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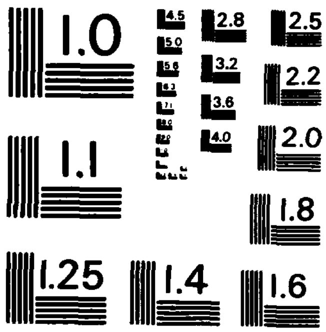

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On the angle for stationary random fields

by

A.G. Miamee and H. Niemi

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ON THE ANGLE FOR STATIONARY RANDOM FIELDS

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Abstract. The angle between past and future for stationary random fields on the lattice points of the plane is defined and it is shown that in contrast with other problems related to the past of random fields the positivity of the angle between past and future is independent of different pasts which have been considered. Most of the known facts concerning the angle for stochastic processes have been extended to the case of random fields.

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1. Introduction. Several authors have studied random fields on the lattice points of the plane. Some of the important results in this field are included in the work of Helson and Lowdenslager [2], where a generalization of Szegö's theorem to half-planes is proved; Chiang [1], where the regularity problem for the half-planes is discussed; Kallianpur and Mandrekar [7], where a Wold-Halmos decomposition Theorem is proved; Korezlioglu and Loubaton [8], where spectral factorizations are considered, Soltani [15], dealing with regularity and quarter-plane moving average representation; and Miamee [10], where an extension of Szegö's theorem for third quadrant is given. Another problem which has been proved to be useful in the prediction theory of stationary stochastic processes is the idea of the angle between past and future. Several authors have worked on this area and revealed its connection with the prediction theory of stationary stochastic processes. cf. Helson and Szegö [4], Hunt, Muckenhoupt and Wheeden [5], Pousson [12], Sarason [14], Pourahmadi [11], and Miamee [9].

In this paper we introduce the definition of the angle for a stationary random field and prove that the crucial properties of the angle in the case of stationary processes have natural extensions to the case of stationary fields. In contrast to other problems concerning the past of random fields, we show that the positivity of the angle between past and future does not depend on the choice of the kind of past one considers. Thus starting with the usual half planes, this fact enables us to use the results of Helson and Lowdenslagers proved for another kind of half-planes that they consider.

2. Preliminaries. In this section we introduce the notations and terminologies needed in the rest of the paper. Let X_{mn}, $(m,n) \in Z^2$, be a double sequence of random variables on a probability space (Ω,B,P) such that

$$EX_{mn} = 0 \qquad E|X_{mn}|^2 < \infty, \text{ for all } (m,n)\in Z.$$

The double sequence X_{mn}, $(m,n) \in Z^2$ is called a *stationary random field* if $EX_{mn}\overline{X}_{rs}$ depends *only* on the differences m-r and n-s; i.e.

$$EX_{mn}\overline{X}_{rs} = \rho(m-r,n-s).$$

In this case the covariance function $\rho(m,n) = EX_{mn}X_{oo}$ is a positive definite function on the group Z^2 of lattice points of plane. It is known (cf. for example Salehi and Scheidt [13]) that there exists a non-negative measure μ, defined on the Borel sets of the torus

$$T = \{\alpha: 0 \leq \alpha \leq \pi\} \times \{\beta: 0 \leq \beta \leq 2\pi\}$$

such that

$$\rho(m,n) = \int e^{-i(m\alpha+n\beta)} d\mu, \text{ for all } (m,n) \in Z^2. \tag{2.1}$$

This measure μ is called the *spectral measure* of the stationary random field X_{mn}. If μ is absolutely continuous with respect to the normalized Lebesgue measure $d\sigma = \frac{d\alpha d\beta}{4\pi^2}$ its Radon-Nikodym derivative w is called the *spectral density* of the field.

L^2_μ will denote the Hilbert space of all functions on the torus which are square summable with respect to the measure μ. From (2.1) it is clear that the operator

$$X_{mn} \to e^{-i(m\alpha+n\beta)}$$

extends to an isomorphism from H_X = the closed linear subspace generated by all X_{mn}'s, onto L^2_μ. This is called the *Kolmogrov isomorphism* between the time domain and spectral domain.

For any subset M of Z^2 we define $H_x(M)$ to be the closed linear subspace of $L^2(\pi,\beta,P) = H$, spanned by all $X_{m,n}$'s with $(m,n) \in M$. The *vertical past-present* P_x^v and the *vertical future* F_x^v of the field X_{mn} is the subspace $H_x(S^v)$, and $H_x(\overline{S^v})$, respectively; where

$$S^v = \{(m,n): m \leq 0,\ n \in Z\}.$$

As a measure of the *angle* between the vertical past-present and future subspaces of the field X_{mn} we take its *vertical-cosine* defined by

$$\rho_x^v = \sup\{|(Y,Z)|: Y \in P_x^v,\ Z \in F_x^v,\ ||Y|| = ||Z|| = 1\};$$

and the subspaces P_x^v and F_x^v are said to be at *positive angle* if $\rho_x^v < 1$.

The *horizontal past-present* subspace P_x^h; the *horizontal future* subspace F_x^h and the *horizontal-cosine* of the angle between these subspaces ρ_x^h, are defined similarly. Finally we define

$$\rho_x = \max(\rho_x^v, \rho_x^h).$$

For any nonnegative measure on the torus ρ_μ^v, ρ_μ^h, and ρ_μ can be defined in the same way. However, if μ is the spectral measure of our stationary random field X_{mn}, then by the Kolmogrov isomorphism it is evident that $\rho_x^v = \rho_\mu^v$, $\rho_\mu^h = \rho_\mu^h$, and $\rho_x = \rho_\mu$.

3. Geometric Characterizations of $\rho_X < 1$. In this section we present some important geometric properties of stationary random fields X_{mn} with $\rho_X < 1$. These include the generalizations of the known geometric characterizations concerning the positivity of the angle between past-present and future of discrete time stationary random processes. At the end of this section we give a set of sufficient spectral conditions which guarantees the validity of the commutative property (to be explained later) which has been used by several authors working on the prediction theory of stationary random fields.

The proof of the following lemma is similar to the corresponding fact in the case of univariate stationary processes and it is hence omitted (cf. Helson and Szegö [4, p. 129]).

3.1 Lemma. *Let* X_{mn} *be a stationary random field. In order for the angle between the vertical past-present and vertical future to be positive it is necessary and sufficient that there exists a constant* N *such that*

$$(3.2) \qquad \left\| \sum_{(m,n)\in S^V} a_{mn} X_{mn} \right\|_H \le N \left\| \sum_{(m,n)\in Z^2} a_{mn} X_{mn} \right\|_H ,$$

where $\{a_{mn}\}$ *is any double sequence of scalars with finitely many non-vanishing elements.*

A similar statement for the horizontal angle is also true. Combining these two facts we get the following Lemma. Proof is again omitted.

3.3 Lemma. *Let* X_{mn} *be a stationary random field. Then the following statements are equivalent*

(i) $\rho_X < 1$

(ii) *There exists a constant* N *such that*

$$\|\sum_{(m,n)\in S^v} a_{mn}X_{mn}\|_H \leq N\|\sum_{(m,n)\in Z^2} a_{mn}X_{mn}\|_H$$

and

$$\|\sum_{(m,n)\in S^h} d_{mn}X_{mn}\|_H \leq N\|\sum_{(m,n)\in Z^2} a_{mn}X_{mn}\|_H,$$

where $\{a_{mn}\}$ *is as in Lemma* 3.1, and S^h is defined similar to S^v.

Now we can prove the following

3.4 Lemma. *Let* X_{mn} *be a stationary random field. Then* $\rho_X < 1$ *iff there exists a constant* M *such that for any double sequence as in Lemma* 3.1 *and any integers* m_0, m_1, n_0, *and* n_1 *we have*

$$\|\sum_{m=m_0}^{m_1}\sum_{n=n_0}^{n_1} a_{mn}X_{mn}\|_H \leq M\|\sum_{(m,n)\in Z^2} a_{mn}X_{mn}\|_H. \tag{3.5}$$

Proof. If $\rho_X < 1$ then $\rho_X^v < 1$. Using Lemma 3.3 and considering the fact that our field is stationary we have

$$\|\sum_{m=-\infty}^{m_1}\sum_{n=-\infty}^{\infty} a_{mn}X_{mn}\|_H \leq N\|\sum_{(m,n)\in Z^2} a_{mn}X_{mn}\|_H.$$

Using Lemma 3.3 again, together with stationarity of X_{mn}, and the fact that the angle between two subspaces is defined symmetrically, we get

$$\|\sum_{m=m_0}^{m_1}\sum_{n=-\infty}^{\infty} a_{mn}X_{mn}\|_H \leq N\|\sum_{m=-\infty}^{m_1}\sum_{n=-\infty}^{\infty} a_{mn}X_{mn}\|_H$$

$$\leq N^2\|\sum_{(m,n)\in Z^2} a_{mn}X_{mn}\|_H.$$

Finally, since $\rho_X < 1$ implies $\rho_X^h < 1$, applying Lemma 3.3 two more times, we get

$$\|\sum_{m=m_0}^{m_1}\sum_{n=n_0}^{n_1} a_{mn}X_{mn}\|_H \leq N^4\|\sum_{(m,n)\in Z^2} a_{mn}X_{mn}\|_H.$$

Thus, one can choose $M = N^4$.

To prove the converse we assume that (3.5) holds. Now since M does not depend on the choice of m_0, m_1, n_0, or n_1 and since $\{a_{mn}\}$ has finitely many non-vanishing elements, if we let $m_0 = n_0 = -\infty$, (3.5) becomes as

$$\|\sum_{(m,n)\in S} a_{mn}X_{mn}\|_H \leq M\|\sum_{(m,n)\in Z^2} a_{mn}X_{mn}\|_H$$

which in turn implies $\rho_x^v < 1$ (by Lemma 3.3). A similar argument shows that $\rho_x^h < 1$; yielding $\rho_x < 1$.

Next we show that for a stationary random field x_{mn} the property $\rho_x < 1$ is equivalent to the fact that x_{mn} is a Schauder basis for H_x. A double sequence e_{mn} is called a *Schauder basis* for a Hilbert space H if for any element Z in H there exists a uniquely determined set of coefficients $C_{mn}(Z)$ such that

$$Z = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{mn}(Z)e_{mn}.$$

We should mention here that by the convergence of a double series $\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{mn}$ to a limit we mean that the double sequence of its partial sums converges to that limit and by the partial sums we mean the rectangular partial sums, defined by

$$S_{m_0n_0}^{m_1n_1} = \sum_{m=m_0}^{m_1} \sum_{n=n_0}^{n_1} a_{mn}$$

Having Lemma 3.4 proved the proof of the following theorem is a standard Schauder basis argument and, in particular can be given similar to the proof given on pages 102 and 103 of [6], and hence it is omitted.

<u>3.6 Theorem</u>. *A stationary random field* X_{mn} *is a Schauder basis for* H_X *if and only if* $\rho_X < 1$.

We are now going to prove the following theorem which generalizes a well-known fact concerning the positivity of the angle between past and future for a stationary sequence. (cf. Helson and Szegö [4] and Hunt; Muckenhoupt; and Wheeden [5] for scalar weight functions and Pousson [12], Pourahmadi [11], and Miamee [9] for the matrical weight functions.)

3.7 Theorem. *Let* X_{mn} *be a stationary random field with a spectral measure* μ *on the torus.* $\rho_X < 1$ *if and only if*

(i) μ *is absolutely continuous with respect to the normalized Lebesgue measure* $d\sigma = \frac{d\alpha d\beta}{4\pi^2}$, *with spectral density* w,

(ii) $L^2_w \subset L^1$, *where* $L^2_w = L^2_{wd\delta} = L^2_\mu$

(iii) *The Fourier series of any* $f \in L^2_w$ *converges to* f *in the norm of* L^2_w.

The proof of Theorem 3.7 is established via a series of Lemmas which we proceed to prove

3.8 Lemma. *Let* X_{mn} *be a stationary random field. If* $\rho_X < 1$, *then* X_{mn} *is horizontally, vertically purely non-deterministic, and strongly non-deterministic, that is*

$$\bigcap_{p\in Z} H_X(\{(m,n)\colon m \leq p,\ n \in Z\}) = \{0\},$$

$$\bigcap_{q\in Z} H_X(\{(m,n)\colon m \in Z,\ n \leq q\}) = \{0\},$$

and

$$\bigcap_{p,q\in Z} H_X\{(m,n)\colon m \leq p \text{ or } n \leq q\} = \{0\}.$$

Proof. We just give the proof of first statement. The proof of the other two statements are similar. Suppose that

$$Y \in \bigcap_{k \in Z} H_X(\{(m,n): m \le k,\ n \in Z\})$$

then $Y \in H_X(\{(m,n): m \le k, n \in Z\})$, for all $k \in Z$. It follows from Theorem 3.6 that X_{mn} is a Schauder basis for H_X; hence $\{X_{mn}, m \le k, n \in Z\}$ is a Schauder basis for $H_X(\{(m,n): m \le k, n \in Z\})$ for each k. Thus for each $k \in Z$ we have a representation as follows

$$Y = \sum_{m=-\infty}^{k} \sum_{n=-\infty}^{\infty} b_{mn}^{k} X_{mn}$$

but since these representations must be unique we conclude that $b_{mn}^{k} = 0$ for all k,m and n. Thus $Y = 0$, which completes the proof of our lemma.

Using Lemma 3.8 above and Theorem 3.4 of Soltani [15] we arrive at the following lemma.

3.9 Lemma. *Let* X_{mn} *be a stationary field with spectral measure* μ. *If* $\rho_X < 1$ *then* μ *is a.c. with respect to the Lebesgue measure, and its spectral density* w *has the property* $\log w \in L^1$.

3.10 Lemma. *Suppose* X_{mn} *is a stationary random field satisfying* $\rho_X < 1$. *Then with the notations of Theorem 3.7, we have*

$$L_w^2 \subset L^1.$$

Proof. The operator I defined by

$$I(P) = \int\int a_{00} w(\alpha,\beta)\, d\sigma,$$

on the polynomials $P = \sum_{(m,n)\in Z^2} a_{mn} e^{i(m\alpha+n\beta)}$ is bounded because

$$|I(P)| = |\iint a_{00}\omega(\alpha,\beta)d\sigma| = |\iint a_{00}\sqrt{w}\ \sqrt{w}\ d\sigma|$$

or

$$|I(P)| \leq (\iint |a_{00}|^2 w d\sigma)^{1/2}(\iint w d\sigma)^{1/2}$$

Hence by Lemma 3.4 and the Kolmogrov isomorphism we get

$$|I(P)| \leq M||P||_{L^2_s} \sqrt{\int w d\sigma}.$$

Thus "I" can be extended to a bounded functional on L^2_w. Hence there exists a function g in L^2_w such that

$$I(f) = (f,g)_{L^2_w}, \quad \text{for all } g \in L^2_w.$$

In particular we have

$$I(e^{i(m\alpha+n\beta)}) = (e^{i(m\alpha+n\beta)},g)_{L^2_w}.$$

On the other hand

$$I(e^{i(m\alpha+n\beta)}) = \begin{cases} 0, & \text{if } (m,n) \neq (0,0) \\ 1, & \text{if } (m,n) = (0,0) \end{cases}$$

Thus we have

$$\iint e^{i(m\alpha+n\beta)} w(\alpha,\beta)\overline{g(\alpha,\beta)}d\sigma = \begin{cases} 0, & \text{if } (m,n) \neq (0,0) \\ 1, & \text{if } (m,n) = (0,0) \end{cases}$$

which means $w\bar{g} \equiv 1$. Hence $w^{-1} = \bar{g} \in L^2_w$; implying $L^2_w \subset L^1$ because for any $h \in L^2_w$ we have

$$\iint h d\sigma = \iint h\sqrt{w}\ \sqrt{w^{-1}}\ d\sigma \leq (\iint h^2 w d\sigma)^{1/2}\ (\iint w^{-1} d\sigma).$$

Remark. Recall that a stationary random field x_{mn} is minimal , if and only if its spectral measure μ is absolutely continuous and, with the notation of Theorem 3.7, $w^{-1} \in L^1$. (cf. Salehi and Scheidt [13]). Thus the argumentation in the proof of Lemma 3.10 shows that the property $\rho_x < 1$ implies that x_{mn} is minimal. That is to say

$$X_{mn} \notin H_X\{(p,q): (p,q) \neq (m,n)\}.$$

This in particular implies that X_{mn} is purely non-deterministic in the Helson-Lowdenslager [2] sense. Hence we have

$$\iint \log w d\sigma > -\infty.$$

<u>3.11 Lemma</u>. *With the notations of Theorem 3.7, if* $\rho_X < 1$, *then the Fourier series of any function* f *in* L^2_w *converges to it in* L^2_w.

<u>Proof</u>. For any fixed non-negative integers m and n, the operator S_{mn} defined on the polynomials by

$$S_{mn}\left(\sum_{(p,q)\in Z^2} a_{pq} e^{i(p\alpha+q\beta)}\right) = \sum_{p=-m}^{m} \sum_{q=-n}^{n} a_{pq}\, e^{i(p\alpha+q\beta)}$$

is bounded (by Lemma 3.4) with a bound M not depending on m,n; and hence it can be extended to a bounded operator on L^2_w, which we will again call S_{mn}, with the same bound M. It can be seen that these operators S_{mn} are just the symmetric Fourier partial sum operators. Let $f \in L^2_w$, to show

$$S_{mn}(f) \to f \quad \text{in } L^2_w,$$

given $\varepsilon > 0$, we take a polynomial P such that

$$\|f - P\|_{L^2_w} < \frac{\varepsilon}{M+1}.$$

We then have

$$||S_{mn}(f) - f||_{L_w^2} \le ||(S_{mn} - I)(f - P)||_{L_w^2} + ||S_{mn}(P) - P||_{L_w^2}$$

$$\le ||S_{mn} - I|| \; ||f - P||_{L_w^2} + ||S_{mn}(P) - P||_{L_w^2}$$

Thus

$$||S_{mn}(f) - f||_{L_w^2} \le (M + 1)\frac{\varepsilon}{M+1} + ||S_{mn}(P) - P||.$$

Now if we take m and n large enough we get $S_{mn}P = P$, and hence we get

$$||S_{mn}(f) - f||_{L_w^2} < \varepsilon.$$

This completes the proof of the lemma.

Proof of Theorem 3.7. If $\rho_x < 1$ then Lemmas 3.9, 3.10, and 3.11 respectively, imply that (i), (ii), and (iii) hold.

On the other hand, suppose that (i), (ii), and (iii) holds. Thus any f in L_w^2 belongs to L^1 and, as such, has a Fourier series

$$f \sim \sum_{(p,q)\in Z^2} a_{pq} e^{i(p\alpha+q\beta)}.$$

We consider the partial sum operator

$$S_{m_0 n_0}^{m_1 n_1} : L_w^2 \to L_w^2$$

defined by

$$S_{m_0 n_0}^{m_1 n_1}(f) = \sum_{p=-m_0}^{m_1} \sum_{q=-n_0}^{n_1} a_{pq} e^{i(p\alpha+q\beta)}.$$

For any $f \in L_w^2$ we can write

$$||S_{m_0 n_0}^{m_1 n_1}(f)||_{L_w^2} = \sum_{p=-m_0}^{m_1} \sum_{q=-n_0}^{m_1} a_{pq} e^{i(p\alpha+q\beta)} ||_{L_w^2}$$

$$\leq \sum_{p=-m_0}^{m_1} \sum_{q=-n_0}^{n_1} ||a_{pq}||_{L_w^2} \ .$$

Hence

$$||S_{m_0 n_0}^{m_1 n_1}(f)||_{L_w^2} = \sqrt{\iint w d\sigma} \sum_{p=-m_0}^{m_1} \sum_{q=-n_0}^{n_1} |a_{pq}| \leq$$

$$(2m_0 m_1 + 2n_0 n_1 + 1)\sqrt{\iint w d\sigma} \ ||f||_{L^1}$$

Now since $L_w^2 \subset L^1$, Miamee [9, Lemma 3.1] implies that there exist a constant K such that

$$||f||_{L^1} \leq K||f||_{L_w^2}$$

This means that all operators $S_{m_0 n_0}^{m_1 n_1}$ are bounded. On the other hand by (iii) for each $f \in L_w^2$ we have

$$S_{m_0 n_0}^{m_1 n_1}(f) \to f \ , \quad \text{in } L_w^2.$$

Hence by uniform boundedness principle there exists a constant M such that

$$||S_{m_0 n_0}^{m_1 n_1}|| \leq M, \qquad \text{for all } m_0, m_1, n_0, n_1 \geq 0.$$

This means that for any $f \in L_w^2$ we have

$$||S_{m_0 n_0}^{m_1 n_1}(f)||_{L_w^2} \leq M||f||_{L_w^2} \ , \qquad \text{for all } m_0, m_1, n_0, n_1 \geq 0.$$

In particular for any polynomial P we have

$$||S_{m_0 n_0}^{m_1 n_1}(P)||_{L_w^2} \leq M||P||_{L_w^2} \ , \quad \text{for all } m_0, n_0, m_1, n_1 \geq 0.$$

But this, up to the Kolmogrov isomorphism, is exactly (3.5). Hence by Lemma 3.4 we deduce that $\rho_X < 1$.

We conclude this section with an application of the results obtained above to the prediction theory of stationary random fields. We are going to present some sufficient condition which guarantees the validity of a commutative property which has been widely assumed by the authors working on prediction of stationary random fields, cf. Kallianpur and Mandrekar [7]. (One can also see Korezlioglu and Loubaton [8], and Miamee [10]). A set of such sufficient conditions is also given by Soltani [15, Theorem 4.5].

We start with the following lemma.

3.12 Lemma. *Let* X_{mn} *be a stationary random field with* $\rho_X < 1$. *Then for any two subsets* A *and* B *of lattice points of* Z^2 *we have* $H_X(A \cap B) = H_X(A) \cap H_X(B)$.

Proof. Since $\rho_X < 1$ by Theorem 3.6 the double sequence X_{mn} forms a Schouder basis for H_X. Now let $Y \in H_X(A) \cap H_X(B)$, then $Y \in H_X(A)$ and $Y \in H_X(B)$. Hence we can write

$$Y = \sum_{(p,q)\in A} a_{pq} X_{pq} \quad \text{and} \quad Y = \sum_{(p,q)\in B} b_{pq} X_{pq}.$$

But by the uniqueness property of representations with respect to Schauder basis X_{mn} we must have

$$Y = \sum_{(p,q)\in A\cap B} a_{pq} X_{pq}.$$

Thus $Y \in H_X(A\cap B)$. This shows that

$$H_X(A) \cap H_X(B) \subset H_X(A\cap B) .$$

The other inclusion being always correct we have proved our lemma.

A stationary random field X_{mn} is said to have the *strong commutative property* if

$$P_{S^v} P_{S^h} = P_{S^h} P_{S^v} = P_Q,$$

where for each subspace M, P_M denotes the projection on the subspace M.

This is a major assumption in results of Kallianpur and Mandrekar [7] concerning their four-fold Halmos decomposition and it is nice to find some spectral characterization for it. Moreover, the next theorem gives some sufficient condition for the validity of this commutativity condition.

3.13 Theorem. *Let* X_{mn} *be a stationary random field with spectral measure* μ. *If* $\rho_X < 1$ *and the Fourier coefficients of* lcg w *(which is in* L^1 *by Lemma 3.8) are zero outside* $Q \cup (-Q) \cup \{(0,0)\}$. *Then* X_{mn} *has the strong commutative property.*

Proof. The proof follows from Lemma 3.8, Lemma 3.9 and Lemma 3.12 together with Theorem 2.4 in Miamee [10].

4. Analytic Characterization of $\rho_X < 1$. In this section we obtain some analytic characterizations for the density w whose corresponding stationary random field X_{mn} has $\rho_X < 1$. This provides an extension of a well-known result due to Helson and Szegö [4 Theorem]. In our proof we would need the following lemma which is of independent interest too.

4.1. Lemma. *If X_{mn} is a stationary random field, then $\rho_X < 1$ if and only if the angle between $H_X(U^v)$ and $H_X(\overline{U^v})$ as well as the angle between $H_X(U^h)$ and $H_X(\overline{U^h})$ are positive, where*

$$U^v = \{(m,n)\colon m \leq -1,\ n \in Z\} \cup \{(0,n)\colon n \leq -1\}$$

and

$$U^h = \{(m,n)\colon m \in Z,\ n \leq -1\} \cup \{(m,0)\colon m \leq -1\}.$$

If this is the case then the angle between $H_X(Q)$ and $H_X(\overline{Q})$ is also positive, where Q is the third quadrant, namely

$$Q = \{(m,n)\colon m \leq 0,\ n \leq 0\} - \{(0,0)\}.$$

Proof. We break the proof of our lemma into the following steps:

Step #1. The angle between $H_X(U^v)$ and $H_X(\overline{U^v})$ is positive if and only if there exists a constant N such that

$$\Big|\Big| \sum_{(m,n)\in U^v} a_{mn} X_{mn} \Big|\Big|_H \leq N \Big|\Big| \sum_{(m,n)\in Z^2} a_{mn} X_{mn} \Big|\Big|_H,$$

where $\{a_{mn}\}$ is any double sequence of scalars with finitely many non-vanishing elements. This statement can be proved similar to Lemma 3.1.

Step #2. The angle between $H_X(U^v)$ and $H_X(\overline{U^v})$ as well as the angle between $H_X(U^h)$ and $H_X(\overline{U^h})$ is positive if and only if there exists a constant L such that

$$(4.2) \qquad \left\| \sum_{(m,n)\in R} a_{mn} X_{mn} \right\|_H \le L \left\| \sum_{(m,n)\in Z^2} a_{mn} X_{mn} \right\|_H ,$$

where a_{mn} is any double sequence of scalars with finitely many non-vanishing elements, and the first summation ranges over any *generalized rectangle* of the form

$$(4.3) \qquad R = U^v_{m_0 n_0} \cap \overline{U^v_{m_1 n_1}} \cap U^h_{p_0 q_0} \cap \overline{U^h_{p_1 q_1}}$$

where

$$U^v_{mn} = \{(r,s):\ r \le m-1,\ s \in Z\} \cup \{(m,s):\ s \le n-1\}$$

and

$$U^h_{m,n} = \{(r,s):\ r \in Z,\ s \le n-1\} \cup \{(r,n):\ r \le m-1\}.$$

The proof of this step is similar to that of Lemma 3.4, and hence it is again omitted.

Step #3. We note that the generalized rectangles contains all the usual rectangles thus to complete the proof of the lemma it suffices to show that (3.2) implies (4.2). To see this we observe that any region R in 4.2, that is any region R of the form (4.3) can be divided as the disjoint union of at most 5 rectangles, say R_i , i=1,2,3,4,5. Thus we can take the L in step #2 to be simply 5M. In fact for any R of the form in (4.2) or (4.3) we can write

$$\begin{aligned} \left\| \sum_{(m,n)\in R} a_{mn} X_{mn} \right\|_H &\le \sum_{i=1}^{5} \left\| \sum_{(m,n)\in R_i} a_{mn} X_{mn} \right\|_H \\ &\le \sum_{i=1}^{5} M \left\| \sum_{(m,n)\in Z^2} a_{mn} X_{mn} \right\|_H \\ &\le 5\, M \left\| \sum_{(m,n)\in Z^2} a_{mn} X_{mn} \right\|_H . \end{aligned}$$

Now we can prove the following generalization of a well-known analytic characterization for positivity of the angle between past and future due to Helson and Szegö [4].

<u>4.4 Theorem</u>. *Let* X_{mn} *be a stationary random field with spectral measure* μ. *In order that* $\rho_X < 1$ *it is necessary and sufficient that* μ *be a.c. with respect to the Lebesgue measure and its spectral density can be written as*

$$w = e^{G^v}|k^v| \quad \text{and} \quad w = e^{G^h}|k^h|$$

with

(i) $G^v(\alpha,\beta)$ *and* $G^h(\alpha,\beta)$ *are bounded real valued functions;*

(ii) $k^v(\alpha,\beta)$ *and* $k^h(\alpha,\beta)$ *belong to* $H^\infty(U^v)$ *and* $H^\infty(U^h)$ *respectively;*

(iii) $|\arg k^v| \leq \frac{\pi}{2} - \varepsilon$ *and* $|\arg k^h| < \frac{\pi}{2} - \varepsilon \quad (\text{mod } 2\pi)$.

<u>Proof</u>. We know that $\rho_X < 1$ if and only if $\rho_X^v < 1$ and $\rho_X^h < 1$. But $\rho_X^v < 1$ if and only if

$$\gamma_X^v = \sup \left| \iint PwF d\sigma \right| < 1,$$

where P and F range over all polynomials on U^v and $\overline{U^v}$, respectively, and have norm less than 1. Now since $\log w \in L^1$ (by Lemma 3.9) we can write

$$\log w = a_{00} + \sum_{(m,n)\in Z^2} a_{mn} e^{-i(m\alpha+n\beta)}$$

taking D^v the corresponding optimal factor

$$D^v = e^{\frac{1}{2}a_{00} + \sum_{(m,n)\in U^v} a_{mn} e^{-i(m\alpha+n\beta)}}$$

we have

$$w(\alpha,\beta) = (D^v(\alpha,\beta))^2 e^{-i\phi(\alpha,\beta)}.$$

Hence

$$\gamma_X^v = \sup \left| \iint (PD^v)\ (FD^v)\ e^{-i\phi}\ d\sigma \right|,$$

with P and F as above. By a result due to Helson and Lowdenslager [2, Theorem 6] these PD's are dense in the unit hall of $H^2(U^v)$. Thus

$$\gamma_X^v = \sup \left| \iint khe^{-i\phi} d\sigma \right|,$$

where h and k range over the unit hall of $H^2(U^v)$. Now this together with another theorem due to Helson and Lowdenslager [2, Theorem 4] gives

$$\gamma_X = \sup \left| \iint ge^{-i\rho} d\sigma \right|,$$

where g ranges over the unit ball of $H^1(U^v)$. Hence γ_X^v is the norm of the functional

$$g \to \iint ge^{-i\phi} d\sigma$$

considered over $H^1(U^v)$. Thus we can write

$$\gamma_X^v = \inf \| e^{-i\phi} - A \|_\infty,$$

where A ranges over $H^\infty(U^v)$. Thus $\rho_X^v < 1$ if and only if $\gamma_X^v < 1$ and this in turn is equivalent to the existence of a positive ε and a function, say A^v, in $H^\infty(U^v)$ such that $|A^v| \geq \varepsilon$ and $|\arg A^v D^v| \leq \frac{\pi}{2} - t \pmod{2\pi}$. Now we can define G^v and k^v as in the one dimensional case. A similar argument for ρ_X^h instead of ρ_X^v completes the proof.

Now if we continue the proof of Theorem 4.4 further as in the proof of corresponding results of Helson and Szegö [4] we arrive at the following theorem. The proof, being similar except for the details, is omitted.

<u>4.5 Theorem</u>. *Let* X_{mn} *be a stationary field with spectral measure* μ. *If* $\rho_X < 1$ *then we can write*

$$w = e^{G^v + \widetilde{s^v}} \text{ and } w = e^{G^h + \widetilde{s^h}},$$

where G^v *is a bounded real valued function and for each* λ, $\widetilde{s^v}$ *is the conjugate of a real valued function* s^v *with* $||s^v||_\infty < \frac{\pi}{2}$ *with similar statements for* G^h, s^h, and $\widetilde{s^h}$.

To state our next theorem we need the following definition

<u>4.6 Definition</u>. *The vertical conjugate operator* C^v *and the horizontal conjugate operator* C^h *and the quarter-plane conjugate operator* C^Q *are defined for any* f *in* L^1 *with*

$$f \sim a_{00} + \sum_{(p,q)\neq(0,0)} a_{pq} e^{i(p\alpha+q\beta)}$$

by

$$C^v(f) = \sum_{(p,q)\neq(0,0)} -i(\operatorname{segm} p) a_{pq} e^{+i(p\alpha+q\beta)}$$

$$C^h(f) = \sum_{(p,q)\neq(0,0)} -i(\operatorname{segm} q) a_{pq} e^{+i(p\alpha+q\beta)}$$

$$C^Q(f) = -i\, C^h C^v.$$

<u>4.7 Theorem</u>. *Let* X_{mn} *be a stationary field with spectral measure* μ *and spectral density* w. (a) *In order that* $\rho_X < 1$ *it is necessary and sufficient that* C^v *and* C^h *be bounded operators on* L^2_w. *If this is the case then* C^Q *is also bounded.*

<u>Proof</u>. Having proved Lemma 3.3 the proof of the first part is as in the case of one variable. The proof of the last statement follows from the definition of C^Q.

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