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TWO RESULTS ON DENSE IMBEDDINGS

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In a recent paper [1], J. U. Kim studied the Cauchy problem for a Bingham fluid in $\mathbb{R}^2$. He points out that the extension of his results to $\mathbb{R}^3$ depends on two lemmas concerning dense imbedding of $C^m_0$-functions in certain spaces. In this note, these lemmas are proved.

**Abstract**

In a recent paper [1], J. U. Kim studied the Cauchy problem for a Bingham fluid in $\mathbb{R}^2$. He points out that the extension of his results to $\mathbb{R}^3$ depends on two lemmas concerning dense imbedding of $C^m_0$-functions in certain spaces. In this note, these lemmas are proved.

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Following Kim's notation, we define

\[ \tilde{F}_p = \{ u \in W^{1,2}(\mathbb{R}^n) \mid \forall \varepsilon \in L^p(\mathbb{R}^n) \} \]

Our first result is the following.

**Lemma 1:**

\[ C_0^\infty(\mathbb{R}^n) \text{ is dense in } \tilde{F}_p \text{ for } 1 \leq p < \infty. \]

**Proof:**

Clearly it suffices to show that functions of compact support are dense; \( C^\infty \)-regularity can easily be achieved by using the Friedrichs mollifier. If we know that \( u \in L^p(\mathbb{R}^n) \) or even that \( u \in L^{p+\varepsilon}(\mathbb{R}^n) \) for sufficiently small \( \varepsilon > 0 \), then we can use the standard cut-off procedure to approximate \( u \) by functions of compact support. I.e., if we set \( u_m(x) = u(x)\psi_m(x) \), where, for example,

\[
\psi_m(x) = \begin{cases} 
1 & , |x| \leq m \\
2 - \frac{|x|}{m} & , m \leq |x| \leq 2m \\
0 & , |x| \geq 2m 
\end{cases}
\]

then it can easily be shown that \( u_m + u \) in \( \tilde{F}_p \). Therefore it suffices to show that \( \tilde{F}_p \cap L^{p+\varepsilon} (\varepsilon > 0 \text{ small}) \) is dense in \( \tilde{F}_p \). If \( p \geq 2 \), it follows from the Sobolev imbedding theorem that \( \tilde{F}_p \subset L^p \), and there is nothing left to prove.

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In the following, we assume \( p < 2 \). We set
\[
\varphi_N(x) = \begin{cases} 
\frac{1}{n^n \Omega_n}, & |x| \leq N, \\
0, & |x| > N.
\end{cases}
\]
Here \( \Omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \). For \( u \in \mathcal{F}_p \), let \( u_N = u - \varphi_N^* u \), where \( * \) denotes convolution. Clearly, \( u_N \) is in \( \mathcal{F}_p \), and we want to show \( u_N + u \) in \( \mathcal{F}_p \). We have \( Vu_N = Vu - \varphi_N^* Vu \), and, if \( p = 1 \), then \( \int_{\mathbb{R}^n} Vu = 0 \), since \( u \in L^2(\mathbb{R}^n) \). Therefore, it suffices to show the following: Let \( 1 \leq r < \infty \), and \( v \in L^r \), for \( r = 1 \), assume in addition that \( \int_{\mathbb{R}^n} v = 0 \). Then \( v_N = v - \varphi_N^* v + v \) in \( L^r \).

To see this, note first that \( \|v_N\|_{L^1} = 1 \), and hence \( \|v_N\|_{L^r} \leq 2\|v\|_{L^r} \), hence it suffices to show \( v_N + v \) for \( v \) in a dense subset of \( L^r \). If \( r > 1 \), take \( v \in L^1 \cap L^r \). Then \( \|v_N^* v\|_{L^r} \leq \|v_N\|_{L^1} \|v\|_{L^r} \), which tends to zero. For \( r = 1 \), let \( v \) have compact support, contained in, say \( \{ |x| \leq R \} \), and assume \( \int_{\mathbb{R}^n} v = 0 \). Then
\[
\begin{aligned}
\|\varphi_N^* v\|_{L^1} &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \varphi_N(x-y)v(y)dy \right| dx \\
&= \int_{-R}^{R} \int_{-R}^{R} |\varphi_N(x-y)v(y)| dy dx \\
&\leq \int_{-R}^{R} \int_{-R}^{R} |\varphi_N(x-y)| |v(y)| dy dx \\
&\leq \int_{-2R}^{2R} |\varphi_N(z)| dz \int_{-R}^{R} |v(y)| dy .
\end{aligned}
\]
This tends to zero as \( N \to \infty \).

It remains to be shown that \( u_N \) lies in \( L^{p+\epsilon}(\mathbb{R}^n) \) for small \( \epsilon > 0 \).

Let \( g \) denote the fundamental solution of the Laplacian. Then an explicit calculation shows that \( g - \varphi_N^* g \) lies in \( L^{1+\delta}(\mathbb{R}^n) \) for small positive \( \delta \), and so do its derivatives. Hence it follows that \( w_N := g^* Vu_N = (g - \varphi_N^* g)^* Vu \) lies in \( L^{p+\epsilon}(\mathbb{R}^n) \) and so does \( u_N = \text{div} w_N \). This completes the proof.
Again following Kim's notation, we now define the following spaces of vector-valued functions.

\[ G_1(\mathbb{R}^3) = \{ f \in (W^{1,2}(\mathbb{R}^3))^3 \mid \epsilon_{ij}(f) := \frac{\partial f_i}{\partial x_j} + \frac{\partial f_j}{\partial x_i} \in L^1(\mathbb{R}^3) \text{ for } i,j = 1,2,3, \text{ div } f = 0 \} \]

\[ S(\mathbb{R}^3) = \{ f \in (C_0^\infty(\mathbb{R}^3))^3 \mid \text{ div } f = 0 \} . \]

The lemma required for Kim's result is the following.

Lemma 2:

\[ S(\mathbb{R}^3) \text{ is dense in } G_1(\mathbb{R}^3). \]

Proof:

Obviously, \( G_1 \) is contained in

\[ G_p = \{ f \in (W^{1,2}(\mathbb{R}^3))^3 \mid \epsilon_{ij}(f) \in L^p(\mathbb{R}^3), \text{ div } f = 0 \} \]

for any \( p \in (1,2] \). Moreover, lemma 1.9 in [1] says that

\[ G_p = F_p = \{ f \in (W^{1,2}(\mathbb{R}^3))^3 \mid \nabla f \in L^p(\mathbb{R}^3), \text{ div } f = 0 \} . \]

for \( p > 1 \). Let \( f_N = f - \varphi_N^*f \) with \( \varphi_N \) as in the proof of lemma 1. Then it follows as before that \( f_N + f \) in \( G_1 \). Let \( a \) be defined by

\[ a = g^*\text{curl} f, \]

where \( g \) is again the fundamental solution of the Laplacian. The convolution makes sense, since we can write \( g = g_1 + g_2 \), where \( g_1 \in L^1(\mathbb{R}^3) \), \( g_2 \in L^2(\mathbb{R}^3) \), and we define \( g_2^*\text{curl} f \) by putting the derivative on \( g_2 \). Clearly, we have \( \text{div } a = 0, \text{curl } a = f \). Now let

\[ a_N = a - \varphi_N^*a \]

so that \( \text{curl } a_N = f_N \), and \( \Delta a_N = \text{curl } f_N \). Since

\[ a_N = (g - \varphi_N^*g)^*\text{curl} f \]

and \( \text{curl } f \in L^{1+\epsilon}(\mathbb{R}^3) \) for \( \epsilon \) small, we conclude as in the proof of lemma 1 that \( a_N \in L^{1+\epsilon}(\mathbb{R}^3) \). From \( a_N \in L^{1+\epsilon}(\mathbb{R}^3) \), \( \Delta a_N \in L^{1+\epsilon}(\mathbb{R}^3) \), it follows that \( a_N \in W^{2,1+\epsilon}(\mathbb{R}^3) \).
It thus remain to be shown that every \( f \in G_1 \) which has the form \( f = \text{curl } a \) with \( a \in W^{2,1+\epsilon}(\mathbb{R}^3) \) can be approximated by functions of compact support. This is easily achieved by multiplying \( a \) with an appropriate cut-off function.
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