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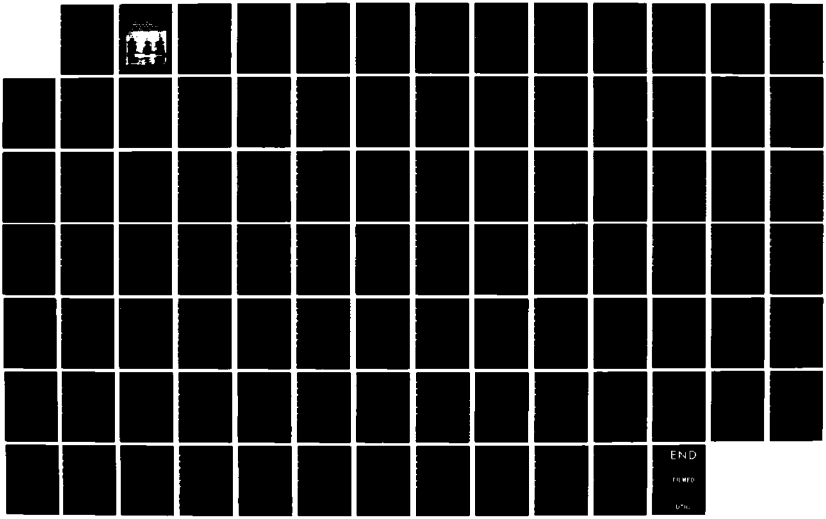

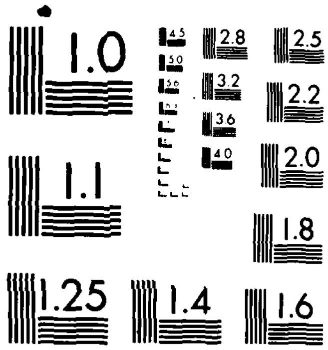

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STABILITY OF A REDUCED ORDER MODEL REFERENCE ADAPTIVE CONTROL SYSTEM WITH PERSISTENT EXCITATION

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

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BY

BRADLEY DEAN RIEDLE

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STABILITY OF A REDUCED ORDER MODEL REFERENCE ADAPTIVE CONTROL SYSTEM WITH PERSISTENT EXCITATION

BY

BRADLEY DEAN RIEDLE

B.S., University of Illinois, 1982

THESIS

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CHAPTER 1

INTRODUCTION

In this chapter, I sketch a short history of work leading to this thesis. Then I give a summary of the results and organization of the thesis.

1.1. Background

Narendra and Valavani (1978) presented a stable model reference adaptive controller. Since then, it has been recognized that some of the assumptions about the plant or about our knowledge of the plant are unrealistic. It has become popular to study model reference adaptive systems which are formed by applying this Narendra-Valavani controller to a plant which meets these unrealistic assumptions and then perturbing the plant without changing the controller.

In this way, researchers have been able to study more realistic applications of adaptive control without having to develop entirely new controller structures. In this thesis, I join the group of researchers who have perturbed the plant by adding high frequency dynamics. The results of a few of the researchers who have preceded me in studying the effects of this perturbation are summarized in the following paragraphs.

Rohrs et al. (1981,1982) studied the effects of an additive disturbance and unmodelled high frequency dynamics when the model reference adaptive system is excited by a constant reference input. They show that the magnitude of the adjustable feedback gains in the controller will eventually become large. If the relative degree of the plant including the high frequency dynamics is greater than two, the linear system with feedback gains fixed at some large values would be unstable. The behavior of the model reference adaptive systems studied by these researchers is characterized by two different intervals of time. During the first interval the output error becomes and remains small while the magnitudes of the adjustable feedback gains drift toward infinity. At some instant the adjustable gains take on values at which the linear system with feedback gains held constant at these values becomes unstable. This is the beginning of the second interval during which the magnitude of output error and the magnitudes of the adjustable feedback gains approach infinity in finite time.

The same group also studied by simulation the effects of high frequency dynamics when the reference input contains high frequencies or the initial conditions on the system states are large. They show that both of these conditions can lead to instability. However, these instabilities are not as easily described as the drift instability.

The drift behavior of the parameters in the presence of an additive disturbance does not depend on the high frequency dynamics. This drift occurs because the control scheme adjusts the feedback gains until the output error is zero. For a constant reference input, there exists an unbounded manifold on which the output error is zero when the disturbance is zero. Except for certain special disturbances, perfect disturbance rejection requires infinite feedback gain. Hence, the feedback gains drift to infinity along the manifold. This behavior is discussed by Egardt (1980) and also by Riedle, Cyr, and Kokotovic (1983).

One obvious way to avoid the drift to infinity is to put a dead zone in the adaptation law around output error equal to zero. One such algorithm was proposed by Peterson and Narendra (1982).

Ioannou and Kokotovic (1983) propose a different solution. They replace the integrator in the original adaptation laws with a first order low pass filter. For a bounded additive disturbance the output error will enter and remain in a small segment containing zero and the adjustable feedback gains will remain bounded. For a perturbation involving high frequency dynamics, they define the term "dominantly rich" input. This is sufficiently rich input which does not excite the high frequency dynamics. Then they show that, when the reference input is dominantly rich and the system is initially in a region of attraction, the output error enters a small segment of the real line containing zero and the adjustable gains remain bounded.

Krause (1983) refines the notion of dominant richness and studies the effect of high frequency dynamics when the reference input is periodic, has at least as many spectral components as unknowns, and is dominantly rich. He finds that when the states of the system and the errors in the values of the adjustable gains are $O(1)$, the adaptation will be in the correct direction if the speed of adaptation is slow enough.

The approach taken in this thesis to make the system robust with respect to perturbations is to make the unperturbed system exponentially stable. Kosut (1983); Kosut, Johnson, and Anderson (1983); and Anderson and Johnstone (1981) have also used this approach. Kosut et al. introduced the concepts of "tuned system" and "tuned error" which are important in this thesis. They showed that persistent excitation which provides exponential stability of the unperturbed system provides robustness with respect to perturbations with bounded effects. However, they did not specify any particular type of perturbation and their method of proof is considerably different from that of this thesis. Anderson and Johnstone considered the discrete time case and used converse Lyapunov function results for difference equations in a manner similar to the way I use converse Lyapunov function results for differential equations.

While none of the results of this thesis are new, I feel that this work represents a good combination of the techniques and ideas mentioned above. Choosing a particular perturbation, high frequency unmodelled dynamics, allows me to study the effects of the perturbation in some detail. With the concepts of tuned system and tuned error I am able to specify and solve a homogeneous problem before attacking the complete problem. Using the singular perturbation approach of Ioannou and Kokotovic to represent the unmodelled high frequency dynamics provides a natural way to partition the system. This natural partitioning of the system can be combined with the converse Lyapunov results to get Lyapunov functions for each subsystem. The final idea is then to use a composite of the Lyapunov functions for the subsystems to prove the desired stability results.

1.2. Summary of Results and Organization of Thesis

This thesis contains two major results, as well as illustrative simulation examples. The first major result is that for a certain class of inputs there exists a unique equilibrium for a model reference adaptive system when the nominal plant is perturbed by high frequency unmodelled dynamics. The second major result is that, for a small enough perturbation, this equilibrium is exponentially stable.

In Chapter 2, I derive the first major result in Theorem 2.1. This chapter also provides a transfer function description of the system and contains a section showing that a small singular perturbation has a small effect on the equilibrium values of the adjustable gains. Chapter 2 ends with remarks which are intended to provide either an intuitive interpretation of the results on the chapter or a comparison with the work of others.

In Chapter 3, I derive several results including the second major result of the thesis. The chapter begins with definitions and theorems to be used in the derivations in the rest of the chapter. Then the differential equations describing the system are presented and an error system is derived. When the tuned error is zero, as it will be for the equilibrium of Theorem 2.1, the second major result of the thesis is stated in Theorem 3.9. After the nonzero tuned error case is handled, Chapter 3 ends with remarks interpreting the results of the chapter.

In Chapter 4, I present two simple examples which show that the system indeed remains exponentially stable when high frequency unmodelled dynamics perturb the original system. The examples also show that estimates of the range of stable perturbations based upon the proof of Theorem 3.9 are so conservative that these estimates are not of practical use.

In Chapter 5, I offer some concluding remarks and suggestions for future research.

CHAPTER 2

EXISTENCE OF AN EQUILIBRIUM

In this chapter I have three major objectives. First, I describe the perturbation from the ideal which I have studied. Next, I show that for a certain class of input signals a unique equilibrium point exists in spite of the perturbation. Finally, I discuss the effects of the perturbation on the parameter values associated with the equilibrium.

2.1. Preliminaries

The following result will be useful in the proof of the uniqueness of the equilibrium point.

Lemma 1: Let $T_1(s)$ be a polynomial of degree $\leq$ n-2, $T_2(s)$ be a polynomial of degree $\leq$ n, R(s) be a polynomial of degree n, and Z(s) be a polynomial of degree n-1. Furthermore, let Z(s) and R(s) be relatively prime. Define Q(s) as a polynomial of degree $\leq$ 2n-1, such that

$$Q(s) = T_1(s)R(s) + T_2(s)Z(s) \quad . \tag{2.1}$$

Let $X(\mu s)$ be a function which can be represented by

$$X(\mu s) = 1 + \sum_{i=1}^{\infty} x_i(\mu s)^i \;\; \forall s \in \{s: |s| < \frac{1}{\mu}\} \quad . \tag{2.2}$$

Further, require that $X(\mu s)$ satisfy

$$0 < 1 - \bar{x}_1\mu|s| < |x(\mu s)| \leq 1 + \bar{x}_2\mu|s| \;\; \forall s \in \{s: |s| < \frac{1}{\mu}\} \quad . \tag{2.3}$$

Define

$$P(\mu,s) = T_1(s)R(s) + T_2(s)Z(s)X(\mu s) \quad . \tag{2.4}$$

Then, there exists $\mu_o \in \mathbb{R}^+$, such that $\forall \mu \in [0,\mu_o)$, if $P(\mu,s) = 0$ for 2n distinct values of s; $s_i \in C$ $i=1,2,\ldots,2n$, then $T_1(s) \equiv 0$ and $T_2(s) \equiv 0$.

Proof: First consider $\mu = 0$. Then $P(0,s) = Q(s)$. Since the degree of Q(s) $\leq 2n-1$ and Q(s) is zero for 2n distinct values of s, then Q(s) must be identically zero. Therefore,

$$T_1(s)R(s) = - T_2(s)Z(s) \;\; \forall s \in C \quad . \tag{2.5}$$

Since R(s) and Z(s) are relatively prime and neither is identically zero, (2.5) implies either $T_1(s) \equiv 0$ and $T_2(s) \equiv 0$, or $T_1(s)$ contains Z(s) as a factor. The latter is clearly impossible because of the degree of Z(s) is n-1 and the degree of $T_1(s)$ is strictly less than n-1. Thus, the lemma is proved for $\mu = 0$.

Now, consider $\mu > 0$. Assume $T_1(s) \not\equiv 0$ or $T_2(s) \not\equiv 0$. Using (2.1), (2.3) and (2.4) write

$$P(\mu,s) = Q(s) + T_2(s)Z(s)\left(\sum_{i=1}^{n} x_i(\mu s)^i\right) \forall s \in \{s: |s| < \frac{1}{\mu}\} \quad . \tag{2.6}$$

By assumption Q(s) has degree greater than n-2 and less than 2n, and, hence, there is at least one s_i such that $Q(s_i) \neq 0$. Without loss of generality let this be s_1. Let

$$\mu_o = \min \left\{\frac{|Q(s_1)|}{\bar{x}|s_1||T_2(s_1)Z(s_1)|}, \frac{1}{|s_1|}\right\} \quad . \tag{2.7}$$

Then, if $\mu \in (0,\mu_o)$

$$|P(\mu,s_1)| \geq |Q(s_1)| - \mu\bar{x}|s_1||T_2(s_1)Z(s_1)| > 0 \quad ; \tag{2.8}$$

this contradicts $P(\mu,s_1)=0$. Therefore, the assumption that $T_1(s)\neq 0$ or $T_2(s)\neq 0$ is false and the lemma is proved.

2.2. The System

The Model Reference Adaptive System to be studied is the Narendra, Valavani (1978) controller applied to a plant with unmodelled high-frequency dynamics. This system has also been studied by Ioannou and Kokotovic (1983), Rohrs et al. (1981,1982), and Krause (1983). The system has the block diagram shown in Figure 2.1.

The linear time invariant plant can be represented by a series connection of two transfer functions, $W_p(s)$ and $W_f(\mu s)$. The transfer function of the nominal or slow part of the plant can be written

$$W_p(s) = k_p \frac{Z_p(s)}{R_p(s)} \quad , \tag{2.9}$$

where k_p is a scalar, and $Z_p(s)$ and $R_p(s)$ are relatively prime monic polynomials of degree n-1 and n, respectively. The transfer function of the fast dynamics of the plant can be written

$$W_f(\mu s) = \frac{Z_f(\mu s)}{R_f(\mu s)} \tag{2.10}$$

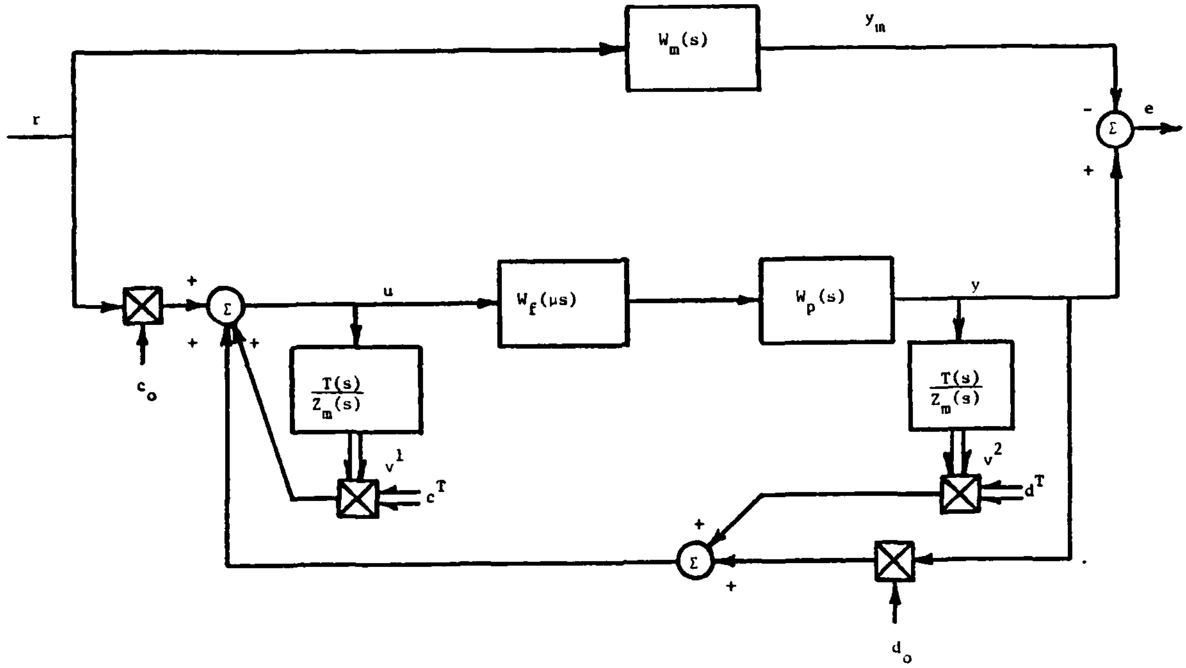

Figure 2.1. Block diagram of MRAS not showing the update mechanism.

where

$$Z_f(\mu s) = 1 + \sum_{k=1}^{N_1} z_{fk}(\mu s)^k \qquad , \tag{2.11}$$

$$R_f(\mu s) = 1 + \sum_{k=1}^{N_1+N_2} r_{fk}(\mu s)^k \qquad . \tag{2.12}$$

The following assumptions are made about the plant:

A1) n is known,

A2) the sign of k_p is known (without loss of generality let it be +),

A3) $Z_p(s)$ is Hurwitz,

A4) $|Z_f(\mu s)| > 1 - z_1\mu|s| > 0 \ \forall s \in \{s : |s| < \frac{1}{\mu}\}$,

A5) if $R_f(\mu s) = 0$ then $Re(s) < -\frac{1}{\mu}$,

A6) $N_2 \geq 1$, $r_{f(N_1 + N_2)} \neq 0$.

The model transfer function can be written

$$W_m(s) = k_m \frac{Z_m(s)}{R_m(s)} \qquad , \tag{2.13}$$

where k_m is a positive scalar, $Z_m(s)$ and $R_m(s)$ are monic Hurwitz polynmials of degree n-1 and n, respectively. $Z_m(s)$ and $R_m(s)$ are chosen so that $W_m(s)$ is a strictly positive real transfer function. The controller has the structure shown in Figure 2.1 with the adjustable parameter vector $[c_o, c^T, d_o, d^T]^T \in \mathbb{R}^{2n}$. $Z_m(s)$ in the controller is the numerator of the model transfer function and T(s) is defined

$$T(s) = \begin{bmatrix} 1 \\ s \\ s^2 \\ \vdots \\ s^{n-2} \end{bmatrix} \qquad . \tag{2.14}$$

2.3. The Equilibrium

In this section, I show that, when the perturbation parameter is sufficiently small and the reference input r belongs to a certain class of signals, an equilibrium exists for the model reference adaptive system described in the previous section. As usual, let the output error be the difference between the plant output and the model output, $e = y - y_m$. If, for some constant value of the parameter vector, the steady state output error is zero, then an equilibrium exists. Hence, when an equilibrium exists, the Fourier transform of the output error will be zero.

Let $W(\mu,s)$ be the transfer function of the controlled plant from the reference input r to the output y for a constant value of the parameter vector,

$$W(\mu,s) = c_o k_p \frac{Z_m(s)Z_p(s)W_f(\mu s)}{Z_m(s)R_p(s) - c^T T(s)R_p(s) - k_p(d_o Z_m(s) + d^T T(s))Z_p(s)W_f(\mu s)} \; . \quad (2.15)$$

If the only restrictions on the reference input, r(t), are boundedness and continuity, then for the steady state output error to be zero, it is necessary that

$$W(\mu,s) = W_m(s) \qquad \forall s \in C \quad . \qquad (2.16)$$

However, when $\mu \neq 0$ the relative degree of $W(\mu,s)$ is not equal to the relative degree of $W_m(s)$. Therefore, no constant parameter vector could assure (2.16).

Clearly, the reference input will have to belong to a more restrictive class than bounded, continuous functions of time. Consider the class of signals which are sums of n distinct sinusoids, that is,

$$r(t) = \sum_{i=1}^{n} r_i \sin(w_i t + \beta_i) \quad , \; r_i \neq 0, w_i \neq 0, w_i - w_j \geq \delta > 0, \; r_i, w_i, \beta_i \in \mathbb{R} \quad . \tag{2.17}$$

Denote the set of s-values in (2.17) by S, that is,

$$S = \{jw_1, -jw_1, \ldots, jw_n, -jw_n\} \quad . \tag{2.18}$$

For an equilibrium to exist with an input signal in this class, it is necessary and sufficient that

$$W(\mu, s) = W_m(s) \text{ for } s \in S \quad . \tag{2.19}$$

This is equivalent to

$$Z_m(s)R_p(s) = c^T T(s)R_p(s) + k_p \left[\frac{c_o}{k_m} R_m(s) + d_o Z_m(s) + d^T T(s)\right] Z_p(s) W_f(\mu s), \quad s \in S \quad . \tag{2.20}$$

<u>Theorem 2.1</u>: There exists $\mu_o \in \mathbb{R}^+$ such that $\forall \mu \in [0, \mu_o)$, the MRAS shown in Figure 1 and described by (2.9)-(2.14) with assumptions A1-A6 has a unique equilibrium for a reference input described by (2.17).

<u>Proof</u>: (2.20) is a set of 2n linear equations in the 2n elements of the parameter vector. Clearly, there exists at least one constant parameter vector which satisfies (2.20). Furthermore, if the 2n equations (2.20) are linearly independent, the constant parameter vector which satisfies (2.20) is unique.

Assume that the 2n equations are not linearly independent. Then there must exist a nonzero constant parameter vector such that

$$0 = c^T T(s)R_p(s) + k_p \left[\frac{c_o}{k_m} R_m(s) + d_o Z_m(s) + d^T T(s)\right] Z_p(s) W_f(\mu s) \quad , \; s \in S \quad . \tag{2.21}$$

By Lemma 1 there exists a $\mu_o \in \mathbb{R}^+$ such that, $\forall\mu \in [0,\mu_o)$, the only solution to (2.21) is the zero vector. Thus, $\forall\mu \in [0,\mu_o)$, the 2n equations must be linearly independent and the theorem is proved.

2.4. Effects of the Perturbation

Let $\theta^*(0)$ be the unique constant parameter vector for which the steady state error is zero in the unperturbed case, $\mu=0$. For $[c_o, c^T, d_o, d^T]^T = \theta^*(0)$ and $\mu=0$, the transfer function of the controlled plant is identically equal to the transfer function of the model. Since the model transfer function is strictly positive real, so is the controlled plant transfer function. In this section, I want to show that $\theta^*(\mu)$, the unique constant parameter vector for which the steady state error is zero in the perturbed plant, satisfies

$$\theta^*(\mu) = \theta^*(0) + O(\mu) \qquad . \tag{2.22}$$

I also want to show that for $[c_o^*, c^{*T}, d_o^*, d^{*T}]^T = \theta^*(\mu)$, the transfer function

$$c_o^* k_p \frac{Z_m(s)Z_p(s)}{Z_m(s)R_p(s) - c^{*T}T(s)R_p(s) - k_p(d_o^* Z_m(s) + d^{*T}T(s))Z_p(s)} \tag{2.23}$$

will be strictly positive real provided μ is small enough. This transfer function is the transfer function of the controlled unperturbed n-th order plant when the parameter vector is held constant at $\theta^*(\mu)$.

From (2.20) $\theta^*(\mu)$ satisfies

$$Z_m(s)R_p(s) = \theta^*(\mu)^T \begin{bmatrix} \frac{k_p}{k_m} R_m(s)Z_p(s)W_f(\mu s) \\ T(s)R_p(s) \\ k_p Z_m(s)Z_p(s)W_f(\mu s) \\ k_p T(s)Z_p(s)W_f(\mu s) \end{bmatrix} , \quad s \in S \quad . \tag{2.24}$$

Define the 2n by 2n matrix

$$H(\mu) = \begin{bmatrix} \frac{k_p}{k_m} R_m(jw_1)Z_p(jw_1)W_f(\mu jw_1) \cdots \frac{k_p}{k_m} R_m(-jw_n)Z_p(-jw_n)W_f(-\mu jw_n) \\ T(jw_1)R_p(jw_1) \cdots\cdots\cdots\cdots T(jw_n)R_p(-jw_n) \\ k_p Z_m(jw_1)Z_p(jw_1)W_f(\mu jw_1) \cdots k_p Z_m(-jw_n)Z_p(-jw_n)W_f(-\ jw_n) \\ k_p T(jw_1)Z_p(jw_1)W_f(\mu jw_1) \cdots k_p T(-jw_n)Z_p(-jw_n)W_f(-\ jw_n) \end{bmatrix} . \tag{2.25}$$

From Theorem 2.1, if $\mu \in [0,\mu_o)$, $H(\mu)$ is nonsingular. Assume

$$\mu_o < \min_{s \in S} \{\frac{1}{|s|}\} \quad . \tag{2.26}$$

Then, $H(\mu)$ has a series representation,

$$H(\mu) = H(0) + \sum_{i=1}^{\infty} H_i \mu^i \quad . \tag{2.27}$$

It follows from (2.27) that

$$H(\mu)^{-1} = H(0)^{-1} + \sum_{i=1}^{\infty} L_i \mu^i \quad . \tag{2.28}$$

Then,

$$\theta^*(\mu) = (H(\mu)^{-1})^T \begin{bmatrix} Z_m(jw_1)R_p(jw_1) \\ \vdots \\ Z_m(-jw_n)R_p(-jw_n) \end{bmatrix}$$

$$= [H(0)^{-1} + \sum_{i=1}^{\infty} L_i\mu^i]^T = \begin{bmatrix} Z_m(jw_1)R_p(jw_1) \\ \vdots \\ Z_m(-jw_n)R_p(-jw_n) \end{bmatrix} \qquad (2.29)$$

$$= \theta^*(0) + \sum_{i=1}^{\infty} \theta_i\mu^i \quad .$$

Thus, the desired property (2.22) holds for μ sufficiently small.

For a transfer function $h(s)$ to be strictly positive real, it must satisfy:

1) $h(s)$ is real for real s,

2) the poles of $h(s)$ should lie in $Re[s] < 0$,

3) for all real w, one had $Re[h(jw)] > 0$, $-\infty < w < \infty$.

Property (3) is equivalent to

3') for all real w, one has $-90° < \text{Phase}\ [h(jw)] < 90°$ $-\infty < w < \infty$.

Theorem 2.2: There exists a $\mu_1 \in \mathbb{R}, \mu_1 > 0$ such that, if $\mu \in [0,\mu_1)$ and

$$[c_o^*, c^{*T}, d_o^*, d^{*T}]^T = \theta^*(\mu) \quad . \qquad (2.30)$$

then the transfer function (2.23) is strictly positive real.

Proof: Condition (1) is met by construction of (2.23). The poles of (2.23) are the zeros of the polynomial

$$Z_m(s)R_p(s) - c^{*T}T(s)R_p(s) - k_p(d_o^* Z_m(s) + d^{*T}T(s))Z_p(s)$$

$$= R_m(s)Z_p(s) - \sum_{i=1}^{\infty} \mu^i \theta_i^T \begin{bmatrix} 0 \\ T(s)R_p(s) \\ k_p Z_m(s)Z_p(s) \\ k_p T(s)Z_p(s) \end{bmatrix} . \qquad (2.31)$$

The zeros of (2.31) are continuous functions of μ. Since the zeros of $R_m(s)Z_p(s)$ lie in $Re[s] < 0$, there exists a μ' such that, if $\mu \in [0,\mu')$, then the poles of (2.23) lie in $Re[s] < 0$. The zeros of (2.23) are not functions of μ. Since the phase is a continuous function of the pole positions, and the pole positions are continuous functions of μ, the phase is also a continuous function of μ. The transfer function (2.23) satisfies condition 3' for μ=o. Hence, there exists $\mu'' > 0$ such that for $\mu \in [0,\mu'')$, condition 3' is satisfied for the transfer function (2.23). Take $\mu_1 = \min[\mu',\mu'']$ and the theorem is proved.

2.5. Discussion

In this chapter, I have shown that for a certain class of inputs and a sufficiently small perturbation in the structure of the plant, a unique equilibrium point exists. Furthermore, it is possible for the controlled nominal plant with the adjustable parameters held constant at the equilibrium values to be strictly positive real. This second fact will be important in determining the stability of this equilibrium point.

I would like to highlight a few points which were not explicitly stated in the derivation of this result.

Remark 2.1: The perturbation of θ^* from $\theta^*(0)$ will be a function of the frequencies w_i in S. This can be seen from (2.25). It can be intuitively seen using the following argument. The equilibrium exists because the Bode plot of the controlled plant transfer function can match that of the model at the n frequencies in the input signal. The contribution of the unmodelled dynamics to the controlled plant Bode plot is a function of frequency. Therefore, the amount by which the Bode plot of the controlled nominal plant is moved from the Bode plot of the model is a function of the frequency. It is not hard to believe, then, that choosing different values of w_i will result in different perturbations of the parameters.

Remark 2.2: Choosing a reference input with more than n distinct frequencies will result in a system that has no equilibrium. This is true because it can be shown in a manner very similar to the proof of Theorem 2.1 that the resulting set of more than 2n equations (2.20) in 2n unknowns is inconsistent. One can also think of this in terms of the Bode plots. The 2n parameters provide 2n degrees of freedom with which to bend the Bode plot of the controlled plant. It is not possible to make the (2n + 1)-th bend to match at an extra point.

Remark 2.3: While the theorems in this chapter prove existence of μ_0, μ_1, they can be used in a different manner. For a given μ which is small enough so that the nominal plant dynamics and the unmodelled dynamics have a reasonable separation, the w_i's of the reference input can be chosen so that the unique equilibrium point exists. This is related to the idea of dominant richness which was introduced by Ioannou and Kokotovic (1983). Krause (1983) discusses how to assure that the reference input is in the dominantly rich range.

CHAPTER 3

STABILITY OF THE EQUILIBRIUM

In this chapter, I study the stability of the equilibrium of Theorem 2.1 when the parameter vector is allowed to be time varying. After some definitions and theorems about stability, I present the differential equations which describe the behavior of the model reference adaptive control system. From this system of equations I derive an error system which describes the variations about the equilibrium.

The error system will have an input which is the difference of the model output and the controlled plant output when $\theta \equiv \theta^*$. In Chapter 2 it was shown that for an input with exactly n sinusoids, the steady state value of this difference will be zero. In this chapter, I will show that when this difference is zero, there exists a $\bar{\mu}$ such that for all $\mu \in [0,\bar{\mu})$, the zero solution of the error system is exponentially stable. Then I will show that this difference is always $O(\mu)$ and that trajectories of the error system which begin in some region of attraction converge exponentially to an $O(\mu)$ residual set containing zero.

3.1. Preliminaries

To insure that the reader and I agree on terminology of stability concepts, I state the following definitions. These definitions are taken from Yoshizawa (1966) and Rouche, Habets, Laloy (1977). Consider a system of differential equations

$$\dot{x} = f(t,x) \quad , \quad (\dot{x}) = \frac{d}{dt} x \quad . \tag{3.1}$$

Suppose that $f(t,x)$ is continuous on $\mathbb{R} \times \mathbb{R}^n$. Assume that (3.1) has solutions which are uniquely determined by the initial condition $(t,x) = (t_o,x_o)$. Let $x(t;x_o,t_o)$ be the solution of (3.1) which passes through the point (t_o,x_o).

Definition 3.1: The solution $x(t) \equiv 0$ of (3.1) is uniform-stable, if for any $\varepsilon > 0$ and $t_o \in \mathbb{R}$, there exists a $\delta(\varepsilon) > 0$ such that if $\|x_o\| < \delta(\varepsilon)$, then $\|x(t;x_o,t_o)\| < \varepsilon$ for all $t \geq t_o$.

Definition 3.2: The zero solution of (3.1) is uniform-asymptotically stable if it is uniform-stable, and if given any $\varepsilon_1 > 0$ and any $t_o \in \mathbb{R}$, there exist a $\delta_1 > 0$ and a $T(\varepsilon_1) > 0$ such that if $\|x_o\| < \delta_1$, then $\|x(t;x_o,t_o)\| < \varepsilon_1$ for all $t \geq t_o + T(\varepsilon_1)$.

Definition 3.3: The zero solution of (3.1) is exponentially stable in the large, if there exists an $\alpha > 0$ and for any $\beta > 0$ there exists a $K(\beta) \geq 1$ such that if $\|x_o\| \leq \beta$

$$\|x(t;x_o,t_o)\| \leq K(\beta) e^{-\alpha(t-t_o)} \|x_o\| \quad . \tag{3.2}$$

Definition 3.4: A function $a : R^+ \to R^+$ is said to be an element of class K if it is continuous, strictly increasing, and $a(0) = 0$. We write $a \in K$.

Let $V(t,x)$ be a continuous scalar function defined on $\mathbb{R} \times \mathbb{R}^n$ and let $V(t,x)$ satisfy locally a Lipschitz condition with respect to x. Then define the function

$$D^+_{(3.1)} V(t,x) = \overline{\lim_{h \to 0^+}} \frac{1}{h} \{V(t+h,\ x+hf(t,x)) - V(t,x)\} \quad . \tag{3.3}$$

If $V(t,x)$ has continuous partial derivatives of the first order,

$$D^{+}_{(3.1)}V(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot F(t,x) \quad , \tag{3.4}$$

where "·" denotes scalar product.

The following theorems will be used to investigate the stability of the equilibrium of Theorem 2.1. The theorems are stated here without proof. Theorems 3.1, 3.2, 3.3 and 3.4 are standard results and proofs can be found in a number of texts. Theorem 3.5 is a well-known result of Lasalle and the proof can be found in Hale (1969). Theorems 3.6 and 3.7 appear in a monograph by Yoshizawa (1966). The proofs of Theorems 3.6 and 3.7 can be found in the appendix to this thesis. Consider the linear system

$$\dot{x} = A(t)x \quad , \tag{3.5}$$

where $A(t)$ is an nxn matrix of continuous functions of time defined on $\mathbb{R}$.

Theorem 3.1: If the zero solution of (3.5) is uniform-asymptotically stable, then it is exponentially stable in the large and K in (3.2) can be chosen independent of β.

Now consider the differential equation (3.1) under the assumptions that $f(t,x)$ is continuous on $0 \leq t < \infty$, $\|x\| < H$, $H > 0$ and $f(t,0) \equiv 0$.

Theorem 3.2: Suppose that there exists a Lyapunov function $V(t,x)$ defined on $0 \leq t < \infty$, $\|x\| < H$ which satisfies the following conditions;

i) $V(t,o) \equiv 0$,

ii) $a(\|x\|) \leq V(t,x) \leq b(\|x\|)$, $a \in K$, $b \in K$,

iii) $D^{+}_{(3.1)}V(t,x) \leq 0$.

Then, the solution $x(t) \equiv 0$ of the system (3.1) is uniform-stable.

Theorem 3.3: Under the same conditions as Theorem 3.2, if

$$D^{+}_{(3.1)}V(t,x) \leq - c(\|x\|), \text{ where } c \in K \quad ,$$

then the solution $x(t) \equiv 0$ of (3.1) is uniform-asymptotically stable.

Corollary 3.3: Under the same conditions as Theorem 3.2, if

$$D^{+}_{(3.1)}V(t,x) \leq - cV(t,x) \quad ,$$

where $c > 0$ is a constant, then the solution $x(t) \equiv 0$ of (3.1) is uniform-asymptotically stable.

Assuming that $f(t,x)$ is continuous on $0 \leq t < \infty$, $x \in \mathbb{R}^n$, and $f(t,0) \times 0$, the following theorem applies.

Theorem 3.4: Suppose there exists a Lyapunov function $V(t,x)$ defined for $0 \leq t < \infty$, $x \in \mathbb{R}^n$ satisfying the following conditions:

i) $\|x\| \leq V(t,x) \leq K(\beta)\ \|x\|$ for $\|x\| \leq \beta$,

ii) $D^{+}_{(3.1)}V(t,x) \leq - \alpha V(t,x)$, where $\alpha > 0$ is a constant.

Then, the zero solution of (3.1) is exponentially stable in the large, that is, $\|x(t;x_o,t_o)\| \leq K(\beta)e^{-\alpha(t-t_o)}\ \|x_o\|$ for $\|x_o\| \leq \beta$.

Theorem 3.5: Let $V(t,x)$ be a continuous Lyapunov function defined on $0 \leq t < \infty$, $x \in G$, where G is an open set in R^n. Suppose that

i) given $x \in \overline{G}$, the closure of G, there is a neighborhood of x, N_x, such that $V(t,x)$ is bounded from below for all $t \geq 0$ and all x in $N_x \cap G$, and

ii) $D^{+}_{(3.1)}V(t,x) \leq - W(x) \leq 0$ for $0 \leq t < \infty$, $x \in G$ and $W(x)$ is continuous on $\overline{G}$.

Define

$$E = \{x: x \in \overline{G} \text{ and } W(x) = 0\} \quad . \tag{3.6}$$

Let $x(t;x_o,t_o)$ be a solution of (3.1) which is bounded and remains in G for $t \geq t_o \geq 0$.

If $W(x)$ has continuous first derivatives on $\bar{G}$ and $\dot{W}_{(3.1)} = \frac{\partial W(x)}{\partial x} \cdot f(t,x)$ is bounded from above along the solution $x(t;x_o,t_o)$, then

$$x(t;x_o,t_o) \to E \text{ as } t \to \infty \quad .$$

Theorem 3.6: Suppose there exists a $K \geq 1$ and a c such that

$$\|x(t;x_o,t_o)\| \leq Ke^{-c(t-t_o)}\|x_o\| \quad , \tag{3.7}$$

where $x(t;x_o,t_o)$ is a solution of (3.5) and c is a constant (≥ 0). Then, there exists a Lyapunov function $V(t,x)$ which satisfies the following conditions:

i) $\|x\| \leq V(t,x) \leq K\|x\|$,

ii) $|V(t,x) - V(t,x')| \leq K\|x - x'\|$,

iii) $D^+_{(3.5)}V(t,x) \leq - cV(t,x)$.

Theorem 3.7: Suppose that $f(t,x)$ of (3.1) is continuous for $0 \leq t < \infty$, $x \in \mathbb{R}^n$ and $f(t,0) \equiv 0$. If $f(t,x)$ satisfies a local Lipschitz condition with respect to x and the zero solution of (3.1) is exponentially stable in the large, i.e., there exists an $\alpha > 0$ and for any $\beta > 0$, there exists a $K(\beta) > 0$ such that if $\|x_o\| \leq \beta$,

$$\|x(t;x_o,t_o)\| \leq K(\beta)e^{-\alpha(t-t_o)} \|x_o\| \text{ for all } t \geq t_o,$$

then there exists a Lyapunov function $V(t,x)$ defined for $0 \leq t < \infty$, $x \in \mathbb{R}^n$ which satisfies the following conditions:

i) $\|x\| \leq V(t,x) \leq K(\beta) \|x\|$ for $\|x\| \leq \beta$,

ii) $|V(t,x) - V(t,x')| \leq L(t,\beta) \|x - x'\|$ for $\|x\| \leq \beta$, $\|x'\| \leq \beta$,

iii) $D^+_{(3.1)}V(t,x) \leq -q\alpha V(t,x)$, where $0 < q < 1$.

3.2. The Error System

For the unperturbed case, μ=0, the proof of stability of the equilibrium $(e,\theta)=(0,\theta^*)$ uses an error system representation of the model reference adaptive control system. In this section, I derive an error system representation for the perturbed case. It is a specific example of the type of error system representation which has been presented recently by Kosut, Johnson, and Anderson (1983).

My derivation will be made in several steps. First, I present the differential equations describing the plant, the control system, and the model. Next, I use a singular perturbation type of transformation on the plant. This explicitly separates the high frequency unmodelled dynamics from the nominal, or slow part, of the plant. The third step is the introduction of a "tuned system." Finally, I replace the original reference model with the tuned system in order to get the desired error system representation.

The differential equations describing the plant are

$$\dot{x} = A_{11}x + b_1 h_2^T z \quad , \qquad x \in \mathbb{R}^n \tag{3.8}$$

$$\mu\dot{z} = A_{22}z + b_2 u \quad , \qquad z \in R^{N_1+N_2} \tag{3.9}$$

$$y = h_1^T x \quad . \tag{3.10}$$

The vectors and matrices in (3.8)-(3.10) satisfy the following equations:

$$h_1^T(sI - A_{11})^{-1}b_1 = W_p(s) \quad , \tag{3.11}$$

$$h_2^T(\mu sI - A_{22})^{-1}b_2 = W_f(\mu s) \quad . \tag{3.12}$$

The equations describing the auxiliary signal generators are

$$\dot{v}^1 = \Lambda v^1 + bu \quad , \qquad v^1 \in R^{n-1} \quad , \tag{3.13}$$

$$\dot{v}^2 = \Lambda v^2 + by \quad , \qquad v^2 \in R^{n-1} \quad , \tag{3.14}$$

where the vector b and matrix Λ satisfy

$$(sI - \Lambda)^{-1}b = \frac{T(s)}{Z_m(s)} \quad . \tag{3.15}$$

Define the parameter vector, θ, and the signal vector, w, by

$$\theta = [c_o, c^T, d_o, d^T]^T \quad , \tag{3.16}$$

$$w = [r, v^{1^T}, y, v^{2^T}]^T \quad . \tag{3.17}$$

The parameter adjustment law is then

$$\dot{\theta} = -\Gamma w(y - y_m) \quad , \qquad \Gamma = \Gamma^T > 0 \quad , \tag{3.18}$$

and the input to the plant is

$$u = \theta^T w \quad . \tag{3.19}$$

Finally, the model is described by

$$\dot{x}_m = A_m x_m + b_m r \quad , \qquad x_m \in R^n \quad , \tag{3.20}$$

$$y_m = h_m^T x_m \quad . \tag{3.21}$$

The vectors b_m, h_m and the matrix A_m satisfy

$$h_m^T (sI - A_m) b_m = W_m(s) \quad . \tag{3.22}$$

Thus, I have a set of differential equations which describe the behavior of the MRAS shown in Figure 2.1.

The choice of a representation for the "unmodelled" part of the plant is the single most important factor in a study of the effects of "unmodelled" dynamics. I have restricted my unmodelled dynamics to the high frequency range. Because of this I was able to choose a system of differential equations, (3.8)-(3.9), in singular perturbation form to represent the plant. Because there are stability results for the unperturbed case, I would like to have my plant represented in a form which appears as an unperturbed plant plus a perturbation. Singular perturbation theory suggests the following method to transform (3.8)-(3.9) to the desired representation. First, set $\mu=0$ in (3.9). This corresponds to making the fast part of the plant infinitely fast. Because A_{22} is assumed stable z converges infinitely fast to

$$\bar{z} = -A_{22}^{-1} b_2 u \quad .$$

Then $\bar{z}$ is considered the "slow" part of z. In order to separate the system into its fast and slow parts, I define a new variable

$$\eta = z - \bar{z} = z + A_{22}^{-1} b_2 u \tag{3.23}$$

to be the fast variable. Now I can rewrite (3.8)-(3.9) as

$$\dot{x} = A_{11} x + b_1 u + b_1 h_2^T \eta \quad , \tag{3.24}$$

$$\mu \dot{\eta} = A_{22} \eta + \mu A_{22}^{-1} b_2 \dot{u} \quad . \tag{3.25}$$

The MRAS using (3.24)-(3.25) has a block diagram shown in Figure 3.1.

The third step in this derivation is the definition of a "tuned system." Define $r^*(t)$ to be an input as in (2.17). Assume that $\mu \in [0, \mu_o)$ so that a unique equilibrium exists. Take $[c_o^*, c^{*T}, d_o^*, d^{*T}]^T$ to be the solution of (2.20) with $r(t)=r^*(t)$. Then, the tuned system is the linear time invariant system formed by the plant plus the controller with $[c_o, c^T, d_o, d^T] \equiv [c_o^*, c^{*T}, d_o^*, d^{*T}]$. An important point is that, whenever $\mu \neq 0$, the tuned system is not positive real. In fact, the order and relative degree of the tuned system are unknown. The tuned system is represented by the equations

$$\dot{x}^* = A_{11} x^* + b_1 u^* + b_1 h_2^T \eta^* \quad , \tag{3.26a}$$

$$\dot{v}^{1*} = \Lambda v^{1*} + b u^* \quad , \tag{3.26b}$$

$$\dot{v}^{2*} = \Lambda v^{2*} + b y^* \quad , \tag{3.26c}$$

$$\mu \dot{\eta}^* = A_{22} \eta^* + \mu A_{22}^{-1} b_2 \dot{u}^* \quad , \tag{3.26d}$$

$$y^* = h_1^T x^* \quad , \tag{3.27}$$

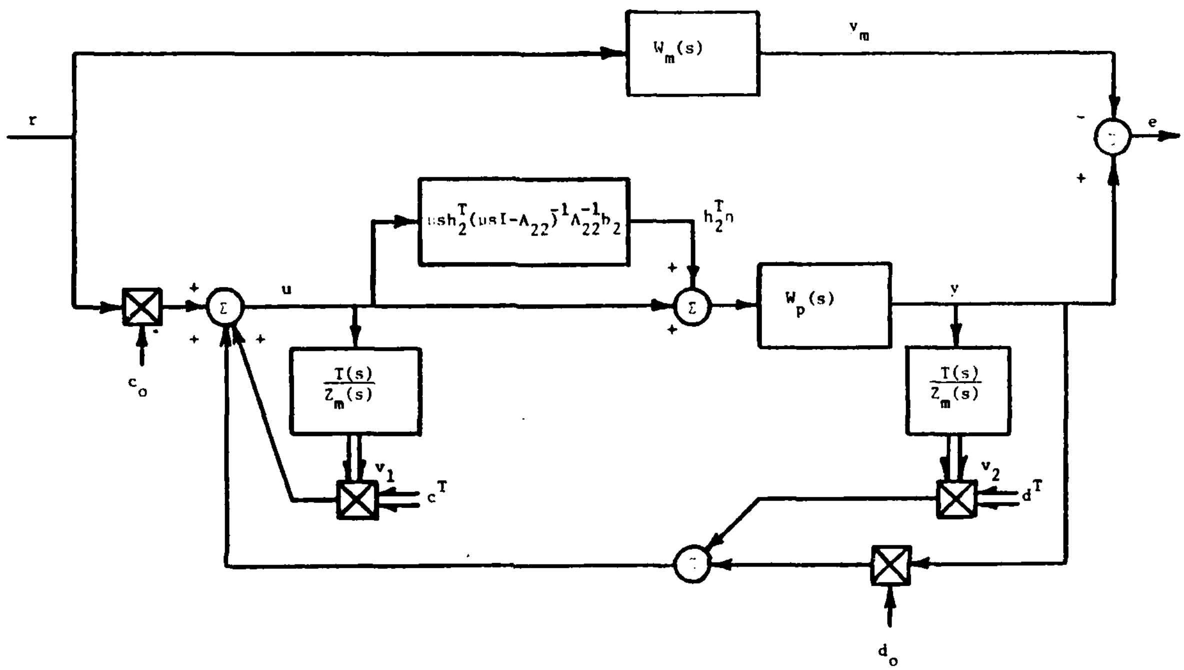

Figure 3.1. Block diagram of MRAS using transformed plant, but not showing the update mechanism.

$$u^* = \theta^{*T} w^* \quad , \tag{3.28}$$

$$w^* = [r, v^{1*T}, y^*, v^{2*T}]^T \quad . \tag{3.29}$$

In order to substitute the tuned system for the reference model, I must introduce the concept of tuned error. The tuned error is denoted by e^* and is defined as the difference between the output of the tuned system and the reference model,

$$e^* \triangleq y^* - y_m \quad . \tag{3.30}$$

Notice that the tuned error is defined for any reference input r(t). Also notice that when $r(t)=r^*(t)$, the steady state value of e^* will be zero. Figure 3.2 illustrates the tuned system and the tuned error.

Now, define the error system variables as

$$x' = x - x^* \quad , \tag{3.31a}$$

$$v^{1'} = v^1 - v^{1*} \quad , \tag{3.31b}$$

$$v^{2'} = v^2 - v^{2*} \quad , \tag{3.31c}$$

$$\theta' = \theta - \theta^* \quad , \tag{3.31d}$$

$$\eta' = \eta - \eta^* \quad . \tag{3.31e}$$

The differential equations describing the behavior of the error system are

$$\dot{x}' = A_{11}x' + b_1u' + b_1h_2^T\eta' \quad , \tag{3.32a}$$

$$\dot{v}^{1'} = \Lambda v^{1'} + bu' \quad , \tag{3.32b}$$

$$\dot{v}^{2'} = \Lambda v^{2'} + by' \quad , \tag{3.32c}$$

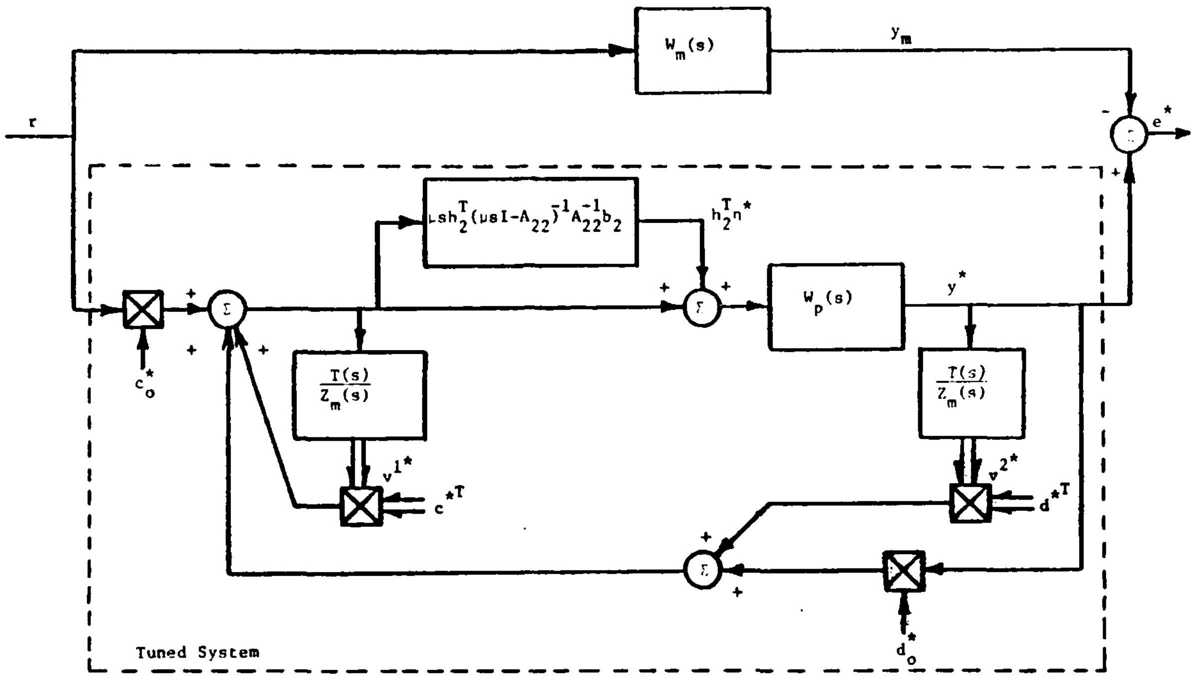

Figure 3.2. Block diagram showing tuned system and tuned error.

$$\dot{\theta}' = -\Gamma w(y' + e^*) \quad , \tag{3.32d}$$

$$\mu\dot{\eta}' = A_{22}\eta' + \mu A_{22}^{-1}b_2\dot{u}' \quad , \tag{3.33}$$

where

$$y' = h_1^T x' = y - y^* \quad , \tag{3.34}$$

$$w' = w - w^* = [0, v^{1'}, y', v^{2'}] \quad , \tag{3.35}$$

$$u' = \theta^{*T} w' + w^T \theta' \quad , \tag{3.36}$$

$$\begin{aligned}
\dot{u}' &= w^T\dot{\theta}' + \dot{w}^{*T}\theta' + \theta^{*T}\dot{w}' + \theta'^T\dot{w}' \\
&= -w^T\Gamma w(y' + e^*) + \dot{w}^{*T}\theta' \\
&\quad + f_1^T[x'^T, v^{1'T}, v^{2'T}, \theta'^T]^T + d_o^* h_1^T b_1 h_2^T \eta' \\
&\quad + [x'^T, v^{1'T}, v^{2'T}, \theta'^T]\begin{bmatrix} J_1 \\ \hline J_2 \end{bmatrix}\theta' + d_o' h_1^T b_1 h_2 \eta' \\
&\quad + d_o' h_1^T b_1 \theta'^T w'
\end{aligned} \tag{3.37}$$

with

$$f_1 = \begin{bmatrix} d_o^* A_{11}^T h_1 + (b^T d^* + d_o^* b^T c^* + h_1^T\, b_1 d_o^{*2}) h_1 \\ \Lambda^T c^* + (b^T c^* + h_1^T\, b_1 d_o^*) c^* \\ \Lambda^T d^* + (b^T c^* + h_1^T\, b_1 d_o^*) \\ (b^T c^* + h_1^T b_1 d_o^*) w^* \end{bmatrix} \quad , \tag{3.38}$$

$$
J_1 = \left[\begin{array}{c|c|c|c}
0 & d_o^* h_1 b^T & A_{11}^T h_1 + (b^T c^* + 2d_o^* h_1^T b_1) h_1 & h_1 b^T \\
\hline
0 & \Lambda^T + c^* b^T + (b^T c^* + d_o^* h_1^T b_1) I & h_1^T b_1 c^* & 0 \\
\hline
0 & d^* b^T & h_1^T b_1 d^* & \Lambda + (b^T c^* + d_o^* h_1^T b_1) I
\end{array}\right], \tag{3.39}
$$

$$
J_2 = \left[\begin{array}{c|c|c|c}
0 & br & h_1^T b_1 r & 0 \\
\hline
0 & \frac{1}{2}(bv^{1*T} + v^{1*T} b^T) & h_1^T x^* b + h_1^T b_1 v^{1*} & bv^{2*T} \\
\hline
0 & 0 & h_1^T b_1 h_1^T x^* & h_1^T b_1 v^{2*T} \\
\hline
0 & 0 & 0 & 0
\end{array}\right]. \tag{3.40}
$$

Figure 3.3 shows a block diagram of the error system.

<u>Remark 3.1</u>: While the expression for $\dot{u}'$ is extremely complicated, notice that a μ multiplies the $\dot{u}'$ term in (3.33) and thus $\dot{u}'$ will have only a small effect as long as $|\dot{u}'| \leq 0(\frac{1}{\mu})$.

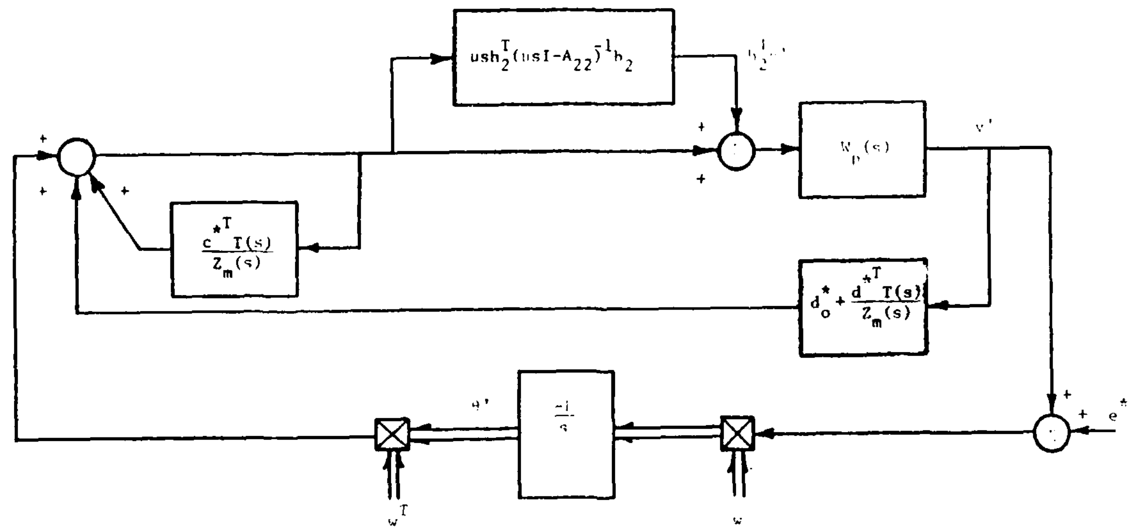

Figure 3.3. Block diagram of the error system.

3.3. Stability of the Equilibrium with $e^* \equiv 0$

Two differences exist between the error sys T derived for the perturbed case and the error system for the unperturbed case. The first is the perturbation itself. Because of the perturbation, the linear time invariant part of the error model is not positive real. The second difference is the tuned error, e^*, which shows up as an extra external input. In this section I study the stability of the zero solution of the error system when $e^* \equiv 0$. I will consider the effects of a nonzero tuned error in Section 3.4.

The results of this section can be summarized as follows. Assume that $e^* \equiv 0$ and the reference input has at least n distinct sinusoids in the low frequency range. Then the zero solution of the error system (3.32), (3.33) is exponentially stable. Furthermore, the magnitude of the region of attraction for this exponentially stable equilibrium is $O(\mu^{-\frac{1}{2}})$ for the slow variables and $O(\mu^{-\frac{3}{2}})$ for the fast variables.

These results are derived in several steps. The first step is to decompose the error system into a fast subsystem and a slow subsystem. The decomposition is performed so that the fast subsystem is a stable linear time invariant one by assumption. The next step is to prove that the slow subsystem is exponentially stable. Then, I apply Theorem 3.7 to the slow subsystem and Theorem 3.6 to the fast subsystem to get a conceptual Lyapunov function for each subsystem. Finally, I use a combination of these two Lyapunov functions to prove that the zero solution of the error system (3.32), (3.33) is exponentially stable in the region of attraction of the zero solution.

I get a good decomposition for this problem using the techniques of singular perturbation theory. This decomposition was set up by the transformation (3.23) which introduced η to serve as a fast variable. To get the slow subsystem from the error system (3.32), (3.33), I set $\mu=0$ in (3.33) which corresponds to making my unmodelled dynamics infinitely fast. Since A_{22} is stable any initial conditions on η' will decay to zero infinitely fast. Hence, my slow subsystem will be (3.32) with $\eta' \equiv 0$. In order to derive the fast subsystem, I first rewrite the error system (3.32), (3.33) in terms of a fast time variable, $\tau = \frac{1}{\mu} t$. Then I set $\mu=0$ which corresponds to making my nominal plant constant at its initial values. The resulting fast subsystem is given by

$$\frac{d}{d\tau}\eta' = A_{22}\eta' \quad . \tag{3.41}$$

In order to study the slow subsystem, I form a system of differential equations from (3.32) by setting $\eta' \equiv 0$ and $e^* \equiv 0$, and then replacing $[x'^T, v^{1'T}, v^{2'T}]^T$ with X and θ' with ϕ. This results in

$$\dot{X} = AX + \bar{b}(w^* + DX)^T \phi, \quad X \in \mathbb{R}^{3n-2} , \tag{3.42a}$$

$$\dot{\phi} = -\Gamma\,(w^* + DX)\bar{h}^T X, \quad \phi \in \mathbb{R}^{2n} , \tag{3.42b}$$

where

$$A = \left[\begin{array}{c|c|c} A_{11} + d_o^* b_1 h_1^T & b_1 c^{*T} & b_1 d^{*T} \\ \hline d_o^* b h_1^T & \Lambda + bc^{*T} & db^{*T} \\ \hline bh_1^T & 0 & \Lambda \end{array}\right] , \tag{3.43}$$

$$\bar{b}^T = [b_1^T, b^T, 0] \quad , \tag{3.44}$$

$$\bar{h}^T = [h_1^T, 0, 0] \quad , \tag{3.45}$$

$$D = \left[\begin{array}{c|c|c} \multicolumn{3}{c}{0_{1x(3n-2)}} \\ \hline 0_{(n-1)xn} & I_{n-1} & 0_{(n-1)xn-1} \\ \hline \bar{h}^T & 0_{1xn-1} & 0_{1x(n-1)} \\ \hline 0_{(n-1)xn} & 0_{(n-1)x(n-1)} & I_{n-1} \end{array}\right] . \tag{3.46}$$

Note that DX replaces w'. From Figure 3.3, it follows that the transfer function of the linear time invariant part of (3.41) is

$$\bar{h}^T(sI-A)^{-1}\bar{b} = c_o^* k_p \frac{Z_m(s)Z_p(s)}{Z_m(s)R_p(s) - c^{*T}T(s)R_p(s) - k_p(d_o^* Z_m(s) + d^{*T}T(s))Z_p(s)} . \tag{3.47}$$

<u>Theorem 3.8</u>: Let $r(t)$ be a signal which satisfies (2.17). Let $\mu_2 = \min\{\mu_o, \mu_1\}$. Then, $\forall \mu \in [0, \mu_2)$, the zero solution of (3.42) is exponentially stable in the large.

<u>Proof</u>: By Theorem 2.2 the transfer function (3.47) is strictly positive real. Then, by the Kalman-Yacubovich lemma, a positive definite symmetric matrix P exists such that

$$A^TP + PA = -qq^T - \epsilon L \ , \ L = L^T > 0, \ \ \epsilon > 0 \quad , \tag{3.48}$$

$$P\bar{b} = \bar{h} \quad , \tag{3.49}$$

and the Lyapunov function

$$W(X, \phi) = X^TPX + \phi^T\Gamma^{-1}\phi \tag{3.50}$$

has the derivative for (3.42)

$$\dot{W}_{(3.42)}(X,\phi) = -X^T(qq^T + \in L)X \leq 0 \quad . \tag{3.51}$$

From (3.50), (3.51) and Theorem 3.2 it follows that the zero solution of (3.42) is uniform-stable.

Because $\dot{W}_{(3.42)}(X,\phi) \leq 0$, every solution of (3.42) initially in the set $G = \{X,\phi\colon W(X,\phi) < \beta\}$ remains in G. Furthermore, by Theorem 3.5

$$X \to 0 \text{ as } t \to \infty \quad . \tag{3.52}$$

Since the elements of w^* are bounded continuous functions of time with bounded derivatives and the states of the system (3.42) are bounded, the states will be continuous functions of time with bounded derivatives. Hence, the elements of

$$\zeta = w^* + DX \tag{3.53}$$

will be bounded continuous functions of time with bounded derivatives. Yuan and Wonham (1977) have shown that since the elements of ζ are bounded continuous functions of time with bounded derivatives, (3.42) and (3.52) imply

$$\zeta^T\phi \to 0 \text{ as } t \to \infty \quad . \tag{3.54}$$

From (3.42b) and (3.52), ϕ converges to a constant as $t \to \infty$. Yuan and Wonham also showed that if, for any $\alpha \in \mathbb{R}^{2n}$,

$$\alpha^T\zeta(t) = 0 \quad t \geq 0 \quad \Rightarrow \quad \alpha = 0 \quad , \tag{3.55}$$

then $\phi \to 0$ as $t \to \infty$. Because of (3.52), (3.55) can be replaced by the condition

$$\alpha^T w^*(t) = 0 \quad t \geq 0 \Rightarrow \alpha = 0 \quad . \tag{3.56}$$

A sufficient condition for (3.56) to hold is that

$$\alpha^T w^*(j\nu) = 0 \quad -\infty < \nu < \infty \Rightarrow \quad \alpha = 0 \tag{3.57}$$

where $w^*(j\nu)$ is the Fourier transform of $w^*(t)$. Because

$$w^*(j\nu) = \begin{bmatrix} \frac{k_p}{k_m} R_m(j\nu)Z_p(j\nu)W_f(j\nu) \\ T(j\nu)R_p(j\nu) \\ k_pZ_m(j\nu)Z_p(j\nu)W_f(j\nu) \\ k_pT(j\nu)Z_p(j\nu)W_f(j\nu) \end{bmatrix} \frac{r(j\nu)}{k_pR_m(j\nu)Z_p(j\nu)W_f(j\nu)} \quad , \tag{3.58}$$

Theorem 2.1, guarantees that condition (3.57) is satisfied.

Since the zero solution of (3.42) is uniform-stable and every solution of (3.42) with bounded initial conditions converges to zero as $t \to \infty$, the zero solution is uniform-asymptotically stable. If (3.42) is rewritten with ζ replacing $w^* + DX$, it has the appearance of a linear system. Hence, every solution of (3.42) with bounded initial conditions can be generated by a uniform-asymptotically stable linear system. Then, by Theorem 3.1, each solution of (3.42) which begins with bounded initial conditions decays exponentially fast to the zero solution and the proof is complete.

Let $Y = [X^T, \phi^T]^T$. Then, from Theorem 3.8 and Definition 3.3, there exists and $\alpha_1 > 0$, and for any $\beta > 0$, there exists a $K_1(\beta) \geq 1$ such that if $Y(t;\ Y_o, t_o)$ is a solution of (3.42), then

$$\|Y(t;\ Y_o,t_o)\| \leq K_1(\beta)e^{-\alpha_1(t-t_o)}\|Y_o\| \quad , \quad \forall t \geq t_o \ . \tag{3.59}$$

By Theorem 3.7, there exists a Lyapunov function $V_1(t,Y)$ which has the following properties:

$$\|Y\| \leq V_1(t,Y) \leq K_1(\beta)\|Y\| \quad , \quad \|Y\| \leq \beta \quad , \tag{3.60}$$

$$|V_1(t,Y_1) - V_1(t,Y_2)| \leq L(t,\beta)\|Y_1-Y_2\| \quad , \quad \|Y_1\| \leq \beta, \|Y_2\| \leq \beta \ , \tag{3.61}$$

$$D^+_{(3.42)}V_1(t,Y) \leq -\ q\ \alpha_1 V_1(t,Y), \quad \text{where } 0 < q < 1 \quad . \tag{3.62}$$

Thus, the existence of a Lyapunov function for the slow subsystem is established.

Define σ such that the real part of each eigenvalue of A_{22} is less than or equal to $-\sigma$, that is

$$\sigma = \inf\{-\mathrm{Re}[\lambda(A_{22})]\} \quad . \tag{3.63}$$

It follows from (3.63) and the linear time invariant nature of (3.41) that there exists a K_2 such that if $\eta'(\tau;\ \eta'_o, \tau_o)$ is a solution of (3.41), then

$$\|\eta'(\tau;\ \eta'_o,\tau_o)\| \leq K_2 e^{-\sigma(\tau-\tau_o)}\|\eta'_o\| \quad , \quad \forall \tau \geq \tau_o \quad . \tag{3.64}$$

Replacing τ with $\frac{1}{\mu}t$ in (3.64) and applying Theorem 3.6 shows that there exists a Lyapunov function $V_2(t,\eta')$ which has the following properties:

$$\|\eta'\| \le V_2(t,\eta') \le K_2\|\eta'\| \quad , \tag{3.65}$$

$$|V_2(t,\eta'') - V_2(t,\eta')| \le K_2\|\eta' - \eta''\| \; , \tag{3.66}$$

$$D^+_{(3.54)}V_2(t,\eta') \le -\frac{\sigma}{\mu} V_2(t,\eta') \quad . \tag{3.67}$$

Thus, a Lyapunov function for the fast subsystem has been found.

Before composing these two Lyapunov functions to form a Lyapunov function for the error system (3.32), (3.33), I need to simplify the expression for $\dot{u}'$. Let

$$X = [x'^T, v^{1'T}, v^{2'T}]^T \quad , \tag{3.68}$$

$$Y = [X^T \,\vdots\, \theta'^T]^T \quad . \tag{3.69}$$

With $e^* \equiv 0$, (3.37) can be rewritten

$$\begin{aligned} \dot{u}' = &- w^{*T}\Gamma w^* \bar{h}^T X - 2w^*\Gamma D X \bar{h}^T X - X^T D^T \Gamma D X \bar{h}^T X \\ &+ \dot{w}^{*T}\theta' + f_1^T Y + d_o^* h_1^T b_1 h_2^T \eta' + Y^T \begin{bmatrix} J_1 \\ \hline J_2 \end{bmatrix} \theta' \\ &+ d_o' h_1^T b_1 h_2 \eta' + d_o' h_1^T b_1 \theta'^T w' \quad . \end{aligned} \tag{3.70}$$

Because r and $\dot{r}$ are bounded, the elements of w^* and $\dot{w}^*$ are bounded and there exist constants $G_1, G_2, G_3, G_4, G_5 \in \mathbb{R}^+$ such that

$$|\dot{u}'| \leq \frac{1}{K_2\|A_{22}^{-1}b_2\|}\,[G_1\|Y\| + G_2\|Y\|^2 + G_3\|Y\|^3 + G_4\|\eta'\| + G_5\|Y\|\|\eta'\|] \quad . \tag{3.71}$$

The main result of this section is contained in Theorem 3.9.

Theorem 3.9: Let $r(t)$ be the same $r(t)$ as in Theorem 3.8. Then, there exists a $\mu_3 \in \mathbb{R}^+$ such that, $\forall\mu \in [0,\mu_3)$ the zero solution of (3.32),(3.33) with $e^* \equiv 0$ is exponentially stable.

Proof: Take $\mu_3 \leq \mu_2$. Then, the Lyapunov function $V_2(t,Y)$ exists and has properties (3.60)-(3.62). Using these properties and (3.65)-(3.67), I can write

$$D^+_{(3.32)e^*\equiv 0}V_1(t,Y) \leq -\,q\,\alpha_1 V_1(t,Y) + K_3(\beta)V_2(t,\eta') \quad , \|Y\| \leq \beta \quad , \tag{3.72}$$

$$D^+_{(3.33)e^*\equiv 0}V_2(t,\eta') \leq -\,(\frac{\sigma}{\mu} - G_4 - G_5V_1(t,Y))V_2(t,\eta') + f(V_1(t,Y))V_1(t,Y), \tag{3.73}$$

where

$$K_3(\beta) = \sup_{t\geq 0}\{L_1(t,\beta)\|b_1\|\|h_2\|\} \quad , \tag{3.74}$$

$$f(V_1(t,Y)) = G_1 + G_2V_1(t,Y) + G_3V_1^2(t,Y) \quad . \tag{3.75}$$

Define $M(Y,\eta')$ as the solution of

$$M(Y,\eta') = K_1(\|Y\|)\|Y\| + \frac{\mu K_3(M(Y,\eta'))}{\sigma - \mu(G_4 + q\,\alpha_1 + G_5M(Y,\eta'))}\,K_2\|\eta'\| \tag{3.76}$$

which satisfies $M(Y,\eta') \simeq K_1(\|Y\|)\|Y\| + \frac{\mu}{\sigma}\,K_3(M(Y,\eta'))K_2\|\eta'\|$.

Let $M_o = M(Y_o,\eta_o')$ and define

$$\alpha_3 = q\,\alpha_1 - \frac{\mu K_3(M_o)}{\sigma - \mu\,G_4 - \mu\,G_5M_o - \mu\,q\,\alpha_1}\,f(M_o) \quad . \tag{3.77}$$

Next, consider the Lyapunov function

$$V_3(t,Y,\eta') = V_1(t,Y) + \frac{\mu K_3(M_o)}{\sigma - \mu\, G_4 - \mu\, G_5 M_o - \mu\, q\, \alpha_1} V_2(t,\eta') \quad . \tag{3.78}$$

The right-side derivative of $V_3(t,Y,\eta')$ along the trajectories of (3.32)-(3.33) with $e^* \equiv 0$ satisfies

$$\begin{aligned} D^+_{(3.32)-(3.33)e^* \equiv 0} V_3(t,Y,\eta') &\le -[q\, \alpha_1 - \frac{\mu K_3(M_o)}{\sigma - \mu\, G_4 - \mu G_5 M_o - \mu q\, \alpha_1} \\ &\qquad \cdot f(V_1(t,Y))]V_1(t,Y) \\ &\quad - K_3(M_o)(\frac{\sigma - \mu\, G_4 - \mu G_5 V_1(t,Y)}{\sigma - \mu\, G_4 - \mu\, G_5 M_o - \mu q\, \alpha_1} - 1)V_2(t,\eta') \\ &= -\alpha_3 V_3(t,Y,\eta') \\ &\quad - \frac{\mu K_3(M_o)}{\sigma - \mu\, G_4 - \mu G_5 M_o - \mu q\, \alpha_1}(f(M_o) - f(V_1(t,Y))V_1(t,Y) \\ &\quad - \frac{\mu K_3(M_o) G_5}{\sigma - \mu\, G_4 - \mu G_5 M_o - \mu q\, \alpha_1}(M_o - V_1(t,Y))V_2(t,\eta') \\ &\quad - (\frac{\mu K_3(M_o)}{\sigma - \mu\, G_4 - \mu G_5 M_o - \mu q\, \alpha_1})^2 f(M_o) V_2(t,\eta') \quad . \end{aligned} \tag{3.79}$$

From (3.60), (3.65), (3.76) and (3.78) it follows that

$$V_3(0,Y_o,\eta'_o) \le M_o \quad . \tag{3.80}$$

From (3.78) it is clear that

$$V_1(t,Y) \le V_3(t,Y,\eta') \quad . \tag{3.81}$$

From (3.79), (3.80) and (3.81), it follows that if $\alpha_3 > 0$, then

$$V_3(t,Y,\eta') \le V_3(0,Y_o,\eta'_o) e^{-\alpha_3 t} \quad . \tag{3.82}$$

Hence, defining

$$\mu_4 = \frac{\sigma q\ \alpha_1}{(q\ \alpha_1)(G_4 + q\ \alpha_1) + K_3(0)G_1} \tag{3.83}$$

and taking $\mu_3 = \min\{\mu_2,\mu_4\}$ completes the proof.

An estimate of the region of attraction is available from the proof of Theorem 3.9. For every $\mu \in [0,\mu_3)$ there exists $M_o^*(\mu) \in \mathbb{R}^+$ such that if $M_o \in [0,M_o^*(\mu))$, then $\alpha_3 > 0$, and if $M_o = M_o^*(\mu)$, then $\alpha_3 = 0$. Then the region of attraction contains the set $\mathcal{D}(\mu)$ defined as

$$\mathcal{D}(\mu) = \{(Y,\eta) : M(Y,\eta') < M_o^*(\mu)\} \quad . \tag{3.84}$$

From (3.77) it is clear that if $K_3(\beta) = 0(1)$ for all $\beta \in \mathbb{R}^+$, then as $\mu \to 0$ $M_o^*(\mu) \to 0(\mu^{-\frac{1}{2}})$. If one further assumes that $K_1(\beta) = 0(1)$ for all $\beta \in \mathbb{R}^+$, then as $\mu \to 0$ the region of attraction includes points which satisfy

$$\|Y_o\| \le 0(\mu^{-\frac{1}{2}}) \quad ,$$

$$\|\eta_o'\| \le 0(\mu^{-\frac{3}{2}}) \quad .$$

Thus, the zero solution of (3.32)-(3.33) with $e^* \equiv 0$ and μ small enough is exponentially stable and the region of attraction is $0(\mu^{-\frac{1}{2}})$ in the slow variables and $0(\mu^{-\frac{3}{2}})$ in the fast variables.

Remark 3.2: In this section I have required that r(t) have exactly n distinct sinusoids in the low frequency range. However, the "exactly n" part of this requirement was used only to choose $\theta^*(\mu)$. If we choose $\theta^*(\mu)$ by some criterion other than making the steady state value of the tuned error equal to zero, then we can replace "exactly n" with "at least n" and the analysis of this section will still hold.

3.4. Effects of $e^* \neq 0$

In this section I show that the effects of $e^* \neq 0$ are only $0(\mu)$. This is accomplished in two steps. First, I show that I can choose initial conditions on the tuned system so that $|e^*| \leq 0(\mu)$ for all $t \geq 0$. Next I show that if M_o^* represents the boundary of the region of attraction when $e^* \equiv 0$, then the boundary of the region attraction when $|e^*| = 0(\mu)$ will be represented by $(1-0(\mu^{\frac{3}{2}}))M_o^*$. At the same time I show that solutions of (3.32)-(3.33) with $|e^*| = 0(\mu)$ beginning in this slightly reduced region of attraction converge to an $0(\mu)$ residual set containing zero.

In order to show that I can choose initial conditions on the tuned system so that $|e^*| \leq 0(\mu)$ for $t \geq 0$, I begin by choosing a different representation of the model. Define

$$\mathbb{A} = \left[\begin{array}{c|c|c} A_{11} & 0 & 0 \\ \hline 0 & \Lambda & 0 \\ \hline bh_1^T & 0 & \Lambda \end{array}\right] + \theta^*(0)^T D \quad . \tag{3.85}$$

Then the model equations (3.20)-(3.21) can be replaced by

$$\dot{X}_m = \mathbb{A}X_m + \bar{b}c_o^*(0)r \ , \qquad X_m(0) = X_{m_o} \tag{3.86}$$

$$y_m = \bar{h}^T X_m \quad . \tag{3.87}$$

Next, I find a Lyapunov function for this representation. Since

$$\bar{h}^T(sI - \mathbb{A})^{-1}\bar{b}c_o^* = k_m \frac{Z_m(s)Z_p(s)Z_m(s)}{Z_m(s)Z_p(s)R_m(s)}$$

and because $Z_m(s)$, $Z_p(s)$ and $R_m(s)$ are Hurwitz, I know that $\mathbb{A}$ is a stable matrix. Define σ_1 so that the real part of each eigenvalue of $\mathbb{A}$ is less than $-\sigma_1$,

$$\sigma_1 = \inf\{-\operatorname{Re}[\lambda(\mathbb{A})]\} \quad . \tag{3.88}$$

Then, if $X_m(t;X_{m_o},t_o)$ is a solution of (3.86) with $r \equiv 0$, there exists a K_4 such that

$$\|X_m(t;X_{m_o},t_o)\| \le K_4 e^{-\sigma_1(t-t_o)}\|X_{m_o}\| \quad , \; t \ge t_o \quad . \tag{3.89}$$

By Theorem 3.6 and (3.89), there exists a Lyapunov function $V_4(t,X_m)$ such that

$$\|X_m\| \le V_4(t,X_m) \le K_4\|X_m\| \quad , \tag{3.90}$$

$$|V_4(t,X_m^1) - V_4(t,X_m^2)| \le K_4\|X_m^1 - X_m^2\| \quad , \tag{3.91}$$

$$D^+_{(3.86)\; r\equiv 0} V_4(t,X_m) \le -\sigma_1 V_4(t,X_m) \quad . \tag{3.92}$$

The first result of this section is contained in Theorem 3.10.

<u>Theorem 3.10</u>: Let $X^* = [x^{*T}, v^{1*T}, v^{2*T}]^T$. Require $r(t)$ and $\dot{r}(t)$ to be bounded. If $X^*(0) = X_m^*(0)$, $\eta^*(0) = 0$, and $\mu \in [0,\mu_3)$, then there exists $\mathfrak{M}_6 \in \mathbb{R}^+$ such that for $t \ge 0$

$$|e^*| \le \mu\mathfrak{M}_6 \quad .$$

Proof: Take $E = X^* - X_m$. Then

$$\dot{E} = AE + (\theta^*(\mu) - \theta^*(0))^T D(E + X_m) + \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix} h_2^T \eta^*, \quad E(0) = 0 \quad , \tag{3.93}$$

$$\dot{\eta}^* = \frac{1}{\mu} A_{22}\eta^* + A_{22}^{-1} b_2 [c_o^* \dot{r} + \theta^*(\mu)^T D(A(E + X_m) + \bar{b}c_o^* r + \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix} h_2^T \eta^*)],$$

$$\eta^*(0) = 0 \quad , \tag{3.94}$$

$$e^* = \bar{h}^T E \quad . \tag{3.95}$$

Because $r(t)$ and $\dot{r}(t)$ are bounded, there exists an $\mathfrak{M}_1$ such that

$$|c_o^*| \, \| A_{22}^{-1} b_2 \| \, |\dot{r} + \theta^*(\mu)^T D\bar{b} r| \leq \mathfrak{M}_1 \quad . \tag{3.96}$$

Since A is stable and $r(t)$ is bounded, there exists an $\mathfrak{M}_2$ such that

$$\| X_m \| \leq \mathfrak{M}_2 \quad . \tag{3.97}$$

Now, define $\mathfrak{M}_3$, $\mathfrak{M}_4$, and $\mathfrak{M}_5$ such that

$$\mathfrak{M}_3 = \| A_{22}^{-1} b_2 \| \, \| \theta^*(\mu)^T D A \| \quad , \tag{3.98}$$

$$\mathfrak{M}_4 = \| b_1 \| \, \| h_2 \| \quad , \tag{3.99}$$

$$\mu \mathfrak{M}_5 = \| (\theta^*(\mu) - \theta^*(0))^T D \| \quad . \tag{3.100}$$

Using (3.65)-(3.67), (3.90)-(3.91), and (3.96)-(3.100), I can write

$$D^+_{(3.93)} V_4(t,E) \leq -(\sigma_1 - \mu K_4 \mathfrak{M}_5) V_4(t,E) + K_4 \mathfrak{M}_4 V_2(t,\eta^*) + \mu K_4 \mathfrak{M}_5 \mathfrak{M}_2 \quad , \tag{3.101}$$

$$D^T_{(3.94)} V_2(t,\eta^*) \leq -(\frac{\sigma}{\mu} - G_4) V_2(t,\eta^*) + K_2 \mathfrak{M}_3 V_4(t,E) + K_2(\mathfrak{M}_1 + \mathfrak{M}_2 \mathfrak{M}_3) \quad . \tag{3.102}$$

Then the Lyapunov function

$$V_5(t,E,\eta^*) = V_4(t,E) + \frac{\mu K_4 \mathfrak{m}_2}{\sigma - \mu G_4 - \mu\sigma_1} V_2(t,\eta^*) \tag{3.103}$$

satisfies

$$D^+_{(3.93)(3.94)} V_5(t,E,\eta^*) \leq -\alpha_5 V_5(t,E,\eta^*) + \mu(K_4 \mathfrak{m}_5 \mathfrak{m}_2 + \frac{K_4 \mathfrak{m}_2 K_2 (\mathfrak{m}_1 + \mathfrak{m}_2 \mathfrak{m}_3)}{\sigma - \mu G_4 - \mu\sigma_1}), \tag{3.104}$$

where

$$\alpha_5 = \sigma_1 - \mu(K_4 \mathfrak{m}_5 + \frac{K_4 \mathfrak{m}_2 K_2 \mathfrak{m}_3}{\sigma - \mu G_4 - \mu\sigma_1}) \quad . \tag{3.105}$$

If $\mu \in [0,\mu_3)$, then $\alpha_5 > 0$. Hence $E^*(0) = 0$, $\eta^*(0) = 0$, (3.90), and (3.104) imply

$$\|E\| \leq V_4(t,E) \leq V_5(t,E,\eta^*) \leq \mu \frac{K_4 \mathfrak{m}_2}{\alpha_5} (\mathfrak{m}_5 + \frac{K_2(\mathfrak{m}_1 + \mathfrak{m}_2 \mathfrak{m}_3)}{\sigma - \mu G_4 - \mu\sigma_1}) \quad . \tag{3.106}$$

The proof is completed by defining

$$\mathfrak{m}_6 = \|h_1\| \frac{K_4 \mathfrak{m}_2}{\alpha_5} (\mathfrak{m}_5 + \frac{K_2(\mathfrak{m}_1 + \mathfrak{m}_2 \mathfrak{m}_3)}{\sigma - \mu G_4 - \mu\sigma_1}) \quad . \tag{3.107}$$

Now that I have a bound on $|e^*|$, I examine the effect of this bounded input on solutions of (3.32)-(3.33). The approach is similar to that when $e^* \equiv 0$. First, I bound $|\dot{u}'|$. Then, I write inequalities for $D^+_{(3.32)} V_1(t,Y)$ and $D^+_{(3.33)} V_2(t,\eta')$. Next, I define some terms needed to state the final result of this chapter. Finally, I summarize the results of this chapter in Theorem 3.11.

As stated, I first find a bound on $|\dot{u}'|$ assuming that $|e^*| \leq \mu\mathfrak{m}_6$. Because $\|w^*\|$ is bounded for all time, there exist constants G_6 and $\bar{G}_1$ such that

$$K_2\|A_{22}^{-1}b_2\|w^{*T}\Gamma w^{*} \leq G_6 \quad , \quad t \geq 0 \quad , \tag{3.108}$$

$$G_1 + \mu 2K_2\|A_{22}^{-1}b_2\| \; \|\Gamma w^{*}\|\mathfrak{M}_6 \leq \bar{G}_1 \quad , \quad t \geq 0 \quad . \tag{3.109}$$

Next, define

$$\bar{G}_2 = G_2 + \mu K_2\|A_{22}^{-1}b_2\| \; \|\Gamma\|\mathfrak{M}_6 \quad . \tag{3.110}$$

Then, I can bound $|\dot{u}'|$ for $t \geq 0$ by

$$|\dot{u}'| \leq \frac{1}{K_2\|A_{22}^{-1}b_2\|} [\bar{G}_1\|Y\| + \bar{G}_2\|Y\|^2 + G_3\|Y\|^3 + G_4\|\eta'\| + G_5\|Y\|\|\eta'\| + \mu G_6\mathfrak{M}_6] \quad . \tag{3.111}$$

Now, I need to write inequalities for $D^{+}_{(3.32)}V1(t,Y)$ and $D^{+}_{(3.33)}V2(t,\eta')$ and define terms I use to state the final theorem. Using (3.60)-(3.62), (3.65)-(3.67), and (3.111) I write

$$D^{+}_{(3.32)}V1(t,Y) \leq -\sigma_2(\beta)V_1(t,Y) + K_3(\beta)V_2(t,\eta') + \mu K_5(\beta)\mathfrak{M}_6 \quad , \tag{3.112}$$

$$D^{+}_{(3.33)}V2(t,\eta') \leq -(\frac{\sigma}{\mu} - G_4 - G_5V_1(t,Y))V_2(t,\eta') + \bar{f}(V_1(t,Y))V_1(t,Y) + \mu G_6\mathfrak{M}_6 \quad , \tag{3.113}$$

where

$$\sigma_2(\beta) = q\alpha_1 - \mu\mathfrak{M}_6\|\Gamma\|K_3(\beta) \quad , \tag{3.114}$$

$$K_5(\beta) = \sup_{t\geq 0} \{L_1(t,\beta)\|\Gamma w^{*}(t)\|\} \quad , \tag{3.115}$$

$$\bar{f}(V_1(t,Y)) = \bar{G}_1 + \bar{G}_2V_1(t,Y) + G_3V_1^2(t,Y) \quad . \tag{3.116}$$

Recall that for the $e^{*} \equiv 0$ case I defined $M(Y,\eta')$ which was used to bound

$\|Y\|$ and estimate the region of attraction. In this case $M(Y,\eta_o')$ will not always be a bound on $\|Y\|$, and therefore, I define M_1 as the solution of

$$M_1 = \frac{\mathfrak{m}_6[K_5(\mu M_1) + \mu \dfrac{K_3(\mu M_1)G_6}{\sigma - \mu[G_4 + \mu G_5 M_1 + q\alpha_1]}]}{\sigma_2(\mu M_1) - \mu \dfrac{K_3(\mu M_1)\bar{f}(\mu M_1)}{\sigma - \mu[G_4 + \mu G_5 M_1 + q\alpha_1]}} \tag{3.117}$$

which satisfies $M_1 \tilde{=} \dfrac{\mathfrak{m}_6 K_4(0)}{\sigma_2(0)}$. Next, I define

$$M_2 = \max\{M(Y_o,\eta_o'),\mu M_1\} \tag{3.118}$$

which I will use to bound $\|Y\|$. The rate of exponential decay will be greater than

$$\alpha_6 = \sigma_2(M_2) - \mu \frac{K_3(M_2)\bar{f}(M_2)}{\sigma - \mu[G_4 + G_5 M_2 + q\alpha_1]} \quad . \tag{3.119}$$

Define $\bar{\mu}_4$ as the solution of

$$0 = \sigma_2(0) - \bar{\mu}_4 \frac{K_3(0)\bar{G}_1}{\sigma - \bar{\mu}_4[G_4 + q\alpha_1]} \tag{3.120}$$

which satisfies $\bar{\mu}_4 \tilde{=} \mu_4$, where μ_4 is defined in the proof of Theorem 3.9. To estimate the region of attraction, define a function $M_2^*:[0,\bar{\mu}_4] \to \mathbb{R}^+$ such that if $M_2 \in [0,M_2^*(\mu))$, then $\alpha_6 > 0$ and if $M_2 = M_2^*(\mu)$, then $\alpha_6 \leq 0$. Next, define

$$M_3 = \frac{\mathfrak{m}_6}{\alpha_6}\,[K_5(M_2) + \mu\,\frac{K_3(M_2)G_6}{\sigma - \mu[G_4 + G_5M_2 + q\alpha_1]}] \quad , \tag{3.121}$$

$$M_4 = \frac{G_6\mathfrak{m}_6}{\sigma - \mu[G_4 + G_5M_2]} + \frac{\bar{f}(\mu M_3)M_3}{\sigma - \mu[G_4 + G_5M_2 + 3q\alpha_1]} \quad . \tag{3.122}$$

Finally, I define three sets which are useful for describing the behavior of solutions of (3.32)-(3.33),

$$\mathcal{D}_1(\mu) = \{Y,\eta' : M(Y,\eta') \le \mu M_3,\ K_2\|\eta'\| \le \mu^2 M_4\} \quad , \tag{3.123}$$

$$\mathcal{D}_2(\mu) = \{Y,\eta' : \|Y\| \le \mu M_3, \|\eta'\| \le \mu^2 M_4\} \quad , \tag{3.124}$$

$$\mathcal{D}_3(\mu) = \{Y,\eta' : M(Y,\eta') < M_2^*(\mu)\} \quad . \tag{3.125}$$

The set $\mathcal{D}_3(\mu)$ will be an estimate of the region of attraction, the set $\mathcal{D}_2(\mu)$ will be a residual set to which all solutions, beginning in $\mathcal{D}_3(\mu)$, converge exponentially, and the set $\mathcal{D}_1(\mu)$ is a region such that solutions beginning in $\mathcal{D}_1(\mu)$ stay in $\mathcal{D}_2(\mu)$ for all $t \ge 0$. Theorem 3.11 shows that, for μ small enough, these statements are true. Theorem 3.11 also provides a more detailed bound on $\|Y\|$ and $\|\eta'\|$.

Theorem 3.11: Let r(t) be the same as in Theorem 3.8. Let $(Y,\eta')(t;Y_o,\eta_o')$ be a solution of (3.32)-(3.33) beginning at (Y_o,η_o') at time $t = 0$. Then, there exists a $\bar{\mu} \in \mathbb{R}^+$ such that for all $\mu \in [0,\bar{\mu})$, if $(Y_o,\eta_o') \in \mathcal{D}_3(\mu)$, then

$$\|Y\| \le M(Y_o,\eta_o')e^{-\alpha_6 t} + \mu M_3(1 - e^{-\alpha_6 t}) \quad , \tag{3.126}$$

$$\|\eta'\| \leq K_2\|\eta_o'\| \exp[-(\frac{\sigma}{\mu} - G_4 - G_5M_2)t]$$
$$+ \mu^2 \frac{G_6\mathfrak{M}_6}{\sigma - \mu[G_4 + G_5M_2 + q\alpha_1]}[1 - \exp[-(\frac{\sigma}{\mu} - G_4 - G_5M_2)t] \qquad (3.127)$$
$$+ \mu \frac{\bar{f}(\bar{M}(Y_o,\eta_o')e^{-\alpha_6 t} + \mu M_3(1 - e^{-\alpha_6 t}))[\bar{M}(Y_o,\eta_o')e^{-\alpha_6 t} + \mu M_3(1 - e^{-\alpha_6 t})]}{\sigma - \mu[G_4 + G_5M_2 + 3q\alpha_1]}$$

for $t \geq 0$.

Proof: The proof takes several steps. First, I define $\bar{\mu}$ so that for all $\mu \in [0, \bar{\mu})$ $\mathcal{D}_1(\mu) \subset \mathcal{D}_3(\mu)$. Next, I assume a bound on $\|Y\|$ exists and show that if $(Y_o, \eta_o') \in \mathcal{D}_3(\mu)$, then M_2 is a bound on $\|Y\|$. Confirmation of M_2 as a bound on $\|Y\|$ leads directly to the bound (3.126). Finally, I use the bound on $\|Y\|$ to get the bound (3.127) on $\|\eta'\|$.

Because $K_3(\beta)$ is a nondecreasing function of β, it follows that $M_2^*(\mu)$ is a decreasing function of μ on $\mu \in [0, \bar{\mu}_4]$. Furthermore, $M_2^*(0) = +\infty$ and $M_2^*(\bar{\mu}_4) = 0$. Clearly, there exists a $\mu_5 \in (0, \bar{\mu}_4)$ such that if $\mu \in [0, \mu_5)$, then $M_2^*(\mu) > \mu M_1$ and $M_2^*(\mu_5) \leq \mu_5 M_1$. Take $\bar{\mu} = \min\{\mu_3, \mu_5\}$. Now, take $\mu \in [0, \bar{\mu})$, and assume that a constant M_5 exists which bounds $\|Y\|$ for $t \geq 0$. Define

$$V_6(t,Y,\eta') = V_1(t,Y,\eta') + \mu \frac{K_3(M_5)}{\sigma - \mu[G_4 + G_5M_5 + q\alpha_1]} V_2(t,\eta') \quad . \qquad (3.128)$$

Using (3.111) and (3.112) I can write

$$
\begin{aligned}
D^{+}_{(3.32)(3.33)}V_6(t,Y,\eta') \le & -(\sigma_2(M_5) - \frac{\mu K_3(M_5)\bar{f}(M_5)}{\sigma - \mu[G_4 + G_5M_5 + q\alpha_1]})V_1(t,Y) \\
& - \frac{\mu K_3(M_5)}{\sigma - \mu[G_4 + G_5M_5 + q\alpha_1]}\, q\alpha_1 V_2(t,\eta') \\
& + \mu\mathfrak{M}_6[K_4(M_5) + \frac{\mu K_3(M_5)G_6}{\sigma - \mu[G_4 + G_5M_2 + q\alpha_1]}] \qquad (3.129) \\
\le & - (\sigma_2(M_5) - \mu\frac{K_3(M_5)\bar{f}(M_5)}{\sigma - \mu[G_4 + G_5M_5 + q\alpha_1]})V_6(t,Y,\eta') \\
& + \mu\mathfrak{M}_6[K_4(M_5) + \frac{\mu K_3(M_5)G_6}{\sigma - \mu[G_4 + G_5M_5 + q\alpha_1]}] \quad .
\end{aligned}
$$

Assume further that $M_5 < M_2^*(\mu)$. Then

$$
\begin{aligned}
V_6(t,Y,\eta') \le & V_6(t,Y_o,\eta_o') \exp[-(\sigma_2(M_5) - \mu\frac{K_3(M_5)\bar{f}(M_5)}{\sigma - \mu[G_4 + G_5M_5 + q\alpha_1]})t] \\
& + \mu\frac{\mathfrak{M}_6[K_4(M_5) + \mu\frac{K_3(M_5)G_6}{\sigma - \mu[G_4 + G_5M_5 + q\ _1]}]}{\sigma_2(M_5) - \mu\frac{K_3(M_5)\bar{f}(M_5)}{\sigma - \mu[G_4 + G_5M_5 + q\alpha_1]}} \quad . \qquad (3.130)
\end{aligned}
$$

In order for M_5 to bound $\|Y\|$ I can choose M_5 to satisfy

$$
M_5 = \max\left\{ K_1(\|Y_o\|)\|Y_o\| + \frac{\mu K_3(M_5)K_2\|\eta_o'\|}{\sigma - \mu[G_4 + G_5M_5 + q\alpha_1]} \ ,\ \frac{\mu\mathfrak{M}_6[K_4(M_5) + \mu\frac{K_3(M_5)G_6}{\sigma - \mu[G_4 + G_5M_5 + q\alpha_1]}]}{\sigma_2(M_5) - \mu\frac{K_3(M_5)\bar{f}(M_5)}{\sigma - \mu[G_4 + G_5M_5 + q\alpha_1]}} \right\} \quad . \qquad (3.131)
$$

If M_5 is equal to the first term on the right-hand side of (3.131), then $M_5 = M(Y_o, \eta_o')$. If M_5 is equal to the second term on the right-hand side of (3.131), then $M_5 = \mu M_1$. Hence, $M_5 = M_2$ and M_2 bounds $\|Y\|$ for $t \geq 0$. For $(Y_o, \eta_o') \in \mathcal{D}_3(\mu)$, then $M_2 < M_2^*(\mu)$ and I can write

$$\|Y\| \leq V_1(t,Y) \leq V_6(t,Y,\eta') \leq M(Y_o,\eta_o')e^{-\alpha_6 t} + \mu M_3(1 - e^{-\alpha_6 t}) \quad (3.132)$$

which provides the bound (3.126) on $\|Y\|$.

Next consider the function

$$V_7(t,Y,\eta') = V_2(t,\eta') - \frac{\mu \bar{f}(V_1(t,Y))V_1(t,Y)}{\sigma - \mu[G_4 + G_5 M_2 + 3q\alpha_1]} \quad . \quad (3.133)$$

Using (3.112) and (3.113), I write

$$D^+_{(3.32)(3.33)} V_7(t,Y,\eta') \leq -(\frac{\sigma}{\mu} - G_4 - G_5 M_2)V_2(t,\eta') + \mu G_6 \mathfrak{M}_6$$

$$+ [\bar{f}(V_1(t,Y)) + \frac{\mu\sigma_2(M_2)(\bar{G}_1 + 2\bar{G}_2 V_1(t,Y) + 3G_3 V_1^2(t,Y))}{\sigma - \mu[G_4 + G_5 M_2 + 3q\alpha_1]}]V_1(t,Y)$$

$$\leq - (\frac{\sigma}{\mu} - G_4 - G_5 M_2)V_2(t,\eta') + \mu G_6 \mathfrak{M}_6 \quad (3.134)$$

$$+ \bar{f}(V_1(t,Y))[1 + \frac{\mu 3\sigma_2(M_2)}{\sigma - \mu[G_4 + G_5 M_2 + 3q\alpha_1]}]V_1(t,Y)$$

$$\leq - (\frac{\sigma}{\mu} - G_4 - G_5 M_2)V_7(t,Y,\eta') + \mu G_6 \mathfrak{M}_6 \quad .$$

From (3.134), it follows that

$$V_7(t,Y,\eta') \le V_7(0,Y_o,\eta_o') \exp[-(\frac{\sigma}{\mu} - G_4 - G_5M_2)t]$$

$$+ \mu^2 \frac{G_6\mathfrak{M}_6}{\sigma - \mu[G_4 + G_5M_2 + 3q\alpha_1]}[1 - \exp[-(\frac{\sigma}{\mu} - G_4 - G_5M_2)t]] \quad . \tag{3.135}$$

Using (3.65), (3.132), (3.133) and (3.135) leads to the bound (3.127) and completes the proof.

In the introduction to this section, I claimed that the effect of the $0(\mu)$ e^* on the region of attraction was very small. To see this, assume that $K_3(\beta) = 0(1)$ for all $\beta \in R^+$, $\|\Gamma\| = 0(1)$, and $\sup_t \|\omega^*(t)\| = 0(1)$. Under these assumptions, $M_2^*(\mu)$ is chosen so that if $M_o \in [0, M_2^*(\mu))$, then

$$0 < q\alpha_1 - \mu M_7(M_o) - \mu \frac{K_3(M_o)}{\sigma - \mu[G_4 + G_5M_o + q\alpha_1]} f(M_o) \tag{3.136}$$

where $M_7 > 0$ is an $0(1)$ valued function of (M_o). Recall that $M_o^*(\mu)$ was chosen so that if $M_o \in [0, M_o^*(\mu))$, then

$$0 < q\alpha_1 - \mu \frac{K_3(M_o)}{\sigma - \mu[G_4 + G_5M_o + q\alpha_1]} f(M_o) . \tag{3.137}$$

Comparing (3.136) and (3.137) we see that as $\mu \to 0$

$$M_2^*(\mu)^2 = (M_o^*(\mu)^2 - 0(\mu)) \tag{3.138}$$

and thus we have $M_2^*(\mu) = M_o^*(\mu)(1 - 0(\mu^{\frac{3}{2}}))$.

<u>Remark 3.3</u>: Notice that Theorem 3.10 does not restrict the number of sinusoids in the reference input. That is, the bound $|e^*| < \mu\mathfrak{M}_6$ does not depend on the number of sinusoids. Using a relaxed version of Theorem 3.8,

as suggested in Remark 3.2, one could relax the hypothesis of Theorem 3.11 to r(t) sufficiently rich and low frequency without changing the conclusions of Theorem 3.11. If we relax the "exactly n" requirement then we must choose $\theta^*(\mu)$ by some criterion other than making the steady state $e^* = 0$. An interesting criterion is $\theta^*(\mu) = \theta^*(0)$. For this choice of θ^*, the previously discussed relaxed version of Theorem 3.11 would be an excellent proof that low frequency sufficient richness, i.e., dominant richness, provides robustness with respect to high frequency unmodelled dynamics. I use robustness here in the sense that in the presence of high frequency unmodelled dynamics, the adaptive control system remains stable and tracking and parameter errors become and remain small, $O(\mu)$.

3.5. Discussion

In this chapter I have shown that trajectories of the model reference adaptive control system excited by n low frequency sinusoids which begin in a region of attraction will converge exponentially to an $O(\mu)$ residual set if μ is sufficiently small. Furthermore, if $r(t) = r^*(t)$, the steady state value of e^* is zero and the equilibrium eventually becomes exponentially stable. It was also shown that if the rate of convergence of the unperturbed MRAS, i.e., when $\mu = 0$, is $O(1)$, then the region of attraction may be $O(\mu^{-\frac{1}{2}})$ in the slow variable and $O(\mu^{-\frac{3}{2}})$ in the fast variables.

Through Remarks 3.2-3.3 I have also detailed a method by which one can prove that robustness with respect to high frequency unmodelled dynamics can be gained via low frequency sufficient richness of the reference input. This method was based upon the idea of fixing the reference input and then showing that for sufficiently small μ, exponential stability is retained. How does one use the ideas in this chapter if μ is fixed and the problem is to choose $r(t)$ and Γ to provide robustness with respect to the perturbation? The first step is that α_3 must be positive. From (3.75) and (3.77) we see that at the very least one needs $G_1 < O(\mu^{-1})$. This leads for one condition to $w^{*T}\Gamma w^* < O(\mu^{-1})$. Another requirement might be that $\mathfrak{m}_6 \le O(\mu^{-\frac{1}{2}})$. This leads to $|\dot{r}| < O(\mu^{-\frac{1}{2}})$ and $|r| < O(\mu^{-\frac{1}{2}})$. We begin to see that keeping $\|\Gamma\|$, $|\dot{r}|$, and $|r|$ all small with respect to $\mu^{-\frac{1}{2}}$ will probably accomplish the desired goals.

Before going on to examples in Chapter 4, I would like to discuss the relationship of this work to that of other authors.

Remark 3.4: While Remark 2.3 suggested that dominant richness was important for the existence of a unique equilibrium, it is easily seen that the stability of this unique equilibrium requires a stronger dominant richness condition. For a given $\mu > 0$, each w_i in the input must be less than $O(\mu^{-1})$ to assure existence. As just discussed, stability can be assured only when the existence condition is met and when each product, $r_i w_i$, is less than $O(\mu^{-\frac{1}{2}})$. This more restrictive condition corresponds exactly with the notion of dominant richness as discussed by Ioannou and Kokotovic (1983) and Krause (1983).

Remark 3.5: Krause (1983) has also found that when the reference input is dominantly rich and μ is sufficiently small, the convergence of the trajectories of the model reference adaptive control system toward $e = 0$, $\theta = \theta^*(0)$ for O(1) parameter errors can be assured by choosing $\|\Gamma\|$ sufficiently small. The results of this thesis agree in the following way. If the assumptions of dominant richness and μ sufficiently small guarantee that $\alpha_6 > 0$ when $\Gamma = 0$, then there must exist Γ with $\|\Gamma\|$ small enough so that $\alpha_6 > 0$ with $\|\Gamma\| > 0$. Since $\alpha_6 > 0$ is sufficient for the convergence of trajectories in $D_3(\mu)$ to $D_2(\mu)$, the trajectories will then behave as previously described.

Remark 3.6: While it is easy to see that the exponential stability of this system with $e^* \equiv 0$ can be used with standard stability results to show robustness with respect to additive disturbances, it is not clear what happens if the reference input is the sum of a dominantly rich input and a high frequency input. Astrom (1983) showed via averaging techniques that for small $\|\Gamma\|$, the low frequency excitation provided robustness with respect to the high frequency excitation. This can be partially seen by realizing that e^* is the difference of the outputs of two low pass filters, and hence, $|e^*|$ will be very small for components of the reference input which are high frequency. Thus the forcing effect of high frequency inputs for slowly varying $\theta(t)$ will be small. However, the techniques of this thesis may not be able to prove this because of the $\dot{w}^*$ coupling term between fast and slow subsystems.

Remark 3.7: As a final comment on this chapter, I must say that the results of this chapter could be interpreted as a special case of the work done by Kosut, Johnson, and Anderson (1983). However, I believe that much can be gained by dealing with the special case I have chosen and by using the Lyapunov approach, which is not the approach of Kosut, et al. First, the special case of high frequency unmodelled dynamics which I have chosen is in itself a fairly general and very important topic of current research in adaptive control. Second, the representation I was able to choose for the unmodeled dynamics allowed for a natural decomposition of the problem which in turn provided the starting point for the composite Lyapunov function proof of the results. Finally, the results help to point out that high frequency inputs and high adaptation gain are the main problems to watch out for when applying model reference adaptive control to a plant with unmodelled high frequency dynamics.

CHAPTER 4

EXAMPLES

In Chapter 3, I derived conditions for stability of a model reference adaptive system with unmodeled high frequency dynamics. In this chapter, I present a sample system of this type and apply the results of Chapter 3 to derive conditions for stability of this sample system with two different sets of parameters. While these two special cases will not illustrate every aspect of the theory presented in this thesis, they provide sufficient insight to appreciate the method suggested by the theory and to discover the main drawbacks of the suggested approach.

This chapter is divided into several sections. In the first section I present the sample system and derive the associated error system following the ideas of Section 3.2. The second section is devoted to a study of the sample system with the parameters chosen so that it becomes a linear time invariant system. In the third section, the parameters are chosen so that the system becomes the simplest nonlinear system possible and the study is repeated for the new case. In the final section I discuss the problems illustrated by the two special cases and some aspects of the theory which are not illustrated by the two special cases.

4.1. The Sample System

In this section, I present a somewhat general model reference adaptive system with high frequency unmodeled dynamics. Then, following Section 3.2, I derive the error system representation of the sample system.

This error system is used with two different sets of parameters to create the examples used in Sections 4.2 and 4.3.

A block diagram of the sample system is shown in Fig. 4.1. The differential equations describing it are

model: $$\dot{x}_m = -x_m + r, \tag{4.1}$$

plant:
$$\dot{x} = ax + b\begin{bmatrix}\sqrt{\frac{5}{2}} & 0\end{bmatrix} z, \tag{4.2a}$$

$$\mu\dot{z} = \begin{bmatrix}-1 & .5 \\ -.5 & -1\end{bmatrix} z + \begin{bmatrix}0 \\ \sqrt{\frac{5}{2}}\end{bmatrix} u , \tag{4.2b}$$

control: $$u = \theta^T w, \qquad w \triangleq [r \quad x]^T, \tag{4.3}$$

parameter update: $$\dot{\theta} = -\begin{bmatrix}\gamma_1 & 0 \\ 0 & \gamma_2\end{bmatrix} w(x-x_m). \tag{4.4}$$

The first step in the derivation of the error system is to make the transformation

$$\eta = z - \sqrt{\frac{2}{5}}\begin{bmatrix}1 \\ 2\end{bmatrix} u. \tag{4.5}$$

The plant equations (4.2) become

$$\dot{x} = ax + bu + b\begin{bmatrix}\sqrt{\frac{5}{2}} & 0\end{bmatrix} \eta, \tag{4.6}$$

$$\mu\dot{\eta} = \begin{bmatrix}-1 & .5 \\ -.5 & -1\end{bmatrix} \eta - \mu\sqrt{\frac{2}{5}}\begin{bmatrix}1 \\ 2\end{bmatrix} \dot{u}. \tag{4.7}$$

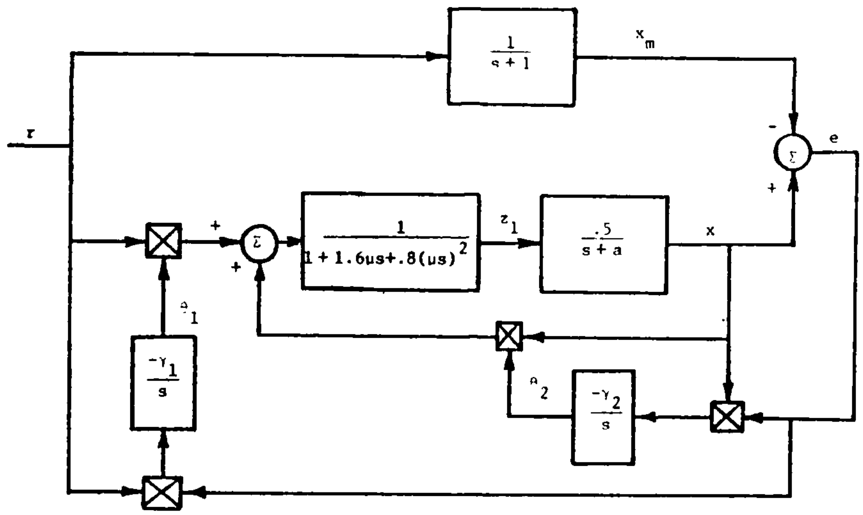

Figure 4.1. Block diagram of sample system, (4.1)-(4.4).

Next, define the tuned system

$$\dot{x}^* = (a + b\theta_2^*)x^* + b\theta_1^* r + b\begin{bmatrix}\sqrt{\frac{5}{2}} & 0\end{bmatrix}\eta^*, \tag{4.8a}$$

$$\mu\dot{\eta}^* = \begin{bmatrix} -1 & .5 \\ -.5 & -1 \end{bmatrix}\eta^* - \mu\sqrt{\frac{2}{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\dot{u}^*, \tag{4.8b}$$

where

$$u^* = \theta^{*T} w^*, \qquad w^* = [r \quad x^*]^T. \tag{4.9}$$

The tuned error is then defined as

$$e^* = x^* - x_m. \tag{4.10}$$

Figure 4.2 shows a block diagram of the tuned system and the tuned error. Finally, making the definitions

$$\begin{aligned} x' &= x - x^*, \\ \theta' &= \theta - \theta^*, \\ \eta' &= \eta - \eta^*, \end{aligned} \tag{4.11}$$

I write the error system equations

$$\dot{x}' = ax' + bu' + b\begin{bmatrix}\sqrt{\frac{5}{2}} & 0\end{bmatrix}\eta', \tag{4.12}$$

$$\mu\dot{\eta}' = -\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} w(x' + e^*), \tag{4.13}$$

$$\mu\dot{\eta}' = \begin{bmatrix} -1 & .5 \\ -.5 & -1 \end{bmatrix}\eta - \mu\sqrt{\frac{2}{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\dot{u}', \tag{4.14}$$

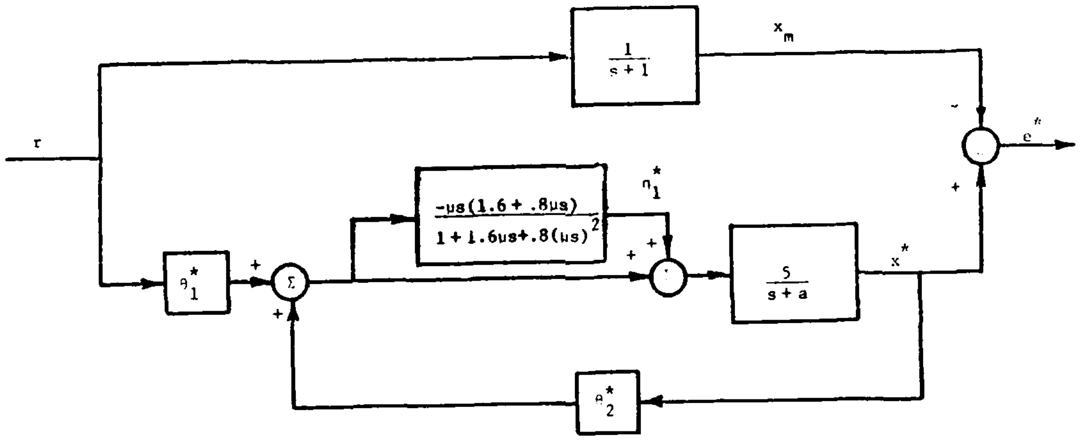

Figure 4.2. Block diagram of the tuned system and tuned error for the sample system.

where

$$u' = w^T\theta' + \theta_2^* x', \tag{4.15}$$

$$\begin{aligned}
\dot{u}' &= w^T\dot{\theta}' + \dot{w}^{*T}\theta' + \theta_2'\dot{x}' + \theta_2^*\dot{x}' \\
&= [x' \;\; \theta_1' \;\; \theta_2'] \begin{bmatrix} \theta_2^*(a+b\theta_2^*) - (\gamma_1 r^2 + \gamma_2 x^{*2} + 2\gamma_2 x^* e^*) \\ br + \dot{r} \\ bx^* + \dot{x}^* \end{bmatrix} \\
&\quad + [x' \;\; \theta_1' \;\; \theta_2'] \begin{bmatrix} -(2\gamma_2 x^* + \gamma_2 e^*) & 0 & \frac{1}{2}(a + 2b\theta_2^*) \\ 0 & 0 & \frac{1}{2} br \\ \frac{1}{2}(a + 2b\theta_2^*) & \frac{1}{2} br & bx^* \end{bmatrix} \begin{bmatrix} x' \\ \theta_1' \\ \theta_2' \end{bmatrix} \\
&\quad + (b\theta_2'^2 - \gamma_2 x'^2)x' + \theta_2^* b \left[\sqrt{\tfrac{5}{2}} \;\; 0 \right] \eta' + \theta_2' b \left[\sqrt{\tfrac{5}{2}} \;\; 0 \right] \eta' \\
&\quad - (\gamma_1 r^2 + \gamma_2 x^{*2}) e^* .
\end{aligned} \tag{4.16}$$

4.2. Example One

The first special case I want to examine has the following set of parameters:

$$\begin{aligned}
a &= -1 \quad, \\
\gamma_2 &= \theta_2^* = \theta_2' = 0, \\
b &= \frac{1}{2} \quad, \\
\theta_1^* &= 2 \quad, \\
\gamma_1 &= 1 \quad.
\end{aligned} \tag{4.17}$$

This corresponds to a plant whose "slow" pole matches the model pole but which has an unknown dc gain of $\frac{1}{2}$. The choice $\theta_1^* = 2$ provides the correct dc gain. The choice of the adaptive gain $\gamma_1 = 1$ is arbitrary and, hopefully, reasonable for this problem.

One of the first things to notice about this system is that $u^* = 2$ is a constant. Therefore $\dot{u}^* = 0$ and hence it follows from (4.1), (4.8), and (4.10) that the choice $x^*(0) = x_m(0)$, $\eta^*(0) = 0$ results in $e^* \equiv 0$. Using this and (4.17) I write $\dot{u}'$ for this case,

$$\dot{u}' = [x' \quad \theta_1'] \begin{bmatrix} -1 \\ .5 \end{bmatrix}. \tag{4.18}$$

Rewriting the error system using the appropriate modifications for this special case and $Y = [x' \quad \theta_1']^T$ results in

$$\dot{Y} = \begin{bmatrix} -1 & .5 \\ -1 & 0 \end{bmatrix} Y + \begin{bmatrix} .5\sqrt{\frac{5}{2}} & 0 \\ 0 & 0 \end{bmatrix} \eta', \tag{4.19a}$$

$$\mu\dot{\eta}' = \mu\sqrt{\frac{2}{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} [1 \quad -.5]Y + \begin{bmatrix} -1 & .5 \\ -.5 & -1 \end{bmatrix} \eta'. \tag{4.19b}$$

Now I want to apply the method developed in Theorem 3.9 to determine a $\hat{\mu}$ such that (4.19) is exponentially stable for all $\mu \in [0, \hat{\mu})$. The first step is to consider the subsystems

$$\dot{Y} = \begin{bmatrix} -1 & .5 \\ -1 & 0 \end{bmatrix} Y \tag{4.20}$$

and

$$\mu\dot{\eta}' = \begin{bmatrix} -1 & .5 \\ -.5 & -1 \end{bmatrix} \eta'. \tag{4.21}$$

Solutions of (4.20) satisfy

$$\|Y\| \leq 2.44e^{-.5t}\|Y(0)\|. \tag{4.22}$$

Hence, there exists $V_1(t,Y)$ such that

$$\begin{aligned} &\|Y\| \leq V_1(t,Y) \leq 2.44\|Y\|, \\ &|V_1(t,Y^1) - V_1(t,Y^2)| \leq 2.44\|Y^1 - Y^2\|, \\ &D^+_{(4.20)}V_1(t,Y) \leq -.5V_1(t,Y). \end{aligned} \tag{4.23}$$

In fact, for this case, one such function is

$$V_1(Y) = \left(\frac{4}{3-\sqrt{5}} Y^T \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} Y \right)^{\frac{1}{2}} \tag{4.24}$$

and $D^+V_1(t,Y) = \frac{d}{dt} V_1(Y)$. Solutions of (4.21) satisfy

$$\|\eta'\| = e^{-\frac{1}{\mu}t} \|\eta'(0)\| \tag{4.25}$$

and we use

$$V_2(\eta') = \|\eta'\|. \tag{4.26}$$

$V_2(\eta')$, of course, satisfies a set of conditions similar to (4.23).

Now I can write

$$D^+_{(4.19a)}V_1(Y) \leq -.5V_1(Y) + (2.44)(.5)(\sqrt{\tfrac{5}{2}})V_2(\eta'), \tag{4.27}$$

$$D^{+}_{(4.19b)}V_2(\eta') \le -\frac{1}{\mu} V_2(\eta') + (\sqrt{\tfrac{2}{5}})(\sqrt{5})(\sqrt{\tfrac{5}{4}})V_1(Y). \tag{4.28}$$

Then using

$$V_3(Y,\eta') = V_1(Y) + \frac{\mu(2.44)(.5)(\sqrt{\tfrac{5}{2}})}{1-\mu(.5)} V_2(\eta') \tag{4.29}$$

I get

$$\begin{aligned} D^{+}_{(4.19)}V_3(Y,\eta') &\le -(.5) \quad V_1(Y) + \frac{\mu(2.44)(.5)(\sqrt{\tfrac{5}{2}})}{1-\mu(.5)} V_2(\eta') \\ &\quad + \mu \frac{(2.44)(.5)(\sqrt{\tfrac{5}{2}})(\sqrt{\tfrac{2}{5}})(\sqrt{5})(\sqrt{\tfrac{5}{4}})}{1-\mu(.5)} V_2(Y) \\ &\le (-.5 + \frac{\mu(1.22)(2.5)}{1-\mu(.5)})V_3(Y,\eta'). \end{aligned} \tag{4.30}$$

Hence, if $\mu \in [0, .15]$, (4.30) guarantees that (4.19) is exponentially stable. Using linear time-invariant system techniques one can show that (4.19) is exponentially stable for $\mu \in [0, 1.2)$. Thus we see that the estimate of range of stable μ's given by the method of Theorem 3.9 is conservative.

4.3. Example Two

Since adaptive control is in general a nonlinear problem I consider a nonlinear example in this section in order to illustrate more fully the ideas of Chapter 3. I will, however, be modest and choose the simplest possible nonlinear problem. The parameters for this simple example are

$$
\begin{aligned}
r &= 1 \quad , \\
a &= 1 \quad , \\
\gamma_2 &= 1 \quad , \\
\theta_2^* &= -4 \quad , \\
b &= \frac{1}{2} \quad , \\
\theta_1^* &= 2 \quad , \\
\gamma_1 &= \theta_1' = 0 \quad .
\end{aligned}
\tag{4.31}
$$

This corresponds to a system for which the high frequency gain of the "slow" part of the plant is known to be $\frac{1}{2}$ but the desired value is 1, and hence I set $\theta_1^* = 2$. Unlike the previous example this plant is unstable without control since $a > 0$. The choice of $\theta_2^* = -4$ provides the correct d.c. gain and the steady state value of e^* is again zero. The choice of $\gamma_2 = 1$ is again arbitrary but, hopefully, reasonable.

This example possesses three difficulties which were missing in the first example. The first difficulty is that because of the feedback θ_2^*, it is possible to make $e^* \equiv 0$ only when $x_m(0) = 1$. The second potential difficulty is that the unperturbed system is nonlinear and determining the constants associated with its exponential decay is not an easy analytic task. The third difficulty is that the region of attraction of the exponentially stable zero solution of the perturbed system is not the whole space as it was for the linear example.

I will discuss the problems associated with the nonzero e^* first since they are the most easily handled of the three difficulties. Noting that for this case $u^* = -4\, x^*$, I can write the tuned system as

$$\dot{x}^* = -x^* + 1 + [.5\sqrt{\tfrac{5}{2}} \quad 0]\eta^*$$

$$\mu\dot{\eta}^* = \begin{bmatrix} -1 & .5 \\ -.5 & -1 \end{bmatrix}\eta^* + \mu 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}[1 \quad 0]\eta^* - \mu 4\sqrt{\tfrac{2}{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}(x^* - 1) \quad . \tag{4.32}$$

Using $e^* = x^* - x_m$, I can write the equations describing e^*,

$$\dot{e}^* = -e^* + [.5\sqrt{\tfrac{5}{2}} \quad 0]\eta^* \tag{4.33a}$$

$$\mu\dot{\eta}^* = \begin{bmatrix} -1 & .5 \\ -.5 & -1 \end{bmatrix}\eta^* + \mu 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}[1 \quad 0]\eta^* - \mu 4\sqrt{\tfrac{2}{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}(x_m - 1) \quad . \tag{4.33b}$$

Choosing $x^*(0) = x_m(0)$ and $\eta^*(0) = 0$, as usual, and using

$$V_4(e^*,\eta^*) = |e^*| + \frac{\mu(.5)(\sqrt{\frac{5}{2}})}{1-\mu(1+2\sqrt{5})}\|\eta^*\| \quad , \tag{4.34}$$

I get

$$D^+_{(4.33)}V_4(e^*,\eta^*) \le -(|e^*| + \frac{\mu(.5)(\sqrt{\frac{5}{2}})}{1-\mu(1+2\sqrt{5})}\|\eta^*\|) + \frac{\mu(.5)(\sqrt{\frac{5}{2}})(4)(\sqrt{\frac{2}{5}})(\sqrt{5})}{1-\mu(1+2\sqrt{5})}|e^*|$$

$$+ \frac{\mu(.5)(\sqrt{\frac{5}{2}})(4)(\sqrt{\frac{2}{5}})(\sqrt{5})}{1-\mu(1+2\sqrt{5})}|x_m - 1| \tag{4.35}$$

$$\le (-1 + \frac{\mu 2\sqrt{5}}{1-\mu(1+2\sqrt{5})})V_4(e^*,\eta^*) + \frac{\mu 2\sqrt{5}}{1-\mu(1+2\sqrt{5})}|x_m - 1| \quad .$$

Since $V_4(e^*,\eta^*) = 0$ at $t=0$ it follows from (4.1), (4.34), (4.35) that

$$|e^*| \le V_4(e^*,\eta^*) \le \frac{\mu 2\sqrt{5}}{1-\mu(1+4\sqrt{5})}\,|x_m(0) - 1|e^{-t} \tag{4.36}$$

for all $\mu \in [0, \frac{1}{1-4\sqrt{5}})$. Clearly, if μ is sufficiently small or $|x_m(0) - 1|$ is sufficiently small e^* will be negligible.

In order to discuss the remaining aspects of this example I need the error system for this special case. Letting $Y = [x', e_2']^T$, the error system can be written

$$\dot{Y} = \begin{bmatrix} -1 & .5x^* \\ -x^* & 0 \end{bmatrix} Y + \begin{bmatrix} .5Y_1Y_2 \\ -Y_1^2 \end{bmatrix} + \begin{bmatrix} .5\sqrt{\frac{5}{2}} & 0 \\ 0 & 0 \end{bmatrix} \eta' , \tag{4.37a}$$

$$\mu\dot{\eta}' = \begin{bmatrix} -1 & .5 \\ -.5 & -1 \end{bmatrix} \eta' - \mu\sqrt{\frac{2}{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \dot{u}' , \tag{4.37b}$$

$$\begin{aligned} \dot{u}' = {} & Y^T \begin{bmatrix} 4 - x^{*2} - 2x^*e^* \\ .5x^* + \dot{x}^* \end{bmatrix} + Y^T \begin{bmatrix} -2x^* + e^* & -\frac{3}{2} \\ -\frac{3}{2} & .5x^* \end{bmatrix} Y \\ & + (.5Y_2^2 - Y_1^2)Y_1 - [2\sqrt{\tfrac{5}{2}} \quad 0]\eta' + Y_2[.5\sqrt{\tfrac{5}{2}} \quad 0]\eta' \\ & - (Y_1 r^2 + {}_2x^{*2})e^* . \end{aligned} \tag{4.38}$$

In order to simplify the problem, first consider the case when $x_m(0) = 1$. Then, $e^* \equiv 0$ and $x^* \equiv 1$. Under these conditions I can bound $|\dot{u}'|$ by

$$|\dot{u}'| \le \frac{1}{\sqrt{2}} [4.3\|Y\| + 3.8\|Y\|^2 + \sqrt{2}\|Y\|^3 + 2\sqrt{5}\|\eta'\| + .5\sqrt{5}\|Y\|\|\eta'\|] . \tag{4.39}$$

Now, I consider the fast and slow subsystems separately in order to find a Lyapunov function for each. Since the fast subsystem is again given by (4.21), I will use $V_2(\eta')$ for this case also. The slow subsystem is given by

$$\dot{Y} = \begin{bmatrix} -1 & .5 \\ -1 & 0 \end{bmatrix} Y + \begin{bmatrix} .5Y_1Y_2 \\ -Y_1^2 \end{bmatrix} \qquad . \tag{4.40}$$

The zero solution of this nonlinear slow subsystem is exponentially stable in the large. However, it is not easy to analytically determine the constants associated with the exponential convergence of (4.40). Therefore, I resorted to numerical techniques. Since I wanted $(x,\theta_2) = (1,0) \Leftrightarrow (x',\theta_2') = (0,4)$ to be in the region of attraction of the zero solution of (4.37), I choose to estimate the exponential rate of convergence of (4.40) for $\|Y_o\| \leq 5$. First, I simulated (4.40) with initial conditions at 16 uniformly spaced points on a circle of radius 5 centered at Y = 0. Second, I plotted the maximum value of the $\|Y\|$ over the 16 trajectories at each sample instant. The integrations were done using the IMSL subroutine DVERK which determines its own step size to remain within a prescribed tolerance in integration error. For plotting purposes I sampled the integration 10 times per second. Finally, I used the plot of the maximum value of $\|Y\|$ versus time to estimate that

$$\|Y\| \leq 1.82\, e^{-0.077t} \|Y(0)\| \quad , \quad \forall\, Y(0) \in S_5 \quad , \tag{4.41}$$

where $S_5 = \{Y: \|Y\| \leq 5\}$. Figure 4.3 shows the plot of the maximum $\|Y\|$ and the bound (4.41). Figure 4.4 shows $\|Y\|$ vs t and Y_1 vs Y_2 for an example trajectory with initial condition $Y(0) = [0,5]^T$.

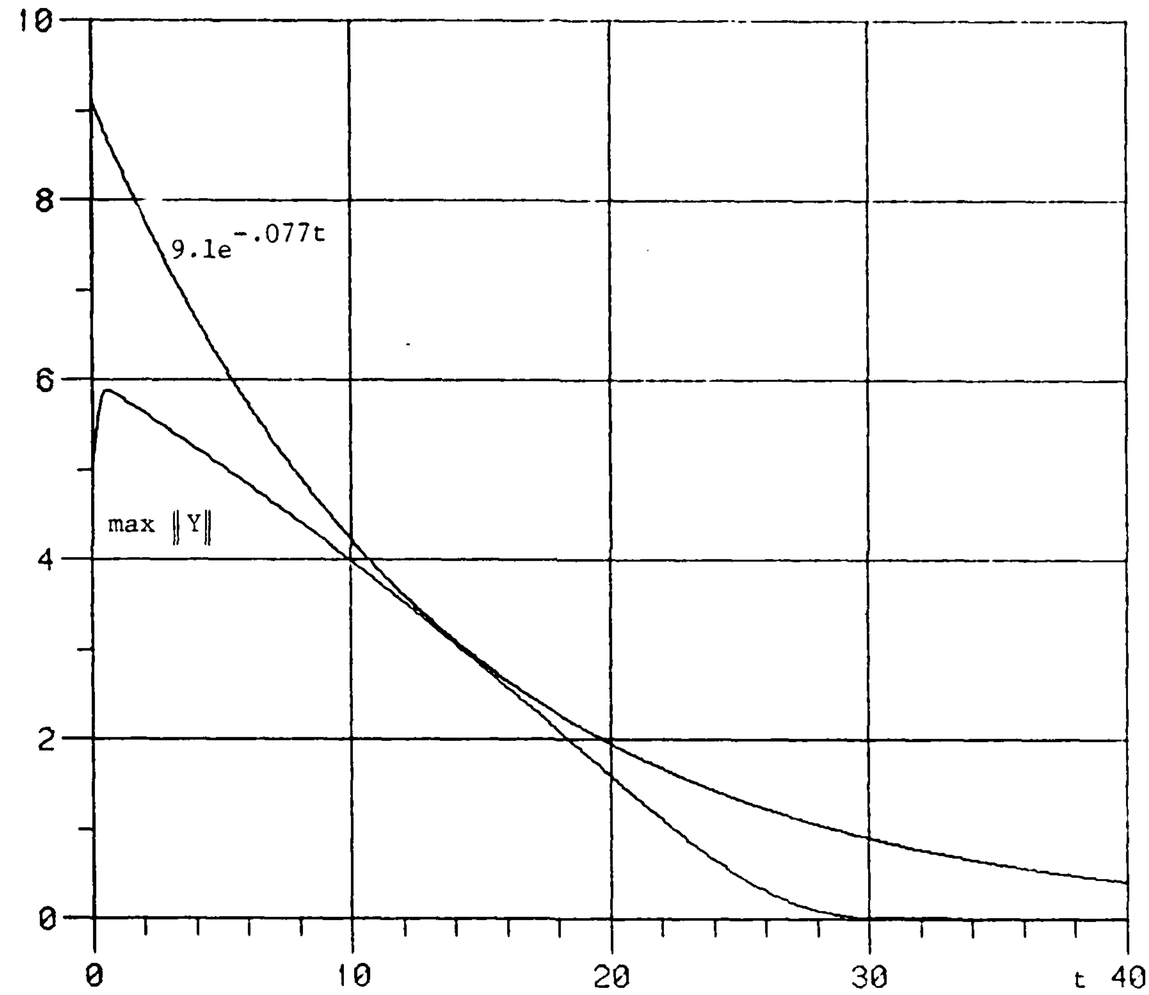

Figure 4.3. Plot of max $\|Y\|$ for (4.40) and the bound (4.41).

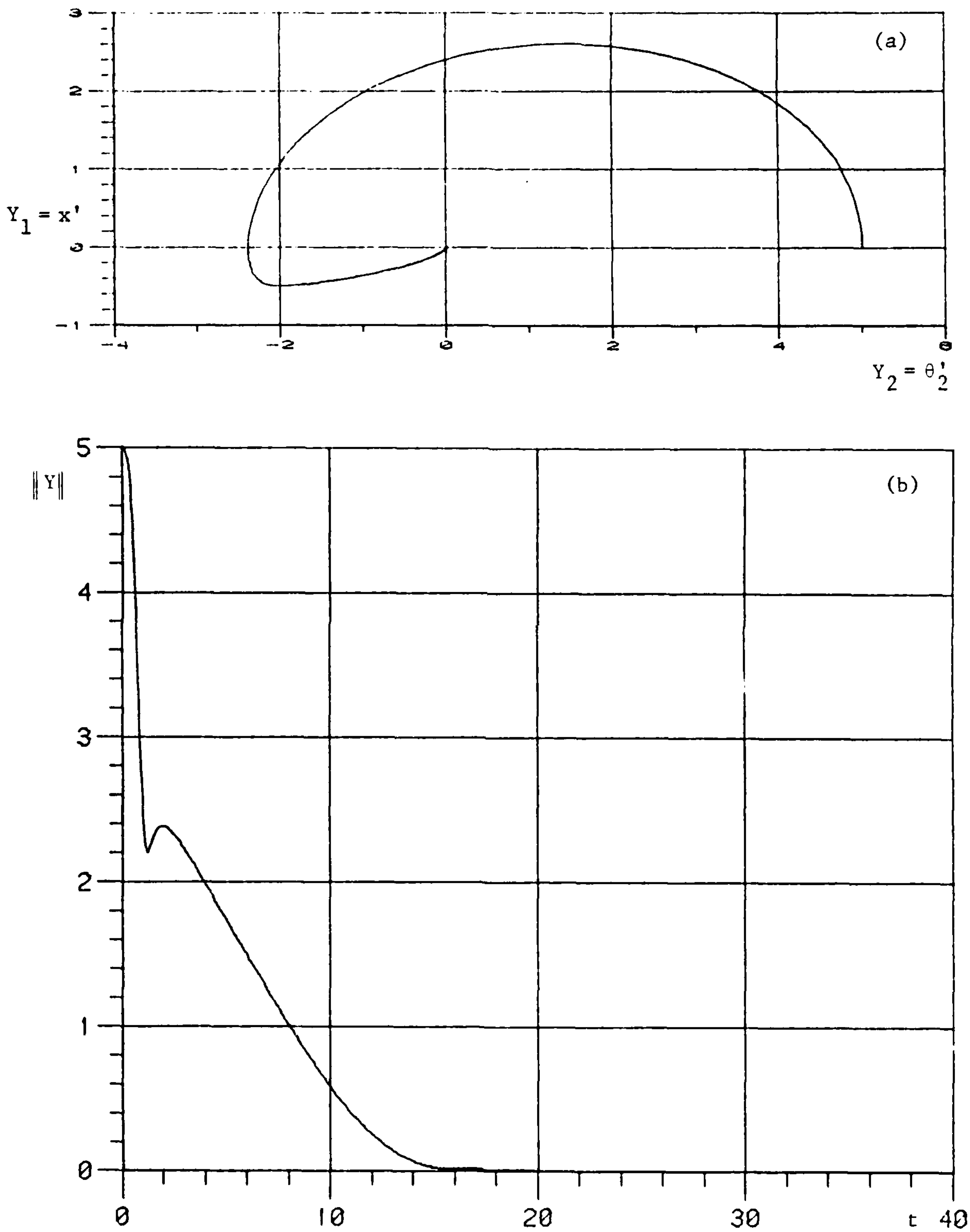

Figure 4.4. (a) Plot of Y_1 vs Y_2 and (b) plot of $\|Y\|$ vs t for sample trajectory of (4.40) beginning at $Y = [0,5]^T$.

From (4.41) and Theorem 3.7, it follows that there exists $V_5(t,Y)$ such that

$$\|Y\| \le V_5(t,Y) \le 1.82\,\|Y\| \quad , \quad \forall Y \in S_5 \quad ,$$

$$|V_5(t,Y^1) - V_5(t,Y^2)| \le L(t,5)\|Y^1 - Y^2\| \quad , \quad Y_1 \in S_5, Y_2 \in S_5 \quad , \tag{4.42}$$

$$D^+_{(4.40)}V_5(t,Y) \le -q(.077)V_5(t,Y) \quad , \quad 0 < q < 1 \quad .$$

For the purposes of this discussion it will suffice to make the unjustified (and optimistic) assumptions that

$$q = 1 \quad , \quad L(t,5) \le 1.82 \quad . \tag{4.43}$$

Using (4.39), (4.42), and (4.43) I write

$$D^+_{(3.37a)e^* \equiv 0}V5(t,Y) \le -\ .077\ V_5(t,Y) + 1.82\ (.5)(\sqrt{\tfrac{5}{2}})V_2(n'), \tag{4.44}$$

$$\begin{aligned} D^+_{(3.37b)e^* \equiv 0}V2(n') \le & \left(-\frac{1}{\mu} + 2\sqrt{5} + (.5)\sqrt{5}\,\|Y\|\right)V_2(n') \\ & + (4.3 + 3.8\,\|Y\| + \sqrt{2}\,\|Y\|^2)V_5(t,Y) \quad . \end{aligned} \tag{4.45}$$

Letting M_o be an upper bound on $\|Y\|$, I define

$$\lambda_6 = .077 - \mu\,\frac{(.91)\sqrt{\frac{5}{2}}\,(4.3 + 3.8\,M_o + \sqrt{2}\,M_o^2)}{1 - \mu(2\sqrt{5} + .077 + .5\sqrt{5}\,M_o)} \quad . \tag{4.46}$$

It is easily shown that if M_o is an upper bound on $\|Y\|$, then

$$V_6(t,Y,n') = V_5(t,Y) + \mu\,\frac{(.91)\sqrt{\frac{5}{2}}}{1 - \mu(2\sqrt{5} + .077 + .5\sqrt{5}\,M_o)}\,V_2(n') \tag{4.47}$$

satisfies

$$D^{+}_{(4.37)e^* \equiv 0} V_6(t,Y,\eta') \leq - \alpha_6 V_6(t,Y,\eta') \quad . \tag{4.48}$$

Hence, if $\alpha_6 > 0$ and M_o is an upper bound on $V_6(0,Y(0),\eta'(0))$, then M_o is an upper bound on $\|Y\|$ since $\|Y\| \leq V_5(t,Y) \leq V_6(t,Y,\eta') \leq V_6(0,Y(0),\eta'(0))$. This is the motivating force behind the definition of $M(Y,\eta')$ as the solution of

$$M(Y,\eta') = 1.82 \|Y\| + \frac{\mu(.91)\sqrt{\frac{5}{2}} \|\eta'\|}{1 - \mu(2\sqrt{5} + .077 + .5\sqrt{5}\, M(Y,\eta')} \quad . \tag{4.49}$$

Then taking $M_o = M(Y(0),\eta'(0))$ ensures that M_o is an upper bound on $V_6(0,Y(0),\eta'(0))$. If I want to guarantee by this method that the set $(Y,\eta') \in \{Y,\eta' : Y \in S_5, \eta' = 0\}$ *is in the region of attraction of the zero* solution of (4.37), I must show $\alpha_6 > 0$ for $M_o = 9.1$. This can be done only for $\mu \in [0,.00034]$.

Returning to the more general problem when $x_m(0) \neq 1$ but for example satisfies $|x_m(0) - 1| \leq 2$, one expects that the requirements on μ will be at least as restrictive as in the $x_m(0) = 1$ case. Thus, start with $\mu \leq .00034$. Clearly $|e^*|$ is then negligible with respect to the other terms in (4.38) so that the bound (4.39) is good with respect to e^* at least. Because $|e^*|$ is negligible

$$x^* \doteq x_m = 1 + (x_m(0) - 1)e^{-t} \quad . \tag{4.50}$$

In order to truly determine the effects of this time varying x^* one may need to find a new bound similar to (4.41) for the slow subsystem

$$\dot{Y} = \begin{bmatrix} -1 & .5x^* \\ -x^* & 0 \end{bmatrix} Y + \begin{bmatrix} .5Y_1Y_2 \\ -Y_1^2 \end{bmatrix} \tag{4.51}$$

with x^* given by (4.50). However, I know that the zero solution of (4.51) is exponentially stable in the large, provided $x^* \neq 0$. Furthermore, $x^* \doteq 1$ for $t > 3$. Thus, for $t > 3$ the previous analysis should hold with $M_o = M(Y(3), \eta'(3))$. With these thoughts in mind one might consider a pseudo worst case analysis in which one assumes that $V_5(t,Y)$ satisfies only $D^+_{(4.57)}V_5(t,Y) \leq 0$ for $0 \leq t \leq 3$. Under this assumption and the assumption that (4.39) is a reasonable bound over $0 \leq t \leq 3$, one gets $\alpha_6 > -.077$ for $\mu \in [0,.00034]$ and $M_o = 9.1$. From this it follows that

$$V(3,Y,\eta') \leq e^{(.077)(3)}V(0,Y,\eta') \quad .$$

Since $M_o = 9.1$ must bound $V(3,Y,\eta')$, this psuedo worst case analysis reveals that the guaranteed region of attraction is reduced by a factor of $e^{-.231} = 0.8$.

How useful is it to know that for $\mu \leq .00034$ I can guarantee that my desired region is contained in the region of attraction? This requires the unmodelled high frequency dynamics to be 3,000 times as fast as the modelled dynamics. This seems like a pretty unrealistic requirement. However, to be accurate, the answer to the first question depends on the answer to: "How big can μ really be before the region of attraction does not contain my desired region?" Working with the desired region $(Y,\eta) \in \{Y,\eta : Y \in S_5, \eta = 0\}$, I tested via simulation whether $\mu = .1$ was small enough so that the desired region was contained in the region of attraction of the zero solution for the $x_m(0) = 1$ case. I found that it was small enough. For this test I integrated (4.37) with $x^* \equiv 1$ and $e^* \equiv 0$ using the same 16 initial conditions on Y as when I studied (4.40) and the initial condition $\eta'(0) = 0$. Then, I again took the maximum of $\|Y\|$ at each sample instant. Figure 4.5 shows the plot of max $\|Y\|$ vs t for both the unperturbed system (4.40) and the perturbed system (4.37). Figure 4.6 shows a sample trajectory of the perturbed system with initial conditions $Y(0) = [0,5]^T$ and $\eta(0) = 0$ and with $\mu = 0.1$. Compaing 0.1 and 0.00034 it is obvious that the estimate provided by the proof of Theorem 3.9 is so conservative that it is of little or no practical use.

4.4. Discussion

I have investigated two simple examples which have illustrated the theory of Chapter 3. That is, for each example there was a $\hat{\mu} > 0$ such that for $\mu \in [0,\hat{\mu})$ the zero solution of the error system with $e^* \equiv 0$ retained its exponential stability. Furthermore, it was shown that $\hat{\mu}$ does not have to be an incredibly small number. However, the examples also showed that, even for these simple systems, the estimates of $\hat{\mu}$ and the region of attraction developed from the techniques of the proofs of the theorems in Chapter 3 are so conservative that they are of little practical use. This conservation arises from the fact that the technique assumes the terms coupling the fast and slow subsystems together always have the maximum detrimental effect upon the stability of the error system while the coupling terms actually have a much smaller effect on the stability of the error system.

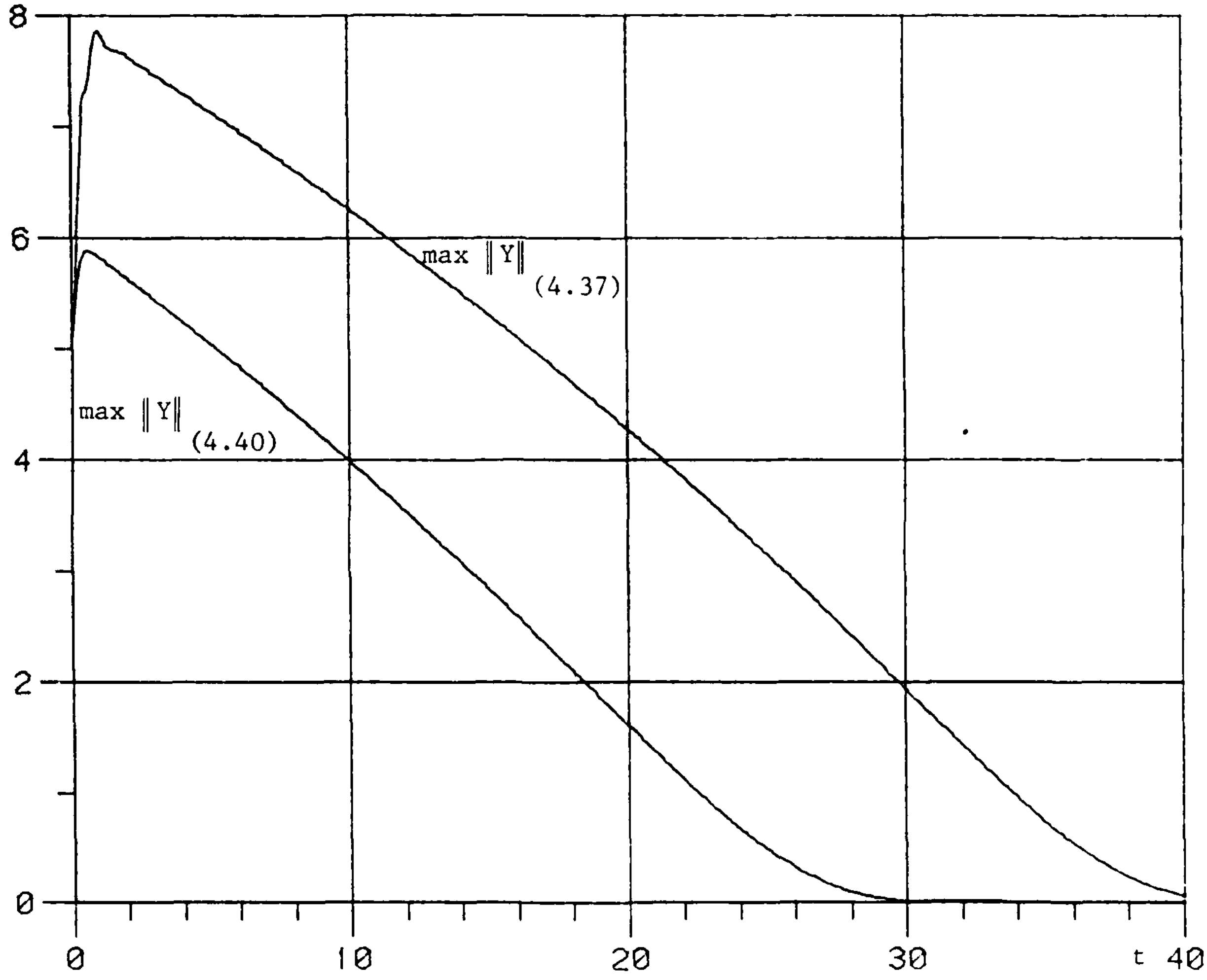

Figure 4.5. Plots of max $\|Y\|$ vs t for (4.37) and (4.40).

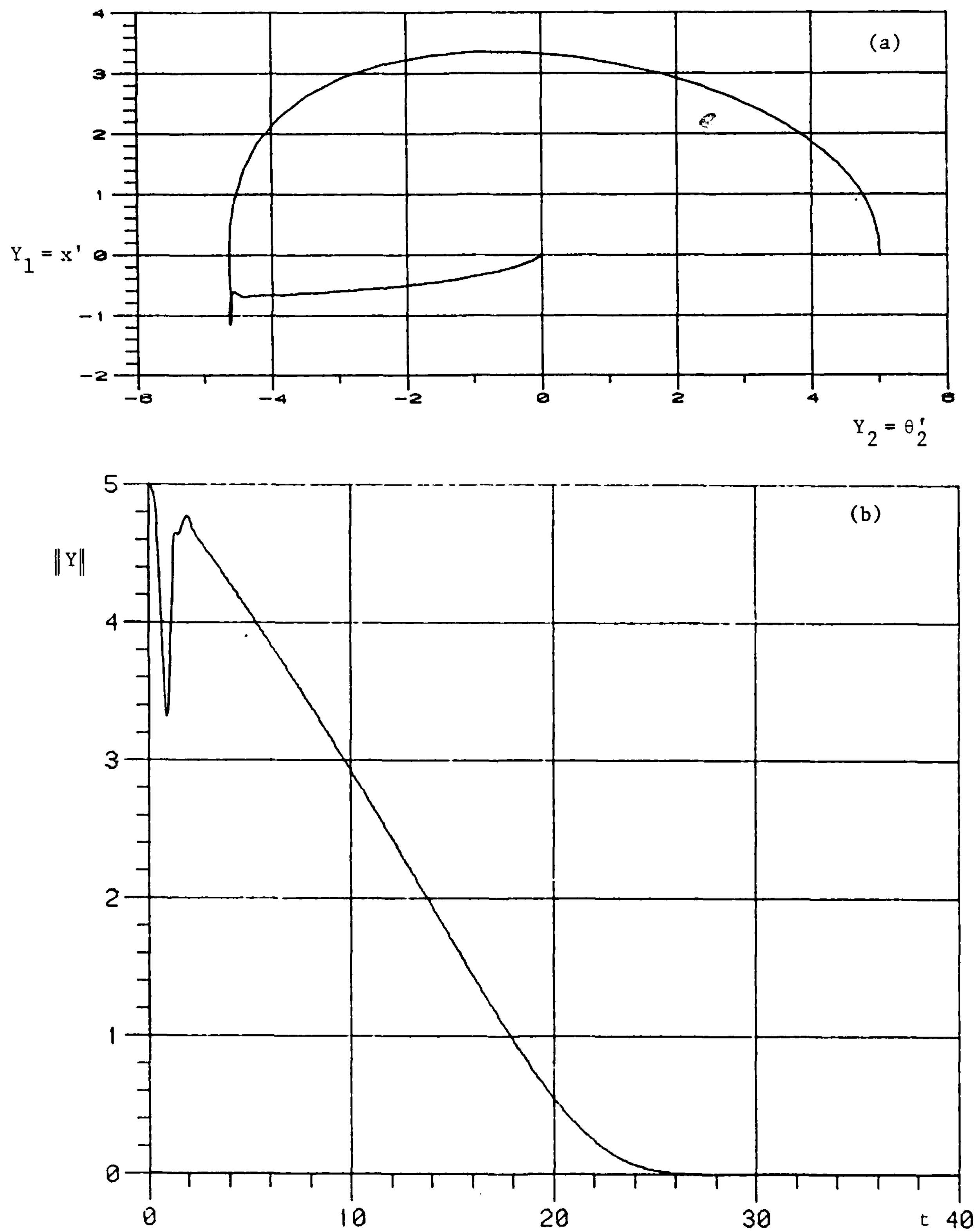

Figure 4.6. (a) Plot of Y_1 vs Y_2 and (b) plot of $\|Y\|$ for sample trajectory of (4.37) beginning at $Y = [0,5]^T$.

As the system gets larger and contains more adjustable parameters, this conservativeness problem will only get worse. This is because of the time-varying nature of the larger systems. First, the technique bounds a time-varying quantity with its maximum absolute value. Then, it applies the maximum detrimental effect idea. This double maximum approach can hurt the accuracy of the estimate very much since the effect of time-varying coupling term may depend upon its average value or the average value of its correlation with another time-varying quantity.

CHAPTER 5

CONCLUSION

This study of model reference adaptive control in the presence of unmodelled high frequency dynamics has shown several interesting results. The first of these results is that for any input in the class of signals which provide exact sufficient richness, the system has a unique $e = 0$ equilibrium provided the high frequency dynamics are at frequencies above the frequencies in the input. Along with this result, it was shown that the equilibrium values of the parameters are only a small distance from the ideal parameters for the unperturbed case.

The remaining results concern the stability of this equilibrium. Taking the two-time scale approach to the problem leads to the satisfying theoretical result that this equilibrium is exponentially stable after the model transients have settled provided that the high frequency unmodelled dynamics are sufficiently high frequency. Through the remarks it was pointed out that this technique can be used to show that the error system converges exponentially to an $O(\mu)$ region around $e = 0$, $\theta = \theta^*$ provided μ is sufficiently small.

Finally, the examples showed that there indeed exists a $\hat{\mu} > 0$ such that the equilibrium is exponentially stable for all $\mu \in [0,\hat{\mu})$. However, the examples also showed that the estimates available from the proofs are not large enough to be of practical use. Clearly, an important goal for new research is to find an approach to this problem which can result in better estimates of the range of μ for which the equilibrium retains its stability. Hopefully, an approach which provides realistic estimates will come close to providing a fundamental understanding of the problem and thus lead to measures for counteracting the destabilizing effects of the unmodelled high frequency dynamics.

APPENDIX

PROOFS OF THEOREMS 3.6 AND 3.7

The following proofs are taken directly from Yoshizawa (1966).

Proof of Theorem 3.6. Let $V(t,x)$ be defined by

$$V(t,x) = \sup_{\tau \geq 0} \|x(t+\tau;x,t)\| e^{c\tau} \quad . \tag{A.1}$$

Then clearly $\|x\| \leq V(t,x)$ and by (3.7)

$$V(t,x) \leq \sup_{\tau \geq 0} Ke^{-c\tau}\|x\| e^{c\tau} = K\|x\| \quad . \tag{A.2}$$

Since the system is linear, we have the relation $x(t+\tau;x,t) - x(t+\tau;x',t) = x(t+\tau;x-x',t)$, and hence,

$$\begin{aligned} |V(t,x) - V(t,x')| &\leq \sup_{\tau \geq 0} \|x(t+\tau;x,t) - x(t+\tau;x',t)\| e^{c\tau} \\ &\leq \sup_{\tau \geq 0} Ke^{-c\tau}\|x-x'\| e^{c\tau} = K\|x-x'\| \quad . \end{aligned} \tag{A.3}$$

Now we shall prove the continuity of $V(t,x)$. Take a $\delta \geq 0$. We have

$$\begin{aligned} |V(t+\delta,x') - V(t,x)| \leq\ & |V(t+\delta,x') - V(t+\delta,x)| \\ &+ |V(t+\delta,x) - V(t+\delta,x(t+\delta;x,t))| \\ &+ |V(t+\delta,x(t+\delta;x,t)) - V(t,x)| \quad . \end{aligned} \tag{A.4}$$

Since $V(t,x)$ is Lipschitzian in x and $x(t+\delta;x,t)$ is continuous in δ, the first two terms are small when $\|x-x'\|$ and δ are small.

Let us consider the third term. Since $x(t+\delta+\tau;x(t+\delta;x,t),t+\delta)$ $= x(t+\delta+\tau;x,t)$, we have

$$\begin{aligned}|V(t+\delta,x(t+\delta;x,t)) - V(t,x)| &= \Big|\sup_{\tau\geq 0} \|x(t+\delta+\tau;x(t+\delta;x,t),t+\delta)\| e^{c\tau} \\ &\quad - \sup_{\tau\geq 0} \|x(t+\tau;x,t)\| e^{c\tau}\Big| \\ &= \Big|\sup_{\tau\geq 0} \|x(t+\tau;x,t)\| e^{c\tau} e^{-c\delta} \\ &\quad - \sup_{\tau\geq 0} \|x(t+\tau;x,t)\| e^{c\tau}\Big| \quad . \qquad \text{(A.5)}\end{aligned}$$

Set $a(\delta) = \sup_{\tau\geq 0} \|x(t+\tau;x,t)\| e^{c\tau}$. Then $a(\delta)$ is nonincreasing and $a(\delta) \to 0$ as $\delta \to 0$, because $\|x(t+\tau;x,t)\| e^{c\tau}$ is a bounded continuous function for all $\tau \geq 0$. Thus,

$$|V(t+\delta,x(t+\delta;x,t)) - V(t,x)| = |a(\delta)e^{-c\delta} - a(0)| \qquad \text{(A.6)}$$

implies that the third term on the right-hand side of (A.4) tends to zero as $\delta \to 0$. Therefore, the continuity of $V(t,x)$ is verified.

Finally, we shall establish condition (iii). Let $x' = x(t+h;x,t)$, $h > 0$. Then,

$$\begin{aligned}V(t+h,x') &= \sup_{\tau\geq 0} \|x(t+h+\tau;x',t+h)\| e^{c\tau} \\ &= \sup_{\tau\geq h} \|x(t+\tau;x,t)\| e^{c\tau} e^{-ch} \leq V(t,x) e^{-ch}, \qquad \text{(A.7)}\end{aligned}$$

which implies

$$\frac{V(t+h;x') - V(t,x)}{h} \leq V(t,x)\,\frac{e^{-ch}-1}{h} \quad . \qquad \text{(A.8)}$$

From this, we obtain $D^{+}_{(3.5)}(t,x) \leq - cV(t,x)$.

Proof of Theorem 3.7. For a constant q such that $0 < q < 1$, define

$$V(t,x) = \sup_{\tau \geq 0} \|x(t+\tau;x,t)\| e^{q\alpha\tau} \quad . \tag{A.9}$$

Then $\|x\| \leq V(t,x)$ and for $x \in S_\beta \overset{\Delta}{=} \{x : \|x\| \leq \beta\}$

$$\|x\| \leq V(t,x) \leq \sup_{\tau \geq 0} K(\beta) e^{-(1-q)\alpha\tau} \|x\| \leq K(\beta)\|x\| \quad . \tag{A.10}$$

Now we shall prove (ii). In the course of the proof, we shall determine $L(t,\beta)$ explicitly. Let $T(\beta)$ be such that $K(\beta) = e^{(1-q)\alpha T(\beta)}$. If $\tau \geq T(\beta)$, then $K(\beta)e^{-(1-q)\alpha\tau}\|x\| \leq \|x\|$ and hence, from (A.10), it follows that $V(t,x)$ must be defined for τ such that $0 \leq \tau \leq T(\alpha)$. Therefore, for $x \in S_\beta$ and $x' \in S_\beta$,

$$|V(t,x) - V(t,x')| \leq \sup_{0 \leq \tau \leq T(\beta)} \|x(t+\tau;x,t) - x(t+\tau;x',t)\| e^{q\alpha\tau}$$

$$\leq e^{q\alpha T(\beta)} \exp\left[\int_t^{t+T(\beta)} M(s,\beta)\,ds\right] \|x - x'\| \quad , \tag{A.11}$$

where $M(s,\beta)$ is a continuous function of s such that $\|f(s,x) - f(s,x')\| \leq M(s,\beta)\, \|x - x'\|$ for $x \in S_{\beta K(\beta)}$ and $x' \in S_{\beta K(\beta)}$.

The remainder of the proof can be verified by the same argument as in the preceding proof.

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