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Special Report 85-8
June 1985

Stefan's problem in a finite domain with constant boundary and initial conditions
Analysis

Shunsuke Takagi

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REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
Special Report 85-8	AD-A158588	

4. TITLE (and Subtitle)	5. TYPE OF REPORT & PERIOD COVERED
STEFAN'S PROBLEM IN A FINITE DOMAIN WITH CONSTANT BOUNDARY AND INITIAL CONDITIONS: Analysis	
	6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)	8. CONTRACT OR GRANT NUMBER(s)
Shunsuke Takagi	

9. PERFORMING ORGANIZATION NAME AND ADDRESS	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
U.S. Army Cold Regions Research and Engineering Laboratory Hanover, New Hampshire 03755-1290	DA Project 4A161102AT24, Task SS, Work Unit 009

11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE
Office of the Chief of Engineers Washington, D.C. 20314	June 1985
	13. NUMBER OF PAGES
	32

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report)
	Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution is unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Frost heave
Mathematical analysis
Stefan's problem

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Stefan's problem in a finite domain is solved under constant boundary and initial conditions. Starting in a semi-infinite domain, the solution passes infinitely many stages of lead times in a finite domain and finally becomes stationary. The singularity at the finite terminal necessitates introduction of lead times. Including lead times, parameters defining the solution vary with time. Only the analytical result is reported in this paper.

PREFACE

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CONTENTS

STEFAN'S PROBLEM IN A FINITE DOMAIN WITH CONSTANT
BOUNDARY AND INITIAL CONDITIONS: ANALYSIS

Shunsuke Takagi

INTRODUCTION

Under any initial and boundary conditions that are given by appro-
priate infinite series, Stefan's problem in a semi-infinite domain can be
solved by utilizing the two infinite series of the elemental temperature
functions, i.e., a general solution for heat conduction in a semi-infinite
domain [1, 2, 3, 4]. Their solution method was brought to completion in
1978 by Tao [4]. He introduced a formula expressing the higher derivatives
of a function. The conditions at the moving boundary can be rewritten by
this formula to a set of simultaneous linear equations of the unknown
parameters. Further progress in the analysis in a semi-infinite domain has
been made by Tao; I mention a few that may be interesting to us. The
density jump that may occur at the moving boundary can be introduced into
the analysis [5]. Analyticity of the interfacial coordinate as a function
of $t^{1/2}$ is proved [6].

In spite of the progress that has been made so far, difficulties in a
finite domain are yet outstanding. Analysis in a finite domain must be
understood for practical applications. The heart of the difficulties seems
to be in analytically formulating the well-established practice in numeric-
al studies (for example [7]), i.e., that the solution of a Stefan's
problem in a finite domain is initially close to that in a semi-infinite
domain but eventually arrives at a final stationary state. In this paper,
we demonstrate how the solution in a finite domain, obtained under the

simplest boundary and initial conditions, transits from the semi-infinite domain solution to the finite domain solution.

The breakthrough is achieved by use of three essential tools. The first is the conversion of a pair of well-used series solutions [1, 2, 3, 4] of heat conduction in a semi-infinite domain to a pair of integral expressions. One of the expressions enable us to rewrite Duhamel's time integral [8] to a space integral. The second is Widder's [9] integral solution for heat conduction in a finite domain. Its use enables us to impose an embedding boundary temperature at the stationary boundary of the new phase in order to formulate the temperature in the old phase. We determine the embedding temperature so as to satisfy the interfacial conditions. The third is an inverse-Laplace-integral type expression of $i^n \mathrm{erfc}\ x$ that is valid for any integer n, positive, zero, or negative. This formula is applicable in the neighborhood of an exponential singularity.

We take a partial sum of the infinite series solution for the temperature in the old phase, and reinterpret the second summand contained in the last term of the partial sum in the following way: This second summand is numerically null initially, but becomes numerically significant at an appropriate time and continues to be so indefinitely after this time. If the last term in the partial sum is the Nth term of the infinite series, we call the time introduced above the Nth lead time. Prior to the 0th lead time, the first term approximates the solution in a semi-infinite domain. At the final stage, where the infinitely many lead times have entered, the temperatures are stationary and linear in both the new and old phases, terminating the phase change.

The first part of this paper consists of the mathematical preliminaries in three sections. In Section 1, four features are presented with regard to the elemental functions. First, the ranges of the indexes of the

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elemental functions are extended to negative integers so that the formula

for the series development of a function of a series may be smoothly

applied in Section 3. Second, the elemental temperature functions are

expressed with integrals so that the solution for heat conduction in a

semi-infinite domain may be given an integral expression in Section 2.

Third, an inverse-Laplace-integral type expression of i^nerfc x, valid for

any integer n, negative, zero, or positive, is derived. This is used in

Sections 7 and 9 for the transformation in the neighborhood of the

exponential singularities located at the finite terminal. Fourth, a con-

nection with Dirac's delta function [10], although well-known, is presented

so that it may be smoothly applied in this paper.

In Section 2, two series constituting the general solution for heat

conduction in a semi-infinite domain [1, 2, 3, 4] are transformed into two

integrals. One of them is the space-integral expression of Duhamel's time

integral [8, p. 30] for solving the boundary-value problem. The other is

the well-known integral expressing evolution from the initial value. In

Section 3, a formula for obtaining a serial development of a function of a

series is presented. The elemental functions defined on the moving bound-

ary are developed into a series of $t^{1/2}$ by this formula. The formula is

applicable at a nonsingular point.

The second part of this paper is the solution of Stefan's problem in a

finite domain. The problem we solve is stated in Section 4. In Section 5,

Widder's [9] solution for heat conduction in a finite domain is transformed

into a form suitable for solving the problem in this paper. In Section 6,

the embedding boundary temperature with unknown coefficients is introduced

at the stationary boundary of the new phase in order to formulate the

temperature in the encroached old phase. In Section 7, the infinite series

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formulating the unknown temperature in the old phase is interpreted as an infinite sequence. Lead times are introduced.

In Section 8, the boundary-value problem during the Nth lead time is solved, where $N \geq 0$. It is proved that, similarly to the Neuman's solution [8, p. 30], each of the solution temperatures consists of a single term. Prior to the entrance of the 0th lead time, this solution is numerically equal to the semi-infinite domain solution. In Section 9, the final stage is analyzed. It is shown that the temperatures in both phases are stationary and linear in space at the final stage. Phase change therefore finally stops.

MATHEMATICAL PRELIMINARIES

1. Elemental Functions

We define $i^n \text{erfc } x$ for a nonnegative integer n by

$$i^n \text{erfc } x = \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{(\lambda - x)^n}{n!} e^{-\lambda^2} \, d\lambda , \tag{1.1}$$

where $-\infty < x < \infty$. For an integer $0 \leq k \leq n$, this definition yields

$$d^k i^n \text{erfc } x / dx^k = (-1)^k i^{n-k} \text{erfc } x . \tag{1.2}$$

Index n may, therefore, be extended to negative integers by defining

$$i^{-n} \text{erfc } x = (-1)^n d^n \text{erfc } x / dx^n . \tag{1.3}$$

Then, (1.2) holds true for any positive integer k. Using the Hermite polynomial, $H_k(x)$, defined by

$$d^k e^{-x^2} / dx^k = (-1)^k e^{-x^2} H_k(x) , \tag{1.4}$$

we transform (1.3) to

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$$i^{-n} \text{erfc } x = \frac{2}{\sqrt{\pi}} e^{-x^2} H_{n-1}(x) . \tag{1.5}$$

Tao [4] defines $G_n(y)$ by

$$G_n(x) = \frac{1}{2} \left\{ i^n \text{erfc}(-x) + (-1)^n i^n \text{erfc } x \right\} . \tag{1.6}$$

Substituting from (1.1), we transform (1.6) to

$$G_n(y) = \frac{1}{n!\sqrt{\pi}} \int_{-\infty}^{\infty} (x-\lambda)^n e^{-\lambda^2} d\lambda , \tag{1.7}$$

which integrates to an nth degree polynomial,

$$G_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} \frac{(2x)^{n-2k}}{k!(n-2k)!} . \tag{1.8}$$

The derivatives of the polynomials $G_n(x)$ are:

$$d^k G_n(x)/dx^k = G_{n-k}(x) \qquad \text{for} \quad k \leq n$$

$$= 1 \qquad \text{for} \quad k = n \tag{1.9}$$

$$= 0 \qquad \text{for} \quad k > n .$$

Index n may, therefore, be extended to negative integers by defining

$$G_{-n}(x) = 0 \qquad \qquad \text{for} \quad n \geq 1 . \tag{1.10}$$

We rewrite

$$k(x, \kappa t) = e^{-x^2/(4\kappa t)}/\sqrt{4\pi\kappa t} \quad , \tag{1.11}$$

to an inverse-Laplace-integral type expression,

$$k(x, \kappa t) = \frac{1}{2\pi i} \frac{1}{\sqrt{4\kappa t}} \int_{c-i\infty}^{c+i\infty} \exp\left(\frac{\lambda^2}{4} - \frac{\lambda x}{\sqrt{4\kappa t}}\right) d\lambda , \tag{1.12}$$

where c is a complex number whose real part is finite. The right side of (1.12) transforms to the right side of (1.11), when the former is integrated with regard to μ, defined by

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$$\mu = \left(\lambda - x/\sqrt{\kappa t}\right)/2 \ .$$

With two phases in Stefan's problem, we use κt in place of t, where κ is the thermal diffusivity of a phase and t is time.

We derive the formula

$$i^n \mathrm{erfc}\ x = \frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} \lambda^{-n-1}\ \exp\left(\frac{\lambda^2}{4} - \lambda x\right) d\lambda \ , \qquad (1.13)$$

valid for any integer n, $-\infty < n < \infty$. For a nonnegative integer n, we keep the real part of c positive, so that the integral is convergent. To prove (1.13) for such an integer n, we begin by transforming (1.1) to

$$i^n \mathrm{erfc}\ \left(x/\sqrt{4\kappa t}\right) = \frac{2}{n!}\ \frac{1}{\left(\sqrt{4\kappa t}\right)^n}\ \int_0^\infty y^n\ k(x+y,\ \kappa t)dy \ . \qquad (1.14)$$

Substituting $k(x+y, \kappa t)$ from (1.12) and evaluating the integral with regard to y, we obtain

$$i^n \mathrm{erfc}\ \frac{x}{\sqrt{4\kappa t}} = \frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} \lambda^{-n-1}\ \exp\left(\frac{\lambda^2}{4} - \frac{\lambda x}{\sqrt{4\kappa t}}\right) d\lambda \ , \qquad (1.15)$$

which is equivalent to (1.13). To prove (1.13) for a negative integer n, we begin by noting that (1.3) is equivalent to

$$i^{-n} \mathrm{erfc}\left(x/\sqrt{4\kappa t}\right) = 2(-1)^{n-1}\ \left(\sqrt{4\kappa t}\right)^n \cdot d^{n-1}\ k(x,\kappa t)/dx^{n-1} \ .$$

Substituting $k(x,\kappa t)$ from (1.12), we find (1.15) for a negative n. In a finite domain $0 \leq x \leq \ell$, we revise (1.15) to a nondimensional form,

$$\left(\frac{\sqrt{4\kappa t}}{\ell}\right)^n i^n \mathrm{erfc}\ \frac{x}{\sqrt{4\kappa t}}\ =\ \frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} \xi^{-n-1}\ \exp\left(\xi^2\ \frac{\kappa t}{\ell^2} - \xi\ \frac{x}{\ell}\right) d\xi \ . \qquad (1.16)$$

Equation (1.12) may transform to an alternative form [9, p. 36],

$$k(x,\kappa t) = \frac{1}{2\pi}\ \int_{-\infty}^\infty \exp(ix\lambda - \kappa t\lambda^2)\ d\lambda \ .$$

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Letting $t = 0$, the above becomes

$$k(x,0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\lambda} \, d\lambda \ .$$

Therefore we find

$$k(x,0) = \delta(x) \ , \tag{1.17}$$

where $\delta(x)$ is Dirac's delta function [10, p. 38].

2. Integral Formulas

The heat equation,

$$\partial T / \partial t = \kappa \, \partial^2 T / \partial x^2 \ , \tag{2.1}$$

has a general solution,

$$T(x,\kappa t) = \sum_{n=0}^{\infty} \left(\sqrt{4\kappa t}\right)^n \left[A_n \, G_n\left(\frac{x}{\sqrt{4\kappa t}}\right) + B_n \, i^n \mathrm{erfc}\frac{x}{\sqrt{4\kappa t}} \right] \ . \tag{2.2}$$

This was used to solve Stefan's problem in a semi-infinite domain [1, 2, 3, 4].

We may use heat functions,

$$u_n(x,\kappa t) = n! \left(\sqrt{4\kappa t}\right)^n i^n \mathrm{erfc}\left(x/\sqrt{4\kappa t}\right) \tag{2.3}$$

and

$$v_n(x,\kappa t) = n! \left(\sqrt{4\kappa t}\right)^n G_n\left(x/\sqrt{4\kappa t}\right) \ . \tag{2.4}$$

The latter is called the heat polynomial by Widder [9, p. 8-9]. By using (1.14) and (1.7), (2.3) and (2.4) may be given integral expressions,

$$u_n(x,\kappa t) = 2 \int_0^{\infty} y^n \, k(x+y, \kappa t) dy \tag{2.5}$$

and

$$v_n(x,\kappa t) = \int_{-\infty}^{\infty} y^n \, k(x-y, \kappa t) dy \ . \tag{2.6}$$

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Therefore, by defining, in the infinite domain $-\infty < x < \infty$, functions

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} A_n x^n \qquad (2.7)$$

and

$$g(x) = 2 \sum_{n=0}^{\infty} \frac{1}{n!} B_n x^n \qquad \text{for} \qquad 0 < x < \infty \qquad (2.8a)$$

$$= 0 \qquad \text{for} \qquad -\infty < x < 0 , \qquad (2.8b)$$

(2.2) transforms to an integral expression

$$T(x, \kappa t) = \int_{-\infty}^{\infty} f(y) \, k(x-y, \kappa t) dy + \int_{-\infty}^{\infty} g(y) \, k(x+y, \kappa t) dy . \qquad (2.9)$$

Use of the delta function (1.17) shows that (2.9) determines the initial value $f(x)$ in the semi-infinite domain $0 < x < \infty$, because $g(-x)$ is null for x in this domain. The initial value $f(x)$ in the integral form (2.9) may be discontinuous and is more general than the serial form (2.2).

We assume $f(x)$ to be odd,

$$f(-x) = - f(x) . \qquad (2.10)$$

Then (2.9) transforms to

$$T(x, \kappa t) = \int_{0}^{\infty} f(y) \left[k(x-y, \kappa t) - k(x+y, \kappa t) \right] dy + \int_{0}^{\infty} g(y) \, k(x+y, \kappa t) dy . \qquad (2.11)$$

The first integral becomes zero at $x = 0$ for any t in the domain $0 < t < \infty$ and the second integral becomes zero at $t = 0$ for any x in the domain $0 < x < \infty$. Therefore, the first and second integrals describe the evolutions from the initial and boundary conditions, respectively, in a semi-infinite domain.

Particularly, we have an expression of the boundary value,

$$T(0, \kappa t) = \int_{0}^{\infty} g(y) \, k(y, \kappa t) dy . \qquad (2.12)$$

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If

$$T(0, \kappa t) = \sum_{n=0}^{\infty} B_n \left(\sqrt{4\kappa t}\right)^n i^n \mathrm{erfc} 0 \; , \qquad (2.13)$$

then the solution $g(y)$ of the integral equation (2.12) is (2.8a).

The solution of the boundary-value problem by use of the integral equation (2.12) is consistent with Duhamel's theorem [8, p. 30]. To show this, we refer to Widder's [9, p.127] solution for heat conduction in a semi-infinite domain. In our notation, it is

$$T(x, \kappa t) = \int_0^{\infty} \left[k(x-y, \kappa t) - k(x+y, \kappa t) \right] T(y, 0) dy +$$

$$+ \int_0^t h\left(x, \kappa(t-\tau)\right) T(0, \kappa\tau) d\tau \; , \qquad (2.14)$$

where

$$h(x, \kappa t) = -2 \, \partial k(x, \kappa t)/\partial x = \left(x/(\kappa t)\right) k(x, \kappa t) \quad . \qquad (2.15)$$

The first integral in (2.14) is the one in (2.11). By substituting $T(0, \kappa t)$ from (2.12), the second integral in (2.14), a product of Duhamel's theorem, becomes the repeated integrals,

$$\int_0^{\infty} g(y) dy \int_0^t h\left(x, \kappa(t-\tau)\right) k(y, \kappa\tau) d\tau \; .$$

The convolution integral in the above simplifies to,

$$\int_0^t h\left(x, \kappa(t-\tau)\right) k(y, \kappa\tau) d\tau = k(x+y, \kappa t) \; , \qquad (2.16)$$

because both sides yield the same Laplace transform,

$$\left(1/\sqrt{4s}\right) \exp\left(-(x+y)/\sqrt{s}\right) \; .$$

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To derive this, we employ κt in place of t in the usual Laplace transforms. Widder's solution is, therefore, the same as ours.

3. Functions Of A Series

At the interface,

$$s(t) = \sum_{n=0}^{\infty} s_n \, t^{(n+1)/2} , \qquad (3.1)$$

the temperatures of the new and old phases need to be expressed as functions of $t^{1/2}$, so that the interfacial conditions may be expressed as functions of $t^{1/2}$. We use a revised version of Faa de Bruno's formula [11, p. 33]: Let functions $z = z(y)$ be analytic and $y = y(x)$ be an infinite series,

$$y = \sum_{n=0}^{\infty} a_n \, x^n . \qquad (3.2)$$

Then the power series expressing the composite function $z(y(x))$ is given by

$$z\big[y(x)\big] = z\big[y(a_o)\big] + \sum_{n=1}^{\infty} x^n \sum_{\nu=1}^{n} S_{n\nu}(a_\nu) \left(\frac{d^\nu z(y)}{dy^\nu}\right)_{x=0} , \qquad (3.3)$$

where

$$S_{n\nu}(a_j) = \sum_{\lambda_1} \frac{1}{\lambda_1! \, \cdots \, \lambda_{n-\nu+1}!} \, a_1^{\lambda_1} \cdots a_{n-\nu+1}^{\lambda_{n-\nu+1}} , \qquad (3.4)$$

for $n \geq 1$ and $\geq z \, \nu \geq 1$, where the summation is over all the sets of at least one nonzero and all nonnegative integers λ_1, λ_2, ..., λ_n that simultaneously satisfy the two equations,

$$\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_{n-\nu+1} = \nu \qquad (3.5a)$$

and

$$\lambda_2 + 2\lambda_3 + \ldots + (n-\nu)\lambda_{n-\nu+1} = n - \nu , \qquad (3.5b)$$

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for every integer ν in the range,

$$1 \leq \nu \leq n \; . \tag{3.5c}$$

The solutions are tabulated in [12, p. 831]. The argument a_j of $S_{n\nu}(a_j)$ stands for $n - \nu + 1$ arguments $a_1 ,--, a_{n-\nu+1}.$

Substituting $s(t)$ from (3.1) for x in $G_n(x/\sqrt{4\kappa t})$ and $i^n\mathrm{erfc}(x/\sqrt{4\kappa t})$, we find the series of τ,

$$G_n\left(\frac{s(t)}{\sqrt{4\kappa t}}\right) = \sum_{k=0}^{\infty} G_k^{(n)}\left(\frac{s_j}{\sqrt{4\kappa}}\right)\tau^k \tag{3.6a}$$

and

$$i^n\mathrm{erfc}\left(\frac{s(t)}{\sqrt{4\kappa t}}\right) = \sum_{k=0}^{\infty} I_k^{(n)}\left(\frac{s_j}{\sqrt{4\kappa}}\right)\tau^k \; . \tag{3.6b}$$

The coefficients in (3.6a) and (3.6b), which we call G-derivatives and I-derivatives, respectively, are given for $k = 0$ by

$$G_0^{(n)}\left(s_j/\sqrt{4\kappa}\right) = G_n\left(s_0/\sqrt{4\kappa}\right) \tag{3.7a}$$

and

$$I_0^{(n)}\left(s_j/\sqrt{4\kappa}\right) = i^n\mathrm{erfc}\left(s_0/\sqrt{4\kappa}\right) \; , \tag{3.7b}$$

and for $k \geq 1$ by

$$G_k^{(n)}\left(\frac{s_j}{\sqrt{4\kappa}}\right) = \sum_{\nu=1}^{k} G_{n-\nu}\left(\frac{s_0}{\sqrt{4\kappa}}\right) \cdot S_{k\nu}\left(\frac{s_j}{\sqrt{4\kappa}}\right) \tag{3.8a}$$

and

$$I_k^{(n)}\left(\frac{s_j}{\sqrt{4\kappa}}\right) = \sum_{\nu=1}^{k} (-1)^{\nu} i^{n-\nu}\mathrm{erfc}\frac{s_0}{\sqrt{4\kappa}} \cdot S_{k\nu}\left(\frac{s_j}{\sqrt{4\kappa}}\right) \; . \tag{3.8b}$$

The index $n - \nu$ can be negative. Due to (1.10), therefore, the number of components of a G-derivative, shown on the right of (3.8a), may be less

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than k or possibly zero. The argument $s_j/\sqrt{4\kappa}$ in the G-derivatives and the I-derivatives stands for k arguments $s_1/\sqrt{4\kappa}$, ... , $s_k/\sqrt{4\kappa}$.

Substituting x in (2.2) with s(t) from (3.1), we find a series of τ,

$$T\big(s(t), \kappa t\big) = \sum_{p=0}^{\infty} T_p^{(s)} \tau^p , \qquad (3.9a)$$

where

$$T_p^{(s)} = \sum_{n=0}^{p} \left(\sqrt{4\kappa}\right)^n \left[A_n \ G_{p-n}^{(n)}\Big(\frac{s_j}{\sqrt{4\kappa}}\Big) + B_n \ I_{p-n}^{(n)}\Big(\frac{s_j}{\sqrt{4\kappa}}\Big)\right] . \qquad (3.9b)$$

Differentiating (2.2) with x, substituting x with s(t) from (3.1), we find another series of τ,

$$\tau \cdot \frac{\partial T}{\partial x} \ \big(s(t), \kappa t\big) = \sum_{p=0}^{\infty} T_p^{(Ds)} \tau^p , \qquad (3.10a)$$

where

$$T_p^{(Ds)} = \sum_{n=0}^{p} \left(\sqrt{4\kappa}\right)^{n-1} \left[A_n \ G_{p-n}^{(n-1)}\Big(\frac{s_j}{\sqrt{4\kappa}}\Big) - B_n \ I_{p-n}^{(n-1)}\Big(\frac{s_j}{\sqrt{4\kappa}}\Big)\right] . \qquad (3.10b)$$

Multiplication of τ by $\partial T/\partial x$ on the left side of (3.10a) makes the series of τ on the right side of (3.10a) conform with the series of τ on the right side of (3.9a).

We call (3.9b) and (3.10b) level-p coefficients of the series (3.9a) and (3.10a), respectively. The highest index of the parameters in the level-p coefficients are p, which we call level-p parameters. Three level-p parameters are contained in the level-p coefficients. Two of them are A_p and B_p. The third is s_p, hidden in $I_p^{(0)}\big(s_j/\sqrt{4\kappa}\big)$ in (3.9b) and in $I_p^{(-1)}\big(s_j/\sqrt{4\kappa}\big)$ in (3.10b), as may be found by use of the formula,

$$S_{n1}(a_j) = a_n , \qquad (3.11)$$

which is a particular case of (3.4). In the equations for the determination of them, which we discuss subsequently, the higher-than-level-0 param-

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eters are all linear terms. Only s_0, one of the level-0 parameters, is nonlinear.

STEFAN'S PROBLEM IN A FINITE DOMAIN

4. Problem

We consider the simplest Stefan's problem in a finite domain $0 \leq x \leq \ell$. We assume freezing starts at $x = 0$ on $t = 0$. The boundary temperature T_A at $x = 0$ is a constant that is lower than the freezing temperature T_F and the initial temperature T_B is a constant that is higher than T_F. The boundary temperature at $x = \ell$ is a constant T_B. We have

$$T_A < T_F < T_B . \tag{4.1}$$

At $t = 0$, a new phase emerges at $x = 0$, whose temperature we express by $T_I(x, \kappa_I t)$, where κ_I is the thermal diffusivity of the new phase. The domain of the new phase is $0 \leq x \leq s(t)$, where $s(t)$ is given by (3.1). The temperature of the old phase is given by $T_{II}(x, \kappa_{II} t)$, where κ_{II} is the thermal diffusivity of the old phase. The domain of the old phase is $s(t) \leq x \leq \ell$. The quantities of the new and old phases are designated by the roman numerals I and II, respectively, used as a sub- or superindex.

We extend the domain

$$s(t) \leq x \leq \ell$$

of the old phase at time t to

$$0 \leq x \leq \ell$$

by introducing the embedding boundary temperature at $x = 0$. It shall be determined so as to satisfy the two interfacial conditions at the freezing front $x = s(t)$,

$$T_{II} = T_F$$

$$K_I \frac{\partial T_I}{\partial x} - K_{II} \frac{\partial T_{II}}{\partial x} = L\rho \frac{ds}{dt} , \tag{4.2}$$

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where K_I and K_{II} are the thermal conductivities of the new and old phases, respectively, ρ is the assumedly equal density of the two phases, and L is the latent heat.

The embedding technique was initiated by Boley [13]. Using elemental functions, ours is much simpler than his.

5. Widder's Solution of Heat Conduction

With unrestricted initial and boundary conditions, Widder's [9, p. 131] solution of heat conduction in a finite domain is applicable to solving the problem of embedding posed in the previous section. In our notation, his solution is

$$T_{II}(x,\kappa_{II}t) = \int_0^\ell \left[\theta(x-y,\kappa_{II}t) - \theta(x+y,\kappa_{II}t) \right] T_{II}(y,0)\ dy +$$

$$+ \int_0^t \phi\left(x,\kappa_{II}(t-\tau)\right) T_{II}(0,\kappa_{II}\tau)d\tau + \tag{5.1}$$

$$+ \int_0^t \phi\left(\ell-x,\kappa_{II}(t-\tau)\right) T_{II}(\ell,\kappa_{II}\tau)d\tau ,$$

where

$$\theta(x,\kappa_{II}t) = \sum_{n=-\infty}^{\infty} k(x + 2n\ell,\kappa_{II}t) \tag{5.2a}$$

and

$$\phi(x,\kappa_{II}t) = \sum_{n=-\infty}^{\infty} h(x + 2n\ell,\kappa_{II}t) . \tag{5.2b}$$

Function $h(x,\kappa_{II}t)$ is given in (2.15). We call the three integrals on the right side of (5.1) the parts due to initial distribution, embedding boundary, and terminal boundary.

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The first integral in (5.1) describes the evolution from the initial distribution $T_\Pi(x, 0)$. To show this, we use function $f(x)$, which is odd, as shown in (2.10), and is periodic with period 2ℓ. We find a relation,

$$\int_{-\infty}^{\infty} f(y) \ k(x-y, \kappa_\Pi t) dy = \int_{0}^{\ell} f(y) \ [\theta(x-y, \kappa_\Pi t) - \theta(x+y, \kappa_\Pi t)] dy \ .$$

We assure that $T_\Pi(x, 0)$ is extended to the infinite domain in the same manner as $f(x)$ is. Letting $t = 0$ on the left side of the above equation and using the delta function (1.17), and noting that by letting $t = 0$ on the right side of (5.1) the second and third integrals both disappear, we find that $T_\Pi(x, 0)$ is the initial value.

The second and third integrals on the right side of (5.1) describe the evolutions from the boundary conditions at $x = 0$ and ℓ, respectively. We clarify this statement in two parts: The former or latter becomes $T_\Pi(0, \kappa_\Pi t)$ or $T_\Pi(\ell, \kappa_\Pi t)$ by letting $x = 0$ or ℓ, respectively. The former or latter reduces to zero by letting $x = \ell$ or 0, respectively. Note that the first integral becomes 0 at $x = 0$ and $x = \ell$ because of its periodic distribution.

We start the proof by formulating the boundary conditions at $x = 0$ and ℓ,

$$T_\Pi(0, \kappa_\Pi t) = \int_{0}^{\infty} g_1(y) \ k(y, \kappa_\Pi t) dy \tag{5.3a}$$

and

$$T_\Pi(\ell, \kappa_\Pi t) = \int_{0}^{\infty} g_2(y) \ k(y, \kappa_\Pi t) dy \ , \tag{5.3b}$$

by applying (2.12) to two semi-infinite domains, $0 < x < \infty$ and $\ell > x > -\infty$, respectively, where y in (5.3a) and (5.3b) stands for x and $\ell-x$, respectively, and functions $g_1(y)$ and $g_2(y)$ are defined to be zero for negative

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values of y. The delta function (1.17) may, therefore, be applied to find the initial temperature.

We substitute (5.3a) or (5.3b) into the second or third integral, respectively, on the right side of (5.1), divide the range of summation on the right side of (5.2b) into the nonnegative integers and the negative integers, and use in the latter the oddity of the h-function, shown by (2.15), so that (2.16), in which the arguments must be positive, may be applied. Thus we find

$$\int_0^t \phi\left(x, \kappa_{\square}(t-\tau)\right) T_{\square}(0, \kappa_{\square}\tau) d\tau$$

$$= \int_0^\infty g_1(y) \sum_{n=0}^\infty \left\{k(2n\ell+x+y, \kappa_{\square}t) - k(2(n+1)\ell-x+y, \kappa_{\square}t)\right\} dy \qquad (5.4a)$$

and

$$\int_0^t \phi\left(\ell-x, \kappa_{\square}(t-\tau)\right) T_{\square}(\ell, \kappa_{\square}\tau) d\tau$$

$$= \int_0^\infty g_2(y) \sum_{n=0}^\infty \left\{k\left((2n+1)\ell-x+y, \kappa_{\square}t\right) - k\left((2n+1)\ell+x+y, \kappa_{\square}t\right) dy \right. . \qquad (5.4b)$$

If we let $x = 0$ or ℓ on the right side of (5.4a) or (5.4b), we find that, after the cancellation, only the $n = 0$ term remains, which is equal to $T_{\square}(0, \kappa_{\square}t)$ by (5.3a) or $T_{\square}(\ell, \kappa_{\square}t)$ by (5.3b), respectively. The first part of the statement is thus proved. The second part immediately follows.

6. Temperature with Embedding Unknowns

We now apply our initial and boundary conditions to Widder's unrestricted solution. To find the part due to the terminal boundary, we let $g_2(y) = 2 T_B$ in (5.4b) and use (1.14) with $n = 0$. We find thus

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$$TB = T_B \sum_{n=0}^{\infty} \left[\mathrm{erfc} \frac{(2n+1)\ell-x}{\sqrt{4\kappa_\square t}} - \mathrm{erfc} \frac{(2n+1)\ell+x}{\sqrt{4\kappa_\square t}} \right] . \tag{6.1a}$$

To integrate the part due to the initial distribution, we let $T_\square(y,0) = T_B$ in the first integral on the right side of (5.1), substitute the θ-functions from (5.2a), change the range of integration from 0 to ℓ to the difference of the one from 0 to ∞ and the one from ℓ to ∞ , carry out the thus defined integrations by use of (1.14) with $n = 0$, and change those with negative argument to those with positive argument by use of (1.6). We find thus

$$ID = T_B \left\{ 1 - \sum_{n=0}^{\infty} (-1)^n \mathrm{erfc} \frac{n\ell+x}{\sqrt{4\kappa_\square t}} + \sum_{n=1}^{\infty} (-1)^n \mathrm{erfc} \frac{n\ell-x}{\sqrt{4\kappa_\square t}} \right\} . \tag{6.1b}$$

Adding (6.1a) and (6.1b), we get

$$TB + ID = T_B \left\{ 1 - \sum_{n=0}^{\infty} \mathrm{erfc} \frac{2n\ell+x}{\sqrt{4\kappa_\square t}} + \sum_{n=1}^{\infty} \mathrm{erfc} \frac{2n\ell-x}{\sqrt{4\kappa_\square t}} \right\} . \tag{6.1c}$$

Using $g(y)$ in (2.8) for $g_1(y)$ in (5.4a), rewriting n and B_n in the initial condition (2.13) to k and $B_k^\square/\ell^k$, respectively, and using (1.14), we integrate the part due to the embedding boundary condition.

Summing the results, we obtain

$$T_\square(x,\kappa_\square t) = T_B +$$

$$\sum_{k=0}^{\infty} B_k^\square \left(\frac{\sqrt{4\kappa_\square t}}{\ell} \right)^k \sum_{n=0}^{\infty} \left\{ i^k \mathrm{erfc} \frac{2n\ell+x}{\sqrt{4\kappa_\square t}} - i^k \mathrm{erfc} \frac{2(n+1)\ell-x}{\sqrt{4\kappa_\square t}} \right\} . \tag{6.2}$$

The coefficients $B_0^\square$, $B_1^\square$,.. are to be determined. Included in the summand for $k = 0$ in (6.2), the formula in the pair of braces in (6.1c) need not be listed. It is easy to directly check that (6.2) satisfies all the conditions thus far imposed. Differentiation yields

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$$\tau \frac{\partial T_{\square}}{\partial x}(x, \kappa_{\square}t) =$$

$$= -\sum_{k=0}^{\infty} \frac{B_k^{\square}}{\sqrt{4\kappa_{\square}}} \left(\frac{\sqrt{4\kappa_{\square}t}}{\ell}\right)^k \sum_{n=0}^{\infty} \left\{ i^{k-1}\text{erfc}\,\frac{2n\ell+y}{\sqrt{4\kappa_{\square}t}} + i^{k-1}\text{erfc}\frac{2(n+1)\ell-x}{\sqrt{4\kappa_{\square}t}} \right\}. \quad (6.3)$$

7. Lead Times

We rewrite (6.2) to a sequence

$$T_{\square}(x, \kappa_{\square}t) = \lim_{N\to\infty} T_{\square}^{(N)}(x, \kappa_{\square}t), \quad (7.1)$$

where

$$T_{\square}^{(N)}(x,\kappa_{\square}t) = T_B +$$

$$+ \sum_{k=0}^{\infty} B_k^{\square N} \left(\frac{\sqrt{4\kappa_{\square}t}}{\ell}\right)^k \sum_{n=0}^{N} \left\{ i^k\text{erfc}\,\frac{2n\ell+x}{\sqrt{4\kappa_{\square}t}} - i^k\text{erfc}\frac{2(n+1)\ell-x}{\sqrt{4\kappa_{\square}t}} \right\}, \quad (7.2)$$

which satisfies the terminal boundary condition and the initial condition. If coefficients $B_k^{\square N}$ are chosen to satisfy the interfacial conditions (4.2), we call (7.2) the Nth stage solution.

To understand the successive emergence of infinitely many stages in (7.1), we introduce a lead time in the Nth stage solution. We define it in a special case such that $B_k^{\square N} = 0$ for all the integers $k \geq K + 1$. Time t that makes $i^K\text{erfc}\left[(2(N+1)\ell - s(t))/\sqrt{4\kappa_{\square}t}\right]$ numerically effective will be called the Nth lead time. Note that this is the least of the values that $i^K\text{erfc}\left[(2(N+1)\ell-x)/\sqrt{4\kappa_{\square}t}\right]$ in the last pair of braces in (7.2) can take for $k = K$ in the domain $s(t) \leq x \leq \ell$ at time t.

This definition is appropriate to the problem we are solving in this paper, where, as we shall show, all the coefficients $B_k^{\square N}$ are zero for $k \geq 1$ and therefore only $B_0^{\square N}$ is nonzero. A lead time is a semi-infinite time interval. Prior to the 0th lead time in our problem, the 0th stage solu-

tion is the semi-infinite domain solution. When the 0th lead time is effective, the second summand exists in the 0th stage. The final stage is arrived at in the infinite stage.

Although not a constant but a time interval, a lead time must be dealt with like a constant in the time differentiation. Except prior to the 0th lead time in the 0th stage, the transcendental equation for the determination of s_0 at the Nth stage includes the lead time, as we shall show, and therefore the root s_0 is a function of a lead time. We express $s(t)$ at the Nth stage by

$$s^{(N)}(t) = \sum_{n=0}^{N} s_n^{(N)} t^{(n+1)/2} , \qquad (7.3)$$

where coefficients $s_n^{(N)}$ are in general functions of the Nth lead time. Similarly to (7.2), we rewrite (6.3) to the expression at the Nth stage,

$$\tau \frac{\partial T_{\amalg}^{(N)}}{\partial x} (x, \kappa_{\amalg} t) =$$

$$= -\sum_{k=0}^{\infty} \frac{B_k^{\amalg N}}{\sqrt{4\kappa_{\amalg}}} \left(\frac{\sqrt{4\kappa_{\amalg} t}}{\ell}\right)^k \sum_{n=0}^{N} \left\{ i^{k-1} erfc \frac{2n\ell + x}{\sqrt{4\kappa_{\amalg} t}} + i^{k-1} erfc \frac{2(n+1)\ell - x}{\sqrt{4\kappa_{\amalg} t}} \right\}. \quad (7.4)$$

To find the series of τ at the interface, we use the formula (1.13) to obtain the integral expression

$$i^k erfc\left(\frac{2n\ell}{\sqrt{4\kappa_{\amalg} t}} \pm n\right) = \frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} \lambda^{-k-1} \exp\left(\frac{\lambda^2}{4} - \lambda \frac{2n\ell}{\sqrt{4\kappa_{\amalg} t}} \pm \lambda n\right) d\lambda , \quad (7.5)$$

where

$$\eta = \sum_{p=0}^{\infty} \frac{1}{\sqrt{4\kappa_{\amalg}}} s_p^{(N)} \tau^p .$$

Use of the development (3.3) yields the expansion

$$e^{\mp \lambda \eta} = \exp\left(\frac{\mp \lambda s_0^{(N)}}{\sqrt{4\kappa_{\amalg}}}\right) \sum_{p=0}^{\infty} \tau^p \sum_{\nu=0}^{p} (\mp \lambda)^\nu S_{p\nu}\left(\frac{s_j^{(N)}}{\sqrt{4\kappa_{\amalg}}}\right), \qquad (7.6)$$

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where the convention,

$$S_{00}(a_j) = 1$$

and

$$S_{n0}(a_j) = 0 \qquad \text{for} \qquad n \geq 1 ,$$

is used. Substituting (7.6) in (7.5) and using (1.13), we obtain the series development,

$$i^k \text{erfc} \, \frac{2n\ell \pm s^{(N)}(t)}{\sqrt{4\kappa_{\Pi} t}} = \sum_{p=0}^{\infty} \tau^p \sum_{\nu=0}^{p} (\mp 1)^{\nu} S_{p\nu}\left(\frac{s_j^{(N)}}{\sqrt{4\kappa_{\Pi}}}\right) i^{k-\nu} \text{erfc} \frac{2n\ell \pm s_0^{(N)}\sqrt{t}}{\sqrt{4\kappa_{\Pi} t}} .$$

$$(7.7)$$

Letting $x = s^{(N)}(t)$ in (7.2) and (7.4) and using (7.7), we find series of τ:

$$T_{\Pi}(s^{(N)}(t), \kappa_{\Pi} t) = \sum_{p=0}^{\infty} T_p^{(\Pi N, s)} \tau^p , \qquad (7.8a)$$

and

$$\tau \cdot \frac{\partial T_{\Pi}}{\partial x} \left(s^{(N)}(t), \kappa_{\Pi} t\right) = \sum_{p=0}^{\infty} T_p^{(\Pi N, Ds)} \tau^p . \qquad (7.8b)$$

We need only the first terms,

$$T_0^{(\Pi N, S)} = T_B +$$

$$+ B_0^{\Pi N} \sum_{n=0}^{N} \left\{ \text{erfc} \, \frac{2n\ell + s_0^{(N)}\sqrt{t}}{\sqrt{4\kappa_{\Pi} t}} - \text{erfc} \frac{2(n+1)\ell - s_0^{(N)}\sqrt{t}}{\sqrt{4\kappa_{\Pi} t}} \right. \qquad (7.9a)$$

and

$$T_0^{(\Pi N, Ds)} =$$

$$- \frac{B_0^{\Pi N}}{\sqrt{4\kappa_{\Pi}}} \sum_{n=0}^{N} \left\{ i^{-1} \text{erfc} \frac{2n\ell + s_0^{(N)}\sqrt{t}}{\sqrt{4\kappa_{\Pi} t}} + i^{-1} \text{erfc} \frac{2(n+1)\ell - s_0^{(N)}}{\sqrt{4\kappa_{\Pi} t}} \right\} . \qquad (7.9b)$$

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Because the coefficients $B_p^{\square IN}$ for $p \geq 1$ are, as we shall show, all equal to zero, the higher terms need not be made explicit.

8. The Nth Stage

The general solution (2.2) gives the Nth stage temperature of the new phase,

$$T_I^{(N)}(x, \kappa_I t) = \sum_{n=0}^{\infty} \left(\sqrt{4\kappa_I t} \right)^n \left[A_n^{IN} \, G_n\!\left(\frac{x}{\sqrt{4\kappa_I t}} \right) + B_n^{IN} \, i^n \text{erfc}\!\left(\frac{x}{\sqrt{4\kappa_I t}} \right) \right] . \qquad (8.1)$$

Substituting x with $s^{(N)}(t)$ from (7.3), and expanding the elemental functions by use of (3.6a) and (3.6b), we rewrite $T_I^{(N)}(s^{(N)}(t), \kappa_I t)$ to a series of τ,

$$T_I^{(N)}\!\left(s^{(N)}(t), \kappa_I t \right) = \sum_{k=0}^{\infty} T_k^{(IN,s)} \tau^k , \qquad (8.2a)$$

where

$$T_k^{(IN,s)} = \sum_{n=0}^{k} \left(\sqrt{4\kappa_I} \right)^n \left[A_n^{IN} \, G_{k-n}^{(n)}\!\left(\frac{s_j}{\sqrt{4\kappa_I}} \right) + B_n^{IN} \, I_{k-n}^{(n)}\!\left(\frac{s_j}{\sqrt{4\kappa_I}} \right) \right] . \qquad (8.2b)$$

Differentiating (8.1) with x, substituting x with $s^{(N)}(t)$ from (7.3), and using (3.6a) and (3.6b), we find

$$\tau \cdot \frac{\partial T_I}{\partial x} \left(s^{(N)}(t), \kappa_I t \right) = \sum_{k=0}^{\infty} T_k^{(IN,Ds)} \tau^k , \qquad (8.3a)$$

where

$$T_k^{(IN,Ds)} = \sum_{n=0}^{k} \left(\sqrt{4\kappa_I} \right)^{n-1} \left[A_n^{IN} \, G_{k-n}^{(n-1)}\!\left(\frac{s_j(N)}{\sqrt{4\kappa_I}} \right) - B_n^{IN} \, I_{k-n}^{(n-1)}\!\left(\frac{s_j(N)}{\sqrt{4\kappa_I}} \right) \right] . \qquad (8.3b)$$

Rewriting the boundary conditions,

$$T_I^{(N)}(0, \kappa_I t) = T_A \qquad (8.4a)$$

and

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$$T_I^{(N)}\left(s^{(N)}(t), \kappa_I t\right) = T_F \tag{8.4b}$$

into series of τ, we can successively determine A_p^N and B_p^N as functions $s_0^{(N)}$,..., $s_p^{(N)}$. The first pair of this sequence is:

$$A_0^{IN} = T_F - T_A \, \mathrm{erfc}\left(s_0^{(N)}/\sqrt{4\kappa_I}\right)\Big/\mathrm{Erf}\left(s_0^{(N)}/\sqrt{4\kappa_I}\right) \tag{8.5a}$$

and

$$B_0^{IN} = -\left(T_F - T_A\right)\Big/\mathrm{Erf}\left(s_0^{(N)}/\sqrt{4\kappa_I}\right) . \tag{8.5b}$$

We write Erf with capital E lest it be confused with erfc x. The higher pairs of the sequence need not be described in our problem, where they are, as we shall show, all equal to zero.

Substituting $T_0^{(\Pi N, S)}$ from (7.9a), the coefficient of τ^0 in equation (4.2a) yields the evaluation of $P_0^{\Pi N}$,

$$B_0^{\Pi N} = -(T_B - T_F)/R_N, \tag{8.6a}$$

where

$$R_N = \sum_{n=0}^{N}\left[\mathrm{erfc}\frac{2n\ell + s_0^{(N)}\sqrt{t}}{\sqrt{4\kappa_\Pi t}} - \mathrm{erfc}\frac{2(n+1)\ell - s_0^{(N)}\sqrt{t}}{\sqrt{4\kappa_\Pi t}}\right] . \tag{8.6b}$$

Substituting $T_0^{(IN, Ds)}$ found from (8.3b) with $k = 0$ and $T_0^{(\Pi N, Ds)}$ from (7.9b), the coefficient of τ^0 in equation (4.2b) yields the transcendental equation for the determination of $s_0^{(N)}$,

$$\frac{K_I(T_F - T_A)}{\sqrt{4\kappa_I}} \frac{i^{-1}\mathrm{erfc}\left(s_0^{(N)}/\sqrt{4\kappa_I}\right)}{\mathrm{Erf}\left(s_0^{(N)}/\sqrt{4\kappa_I}\right)} - \frac{K_\Pi(T_B - T_F)}{\sqrt{4\kappa_\Pi}} \frac{Q_N}{R_N} = \frac{1}{2} L\rho \, s_0^{(N)} \tag{8.7}$$

where

$$Q_N = \sum_{n=0}^{N} \left[i^{-1}\text{erfc}\frac{2n\ell + s_0^{(N)}\sqrt{t}}{\sqrt{4\kappa_{\text{II}}t}} + i^{-1}\text{erfc}\frac{2(n+1)\ell - s_0^{(N)}\sqrt{t}}{\sqrt{4\kappa_{\text{II}}t}} \right] . \tag{8.8}$$

Thus determined, $s_0^{(N)}$ is a function of the Nth lead time. Then, A_0^{IN}, B_0^{IN}, and B_0^{IIN} are also functions of the Nth lead time.

We have thus evaluated the level-0 parameters. We call the equation for determining the level-n parameters level-n equations. For $n \geq 1$, the level-n parameters are linear in the level-n equations. The level-1 equations are homogeneous with respect to the level-1 parameters, which are therefore all equal to zero. Proceeding this way, we find that the higher-than-level-0 parameters are all equal to zero.

Thus we find

$$T_I^{(N)}(x,\kappa_I t) = T_A + \left(T_F - T_A\right) \text{Erf}\frac{x}{\sqrt{4\kappa_I t}} \Big/ \text{Erf}\frac{s_0^{(N)}}{\sqrt{4\kappa_I}} \tag{8.9a}$$

and

$$\tau \cdot \frac{\partial T_I^N}{\partial x}(x,\kappa_I t) = -\frac{1}{\sqrt{4\kappa_I}} B_0^{\text{IN}} \left[i^{-1}\text{erfc}\frac{x}{\sqrt{4\kappa_I t}} \right] \tag{8.9b}$$

for the new phase. Also we find

$$T_{\text{II}}^{(N)}(x,\kappa_{\text{II}} t) = T_B + B_0^{\text{IIN}} \sum_{n=0}^{N} \left[\text{erfc}\frac{2n\ell+x}{\sqrt{4\kappa_{\text{II}}t}} - \text{erfc}\frac{2(n+1)\ell-x}{\sqrt{4\kappa_{\text{II}}t}} \right] \tag{8.10a}$$

and

$$\tau \cdot \frac{\partial T_{\text{II}}^N}{\partial x}(x,\kappa_{\text{II}} t) = -\frac{B_0^{\text{IIN}}}{\sqrt{4\kappa_{\text{II}}}} \sum_{n=0}^{N} \left[i^{-1}\text{erfc}\frac{2n\ell+x}{\sqrt{4\kappa_{\text{II}}t}} + i^{-1}\text{erfc}\frac{2(n+1)\ell-x}{\sqrt{4\kappa_{\text{II}}t}} \right]$$

$$\tag{8.10b}$$

for the old phase.

9. The Final Stage

To formulate the final stage, we transform the infinite series,

$$L_k = \left(\frac{\sqrt{4\kappa t}}{\ell}\right)^k \sum_{n=0}^{\infty} \left[i^k \text{erfc}\frac{2n\ell+x}{\sqrt{4\kappa t}} - i^k \text{erfc}\frac{2(n+1)\ell-x}{\sqrt{4\kappa t}}\right] \qquad (9.1)$$

and

$$M_k = \left(\frac{\sqrt{4\kappa t}}{\ell}\right)^k \sum_{n=0}^{\infty} \left[i^k \text{erfc}\frac{2n\ell+x}{\sqrt{4\kappa t}} + i^k \text{erfc}\frac{2(n+1)\ell-x}{\sqrt{4\kappa t}}\right], \qquad (9.2)$$

contained in (6.2) and (6.3), respectively. We drop the subscript II in this computation. Although our problem is concerned with L_0 and M_{-1}, we transform the general cases L_k and M_k at this stage of the exposition.

We begin with the transformation of the infinite series,

$$\left(\frac{\sqrt{4\kappa t}}{\ell}\right)^k \sum_{n=0}^{\infty} i^k \text{erfc}\frac{2n\ell+x}{\sqrt{4\kappa t}}.$$

Using formula (1.16), this becomes

$$= \frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} \xi^{-k-1} \exp\left(\xi^2 \frac{\kappa t}{\ell^2} - \xi \frac{x}{\ell}\right) \sum_{n=0}^{\infty} e^{-2n\xi} d\xi$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \xi^{-k-1} \exp\left(\xi^2 \frac{\kappa t}{\ell^2} + \xi \frac{\ell-x}{\ell}\right) \text{Cosech}\,\xi\, d\xi.$$

We write Cosech x with capital C, lest it be confused with cosec x. Similarly,

$$\left(\frac{\sqrt{4\kappa t}}{\ell}\right)^k \sum_{n=0}^{\infty} i^k \text{erfc}\frac{2(n+1)\ell-x}{\sqrt{4\kappa t}}$$

transforms to

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \xi^{-k-1} \exp\left(\xi^2 \frac{\kappa t}{\ell^2} - \xi \frac{\ell-x}{\ell}\right) \text{Cosech}\,\xi\, d\xi.$$

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Thus we find

$$L_k = \frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} \xi^{-k-1} \exp\left(\xi^2 \frac{\kappa t}{\ell^2}\right) \operatorname{Sinh}\left(\xi \frac{\ell-x}{\ell}\right) \operatorname{Cosech}\xi \, d\xi . \qquad (9.3a)$$

Expanding the Sinh function by use of the addition formula, and using (1.16) and (1.6), we may give (9.3a) another form,

$$L_k = \left(\frac{\sqrt{4\kappa t}}{\ell}\right)^k \left[G_k\left(\frac{x}{\sqrt{4\kappa t}}\right) + \frac{1-(-1)^k}{2} \, i^k \operatorname{erfc}\frac{x}{\sqrt{4\kappa t}} \right] -$$

$$- \frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} \xi^{-k-1} \exp\left(\xi^2 \frac{\kappa t}{\ell^2}\right) \operatorname{Sinh}\left(\frac{\xi x}{\ell}\right) \operatorname{Coth}\xi \, d\xi . \qquad (9.3b)$$

When x is substituted for s(t) from (3.1), the latter transforms to a series of τ, but the former does not. Similarly, we find

$$M_k = \frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} \xi^{-k-1} \exp\left(\xi^2 \frac{\kappa t}{\ell^2}\right) \operatorname{Cosh}\left(\xi \frac{\ell-x}{\ell}\right) \operatorname{Cosech}\xi \, d\xi \qquad (9.4a)$$

and

$$M_k = - \left(\frac{\sqrt{4\kappa t}}{\ell}\right)^k \left[G_k\left(\frac{x}{\sqrt{4\kappa t}}\right) - \frac{1+(-1)^k}{2} \, i^k \operatorname{erfc}\frac{x}{\sqrt{4\kappa t}} \right] +$$

$$+ \frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} \xi^{-k-1} \exp\left(\xi^2 \frac{\kappa t}{\ell^2}\right) \operatorname{Coth}\xi \operatorname{Cosh}\frac{\xi x}{\ell} \, d\xi . \qquad (9.4b)$$

In our problem, the temperatures at the final stage are stationary and linear in space. To show this, we begin by letting

$$\phi(\xi) = \exp\left(\xi^2 \frac{\kappa t}{\ell^2}\right) \operatorname{Sinh}\left(\xi \frac{\ell-x}{\ell}\right) \Big/ \operatorname{Sinh} \xi \qquad (9.5a)$$

in (9.3a) and

$$\phi(\xi) = \exp\left(\xi^2 \frac{\kappa t}{\ell^2}\right) \operatorname{Cosh}\left(\xi \frac{\ell-x}{\ell}\right)(\xi/\operatorname{Sinh}\xi) \qquad (9.5b)$$

in (9.4a). Noting that the $\phi(\xi)$'s are even functions, we find that L_0

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and M_{-1} are given by the respective residues at $\xi = 0$. Thus we find

$$L_0 = 1 - x/\ell \tag{9.6a}$$

and

$$M_{-1} = 1 . \tag{9.6b}$$

To formulate the final temperature, we first note that

$$\lim_{N \to \infty} s_0^{(N)} = 0$$

and that the final interface coordinate, $s(\infty)$, is given by

$$s(\infty) = \lim_{N \to \infty} s_0^{(N)} \sqrt{t} . \tag{9.7}$$

Note that as $N \to \infty$, also $t \to \infty$. Taking the limit in (8.9a), and noting that

$$\text{Erf}(x) \doteq \frac{2}{\sqrt{\pi}} x , \tag{9.8}$$

for a small x, we find that the final temperature in the new phase is given by

$$T_I^{(\infty)}(x, \kappa_I t) = T_A + (T_F - T_A) x \Big/ s(\infty) . \tag{9.9}$$

Using M_{-1} in (9.6b) and L_0 in (9.6a), we find

$$\lim_{N \to \infty} Q_N = \sqrt{4 \kappa_{\amalg} t} / \ell \tag{9.10a}$$

and

$$\lim_{N \to \infty} R_N = 1 - s(\infty)/\ell . \tag{9.10b}$$

Dividing all the terms in (8.7) by $t^{1/2}$, and letting $t \to \infty$, we find that the limit of (8.7) is stationary, given by

$$\frac{K_I(T_F - T_A)}{s(\infty)} - \frac{K_{\amalg}(T_B - T_F)}{\ell - s(\infty)} = 0 . \tag{9.11}$$

The phase change therefore finally stops. Using (9.10b) in (8.6a), we find that the limit of (8.10a) is given by

$$T_{\square}^{(\infty)}(x, \kappa_{\square} t) = T_B - (T_B - T_F)(\ell - x) \big/ (\ell - s(\infty)). \qquad (9.12)$$

CONCLUSIONS

By using the simplest boundary and initial conditions we have shown that Stefan's problem in a finite domain can be solved. To solve the problem, we have employed the solution for heat conduction in a finite domain whose elemental temperature functions are members of the family of the error function. The effect of the finite terminal is the introduction of singularities. Starting at a solution in the semi-infinite domain and passing through infinitely many lead times, the solution in the finite domain arrives at the final stationary stage.

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