},\)

\[
\frac{d\varepsilon}{d\theta} = \frac{s_{r\theta,\theta} \sqrt{3}}{H} = \frac{1}{\sqrt{3}} \frac{d\gamma_{r\theta}}{d\theta}.
\]  
(19)

Differentiating (10) gives:

\[
\frac{d\varepsilon}{d\theta} = \frac{F'' + F}{\sqrt{3} F}.
\]  
(20)

Introducing (19) into (20) gives finally

\[
F'' + k^2 F = 0,
\]  
(21)

where

\[
k^2 = 1 + \frac{3 s_{r\theta,\theta}}{H} = 1 + \frac{d\gamma_{r\theta}}{d\theta}
\]  
(22)

The solution of (21) is

\[
F(\theta) = A \cos k \theta + B \sin k \theta.
\]  
(23)

Referring to Fig. 2a we denote by \(V_r, V_u\) the (rigid body) velocities at the lower and upper boundaries of the deforming region, which are at angles \(\theta_l\) and \(\theta_u\) respectively. Then the boundary conditions are:

at the lower boundary,

\[
u_r = -V_r \cos \theta_l, \quad \text{and by (2),} \quad F'(\theta_l) = -V_r \cos \theta_l,
\]  
(24)

\[
u_\theta = V_r \sin \theta_l, \quad \text{and by (3),} \quad F(\theta_l) = -V_r \sin \theta_l;
\]  
(25)
similarly, at the upper boundary,

\[ F'(\theta_u) = -V_u \cos(\theta_u + \omega), \quad (26) \]

\[ F(\theta_u) = -V_u \sin(\theta_u + \omega). \quad (27) \]

Solving (23)-(27) for \( \omega \) in terms of \( \theta_p \), \( \theta_u \) gives

\[ \omega = \tan^{-1}\left[ \frac{P - \tan \theta_i}{k(1 + P \tan \theta_i)} \right] - \theta_u, \quad (28) \]

where

\[ P = \frac{k \tan \theta_i + \tan \theta_u}{1 - k \tan \theta_i \tan \theta_u}. \quad (29) \]

Substituting back into the boundary conditions gives

\[ \frac{V_u}{V_l} = \frac{k \sin \theta_i \cos \theta_u + \cos \theta_i \sin \theta_u}{k \sin(\theta_u + \omega) \cos \theta_i + \cos(\theta_u + \omega) \sin \theta_i}, \quad (30) \]

\[ A = \frac{(V_u/V_l) \sin(\theta_u + \omega) \sin \theta_i - \sin \theta_u \sin \theta_i}{\sin(\theta_u - \theta_i)}, \quad (31) \]

\[ B = \frac{\sin \theta_i \cos \theta_u - (V_u/V_l) \sin(\theta_u + \omega) \cos \theta_i}{\sin(\theta_u - \theta_i)}. \quad (32) \]

Assume now a critical strain \( \gamma_u \) at the upper boundary. Then

\[ \frac{d\gamma_{\theta_0}}{d\theta} = \gamma_u/(\theta_u - \theta_i), \quad (33) \]

and

\[ k^2 = 1 + \frac{\gamma_u}{\theta_u - \theta_i}. \quad (34) \]

The streamlines for a particular example, and the equation for the rotation of elements are given in the Appendix.
According to Hill [7], the infinite mean normal stress by (17) cannot be sustained at the rigid flank and this will lead to plastic yielding. For further insight, turn to the approximate superposition of singularities for stationary cracks.

SUPERPOSITION OF STATIONARY SINGULARITIES

Shih [3] solved the mixed Mode I and II singular fields for the stationary crack field, extending the Mode I field of Hutchinson [4], Rice and Rosengren [5] (HRR). In terms of a stress-strain law of the form $\sigma = \sigma_i \epsilon^p$, for a far field defined by the path independent integral $J$ with Mode I mixity parameter $M_P$ and the scalar function $I_{1/a}(M_P)$, the displacement and strain components at $r, \theta$ for the fully plastic parameters of interest here are (see e.g. McClintock [5]):

$$u_i(r, \theta) = \left[ \frac{J}{\sigma_i I_{1/a}(M_P) r} \right]^{1/(n+1)} u_i(\theta, 1/n, M_P), \quad (35)$$

$$\epsilon_{ij}(r, \theta) = \left[ \frac{J}{\sigma_i I_{1/a}(M_P) r} \right]^{1/(n+1)} \epsilon_{ij}(\theta, 1/n, M_P). \quad (36)$$

Superposition does not strictly apply to (35) for two reasons: it does not take the convection of hardened material into account and it is a non-linear relation between displacement and $J$. Qualitative insights may be obtained, however, by assuming, following (35), that the displacement increments vary with radius according to

$$u_i \propto r^{n/(n+1)}, \quad (37)$$

and that, correspondingly, the strain rates vary as

$$\epsilon_{ij} \propto r^{-1/(n+1)}. \quad (38)$$

For the non-hardening material, $n=0$, (37) correctly indicates displacement rates.
independent of radius, which for a growing crack, integrates to displacements increasing linearly away from the tip. For a power law material the displacement rate increases as a fractional root of the distance away from the tip, and its integral gives displacements of the order

\[ u_\theta = \int u_\theta dr \propto r^{1+n/(n+1)}, \]  

which indicates a cusp.

Correspondingly, integration of the strain rates from (38) with respect to the distance as the material sweeps by the tip of the crack gives strains varying as

\[ \epsilon \propto r^{n/(n+1)}. \]  

Thus, due to convection, the strains increase continuously behind the crack tip, whereas the stationary Shih solution gave strains that decrease. That is, at any instant during the integration, the material behind the crack is actually harder than assumed for the displacement and strain rates of (37) and (38). Therefore, the above superposition exaggerates any cusp. Furthermore, the nonhardening solution with flank yielding, for bending and tensile doubly grooved specimens gives linear displacement increments which, when integrated, would show an increasing crack opening angle near the crack tip [6]. For example, the normal component of the displacement field at the flank for the doubly grooved specimens and the resulting from integration flank shape is shown in Fig. 3.

For further insight, the relative dominance of the field for a cusp will be considered. Rigid-plastic flow past a cusp-like crack would not exhibit singularities in the strain but, instead, in higher order terms like strain rates, as follows. For no crack tip opening angle, the displacement rates are of the form:
\[ u_i = O(r^m) + \text{rigid body motion}. \] (41)

The stream function can thus be regarded as being a superposition of steady state rigid body translation of the material past the crack tip and a strain rate singularity:

\[ \psi = r^{m+1}F(\theta) - cr \sin \theta. \] (42)

Then the velocities are:

\[ \dot{u}_r = -\frac{1}{r} \frac{\partial \psi}{\partial \theta} = r^m F'(\theta) - c \cos \theta, \] (43)

\[ \dot{u}_\theta = -\frac{\partial \psi}{\partial r} = -(m+1)r^m F(\theta) + c \sin \theta. \] (44)

Differentiating the displacement rates would give strain rates, and hence the equivalent strain rate, of the order \( O(r^{m-1}) \). The accumulated equivalent strain is found by integrating along a streamline, with the time interval to traverse a given angular increment along a streamline expressed in terms of the tangential velocity:

\[ \bar{\varepsilon} = \int_0^\theta \frac{\varepsilon}{\dot{u}_\theta} rd\theta. \] (45)

For small \( r \), \( \dot{u}_\theta \) is of order \( r^0 \) from (44) while \( \varepsilon \) is of order \( r^{m-1} \) and so the integrand in (45) is of order \( r^m \). Thus

\[ \bar{\varepsilon}_{\theta} = O(r^m), \] (46)

and vanishes for small \( r \) (unless \( m=0 \), which turns out to be the nonhardening case) and the strains are nonsingular. Notice that no stress-strain relationship has been used yet, which means that for any rigid-plastic law the strains are nonsingular for a zero crack opening angle. This field would thus be dominated by any field that exhibits any nonzero crack opening angle. For example the elastic-perfectly plastic field shows logarithmic singularities in the strains but \( d\varepsilon/dr \) is unbounded as \( r \to 0 \),
giving a vertical tangent at the crack tip [10]. The strains for that field are of the order:

$$\gamma^p = O[\ln(R/r)] + O(r^0).$$

which goes to infinity for $r \to 0$, thus dominates the local strain. The large-scale view of a fracture, however, may look like a cusp and since the strains for the cusp field are $O(r^m)$, increasing with $r$, they may become larger at a sufficiently distant point.

The tendency shown above for flank yielding with any finite opening angle leads to the need for an exact solution of the growing crack, where its shape is unknown and the flank, a part of the deforming boundary, is traction-free. Finally, connecting the steady-state continuum mechanics solution to the micromechanics of hole growth would require a transition to non-steady analysis.

DISCUSSION

In machining, a shear band with an undetermined rigid-plastic boundary breaks through to a free surface. The problem is similar to mixed mode crack growth, except that the deformation is larger. Christopherson et al [2] tried to assess the effect of work hardening in the mechanics of orthogonal machining. By modifying the slip line equations and estimating roughly the magnitude of the added term, they pointed out that, due to hardening, the hydrostatic stress changes from compressive at the free surface to tensile near the tool point. What they found was essentially the qualitative effect of the logarithmic singularity derived above for fully plastic flow. In fact, we can also deduce that, for a certain change in the flow strength between the chip and the parent material, if the deforming region is narrower, the angular change of the shear stress (i.e. $s_{r\theta,\theta}$ in (14)) is bigger and, consequently, the
singularity stronger, in accordance with their observation that the work-hardening effect becomes more pronounced as the plastic zone gets narrower.

It is worth considering now the region of dominance of the logarithmic singularity in mean normal stress that would characterize the flow past rigid flanks. Using typical data $d\sigma/d\varepsilon = H \approx Y$ for 1020 steel and $d\varepsilon/d\theta \approx 0.8$, gives from (8)

$$s_{\theta,\theta} = \sqrt{3} \frac{d\bar{r}_1}{d\bar{r}} = \sqrt{3} \frac{d\bar{r}}{d\theta} \approx 1.38 Y.$$  \hspace{1cm} (47)

From the fully plastic flow field of Prandtl for tension of grooved plane strain specimens (see e.g. McClintock [8]) $\sigma \approx 2.8Y$ and assuming that $R$ is the radius at which $\sigma(r,0)$ changes sign, gives from (17), $r/R \approx 0.1$. The distance $R$ is within the macroscopic scale as is evident from the approximate study for machining field done by Christopherson et al. [9]. According to their slip line theory modified to include hardening the change in the mean normal stress $\Delta\sigma$ from the free surface to a point in the band is roughly estimated in terms of the flow strengths in work-piece and chip, $Y_w$ and $Y_c$, the distance $s$ from the free surface, and the local width $t$ of the slipband:

$$\Delta\sigma \approx (Y_c - Y_w) \frac{s}{t\sqrt{3}}.$$  \hspace{1cm} (48)

In machining mild steel, $Y_c$ may be 40% more than $Y_w$, so $Y_c - Y_w \approx 0.4k$. Since at the free surface $\sigma \approx -Y/\sqrt{3}$, the mean normal stress becomes positive at about $s/t = 3$, which for $10^0$ angular width happens at a radius $R$ approximately $1/3$ the total shear band length. Thus the singularity in the mean normal stress dominates in a significant region.

Now that tension has been shown to exist near the tool point, it is possible that brittle (or ductile) fracture may occur at a particular history of stress and this
could give rise to the characteristic fracture running ahead of the tool point and the formation of a built-up edge or a discontinuous chip [9]. In particular, according to the "tensile" field, the maximum strain occurs at the boundary with the chip (upper boundary of the shear band), where cracking could occur. It should be noted, however, that nonsteady effects have not been considered.

CONCLUSIONS

A logarithmic tensile singularity in the mean normal stress has been found for rigid-plastic flow past a growing crack of finite angle with rigid flanks under combined shear and tension. Applied to the machining problem, this result helps to explain the formation of a discontinuous chip or the precracking ahead of the tool.

The tensile singularity predicts yielding of the crack flanks. Approximate solutions for flank yielding give contradictory indications. A tendency to form a cusp has been found from an order-of-magnitude analysis on the quasi-steady integral of the extended near tip HRR singular field. That result indicates decreasing strain behind the crack tip and hence overestimates any cusp. From non-hardening solutions with flank yielding for bending and tension of doubly grooved specimens there is an increasing crack opening angle near the tip. Furthermore, the strains for any cusp would be dominated by the elastic-plastic singularity which gives a blunt tip. Thus there is a need for an exact solution of a crack growing with deforming flanks into strain-hardening material.

REFERENCES

1. McCINTOCK F.A. and SLOCUM A.H., "Predicting Fully Plastic Mode II Crack Growth from an Asymmetric Defect", *Int. J. Fract.*, 27,


**APPENDIX**

Taking an example from machining (Fig. 4) for $\gamma_u = 1.3$, $\theta_l = 40^0$, $\theta_u = 50^0$

we find by using equations (28) - (34):

$\omega = 59^0$, $V_u/V_l = 0.73$, $d\gamma_u/d\theta = 7.44$

and, for $\theta$ in radians,
\[ F(\theta)/V_i = 0.59 \cos(2.9\theta) - 0.459 \sin(2.9\theta). \]

A streamline, resulting from (1) for this particular example, has been sketched in Fig. 4. The velocity triangle shown in Fig. 2a defines a "slip" angle \( \theta_s \):

\[ \theta_s = \sin^{-1} \left[ \frac{(V_u/V_i)\sin\omega}{[(V_u/V_i)^2 + 1 - 2(V_u/V_i)\cos\omega]^{1/2}} \right]. \]  
\[(49)\]

For this particular example \( \theta_s = 45.06^\circ \).

A second example of a growing crack with \( \gamma_u = 0.25, \theta_i = 30^\circ, \theta_u = 40^\circ \), gives \( \omega = 5^\circ, \theta_s = 35.20^\circ, V_u/V_i = 0.88, d\gamma_{rel}/d\theta = 1.43 \)

and

\[ F(\theta)/V_i = 0.0623 \cos(1.55\theta) - 0.744 \sin(1.55\theta). \]

Finally, the rotation of the material element relative to that of the stress field is important in hole growth and thus is worth considering. The rotation of the element is

\[ \phi_m = \frac{1}{2} \left( \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right). \]  
\[(50)\]

and from (2) and (3)

\[ \phi_m = -\frac{1}{2r}(F + F''). \]  
\[(51)\]

while that of the stress field is

\[ \phi_f = \frac{u_\theta}{r} = -\frac{F}{r}. \]  
\[(52)\]

giving a relative rotation

\[ \phi_{rel} = \frac{1}{2r}(F - F''). \]  
\[(53)\]

For \( F(\theta) \) given by (23) and since \( k>1 \) by (22), it is found that rotation and shear
strain (given by (5)) are of different sign. The effect is to open up the holes and thus to increase the damage.
Fig. 1. Symmetric and asymmetric shear from cracks in a plate.
Fig. 2a. The flow field for tension in the band. The machining case is illustrated; otherwise \( \omega \) is the crack opening angle.
Fig. 2b. The field for compression in the band.
Fig. 2c Field with discontinuity in stress across $\theta = \theta_\infty$. 
Fig. 3. Flank normal displacement increments and integrated flank shape for the doubly grooved specimen.
Fig. 4. A typical streamline predicted by the stream function (1) for the singular region in machining.
CHAPTER SEVEN

THE ASYMMETRIC (MIXED MODE I AND II) FULLY PLASTIC FRACTURE - OVERVIEW

TABLE OF SYMBOLS

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<th>Definition</th>
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<tbody>
<tr>
<td>COA</td>
<td>crack opening angle</td>
</tr>
<tr>
<td>D_g</td>
<td>crack ductility (eq. 1)</td>
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<tr>
<td>D_{AC,u}</td>
<td>apparent crack ductility (upper flank)</td>
</tr>
<tr>
<td>D_{AC,l}</td>
<td>apparent crack ductility (lower flank)</td>
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<td>amount of fracture in shear band model</td>
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<td>J-integral</td>
</tr>
<tr>
<td>k</td>
<td>shear yield</td>
</tr>
<tr>
<td>l_0</td>
<td>initial ligament</td>
</tr>
<tr>
<td>l_{l_1}</td>
<td>projected lower flank length</td>
</tr>
<tr>
<td>l_{u_1}</td>
<td>projected upper flank length</td>
</tr>
<tr>
<td>M_s</td>
<td>mixity parameter (eq. 4)</td>
</tr>
<tr>
<td>n</td>
<td>strain hardening exponent</td>
</tr>
<tr>
<td>P</td>
<td>load</td>
</tr>
<tr>
<td>s_l</td>
<td>amount of slip along lower plane in shear band model</td>
</tr>
<tr>
<td>s_u</td>
<td>amount of slip along upper plane in shear band model</td>
</tr>
<tr>
<td>T.S.</td>
<td>tensile strength</td>
</tr>
<tr>
<td>T</td>
<td>tearing modulus</td>
</tr>
<tr>
<td>T_{ASY}</td>
<td>eq. 2</td>
</tr>
<tr>
<td>T_{TASY}</td>
<td>eq. 2</td>
</tr>
<tr>
<td>u_i</td>
<td>idealized initiation displacement</td>
</tr>
<tr>
<td>u_i</td>
<td>initiation displacement</td>
</tr>
<tr>
<td>u_g</td>
<td>growth displacement</td>
</tr>
<tr>
<td>v_i</td>
<td>total displacement vector</td>
</tr>
<tr>
<td>v_i</td>
<td>growth displacement vector</td>
</tr>
<tr>
<td>p</td>
<td>mean inclusion spacing</td>
</tr>
<tr>
<td>\phi</td>
<td>angle of total displacement vector from transverse</td>
</tr>
<tr>
<td>\gamma_c</td>
<td>crack direction from transverse.</td>
</tr>
<tr>
<td>\theta_f</td>
<td>fracture plane</td>
</tr>
<tr>
<td>\theta_{su}, \theta_{sl}</td>
<td>upper and lower slip plane</td>
</tr>
<tr>
<td>\chi</td>
<td>fracture parameter (=f/s_u)</td>
</tr>
<tr>
<td>\xi</td>
<td>shearing parameter (=s_l/s_u)</td>
</tr>
<tr>
<td>\theta_u, \theta_l</td>
<td>upper and lower flank angle from transverse</td>
</tr>
</tbody>
</table>
In symmetric singly grooved tensile specimens the crack advances into the relatively undamaged region between two plastic shear zones. A crack near a weld or shoulder, loaded into the plastic range, may have only a single shear band, along which the crack grows into prestrained and damaged material with less ductility than the symmetrical unconstrained configurations. In this chapter, work that deals with the effect of asymmetry in crack propagation of unconstrained fully plastic singly grooved tensile specimens is summarized. A crack growth ductility is defined as the minimum displacement per unit crack growth. Tests of six alloys showed that, for the low-hardening alloys, the crack ductility in the asymmetric case is less than a third that of the symmetric. In the higher hardening alloys the crack ductility in the asymmetric case is smaller by a factor of 1.2 at most. A noteworthy result is the presence of a Mode I opening component even with asymmetry, as is shown by the far field displacement vector being more than 45° from the transverse direction. The crack direction is less than 45°, indicating the effect of triaxiality on cracking. A macro-mechanical model for crack advance by combined fracture and sliding off along two slip planes gives the independent physical parameters (cracking and two shear directions, relative amounts of cracking and shearing) in terms of the observable quantities of the macroscopic fracture (flank angles, flank lengths, back angle) for both the symmetric and asymmetric specimens. A finite element study of the asymmetric specimens gave a crack direction within two degrees and a far field displacement vector at initiation within three degrees of that experimentally found. Stress and strain fields indicate the presence of a Mode I component. Early growth, studied by successive removal of the most damaged element, resulted in crack
growth rate for the lower hardening case about twice that of the higher hardening one.

INTRODUCTION

In symmetric singly grooved tensile specimens the crack advances into the relatively undamaged region between two symmetric shear zones. In the fully plastic case these zones narrow into bands that traverse the section. An asymmetry, introduced through a weld fillet or a harder, heat-affected zone or a shoulder on one side of the crack (Fig. 1) suppresses one of the two shear bands that would appear in a symmetrical specimen. In that case the crack advances asymmetrically, along the remaining active slip band into previously damaged material. Thus one might expect that the ductility would be less than that of pure Mode I unconstrained symmetric case.

Near the tip of the growing crack, strain hardening will cause the deformation field to fan out. For power law creep or deformation theory plasticity and a stationary crack, the asymptotic stress and strain distribution may be found from the extended by Shih's [1] HRR [2,3] fields for the general mixed mode case. Notice, however, that such a superposition of stationary singularities does not take into account the hardening of the material left behind the growing crack. Indeed, the stress and strain fields near the tips of growing cracks in ductile materials are known to differ from the stress and strain state around stationary cracks in the same materials as is shown from asymptotic solutions [4,5,6,7], supplemented through finite element calculations [8,9]. Thus, more accurately, a solution for the distribution of strain increments of a growing mixed mode crack should be used; however such a solution is not yet available.
A test with pure shear (Mode II) loading was carried out by Chant et al. [10] of high hardening carbon manganese steel (B.S. 1501-151-430A, Y.S. = 329 MN/m², T.S. = 490 MN/m²). Small specimens were subjected to both Mode II and Mode I testing but the ductility, measured by dJ/da, was practically the same although the microscopic features for the pure shear specimens are different than those observed in the Mode I specimens.

Representing ductile crack propagation has been in general based on the introduction of d(COD)/da [11,12,13] and the tearing modulus T or dJ/da [14,15] concepts. Such single-parameter measures are incomplete since the triaxiality and the local distribution of strain are affected by the geometry and mode of loading. The triaxiality and strain distribution in turn strongly affect the cleavage and hole growth mechanisms of crack growth.

The objective of the current chapter is to summarize the important findings of the experimental, analytical, and numerical work that deals with the effect of an asymmetry in crack propagation of unconstrained fully plastic singly grooved tensile configurations. First, approximate solutions based on the superposition of stationary singularities are presented. Next, test results on symmetric and asymmetric specimens of six alloys are summarized, along with a method for quantifying and representing the ductility. In addition, a macro-mechanical model of crack growth by combined fracture on one plane and sliding off along two others, describes, for this idealization of the physical mechanisms, the ductile crack growth for both the asymmetric and symmetric specimens. To account for the effect of the finite width of the shear band and study the stress and strain fields at initiation, a finite element investigation of the asymmetric specimens is performed. Early growth is also studied by successive removal of the critical elements.
INTEGRATED STATIONARY SOLUTIONS

1. Pure Mode II approximation.

A formulation for the accumulation of damage directly ahead of an asymmetric crack, based on strain increments adapted from Shih’s [1] analysis was developed [16]. Strain increments, following a power law relation were integrated both numerically and quasi-steadily. The crack was assumed to follow the center of a 45° shear band of infinitesimal width with the far field displacement, u, being parallel to the shear band (Fig. 2a). The critical fracture strain is determined from the fracture criterion by McClintock, Kaplan and Berg [17].

The predicted displacement to crack initiation is found $u_i = O(\rho)$, of the order of mean incision spacing $\rho$. The crack growth per unit displacement was predicted $dc/du = O[\ln(c/\rho)]$, i.e. to increase approximately as the logarithm of the total crack advance per inclusion spacing $\rho$. The growth rate was found to be practically unaffected by strain hardening. The increasing crack growth rate, associated with the strain distribution flattening out in front of the crack at a decreasing rate that does not reach a steady state, leads to size effects in crack growth.

2. Directional effects.

Due to the higher triaxiality on one side, there is a tendency for fracture in that direction. Thus the previous pure Mode II work was extended to include sites at several angles ahead of the crack. Far field displacement is again assumed to take place parallel to the shear band (Fig. 2b). Strain increments are approximated from the mixed mode, power-law elastic solution for a stationary crack [1] and used with the fracture criterion for hole growth in shear bands [17] to predict the critical
direction. The crack is assumed to advance to advance in the direction that requires the minimum far field displacement to reach critical damage.

At initiation, several sites around the tip are considered. The strain and hence the required displacement for damage of unity is found. The critical direction is that which minimizes the required displacement. In growth, the accumulated damage due to initiation and prior growth is found and then the required increment in damage and hence far field displacement is determined.

For a shear band at $45^0$ the crack progresses at an angle of $21^0-30^0$ from the transverse, depending on the strain hardening, indicating the effect of higher triaxiality. The crack growth rate is about 8-15% higher than with growth along the shear band. Lower strain-hardening results in the final crack orientation being closer to the $45^0$ shear band and the higher crack growth rate.

EXPERIMENTAL STUDY

Tests were performed on fatigue precracked asymmetric (Fig. 3) and symmetric (Fig. 4) specimens of six alloys: 1018 cold finished, 1018 normalized, A36 hot rolled, HY80, HY100 steel, 5086-H111 aluminum. The low-hardening alloys are the 1018 CF, HY80, HY100 steel ($n \approx 0.10$) and the high-hardening alloys are the A36 HR and 1018 normalized steel ($n \approx 0.23$). In addition to the load-displacement data, the topographies of the fracture surfaces were plotted using a metallurgical microscope with a travelling stage. A schematic of the fracture surface profile is shown in Fig. 6. These profiles allow determining the growth displacement vector $V_g^r$, the total displacement $V_i$, and hence the initiation displacement $V_i = V_i - V_g$, as well as the geometry of the fracture (flank angles, flank lengths, crack orientation).
1. Quantifying crack initiation and growth.

Initiation. As a convenient measure of crack initiation displacement from the load-displacement curves, define the "idealized initiation displacement", \( u^I_i/l_0 \) as the normalized extension between initial elastic loading and steepest unloading lines at maximum load (Fig. 5). The normalized form is used in the plots for convenience in correlating crack growth; multiplied by \( l_0 \) it becomes analogous to the more familiar COD. The tests gave:

\[
(u^I_i/l_0)_{ASY} \approx (u^I_i/l_0)_{SY}, \tag{1}
\]

and

\[
(u^I_i/l_0)_{high\, n} \approx (2-4)(u^I_i/l_0)_{low\, n}. \tag{2}
\]

The axial component of the initiation displacement, as measured from the profiles of the fracture surfaces, \( u_i/l_0 \), has the same behavior as the previously defined "idealized initiation displacement", \( u^I_i/l_0 \), i.e., \( u_i/l_0 \) is not different between asymmetric and symmetric cases. In addition, for the high hardening alloys it is about two to four times that of the low hardening ones for both geometries. A noteworthy observation is the fair amount of blunting occurring in both geometries (more blunting with higher \( n \)). An approximate relation can also be observed:

\[
(u^I_i/l_0) \approx (1.5-2.2)(u_i/l_0). \tag{3}
\]

In short, initiation displacement is almost the same same in both asymmetric and symmetric cases; strain hardening affects initiation in both symmetric and asymmetric specimens.

Growth
(i) For a measure of crack growth resistance, define the crack ductility, \( D_g \), as the minimum displacement, \( du_c \), per unit ligament reduction \( dl \). The displacement \( du_c \) is associated with the crack opening stretch and consists of the gauge displacement, \( du \), and the elastic unloading \( du_{\text{unl}} \) (Fig. 5). The ligament reduction, \( dl \), is approximated from the relative load drop, \( dl \approx (dP/P_{\text{max}})l_0 \). Notice that thinning of the ligament from the far side in fully plastic flow makes the reduction in ligament rather than crack advance the appropriate measure of load drop. Thus

\[
D_g = \left( \frac{du_c/l_0}{dP/P_{\text{max}}} \right)_{\text{min}} \approx \left( \frac{du_c}{dl} \right)_{\text{min}}
\]

The crack ductility \( D_g \) is analogous to \( d(COD)/da \) and is related to the crack opening angle (COA):

\[
D_g \approx \text{COA}/\cos^2 \theta_c,
\]

where \( \theta_c \) is the crack orientation. It is also the normalized maximum axial compliance allowed for stability:

\[
\text{compliance allowed} < D_g l_0/P_{\text{max}}.
\]

Tests showed that the crack ductility of the asymmetric specimens vs. that of the symmetric ones is primarily affected by strain hardening. For example,

\[
(D_g)_{\text{SY}}/(D_g)_{\text{ASY}} = 3.37 \text{ HY-100 steel (n=0.10)}
\]

\[
= 1.06 \text{ A36 HR steel (n=0.24)}
\]

In short, substantial reduction in crack growth ductility with asymmetry occurs in low hardening alloys. High hardening alloys are almost as ductile in the asymmetric configuration as in the symmetric one.

(ii) Other possible measures of growth are related to \( D_g \). A parameter, \( T^* \), analogous to tearing modulus, \( T=(E/\sigma_0^2)(dJ/da) \), can be defined. By approximating
we can define $T^*$ in terms of $D_g$, the tensile strength, T.S., and the modulus of elasticity, E:

$$T^*_{ASY} = D_g\left(\frac{E}{\sqrt{3}}\right)/T.S., \quad T^*_{SY} = D_g\left(\frac{2E}{\sqrt{3}}\right)/T.S.$$  \hfill (8)\]

Tests gave:

$$(T^*_{ASY})_{\text{high}}/\left(T^*_{ASY}\right)_{\text{low}} > 3$$

$$(T^*_{SY})_{\text{high}} \simeq \left(T^*_{SY}\right)_{\text{low}}$$

In conclusion, strain hardening does not affect the ductility of symmetric specimens; it does affect the ductility of asymmetric specimens.

(iii) The growth displacement as measured from the profiles of the fracture surfaces, $u_g/\ell_0$, has the same behavior as the crack ductility $D_g$: for the low hardening alloys it is smaller in the asymmetric configuration than the symmetric by a factor of more than three whereas in the high hardening alloys it is reduced by a factor of 1.2 at most.

(iv) The displacement vector in the asymmetric specimens is more axial than $45^0$, suggesting a Mode I component. The angle from transverse, $\phi$, is between $51^0$ for the 1018 CF steel and $63^0$ for the 1018 normalized steel.

(v) The crack direction in the asymmetric specimens is less than $45^0$, indicating the effect of triaxiality. The angle $\theta_c$ is $38^0-41^0$ from transverse, larger values for the lower hardening alloys.
In conclusion, the experiments showed that while the crack initiation displacements are similar, the growth displacement for the low hardening alloys in the asymmetric case is much less than that of the symmetric. Triaxiality on one side of the asymmetric crack diverts it from $45^0$ to $38^0-41^0$ while the far field displacement vector is more axial than $45^0$, at $51^0-63^0$, suggesting a Mode I component, even with asymmetry.

Table 1 compares the experimental findings with the predictions of the integrated stationary crack field solutions. The initiation displacement is an order of magnitude larger than the theoretical one, apparently due to blunting. The incremental models cannot account for the big effect of strain hardening in crack growth; notice that these models are based on a superposition of stationary singularities and thus do not take the convection of hardened material into account. The size effects in fully plastic crack growth that are predicted from the incremental pure Mode II analysis are associated with the transient behavior (increasing crack growth per unit displacement).

Finally, fractographic observations show as noteworthy features in the asymmetric specimens the "shear type" fracture, more evident in the lower hardening alloys and a larger amount of sliding off in the lower flank. The symmetric specimens, with the larger ductility, show in turn the "normal type" fracture with more equiaxed voids than the corresponding asymmetric specimens.

SHEAR BAND CHARACTERIZATION OF CRACK GROWTH

To provide a physical basis for interpreting the test data, a macro-mechanical model for crack advance by sliding off and fracture was developed. The model
assumes in the general mixed mode case sliding off along two slip planes and fracture on a third and gives the independent parameters (shear and cracking directions, relative amounts of cracking and shearing) in terms of the observable quantities of the macroscopic fracture (flank angles, flank lengths, back angle).

To describe the development of deformation, assume cycles of sliding off on an upper slip plane at $\theta_{su}$ through a distance $s_u$, then on a lower at $\theta_{sl}$ through $s_l$ and finally fracture at $\theta_f$ over a distance $f$ (Fig. 7). The limiting Mode I case with two symmetric slip planes corresponds to $\theta_{su} = -\theta_{sl}$, $\theta_f = 0^0$, $s_u = s_l$, and the limiting Mode II single slip plane case corresponds to $s_l = 0$. Thus there are 5 independent physical variables: the slip and fracture angles, $\theta_{su}$, $\theta_{sl}$, $\theta_f$, the cracking ratio $\chi = f/s_u$ and the shearing ratio $\xi = s_l/s_u$. Observable quantities that allow solving for the physical variables are the flank angles, $\theta_h$, $\theta_u$, the flank lengths normalized with the initial ligament, $l_u/l_0$, $l_l/l_0$, and the back angle, $\beta_u$, defined as the angle the deformed upper back surface makes to the load axis. Closed-form expressions are given in chapter four.

Examples (HY-100 and 1018 normalized steel) are shown in Table 2. For the asymmetric specimens the shearing ratio $\xi$ is found to be about 0.5 indicating shearing in lower flank twice that in upper flank. SEM fractographs have confirmed that the lower flank shows indeed more "shear type" fracture than the upper one. The slip angle difference $\theta_{sl} - \theta_{su}$ is a measure of the spreading out of deformation and is found to be in the high hardening alloys $4^0-6^0$ as opposed to $1^0-2^0$ for the low hardening ones. The cracking ratio $\chi$ is a measure of the relative amount of fracture and sliding off on the upper surface and allows comparing with the "apparent crack ductility", $D_{AC}$, as observed fractographically and defined as the sliding off to total area including fracture. Thus,
In upper flank $D_{AC,u} = s_f/(f+s_f) = 1/(\chi/\xi+1)$.

In lower flank $D_{AC,l} = s_u/(f+s_u) = 1/(\chi+1)$.

A comparison with values observed from SEM fractographs (Table 3), shows that the values from the fractographs are bigger by about a factor of two. Considering the idealization of the complex hole-crack tip interaction and the difficulty in measuring $D_{AC}$ (from the extent of void growth) in the fractographs, the agreement is encouraging, giving the right trend (low hardening alloys less ductile in the asymmetric configuration than the symmetric but high hardening alloys almost equally ductile in both geometries).

**FINITE ELEMENT STUDY OF THE ASYMMETRIC SPECIMENS**

For further insight, a finite element study of the asymmetric specimens is performed. This work is needed to relax the assumption of the far field displacement being parallel to shear band (as was presumed in the superposition of stationary singularities), to account for the finite width of the shear band and to allow describing the Mode $I$ component at initiation. In this finite element work (mesh is shown in Fig. 8) blunting was neglected. Besides initiation, early growth was studied by successive removal of elements reaching unit damage. To describe the Mode $I$ component, use the Mode $I$ mixity parameter $M^P$ based on stresses [1].

$$M^P = \frac{2}{\pi} \tan^{-1} \left( \lim_{r \to 0} \frac{\sigma_{\theta}(r, \theta = \theta_{\text{crack}})}{\sigma_{r}(r, \theta = \theta_{\text{crack}})} \right).$$

Results, compared with the superposition of stationary singularities and test data are shown in Table 4. Notice the presence of a large Mode $I$ component with the far field displacement vector not along the $45^0$ shear band but at an angle about $68^0$ from the transverse. In addition stress and strain fields are found consistent
with the solutions for the mixed mode extended HRR fields. Displacement to crack
initiation is of the order of the fracture process zone size. The critical direction is
predicted at an angle of \(39^0-43^0\) from the transverse, increasing for a lower strain
hardening exponent. Finite element study of early growth resulted in extension rate
for the lower hardening case about half that of the higher hardening one.

**CONCLUSIONS**

1) Low hardening asymmetric specimens are substantially less ductile than the
symmetric ones. For the crack ductility, \(D_g\), defined as the minimum displacement
per ligament reduction,

\[
\frac{(D_g)_{SY}}{(D_g)_{ASY}} > 3 \text{ for lower hardening alloys} \\
< 1.2 \text{ for higher hardening alloys}
\]

Thus, there is a significant effect of strain hardening in mixed Mode I and II fully
plastic crack growth.

2) The initiation displacement is insensitive to geometry; however it depends
on strain hardening. Blunting of the order of \(10\rho\) or 0.1mm occurs in both
generics.

3) The displacement vector is more axial than \(45^0\), at \(51^0-63^0\) from transverse,
suggesting a Mode I component even with asymmetry (where nonhardening solutions
give pure shear).

4) The crack direction is less axial than \(45^0\), at \(38^0-41^0\) from transverse (closer
to \(45^0\) with less hardening); this indicates the effect of triaxiality.

5) A superposition of stationary singularities gives practically no effect of
strain hardening; it overestimates the ductility of low hardening asymmetric
specimens.
6) Finite element study of the asymmetric specimens, neglecting blunting, predicts at initiation a critical direction at about 40°, a far field displacement at about 68°, gives HRR consistent stress and strain fields and describes the Mode I component.

7) Early growth, studied by successive removal of critical elements shows an effect of strain hardening; the crack growth rate for \( n=0.12 \) was twice that of \( n=0.24 \).

8) A shear band model by fracture and sliding off on two planes describes mixed mode crack growth; provides a physical basis of interpreting results.

SIGNIFICANCE AND RECOMMENDATIONS

Although much work has been done in the elastic and elastic-plastic fracture mechanics, less is known for fracture under fully plastic conditions. In structures, fully plastic flow before fracture is desirable since large deformations help detect impending fracture as well as help stabilize it by load shifting. Most fracture tests use symmetric specimens (e.g. bend, compact tensile specimens). An asymmetric configuration, however, may arise due to a weld fillet or a harder, heat-affected zone or a shoulder on one side of the crack. The effect of asymmetry on unconstrained tensile specimens has been quantified and analyzed for several alloys. Results reported here show that asymmetric (mixed mode I and II) fully plastic configurations in low hardening alloys may be less ductile than the corresponding symmetric singly grooved tensile specimens by more than a factor of 3, increasing the stiffness requirements for fracture-stable design. In addition a standard way of representing tests and comparing the two geometries is suggested. In the fully plastic state, since geometry and mode of loading can affect the triaxiality and the local strain fields, the crack growth ductility will not be a single parameter but a set.
each referring to a certain configuration and triaxiality.

Further work should include studying the effect of triaxiality by performing constrained asymmetric tests. For example, tensile testing on doubly-grooved specimens with the asymmetry introduced through varying notch angles and positions; or laterally constrained singly-grooved tensile tests. Fully plastic fracture under high triaxiality could be studied by wedge-splitting of a doubly grooved specimen. Another extension could involve testing part-through cracks in plates with asymmetric shoulders.

On the analytical side, there is a need for an asymptotic solution of mixed mode growing cracks, coupled with finite element solutions that connect far-field and near-field parameters; and possibly finding a rigid-plastic singularity for a growing crack with deforming flanks.

REFERENCES


### TABLE 1

**Asymmetric fully plastic cracks**

<table>
<thead>
<tr>
<th>Approximate Mode II soln.</th>
<th>Superposition of Shih soln.</th>
<th>Tests</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initiation displacement:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$u_1 = (0.8-0.6)\rho$</td>
<td>$u_1 = (0.6-0.4)\rho$</td>
<td>$u_1 = (0.39)\rho$ due to blunting</td>
</tr>
<tr>
<td>Crack direction:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta_c = 45^0$</td>
<td>$\theta_c = 21^0-30^0$</td>
<td>$\theta_c = 38^0-41^0$</td>
</tr>
<tr>
<td>(assumed)</td>
<td>$\uparrow \theta_c$ with $n \downarrow$</td>
<td>$\uparrow \theta_c$ with $n \downarrow$</td>
</tr>
<tr>
<td>Growth Ductility ($D_g = du_c/dl$):</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.190-0.200</td>
<td>0.180-0.170</td>
<td>0.215-0.072</td>
</tr>
<tr>
<td>no effect of $n$</td>
<td>little effect of $n$</td>
<td>strong effect of $n$</td>
</tr>
</tbody>
</table>

Size effects in fully plastic growth:

1.5in. dia. specimens vs. 0.5in. dia. specimens:

$$\frac{(D_g)_{0.5in}}{(D_g)_{1.5in}} = 1.20$$

$$\frac{(D_g)_{0.5in}}{(D_g)_{1.5in}} = 1.04$$
TABLE 2  
Deformation of singly-grooved  
asymmetric and symmetric specimens

<table>
<thead>
<tr>
<th>Alloy:</th>
<th>HY-100</th>
<th>1018 normalized</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>n=0.10</td>
<td>n=0.24</td>
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<table>
<thead>
<tr>
<th>Observations</th>
<th>Asy</th>
<th>Sy</th>
<th>Asy</th>
<th>Sy</th>
</tr>
</thead>
<tbody>
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<td>Length ratio, $l_u/l_0$</td>
<td>0.820</td>
<td>0.780</td>
<td>0.750</td>
<td>0.740</td>
</tr>
<tr>
<td>Length ratio, $l_1/l_0$</td>
<td>0.900</td>
<td>0.780</td>
<td>0.870</td>
<td>0.740</td>
</tr>
<tr>
<td>Flank angle, $\theta_u$</td>
<td>39°</td>
<td>-14°</td>
<td>36°</td>
<td>-12°</td>
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<tr>
<td>Flank angle, $\theta_1$</td>
<td>41°</td>
<td>14°</td>
<td>42°</td>
<td>12°</td>
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<tr>
<td>Back angle, $\beta_u$</td>
<td>14°</td>
<td>(13°)</td>
<td>13°</td>
<td>(16°)</td>
</tr>
</tbody>
</table>

Corresponding slip and fracture parameters

| Slip angle, $\theta_{sl}$             | 53°     | -41°            | 52°    | -31°            |
| Slip angle, $\theta_{su}$              | 54°     | -41°            | 58°    | 31°             |
| Cracking angle, $\theta_f$             | 37°     | 0°              | 31°    | 0°              |
| Shearing ratio, $\xi$                  | 0.536   | 1.00            | 0.415  | 1.00            |
| Cracking ratio, $\chi$                 | 2.012   | 1.907           | 1.518  | 1.579           |
### TABLE 3
Apparent crack ductility

<table>
<thead>
<tr>
<th>Shear band model</th>
<th>HY100 steel</th>
<th>1018 normalized steel</th>
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<tr>
<td></td>
<td>0.64</td>
<td>0.67</td>
</tr>
<tr>
<td></td>
<td>0.64</td>
<td>0.67</td>
</tr>
</tbody>
</table>
TABLE 4
Comparing with the finite element results.

<table>
<thead>
<tr>
<th>Finite Element</th>
<th>Superposition of Shih soln.</th>
<th>Tests due to blunting</th>
</tr>
</thead>
<tbody>
<tr>
<td>(blunting neglected)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Initiation displacement:

\[ u_i = (0.5-0.6) \rho \]

\[ u_i = (0.6-0.4) \rho \]

\[ u_i = (0.3-0.9)_\rho \]

Mode I component at initiation, MP:

relative to initial crack direction

\[ \approx 0.93 \]

relative to final crack direction

\[ \approx 0.71 \]

Displacement vector at initiation:

\[ \phi \approx 68^0 \]

\[ \phi = 45^0 \]

\[ \phi \approx 69^0-58^0 \]

(assumed)

Crack direction:

\[ \theta_c = 39^0-43^0 \]

\[ \theta_c = 21^0-30^0 \]

\[ \theta_c = 38^0-41^0 \]

\[ \theta_c \uparrow \text{ with } n \]

\[ \theta_c \uparrow \text{ with } n \]

\[ \theta_c \uparrow \text{ with } n \]

Gauge extension rate

0.158-0.075

(0.180-0.170)_\text{min}

(0.153-0.060)_\text{min}

(Early growth)

strong effect of n

little effect of n

strong effect of n
Figure 1. Asymmetric crack from a defect near a weld or a shoulder. The symmetric case is shown above.
Figure 2.  
(a) Pure Mode II case. 
(b) Crack at an angle to the shear band.
Figure 3. Second machining (after fatigue precracking) for the asymmetric specimen. \( a \) is the fatigue precrack.
Figure 4  Second machining (after fatigue precracking) for the symmetric specimens: a is the fatigue precrack
Figure 5. Schematic of the load vs. axial extension curve. The displacements $u_1$ and $u_2$ are measured after fracture.
Figure 6. Schematic of the Fracture Surface Profile of the Asymmetric and Symmetric specimens.
Figure 7. Development of deformation for the two-slip plane model.
Figure 8. The finite element mesh.
APPENDIX A - Experimental Techniques

In this section the experimental techniques used (tensile testing and extensometer connections, microscopic surface topography mapping, fatigue-precracking) are described.

1. TENSILE TESTS WITH MTS, 50 METRIC TON MACHINE

Preliminary
Set Console Power switch On (Master Control Panel 413).
Set Hydraulic Pump switch in Room 1-014 ON.
Check that Feedback Selector (Model 440-32 behind the 442 Controller Panel) is in LOCAL.
Set switch in PDP-11 behind the panel under the disk drives to OUTPUT.

410 - Digital Function Generator
Set to LOCAL (to start by pressing START - if set to REMOTE then you must start from RUN in 413).
Select rates of loading and unloading, e.g.
For 20 mm stroke range,
Rate 1 = 2.4E3 sec. means 2400 secs for 20 mm loading
Rate 2 = 2.4E2 sec. means 240 secs for 20 mm unloading
For 50 mm stroke range, same rates of loading and unloading require
Rate 1 = 6E3 sec.,
Rate 2 = 6E2 sec.
PRESS RAMP, DUAL SLOPE for monotonic loading and unloading.
Set BREAKPOINT to NORMAL and BRKPNT PERCENT to 100.
**442 - Controller**

Press STROKE (for stroke control).

Set STROKE range as desired (e.g. ±20 mm) by turning the RANGE knob behind the panel.

Set LOAD Range e.g. to 10K by turning the RANGE knob on stroke module behind the panel; 10K means 10V output correspond to 10,000 kgf=22,000 lb

Set SPAN 1 to 100%

Zero load indication (see digital indicator channel 1 in 430 panel) before inserting specimen by turning the ZERO knob on Load module (Model 440.21) behind the 442 panel.

Press RETURN TO ZERO (in 410) to zero out any pre-existing function generator signal.

Set METER at DC ERROR. Zero out error by turning SET POINT right if pointer is right; left if pointer is left. If whole range is not enough, use ZERO knob on stroke module behind 442 panel.

Press INTERLOCK RESET.

**413 - Master Control Panel**

Press RESET.

Press HYDRAULIC PRESSURE LOW, then HIGH.

Put SET POINT in 442 to 5.0 - Ram will move. (Digital indicator channel 3 should read 0.0). To further move the ram use ZERO knob on stroke module (Model 440.21) behind 442 (Controller) panel.

Notes: 1. Ram moves DOWN when turning SET POINT right,

2. to increase stiffness move ram up.

As you tighten the locknuts use SET POINT to relieve any compression: watch digital indicator channel 1 for load (channel 3 is stroke).
Press START in 410.

**End of Test**

TO STOP: Press HYDRAULIC PRESSURE LOW then HYD OFF on 413.
To LEAVE: Set Hydraulic Pump switch in Room 1-014 OFF.
Set Console Power OFF on 413.

**Intermediate Manual Unloading and Reloading**

To UNLOAD: Press HOLD in 410 (holds the test), then turn SET POINT in 442 left to unload.
TO RELOAD: Turn SET POINT right.
To CONTINUE with preset rate: Press HOLD once more.

**Moving Crosshead**

Press HYDRAULIC LOW, then HIGH on 413.
Set switch behind the MTS to UNLOCK.
Move crosshead by turning the UP or DOWN handle as you wish.
Set switch behind the MTS to LOCK.
Press RESET on 413.

**MTS Plotter**

3 channels, X,Y1 and Y2. Y1 not working

* e.g. Using X channel for stroke, selecting 5% of range/in with 20 mm stroke range corresponds to 1 mm/in on the plot.

Using Y2 channel for load, selecting 20% of range/in with 10K load range corresponds to 2000 kgf/in on the plot.

**Stiffness Data of the MTS.**
Load Cell 33.0E6 lb/in
Load Frame 6.0E6 lb/in
2. EXTENSOMETER CONNECTIONS WITH THE VISHAY AMPLIFIER/CONDITIONER

Wiring Correspondence

**AXIAL Extensometer - Full Bridge**

<table>
<thead>
<tr>
<th>Extens.</th>
<th>Vishay</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>RED D out +</td>
</tr>
<tr>
<td>B</td>
<td>GREEN C exc -</td>
</tr>
<tr>
<td>C</td>
<td>YELLOW A out -</td>
</tr>
<tr>
<td>D</td>
<td>BLUE B exc +</td>
</tr>
<tr>
<td>E</td>
<td>BLACK F ground</td>
</tr>
</tbody>
</table>

CA. A corresponds to 0.0275 in. (0.6985 mm)

**TRANSVERSE extensometer - Half Bridge**

<table>
<thead>
<tr>
<th>Extens.</th>
<th>Vishay</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A out +</td>
</tr>
<tr>
<td>B</td>
<td>C exc - , out -</td>
</tr>
<tr>
<td>D</td>
<td>B exc +</td>
</tr>
<tr>
<td>E</td>
<td>F ground</td>
</tr>
</tbody>
</table>

CA. A corresponds to 0.055 in. (1.397 mm)
3. SURFACE TOPOGRAPHY MAPPING WITH THE BAUSCH AND LOMB TRAVELLING MICROSCOPE.

The apparatus consists of the microscope, linear potentiometers, stage extension bar and assorted rubber bands and C-clamps for fixing the potentiometers on the microscope. One potentiometer is clamped to the side of the microscope and one to the travelling stage.

Rubber bands are used to secure the stage extension bar to the microscope knee. Note that some of the rubber bands go around the back of the microscope. They serve two purposes; they keep the stage extension bar firmly against the microscope and they act to offset its weight.

Adjust the potentiometer and the stage extension bar so that the points of the potentiometer are reasonably centered on the bottom of the cap screws in the stage extension bar. Use rubber bands to secure the potentiometer ends on the stage extension bar screws.

The electric circuit employs a $V_0=3V$ battery and a balancing 10-turn potentiometer for each linear pot. The two green wires from the linear pot go to the battery terminals hooked with the balancing pot as in Fig. 1. The blue wire from the linear pot and the remaining wire from the balancing pot go to the plotter. Solving the circuit gives

$$V_1 - V_2 = \left( \frac{R_2}{R_1 + R_2} - \frac{R_4}{R_3 + R_4} \right) V_0.$$  

Zero the output $V_1-V_2$ at the starting point by using the balancing 10-turn potentiometer (i.e. adjust $R_4$ so that $V_1-V_2=0$).
To plot the topography use the 10x power on the microscope. The higher magnifications require the objective to be quite close to the surface of the specimen and one could easily hit the objective on a peak of the viewed surface when trying to focus into a valley. Move slowly your specimen along the horizontal axis and get the corresponding vertical coordinate by having the centerline of the specimen always in focus. Notice that the vertical fine scale on the microscope is 10 rev=1 mm.
4. FATIGUE PRE‐CRACKING

Specimen precracking was done on the SF‐1 Fatigue Machine, which is a fixed frequency (3600 rpm) rotating mass machine. It was used with the bending fixtures. The specimens were subjected to four‐point bending. A uniform bending moment \( M \) is applied across the span of the specimen, given in terms of the load moment arm \( R \) (distance between rocker bearings, \( R = 3 \) or \( 6 \) inches) and the total applied force \( P \) from

\[
M = PR/2 .
\]  

(1)

The nominal alternating stress \( \sigma_a \) can be found by using the moment of inertia

\[
I = bh^3/12 \quad (b, h \text{ are specimen width, and thickness}) \text{ from from} \]

\[
\sigma_a = \frac{M(h/2)}{I} = \frac{3PR}{bh^2} = \frac{M}{bh^2/6} . \]  

(2)

Notice that \( \sigma_a \) does not depend on the specimen length. In terms of the specimen length between grips, \( L \) (\( L = 3, 5, 6 \) or \( 8 \) inches) and the modulus of elasticity of the specimen, \( E \), the amplitude of the vibrating platen \( Y \) is

\[
Y = R \frac{ML}{2EI} = \frac{3PR^2L}{Ebh^3} . \]  

(3)

The following restrictions apply

Maximum applied force, \( P = 1,000 \) lb

Maximum amplitude of reciprocating platen \( Y = 0.4 \) inches

As a rough approximation assume \( \sigma_a \approx \text{T.S.} \). Then from (2) find the necessary

\[ M = (\text{T.S.})bh^2/6. \]  

For a chosen \( R \) find the required \( P \) from (1), \( P = 2M/R \). Next, for the chosen specimen length \( L \) check that the resulting \( Y \) from (3) is less than 0.4 inches. Notice that \( P \) should be less than 1,000 lb.
Indicative data for fatigue pre-cracking on the SF-1 with R=3 inches are:

1018 CF steel (HBN=157) 0.50” dia. needs about 10,000 cycles with P=90 lb to grow 0.050” fatigue crack.
A36 steel (HBN=105) 0.50” dia. needs about 7,000 cycles, P=80 lb to grow 0.050” fatigue crack.
5086-H111 aluminum (HBN=70) 1.50” dia. needs about 12,000 cycles, P=1,000 lb to grow 0.150” fatigue crack.

To operate the machine (for details see instruction manual):

Turn the CONTROLLER POWER switch to STANDBY.
Turn on main power switch on wall behind machine. Wait for at least half hour to allow warmup.
Attach tuning weights to the studs on either side of the orange cage.
Set the oscillating load P.
Check that knob of variable transformer is at zero.
Press START button. Gradually turn knob of variable transformer, increasing the motor speed to the extreme 100 position. This should take from 20 to 80 sec.
Turn CONTROLLER POWER switch ON.
To STOP press the STOP button and turn the variable transformer back to zero.
Figure 1. Surface topography mapping circuit.
APPENDIX B - MESH GENERATOR

The following FORTRAN program generates the finite element mesh for the asymmetric specimens for any desirable radial size ratio and angular spacings. Bar denotes user input.
This program creates the node nos., element nos., and the corresponding MPC constraints for the fan. We have maximum Nr segments radially and N6 radial lines around; here Nr=32 and N6=19.

RG=10 microns (one inclusion spacing) = 0.01 mm

s is the size ratio in the log circular mesh=1.155

Angular spacings are (larger to smaller) theta, theta1, theta2, theta3; here theta=30, theta1=15, theta2=7.5, theta3=3.75 deg.

L is the node no. with coords x(L), y(L); M is the element no.

with nodes Nod1(M), Nod2(M), ..., Nod8(M).

File no. 15 contains the nodes, no. 16 contains the element data and no. 17 contains the MPC constraints.

DIMENSION x(15000), y(15000), Nod1(15000), phi(100),
- Nod2(15000), Nod3(15000), Nod4(15000), R(100),
- Nod5(15000), Nod6(15000), Nod7(15000), Nod8(15000)

Note: Nr should be a multiple of 8 (so that the min. no.

c of segments radially, corresp. to the largest angle, is Nr/8)

c Rmax = RG(s**Nr-1)/(s-1)

RO = 0.01
s = 1.155
pi = 3.14159
theta = 30.*pi/180.

theta1 = 15.*pi/180.
theta2 = 7.5*pi/180.

R30 = RO*(1+s+s**2+s**3+s**4+s**5+s**6+s**7)
s30 = s**8

N30 = Nr/8
R15 = RO*(1+s+s**2+s**3)
s15 = s**4
N15 = Nr/4
R75 = RO*(1+s)
s75 = s**2

N75 = Nr/2

1001 FORMAT (I5,F10.7,F10.7)

Dtheta for radial lines 1-N1 is 15 deg.; N1-N2: 7.5 deg.;
N2-N3: 3.75 deg.; N3-N4: 7.5 deg.;
N4-N5: 15 deg.; N5-N6: 30 deg.; N7: crack flank.

N1 = 3
N2 = 4
N3 = 8
N4 = 9
N5 = 11
N6 = 19
N7 = 14

Input Nodes for the 15 deg. sectors
DO 90 I=1, N1-1
  phia = (I-1)*theta1
  phi(I) = phia
DO 100 J=1, N75
  R1 = R75*(S75**J - 1)/(S75-1)
  L = 500*I + 4*J
  x(L) = R1*cos(phi)
  y(L) = R1*sin(phi)
  WRITE(15,1001) L, x(L), y(L)
100 CONTINUE
90 CONTINUE
DO 911 I=1, N1-1
phia = (I-1)*thet1 + thet1/2.
DO 922 J=1, N15
   R1 = R15*(s15**J - 1)/(s15-1)
   L = 500*I + 200 + 8*J
   x(L) = R1*cos(phia)
   y(L) = R1*sin(phia)
   WRITE(15, 1001) L, x(L), y(L)
922 CONTINUE
911 CONTINUE

* Input Nodes for the 7.5 deg. sector
DO 82 I=N1, N2-1
   phia = (N1-1)*thet1 + (I-N1)*thet2
   phi1(I) = phia
   DO 101 J=1, Nr
      R1 = RO*(s**J - 1)/(s-1)
      L = 500*I + 2*J
      x(L) = R1*cos(phia)
      y(L) = R1*sin(phia)
      WRITE(15, 1001) L, x(L), y(L)
101 CONTINUE
82 CONTINUE
DO 912 I=N1, N2-1
   phia = (N1-1)*thet1 + (I-N1)*thet2 + thet2/2.
   DO 923 J=1, N75
      R1 = R75*(s75**J - 1)/(s75-1)
      L = 500*I + 200+4*J
      x(L) = R1*cos(phia)
      y(L) = R1*sin(phia)
      WRITE(15, 1001) L, x(L), y(L)
923 CONTINUE
912 CONTINUE

* Input nodes for the 3.75 deg. sectors
DO 83 I=N2, N3
   phia = (N1-1)*thet1 + (N2-N1)*thet2 + (I-N2)*thet3
   phi1(I) = phia
   R1p = 0.
   DO 132 J=1, Nr
      R1 = RO*(s**J - 1)/(s-1)
      L = 500*I + 2*J
      x(L) = R1*cos(phia)
      y(L) = R1*sin(phia)
      WRITE(15, 1001) L, x(L), y(L)
      L = 500*I + (2*J-1)
      R2 = (R1+R1p)/2.
      x(L) = R2*cos(phia)
      y(L) = R2*sin(phia)
      WRITE(15, 1001) L, x(L), y(L)
      R1p = R1
132 CONTINUE
83 CONTINUE
DO 913 I=N2, N3-1
   phia = (N1-1)*thet1 + (N2-N1)*thet2 + (I-N2)*thet3
   + (I-N2)*thet3/2.
   DO 924 J=1, Nr
      R1 = RO*(s**J - 1)/(s-1)
      L = 500*I + 200 +2*J
      x(L) = R1*cos(phia)
      y(L) = R1*sin(phia)
      WRITE(15, 1001) L, x(L), y(L)
924 CONTINUE
913 CONTINUE

914 CONTINUE

c Input nodes for the 7.5 deg sector
DO 84 I=N3+1, N4
   phia = (N1-1)*thet1 + (N2-N1)*thet2
   + (N3-N2)*thet3 + (I-N3)*thet2
   phi(I) = phia
DO 133 J=1, Nr
   R1 = R0*(s**J - 1)/(s-1)
   L = 500*I + 2*J
   x(L) = R1*cos(phi)
   y(L) = R1*sin(phi)
   WRITE(15,1001) L, x(L), y(L)
133 CONTINUE
84 CONTINUE

915 CONTINUE

916 CONTINUE

c Input nodes for the 15 deg. sectors
DO 85 I=N4+1, N5
   phia = (N1-1)*thet1 + (N2-N1)*thet2
   + (N3-N2)*thet3 + (I-N3)*thet2 + thet2/2.
   DO 925 J=1, N75
      R1 = R75*(s75**J - 1)/(s75-1)
      L = 500*I + 200 + 4*J
      x(L) = R1*cos(phi)
      y(L) = R1*sin(phi)
      WRITE(15,1001) L, x(L), y(L)
925 CONTINUE
85 CONTINUE

917 CONTINUE

918 CONTINUE

c Input nodes for the 30 deg. sectors
DO 86 I=N5+1, N6
   phia = (N1-1)*thet1 + (N2-N1)*thet2 + (N3-N2)*thet3
   + (N4-N3)*thet2 + (N5-N4)*thet1 + (I-N5)*thet1/2.
   DO 926 J=1, N15
      R1 = R15*(s15**J - 1)/(s15-1)
      L = 500*I + 200 + 8*J
      x(L) = R1*cos(phi)
      y(L) = R1*sin(phi)
      WRITE(15,1001) L, x(L), y(L)
926 CONTINUE
86 CONTINUE

919 CONTINUE
CONTINUE
CONTINUE
DO 916 I=N5, N6
   phia = (N1-I)*thet1 + (N2-N1)*thet2
       + (N3-N2)*thet3 + (N4-N3)*thet2
       - (N5-N4)*thet1 + (I-N5)*thet + thet/2.
   DO 927 J=1, N30
      R1 = R30*(s30**J - 1)/(s30-1)
      L = 500*I + 200 + 16*J
      x(L) = R1*cos(phia)
      y(L) = R1*sin(phia)
      WRITE(15,1001) L, x(L), y(L)
927 CONTINUE
916 CONTINUE
C Nodes for crack flank
I=N7
   phia = phi
   phi(I) = phia
   DO 11 J=1, N15
      R1 = R15*(s15**J - 1)/(s15-1)
      L = 500*I + 100 + 8*J
      x(L) = R1*cos(phia)
      y(L) = R1*sin(phia)
      WRITE(15,1001) L, x(L), y(L)
11 CONTINUE
C Nodes for crack tip
DO 87 I=1, N6
   L = 500*I
   x(L) = 0.
   y(L) = 0.
   WRITE (15,1001) L, x(L), y(L)
   L = 500*I + 200
   x(L) = 0.
   y(L) = 0.
   WRITE (15,1001) L, x(L), y(L)
87 CONTINUE
C Elements - M is the element no.
1002 FORMAT(15,8I5)
C
N8 = 12
N9 = 16
N99 = 17
N10 = 18
xc = 3.81
phia = (N7-N8)*thet
Rcy = 3.81/cos(phia)
yc = Rcy*sin(phia)
C Notice: For the shoulder, the following constraint
C should be fulfilled:
C (xc1 + rad - xc)**2 + dy**2 = rad**2
C with dy=xc1*tan(thet)
rad = 1.49
alpha = 1. + tan(thet)*tan(thet)
beta = xc - rad
gamma = (xc-rad)*(xc-rad) - rad*rad
\[
delta = \beta \cdot \beta - \alpha \cdot \gamma
\]
\[
x_{cl} = (\beta + \sqrt{\delta}) / \alpha
\]
\[
dy = x_{cl} \cdot \tan(\theta)
\]
\[
\text{Nadd} = 2 \cdot (N_r + 1)
\]
\[
\text{Nadd} = 2 \cdot N_r - 1
\]

In each sector, first determine boundary nodes and then input elements.

DO 151 I=N1, N2-1

Rc(I) = x_{cl} / \cos(\phi(I))

arg = (Rc(I) * (s75-1) / R75) + 1

N_r = NINT( \log(\text{arg}) / \log(s75) )

L_1 = 500 \cdot I + \text{Nadd}

x(L_1) = x_{cl}

y(L_1) = Rc(I) * \sin(\phi(I))

WRITE (15,1001) L_1, x(L_1), y(L_1)

L_2 = 500 \cdot (I+1) + \text{Nadd}

Rc(I+1) = x_{cl} / \cos(\phi(I+1))

x(L_2) = x_{cl}

y(L_2) = Rc(I+1) * \sin(\phi(I+1))

WRITE (15,1001) L_2, x(L_2), y(L_2)

L_3 = L_1 + 200

x(L_3) = (x(L_1) + x(L_2)) / 2.

y(L_3) = (y(L_1) + y(L_2)) / 2.

WRITE (15,1001) L_3, x(L_3), y(L_3)

DO 141 J = 1, N_r-1

M = 100 \cdot J + I

Nod1(M) = 500 \cdot I + 4 \cdot (J-1)

Nod2(M) = 500 \cdot I + 4 \cdot J

Nod3(M) = 500 \cdot (I+1) + 4 \cdot J

Nod4(M) = 500 \cdot (I+1) + 4 \cdot (J-1)

Nod5(M) = 500 \cdot I + 2 \cdot (2 \cdot J-1)

Nod6(M) = 500 \cdot I + 200 + 4 \cdot J

Nod7(M) = 500 \cdot (I+1) + 2 \cdot (2 \cdot J-1)

Nod8(M) = 500 \cdot I + 200 + 4 \cdot (J-1)

WRITE (16,1002) M, Nod1(M), Nod2(M), Nod3(M),

Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)

CONTINUE

J = N_r

L_1 = 500 \cdot I + 4 \cdot (J-1)

L_2 = 500 \cdot I + \text{Nadd}

L_3 = 500 \cdot (I+1) + \text{Nadd}

L_4 = 500 \cdot (I+1) + 4 \cdot (J-1)

L_{12} = 500 \cdot I + \text{Nadd}

x(L_{12}) = (x(L_1) + x(L_2)) / 2.

y(L_{12}) = (y(L_1) + y(L_2)) / 2.

WRITE (15,1001) L_{12}, x(L_{12}), y(L_{12})

L_{34} = 500 \cdot (I+1) + \text{Nad1}

x(L_{34}) = (x(L_3) + x(L_4)) / 2.

y(L_{34}) = (y(L_3) + y(L_4)) / 2.

WRITE (15,1001) L_{34}, x(L_{34}), y(L_{34})

M = 100 \cdot J + I

Nod1(M) = L_1

Nod2(M) = L_2

Nod3(M) = L_3

Nod4(M) = L_4

Nod5(M) = L_{12}

Nod6(M) = 500 \cdot I + 200 + \text{Nadd}

Nod7(M) = L_{34}

Nod8(M) = 500 \cdot I + 200 + 4 \cdot (J-1)

WRITE (16,1002) M, Nod1(M), Nod2(M), Nod3(M)
MIXED MODE I AND II FULLY PLASTIC CRACK GROWTH FROM SIMULATED WELD DEFECTS(U) MASSACHUSETTS INST OF TECH CAMBRIDGE DEPT OF MECHANICAL ENGIN. G A KARDOMATEAS UNCLASSIFIED 23 OCT 85 N00014-82-K-0025 F/G 13/8 NL
CONTINUE
DO 152 I=N2, N3-1
Rc(I) = xci/cos(phi(I))
arg = ( Rc(I)*(s-1)/RO ) + 1
Nr1 = NINT( LOG(arg)/LOG(s) )
L1 = 500*I + Nadd
x(L1) = xcl
y(L1) = Rc(I)*sin(phi(I))
WRITE(15,1001) L1, x(L1), y(L1)
L2 = 500*(I+1) + Nadd
Rc(I+1) = xcl/cos(phi(I+1))
arg = ( Rc(I+1)*(s-1)/RO ) + 1
Nr1 = NINT( LOG(arg)/LOG(s) )
L1 = 500*I + Nadd
x(L1) = xcl
y(L1) = Rc(I+1)*sin(phi(I+1))
WRITE(15,1001) L1, x(L1), y(L1)
L3 = L1 + 200
x(L3) = ( x(L1) + x(L2) )/2.
y(L3) = ( y(L1) + y(L2) )/2.
WRITE(15,1001) L3, x(L3), y(L3)
DO 142 J=1, Nr1-1
M = 100*J + I
Nod1(M) = 500*I + 2*(J-1)
Nod2(M) = 500*I + 2*J
Nod3(M) = 500*(I+1) + 2*J
Nod4(M) = 500*(I+1) + 2*(J-1)
Nod5(M) = 500*I + (2*J-1)
Nod6(M) = 500*I + 200 + 2*J
Nod7(M) = 500*(I+1) + (2*J-1)
Nod8(M) = 500*I + 200 + 2*(J-1)
WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M), Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)
CONTINUE
DO 153 I=N3, N4-1
Rc(I) = xci/cos(phi(I))
arg = ( Rc(I)*(s75-1)/R75 ) + 1

CONTINUE
DO 153 I=N3, N4-1
Rc(I) = xci/cos(phi(I))
arg = ( Rc(I)*(s75-1)/R75 ) + 1
Nr1 = NINT( LOG(arg)/LOG(s30) )
L1 = 500*I + Nadd
x(L1) = xcl
y(L1) = Rc(I)*sin(phi(I))
WRITE(15,1001) L1, x(L1), y(L1)
L2 = 500*(I+1) + Nadd
Rc(I+1) = xcl/cos(phi(I+1))

y(L2) = Rc(I+1)*sin(phi(I+1))
WRITE(15,1001) L2, x(L2), y(L2)
L3 = L1 + 200
x(L3) = ( x(L1) + x(L2) )/2.
y(L3) = ( y(L1) + y(L2) )/2.
WRITE(15,1001) L3, x(L3), y(L3)

DO 143 J=1, Nr1-1

M = 100*J + I

Nod1(M) = 500*I + 4*(J-1)
Nod2(M) = 500*I + 4*J
Nod3(M) = 500*(I+1) + 4*J
Nod4(M) = 500*(I+1) + 4*(J-1)
Nod5(M) = 500*I + 2*(2*I-1)
Nod6(M) = 500*I + 200 + 4*J
Nod7(M) = 500*(I+1) + 2*(2*I-1)
Nod8(M) = 500*I + 200 + 4*(J-1)
WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M),

... Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)

CONTINUE

J = Nr1
L1 = 500*I + 4*(J-1)
L2 = 500*(I+1) + Nadd
L3 = 500*(I+1) + Nadd
L4 = 500*(I+1) + 4*(J-1)
L12 = 500*I + Nadi
x(L12) = ( x(L1) + x(L2) )/2.
y(L12) = ( y(L1) + y(L2) )/2.
WRITE(15,1001) L12, x(L12), y(L12)
L34 = 500*(I+1) + Nad1
x(L34) = ( x(L3) + x(L4) )/2.
y(L34) = ( y(L3) + y(L4) )/2.
WRITE(15,1001) L34, x(L34), y(L34)

M = 100*J + I

Nod1(M) = L1
Nod2(M) = L2
Nod3(M) = L3
Nod4(M) = L4
Nod5(M) = L12
Nod6(M) = 500*I + 200 + Nadd
Nod7(M) = L34
Nod8(M) = 500*I + 200 + 4*(J-1)
WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M),

... Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)

CONTINUE

DO 154 I=NS, N8-1

Rc(I) = yc/sin(phi(I))
arg = ( Rc(I)+(s30-1)/R30 ) + 1
Nr1 = NINT( LOG(arg)/LOG(s30) )
L1 = 500*I + Nadd
x(L1) = Rc(I)*cos(phi(I))
y(L1) = yc
WRITE(15,1001) L1, x(L1), y(L1)
L2 = 500*(I+1) + Nadd
Rc(I+1) = yc/sin(phi(I+1))
\( x(L2) = Rc(I+1) \cdot \cos(\phi(I+1)) \)
\( y(L2) = yc \)
WRITE(15,1001) L2, x(L2), y(L2)
L3 = L1 + 200
\( x(L3) = \left( x(L1) + x(L2) \right) / 2. \)
\( y(L3) = \left( y(L1) + y(L2) \right) / 2. \)
WRITE(15,1001) L3, x(L3), y(L3)
DO 144 J=1, Nr1-1
M = 100*J + I
Nod1(M) = 500*I + 16*(J-1)
Nod2(M) = 500*I + 16*J
Nod3(M) = 500*(I+1) + 16*J
Nod4(M) = 500*(I+1) + 16*(J-1)
Nod5(M) = 500*I + 8*(2*J-1)
Nod6(M) = 500*I + 200 + 16*J
Nod7(M) = 500*(I+1) + 8*(2*J-1)
Nod8(M) = 500*I + 200 + 16*(J-1)
WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M), Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)
144 CONTINUE
J = Nr1
L1 = 500*I + 16*(J-1)
L2 = 500*I + Nadd
L3 = 500*(I+1) + Nadd
L4 = 500*(I+1) + 16*(J-1)
L12 = 500*I + Nad1
\( x(L12) = \left( x(L1) + x(L2) \right) / 2. \)
\( y(L12) = \left( y(L1) + y(L2) \right) / 2. \)
WRITE(15,1001) L12, x(L12), y(L12)
L34 = 500*(I+1) + Nad1
\( x(L34) = \left( x(L3) + x(L4) \right) / 2. \)
\( y(L34) = \left( y(L3) + y(L4) \right) / 2. \)
WRITE(15,1001) L34, x(L34), y(L34)
M = 100*J + I
Nod1(M) = L1
Nod2(M) = L2
Nod3(M) = L3
Nod4(M) = L4
Nod5(M) = L12
Nod6(M) = 500*I + 200 + Nadd
Nod7(M) = L34
Nod8(M) = 500*I + 200 + 16*(J-1)
WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M), Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)
154 CONTINUE
DO 155 I=NB, N7-1
Rc(I+1) = -xc/cos(\phi(I+1))
arg = ( Rc(I+1)*(s30-1)/R30 ) + 1
Nr1 = NINT( LOG(arg)/LOG(s30) )
L1 = 500*(I+1) + Nadd
\( x(L1) = -xc \)
\( y(L1) = Rc(I+1) \cdot \sin(\phi(I+1)) \)
WRITE(15,1001) L1, x(L1), y(L1)
L2 = 500*I + Nadd
Rc(I) = -xc/cos(\phi(I))
\( x(L2) = -xc \)
\( y(L2) = Rc(I) \cdot \sin(\phi(I)) \)
WRITE(15,1001) L2, x(L2), y(L2)
L3 = L2 + 200
x(L3) = ( x(L1) + x(L2) ) / 2.
y(L3) = ( y(L1) + y(L2) ) / 2.
WRITE (15,1001) L3, x(L3), y(L3)
DO 145 J=1, Nr1-1
M = 100*J + I
Nod1 (M) = 500*I + 16*(J-1)
Nod2 (M) = 500*I + 16*J
Nod3 (M) = 500*(I+1) + 16*J
Nod4 (M) = 500*(I+1) + 16*(J-1)
Nod5 (M) = 500*I + 8*(2*J-1)
Nod6 (M) = 500*I + 200 + 16*J
Nod7 (M) = 500*(I+1) + 8*(2*J-1)
Nod8 (M) = 500*I + 200 + 16*(J-1)
WRITE (16,1002) M, Nod1 (M), Nod2 (M), Nod3 (M),
           Nod4 (M), Nod5 (M), Nod6 (M), Nod7 (M), Nod8 (M)
CONTINUE
J = Nr1
L1 = 500*I + 16*(J-1)
L2 = 500*I + Nadd
L3 = 500*(I+1) + Nadd
L4 = 500*(I+1) + 16*(J-1)
L12 = 500*I + Nod1
x(L12) = ( x(L1) + x(L2) ) / 2.
y(L12) = ( y(L1) + y(L2) ) / 2.
WRITE (15,1001) L12, x(L12), y(L12)
L34 = 500*(I+1) + Nod1
x(L34) = ( x(L3) + x(L4) ) / 2.
y(L34) = ( y(L3) + y(L4) ) / 2.
WRITE (15,1001) L34, x(L34), y(L34)
M = 100*J + I
Nod1 (M) = L1
Nod2 (M) = L2
Nod3 (M) = L3
Nod4 (M) = L4
Nod5 (M) = L12
Nod6 (M) = 500*I + 200 + Nadd
Nod7 (M) = L34
Nod8 (M) = 500*I + 200 + 16*(J-1)
WRITE (16,1002) M, Nod1 (M), Nod2 (M), Nod3 (M),
           Nod4 (M), Nod5 (M), Nod6 (M), Nod7 (M), Nod8 (M)
CONTINUE
DO 155 I=N7+1, N9-1
Rc(I) = -xc/cos(phi(I))
arg = ( Rc(I)*(s30-I)/R30 ) + 1
Nr1 = NINT( LOG(arg)/LOG(s30) )
L1 = 500*I + Nadd
x(L1) = -xc
y(L1) = Rc(I)*sin(phi(I))
WRITE (15,1001) L1, x(L1), y(L1)
L2 = 500*(I+1) + Nadd
Rc(I+1) = -xc/cos(phi(I+1))
x(L2) = -xc
y(L2) = Rc(I+1)*sin(phi(I+1))
WRITE (15,1001) L2, x(L2), y(L2)
L3 = L1 + 200
x(L3) = ( x(L1) + x(L2) ) / 2.
y(L3) = ( y(L1) + y(L2) ) / 2.
WRITE (15,1001) L3, x(L3), y(L3)
DO 146 J=1, Nr1-1
\[ M = 100 \times J + I \]
\[ \text{Nod1}(M) = 500 \times I + 16 \times (J-1) \]
\[ \text{Nod2}(M) = 500 \times I + 16 \times J \]
\[ \text{Nod3}(M) = 500 \times (I+1) + 16 \times J \]
\[ \text{Nod4}(M) = 500 \times (I+1) + 16 \times (J-1) \]
\[ \text{Nod5}(M) = 500 \times I + 8 \times (2 \times J-1) \]
\[ \text{Nod6}(M) = 500 \times I + 200 + 16 \times J \]
\[ \text{Nod7}(M) = 500 \times (I+1) + 8 \times (2 \times J-1) \]
\[ \text{Nod8}(M) = 500 \times I + 200 + 16 \times (J-1) \]
\[ \text{WRITE}(16,1002) \ M, \ \text{Nod1}(M), \ \text{Nod2}(M), \ \text{Nod3}(M), \]
\[ \ \ \ \text{Nod4}(M), \ \text{Nod5}(M), \ \text{Nod6}(M), \ \text{Nod7}(M), \ \text{Nod8}(M) \]

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CONTINUE
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M = 100 \times J + I
```
\[ \text{Nod1}(M) = 500 \times I + 16 \times (J-1) \]
\[ \text{Nod2}(M) = 500 \times I + 16 \times J \]
\[ \text{Nod3}(M) = 500 \times (I+1) + 16 \times J \]
\[ \text{Nod4}(M) = 500 \times (I+1) + 16 \times (J-1) \]
\[ \text{Nod5}(M) = 500 \times I + 8 \times (2 \times J-1) \]
\[ \text{Nod6}(M) = 500 \times I + 200 + 16 \times J \]
\[ \text{Nod7}(M) = 500 \times (I+1) + 8 \times (2 \times J-1) \]
\[ \text{Nod8}(M) = 500 \times I + 200 + 16 \times (J-1) \]

```
CONTINUE
```

\[ J = N_{ri} \]
\[ L_1 = 500 \times I + 16 \times (J-1) \]
\[ L_2 = 500 \times I + \text{Nadd} \]
\[ L_3 = 500 \times (I+1) + \text{Nadd} \]
\[ L_4 = 500 \times (I+1) + 16 \times (J-1) \]
\[ L_{12} = 500 \times I + \text{Nadd} \]
\[ x(L_{12}) = \left( x(L_1) + x(L_2) \right) / 2. \]
\[ y(L_{12}) = \left( y(L_1) + y(L_2) \right) / 2. \]
\[ \text{WRITE}(15,1001) \ L_{12}, \ x(L_{12}), \ y(L_{12}) \]
\[ L_{34} = 500 \times (I+1) + \text{Nadd} \]
\[ x(L_{34}) = \left( x(L_3) + x(L_4) \right) / 2. \]
\[ y(L_{34}) = \left( y(L_3) + y(L_4) \right) / 2. \]
\[ \text{WRITE}(15,1001) \ L_{34}, \ x(L_{34}), \ y(L_{34}) \]
\[ M = 100 \times J + I \]
\[ \text{Nod1}(M) = L_1 \]
\[ \text{Nod2}(M) = L_2 \]
\[ \text{Nod3}(M) = L_3 \]
\[ \text{Nod4}(M) = L_4 \]
\[ \text{Nod5}(M) = L_{12} \]
\[ \text{Nod6}(M) = 500 \times I + 200 + \text{Nadd} \]
\[ \text{Nod7}(M) = L_{34} \]
\[ \text{Nod8}(M) = 500 \times I + 200 + 16 \times (J-1) \]
\[ \text{WRITE}(16,1002) \ M, \ \text{Nod1}(M), \ \text{Nod2}(M), \ \text{Nod3}(M), \]
\[ \ \ \ \text{Nod4}(M), \ \text{Nod5}(M), \ \text{Nod6}(M), \ \text{Nod7}(M), \ \text{Nod8}(M) \]

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CONTINUE
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\[ \text{DO 147 J}=1, N_{ri}-1 \]
\[ M = 100 \times J + I \]
\[ \text{Nod1}(M) = 500 \times I + 16 \times (J-1) \]
\[ \text{Nod2}(M) = 500 \times I + 16 \times J \]
\[ \text{Nod3}(M) = 500 \times (I+1) + 16 \times J \]
\[ \text{Nod4}(M) = 500 \times (I+1) + 16 \times (J-1) \]
Nod5(M) = 500*I + 8*(2*J-1)
Nod6(M) = 500*I + 200 + 16*J
Nod7(M) = 500*(I+1) + 8*(2*J-1)
Nod8(M) = 500*I + 200 + 16*(J-1)
WRITE (16,1002) M, Nod1(M), Nod2(M), Nod3(M),
 Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)
CONTINUE
J = Nr1
L1 = 500*I + 16*(J-1)
L2 = 500*I + Nadd
L3 = 500*(I+1) + Nadd
L4 = 500*(I+1) + 16*(J-1)
L12 = 500*I + Nadd
x(L12) = ( x(L1) + x(L2) ) / 2.
y(L12) = ( y(L1) + y(L2) ) / 2.
WRITE(15,1001) L12, x(L12), y(L12)
L34 = 500*(I+1) + Nadd
x(L34) = ( x(L3) + x(L4) ) / 2.
y(L34) = ( y(L3) + y(L4) ) / 2.
WRITE(15,1001) L34, x(L34), y(L34)
M = 100*J + I
Nod1(M) = L1
Nod2(M) = L2
Nod3(M) = L3
Nod4(M) = L4
Nod5(M) = L12
Nod6(M) = 500*I + 200 + Nadd
Nod7(M) = L34
Nod8(M) = 500*I + 200 + 16*(J-1)
WRITE (16,1002) M, Nod1(M), Nod2(M), Nod3(M),
 Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)
CONTINUE
DO 158 J=N99, N10-1
Re(I) = -yc/sin(phi(I))
arg = ( Re(I)*(s30-1)/R30 ) + 1
Nr1 = NINT( LOG(arg)/LOG(s30) )
L1 = 500*I + Nadd
x(L1) = Re(I)*cos(phi(I))
y(L1) = -yc
WRITE(15,1001) L1, x(L1), y(L1)
L2 = 500*(I+1) + Nadd
Re(I+1) = -yc/sin(phi(I+1))
x(L2) = Re(I+1)*cos(phi(I+1))
y(L2) = -yc
WRITE(15,1001) L2, x(L2), y(L2)
L3 = L1 + 200
x(L3) = ( x(L1) + x(L2) )/2.
y(L3) = ( y(L1) + y(L2) )/2.
WRITE(15,1001) L3, x(L3), y(L3)
DO 148 J=1, Nr1-1
M = 100*J + I
Nod1(M) = 500*I + 16*(J-1)
Nod2(M) = 500*I + 16*J
Nod3(M) = 500*(I+1) + 16*J
Nod4(M) = 500*(I+1) + 16*(J-1)
Nod5(M) = 500*I + 8*(2*J-1)
Nod6(M) = 500*I + 200 + 16*J
Nod7(M) = 500*(I+1) + 8*(2*J-1)
Nod8(M) = 500*I + 200 + 16*(J-1)
WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M),
CONTINUE

J = Nr1
L1 = 500*I + 16*(J-1)
L2 = 500*I + Nadd
L3 = 500*(I+1) + Nadd
L4 = 500*(I+1) + 16*(J-1)
L5 = 500*I + Nadd
x(L12) = ( x(L1) + x(L2) ) / 2.
y(L12) = ( y(L1) + y(L2) ) / 2.
WRITE(15,1001) L12, x(L12), y(L12)
L34 = 500*(I+1) + Nadd
x(L34) = ( x(L3) + x(L4) ) / 2.
y(L34) = ( y(L3) + y(L4) ) / 2.
WRITE(15,1001) L34, x(L34), y(L34)

M = 100*J + I
Nod1(M) = L1
Nod2(M) = L2
Nod3(M) = L3
Nod4(M) = L4
Nod5(M) = L5
Nod6(M) = 500*I + 200 + Nadd
Nod7(M) = L34
Nod8(M) = 500*I + 200 + 16*(J-1)
WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M),
Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)

CONTINUE

DO 159 I=N10, N6-1
Rc(I+1) = xc/cos(phi(I+1))
arg = ( Rc(I+1)*(S30-1)/R30 ) + 1
Nr1 = NINT( LOG(arg)/LOG(S30) )
L1 = 500*(I+1) + Nadd
x(L1) = xc
y(L1) = Rc(I+1)*sin(phi(I+1))
WRITE(15,1001) L1, x(L1), y(L1)
L2 = 500*I + Nadd
Rc(I) = xc/cos(phi(I))
X(L2) = xc
y(L2) = Rc(I)*sin(phi(I))
WRITE(15,1001) L2, x(L2), y(L2)
L3 = L2 + 200
x(L3) = ( x(L1) + x(L2) )/2.
y(L3) = ( y(L1) + y(L2) )/2.
WRITE(15,1001) L3, x(L3), y(L3)

DO 149 J=1, Nr1-1
M = 100*J + I
Nod1(M) = 500*I + 16*(J-1)
Nod2(M) = 500*I + 16*J
Nod3(M) = 500*(I+1) + 16*J
Nod4(M) = 500*(I+1) + 16*(J-1)
Nod5(M) = 500*I + 8*(2*J-1)
Nod6(M) = 500*I + 200 + 16*J
Nod7(M) = 500*(I+1) + 8*(2*J-1)
Nod8(M) = 500*I + 200 + 16*(J-1)
WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M),
Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)

CONTINUE

J = Nr1
L1 = 500*I + 16*(J-1)
L2 = 500*I + Nadd
L3 = 500*(I+1) + Nadd
L4 = 500*(I+1) + 16*(J-1)
L12 = 500*I + Nad1
x(L12) = (x(L1) + x(L2))/2.
y(L12) = (y(L1) + y(L2))/2.
WRITE(15,1001) L12, x(L12), y(L12)
L34 = 500*(I+1) + Nad1
x(L34) = (x(L3) + x(L4))/2.
y(L34) = (y(L3) + y(L4))/2.
WRITE(15,1001) L34, x(L34), y(L34)
M = 100*J + I
Nod1(M) = L1
Nod2(M) = L2
Nod3(M) = L3
Nod4(M) = L4
Nod5(M) = L12
Nod6(M) = 500*I + 200 + Nadd
Nod7(M) = L34
Nod8(M) = 500*I + 200 + 16*(J-1)
WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M),
Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)
CONTINUE
I=N7
Rc(I) = -xc/cos(ph1(I))
arg = (Rc(I)*(s30-1)/R30 ) + 1
Nri = NINT(LOC(arg)/LOC(s30))
L1 = 500*I + 100 + Nadd
x(L1) = -xc
y(L1) = Rc(I)*sin(ph1(I))
WRITE(15,1001) L1, x(L1), y(L1)
L2 = 500*(I+1) + Nadd
Rc(I+1) = -xc/cos(ph1(I+1))
x(L2) = -xc
y(L2) = Rc(I+1)*sin(ph1(I+1))
WRITE(15,1001) L2, x(L2), y(L2)
L3 = 500*I + 200 + Nadd
x(L3) = (x(L1) + x(L2))/2.
y(L3) = (y(L1) + y(L2))/2.
WRITE(15,1001) L3, x(L3), y(L3)
DO 161 J=1, Nri-1
M = 100*J + I
Nod1(M) = 500*I + 100 + 16*(J-1)
Nod2(M) = 500*I + 100 + 16*J
Nod3(M) = 500*(I+1) + 16*J
Nod4(M) = 500*(I+1) + 16*(J-1)
Nod5(M) = 500*I + 100 + 8*(2*J-1)
Nod6(M) = 500*I + 200 + 16*J
Nod7(M) = 500*(I+1) + 8*(2*J-1)
Nod8(M) = 500*I + 200 + 16*(J-1)
WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M),
Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)
CONTINUE
J = Nri
L1 = 500*I + 100 + 16*(J-1)
L2 = 500*I + 100 + Nadd
L3 = 500*(I+1) + Nadd
L4 = 500*(I+1) + 16*(J-1)
L12 = 500*I + 100 + Nad1
x(L12) = (x(L1) + x(L2))/2.
y(L12) = (y(L1) + y(L2))/2.
WRITE(15,1001) L12, x(L12), y(L12)
L34 = 500*(I+1) + Nadi
x(L34) = ( x(L3) + x(L4) ) / 2.
y(L34) = ( y(L3) + y(L4) ) / 2.
WRITE(15,1001) L34, x(L34), y(L34)
M = 100*J + I
Nod1(M) = L1
Nod2(M) = L2
Nod3(M) = L3
Nod4(M) = L4
Nod5(M) = L12
Nod6(M) = 500*I + 200 + Nadd
Nod7(M) = L34
Nod8(M) = 500*I + 200 + 16*(J-1)
WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M),
Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)
I=N6
Rc(1) = xc/cos(phi(1))
arg = ( Rc(1)*(s30-1)/R30 ) + 1
Nr1 = NINT( LOG(arg)/LOG(s30) )
L1 = 500*I + Nadd
x(L1) = xc
y(L1) = Rc(1)*sin(phi(1))
WRITE(15,1001) L1, x(L1), y(L1)
L2 = 500*I + Nadd
Rc(I) = xc/cos(phi(I))
x(L2) = xc
y(L2) = Rc(I)*sin(phi(I))
WRITE(15,1001) L2, x(L2), y(L2)
L3 = L2 + 200
x(L3) = ( x(L1) + x(L2) ) / 2.
y(L3) = ( y(L1) + y(L2) ) / 2.
WRITE(15,1001) L3, x(L3), y(L3)
DO 162 J=1, Nr1-1
M = 100*J + I
Nod1(M) = 500*I + 16*(J-1)
Nod2(M) = 500*I + 16*I
Nod3(M) = 500*I + 16*I
Nod4(M) = 500*I + 16*(J-1)
Nod5(M) = 500*I + 8*(2*J-1)
Nod6(M) = 500*I + 200 + 16*I
Nod7(M) = 500*I + 8*(2*J-1)
Nod8(M) = 500*I + 200 + 16*(J-1)
WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M),
Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)
CONTINUE
J = Nr1
L1 = 500*I + 16*(J-1)
L2 = 500*I + Nadd
L3 = 500*I + Nadd
L4 = 500*I + 16*(J-1)
L12 = 500*I + Nadi
x(L12) = ( x(L1) + x(L2) ) / 2.
y(L12) = ( y(L1) + y(L2) ) / 2.
WRITE(15,1001) L12, x(L12), y(L12)
L34 = 500*I + Nadi
x(L34) = ( x(L3) + x(L4) ) / 2.
y(L34) = ( y(L3) + y(L4) ) / 2.
WRITE(15,1001) L34, x(L34), y(L34)
M = 100*J + I
Nod1(M) = L1
Nod2(M) = L2
Nod3(M) = L3
Nod4(M) = L4
Nod5(M) = L12
Nod6(M) = 500*I + 200 + Nadd
Nod7(M) = L34
Nod8(M) = 500*I + 200 + 16*(J-1)
WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M),
- Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)

I=N5-1
Re(I) = yc/sin(phi(I))
arg = ( Re(I)*(s15-1)/R15 ) + 1
Nr1 = NINT( LOG(arg)/LOG(s15) )
L1 = 500*I + Nadd
dx(L1) = Re(I)*cos(phi(I))
y(L1) = yc
WRITE(15,1001) L1, x(L1), y(L1)
Re(I+1) = yc/sin(phi(I+1))
L2 = 500*(I+1) + Nadd
dx(L2) = Re(I+1)*cos(phi(I+1))
y(L2) = yc
WRITE(15,1001) L2, x(L2), y(L2)
L3 = L1 + 200
dx(L3) = ( x(L1) + x(L2) )/2.
y(L3) = ( y(L1) + y(L2) )/2.
WRITE(15,1001) L3, x(L3), y(L3)
DO 163 J=1, Nr1-1
   M = 100*J + I
   Nod1(M) = 500*I + 8*(J-1)
   Nod2(M) = 500*I + 8*J
   Nod3(M) = 500*(I+1) + 8*J
   Nod4(M) = 500*(I+1) + 8*(J-1)
   Nod5(M) = 500*I + 4*(2*J-1)
   Nod6(M) = 500*I + 200 + 8*J
   Nod7(M) = 500*(I+1) + 4*(2*J-1)
   Nod8(M) = 500*I + 200 + 8*(J-1)
   WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M),
- Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)
   CONTINUE

163
J = Nr1
L1 = 500*I + 8*(J-1)
L2 = 500*I + Nadd
L3 = 500*(I+1) + Nadd
L4 = 500*(I+1) + 8*(J-1)
L12 = 500*I + Nad1
dx(L12) = ( x(L1) + x(L2) )/2.
y(L12) = ( y(L1) + y(L2) )/2.
WRITE(15,1001) L12, x(L12), y(L12)
L34 = 500*(I+1) + Nad1
dx(L34) = ( x(L3) + x(L4) )/2.
y(L34) = ( y(L3) + y(L4) )/2.
WRITE(15,1001) L34, x(L34), y(L34)
M = 100*J + I
Nod1(M) = L1
Nod2(M) = L2
Nod3(M) = L3
Nod4(M) = L4
Nod5(M) = L12
Nod6(M) = 500*I + 200 + Nadd
Nod7(M) = L34
Nod8(M) = 500*I + 200 + 8*(J-1)
WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M),
                   Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)
I=N4
Rc(I) = xc1/cos(phil(I))
arg = (Rc(I)*(s15-1)/R15 ) + 1
Nri = NINT( LOG(arg)/LOG(s15) )
L1 = 500*I + Nadd
x(L1) = xc1
y(L1) = Rc(I)*sin(phil(I))
WRITE(15,1001) L1, x(L1), y(L1)
L2 = 500*(I+1) + 8*Nri
x(L2) = Rc(I)*cos(phil(I+1))
y(L2) = Rc(I)*sin(phil(I+1))
WRITE(15,1001) L2, x(L2), y(L2)
L3 = L1 + 200
x(L3) = (x(L1) + x(L2))/2.
y(L3) = (y(L1) + y(L2))/2.
WRITE(15,1001) L3, x(L3), y(L3)
L4 = 500*I + 2*(Nri+3)
x(L4) = xc1
y(L4) = yc
WRITE(15,1001) L4, x(L4), y(L4)
L5 = 500*(I+1) + Nadd
L6 = L4 + 1
x(L6) = (x(L4) + x(L1))/2.
y(L6) = (y(L4) + y(L1))/2.
WRITE(15,1001) L6, x(L6), y(L6)
L7 = L4 + 2
x(L7) = (x(L4) + x(L5))/2.
y(L7) = (y(L4) + y(L5))/2.
WRITE(15,1001) L7, x(L7), y(L7)
L8 = L2 + 8
x(L8) = (x(L2) + x(L5))/2.
y(L8) = (y(L2) + y(L5))/2.
WRITE(15,1001) L8, x(L8), y(L8)
DO 166 J=1, Nri-1
M = 100*J + I
Nod1(M) = 500*I + 8*(J-1)
Nod2(M) = 500*I + 8*J
Nod3(M) = 500*(I+1) + 8*J
Nod4(M) = 500*(I+1) + 8*(J-1)
Nod5(M) = 500*I + 4*(2*J-1)
Nod6(M) = 500*I + 200 + 8*J
Nod7(M) = 500*(I+1) + 4*(2*J-1)
Nod8(M) = 500*I + 200 + 8*(J-1)
WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M),
                   Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)
166    CONTINUE
J = Nri
M = 100*J + I
Nod1(M) = 500*I + 8*(J-1)
Nod2(M) = 500*I + Nadd
Nod3(M) = 500*(I+1) + 8*J
Nod4(M) = 500*(I+1) + 8*(J-1)
Nod5(M) = 500*I + 4*(2*J-1)
Nod6(M) = 500*I + 200 + Nadd
Nod7(M) = 500*(I+1) + 4*(2*J-1)
Nod8(M) = 500*I + 200 + 8*(J-1)
WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M),
          Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)
J = Nr1 + 1
M = 100*J + I
Nod1(M) = L2
Nod2(M) = L1
Nod3(M) = L4
Nod4(M) = L5
Nod5(M) = L3
Nod6(M) = L6
Nod7(M) = L7
Nod8(M) = L8
WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M),
          Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)
DO 164 11., Ni-i
beta = (xcl+rad)*cos(phi(I+i)) + dy*sin(phi(I+i))
gamma = (xcl+rad)*(xcl+rad) + dy*dy - rad*rad
Rc(I+i) = beta - SQRT(beta*beta-gamma)
arg = (Rc(I+i)*(s15-1)/R15 ) + 1
Nr1 = NINT(LOG(arg)/LOG(s15) )
L1 = 500*(I+i) + Nadd
x(L1) = Rc(I+i)*cos(phi(I+i))
y(L1) = Rc(I+i)*sin(phi(I+i))
WRITE(15,1001) L1, x(L1), y(L1)
L2 = 500*I + Nadd
phi3 = ( phi(I) + phi(I+1) ) / 2.
beta = (xcl+rad)*cos(phi3) + dy*sin(phi3)
gamma = (xcl+rad)*(xcl+rad) + dy*dy - rad*rad
Rc3 = beta - SQRT(beta*beta-gamma)
L3 = L2 + 200
x(L3) = Rc3*cos(phi3)
y(L3) = Rc3*sin(phi3)
WRITE(15,1001) L3, x(L3), y(L3)
DO 165 J=1, Nr1-1
M = 100*J + I
Nod1(M) = 500*I + 8*(J-1)
Nod2(M) = 500*I + 8*J
Nod3(M) = 500*(I+1) + 8*J
Nod4(M) = 500*(I+1) + 8*(J-1)
Nod5(M) = 500*I + 4*(2*J-1)
Nod6(M) = 500*I + 200 + 8*J
Nod7(M) = 500*(I+1) + 4*(2*J-1)
Nod8(M) = 500*I + 200 + 8*(J-1)
WRITE(16,1002) M, Nod1(M), Nod2(M), Nod3(M),
          Nod4(M), Nod5(M), Nod6(M), Nod7(M), Nod8(M)
165 CONTINUE
J = Nr1
L1 = 500*I + 8*(J-1)
L2 = 500*I + Nadd
L3 = 500*(I+1) + Nadd
L4 = 500*(I+1) + 8*(J-1)
L12 = 500*I + Nad1
x(L12) = ( x(L1) + x(L2) ) / 2.
y(L12) = ( y(L1) + y(L2) ) / 2.
WRITE(15,1001) L12, x(L12), y(L12)
L34 = 500*(I+1) + Nad1
x(L34) = ( x(L3) + x(L4) ) / 2.
y(L34) = ( y(L3) + y(L4) ) / 2.
WRITE(15,1001) L34, x(L34), y(L34)
M = 100*J + I
DO 111 J=1, Nr1-1
   L1 = 500*I + 4*(4*J-3)
   L11 = 500*I + 4*(4*J-1)
   L2 = 500*I + 16*(J-1)
   L3 = 500*I + 16*J
   L4 = 500*I + 8*(2*J-1)
   WRITE(17,2001) L, L1, L2, L3, L4
   WRITE(17,2001) L, L11, L2, L3, L4
111 CONTINUE
I = 1
Nr1 = 4
DO 12 J=1, Nr1-1
   L1 = 500*I + 2*(4*J-3)
   L11 = 500*I + 2*(4*J-1)
   L2 = 500*I + 8*(J-1)
   L3 = 500*I + 8*J
   L4 = 500*I + 4*(2*J-1)
   WRITE(17,2001) L, L1, L2, L3, L4
   WRITE(17,2001) L, L11, L2, L3, L4
12 CONTINUE

C 7.5-15 interface
C Nr1 refers to the 15 i.e. 7 elements
I = 3
Nr1 = 7
DO 13 J=1, Nr1-1
   L1 = 500*I + 2*(4*J-3)
   L11 = 500*I + 2*(4*J-1)
   L2 = 500*I + 8*(J-1)
   L3 = 500*I + 8*J
   L4 = 500*I + 4*(2*J-1)
   WRITE(17,2001) L, L1, L2, L3, L4
   WRITE(17,2001) L, L11, L2, L3, L4
13 CONTINUE
I = 9
Nr1 = 8
DO 14 J=1, Nr1-1
   L1 = 500*I + 2*(4*J-3)
   L11 = 500*I + 2*(4*J-1)
   L2 = 500*I + 8*(J-1)
L3 = 500*I + 8*J
L4 = 500*I + 4*(2*J-1)
WRITE(17,2001) L, L1, L2, L3, L4
WRITE(17,2001) L, L11, L2, L3, L4

14 CONTINUE

! 3.75-7.5 interface
! Nri refers to the 7.5 i.e. 13 elements
I = 4
Nri = 13
DO 15 J=1, Nri-1
   L1 = 500*I + (4*J-3)
   L11 = 500*I + (4*J-1)
   L2 = 500*I + 4*(J-1)
   L3 = 500*I + 4*J
   L4 = 500*I + 2*(2*J-1)
WRITE(17,2001) L, L1, L2, L3, L4
WRITE(17,2001) L, L11, L2, L3, L4
15 CONTINUE

I = 8
Nri = 15
DO 16 J=1, Nri-1
   L1 = 500*I + (4*J-3)
   L11 = 500*I + (4*J-1)
   L2 = 500*I + 4*(J-1)
   L3 = 500*I + 4*J
   L4 = 500*I + 2*(2*J-1)
WRITE(17,2001) L, L1, L2, L3, L4
WRITE(17,2001) L, L11, L2, L3, L4
16 CONTINUE

STOP
END.