VARIATIONAL PRINCIPLE FOR PENETRATOR DYNAMICS USING BILINEAR FUNCTIONAL AND ADJOINT FORMULATION

C. N. SHEN

JUNE 1985

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT CENTER
LARGE CALIBER WEAPON SYSTEMS LABORATORY
BENET WEAPONS LABORATORY
WATERVLIET N.Y. 12189

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED
DISCLAIMER

The findings in this report are not to be construed as an official Department of the Army position unless so designated by other authorized documents.

The use of trade name(s) and/or manufacture(s) does not constitute an official indorsement or approval.

DISPOSITION

Destroy this report when it is no longer needed. Do not return it to the originator.
Variational Principle for Penetrator Dynamics Using Bilinear Functional and Adjoint Formulation

C. N. Shen

US Army Armament Research & Development Center
Benet Weapons Laboratory, SMCAR-LCB-TL
Watervliet, NY 12189-5000

US Army Armament Research & Development Center
Large Caliber Weapon Systems Laboratory
Dover, NJ 07801-5001


Variational Principle
Penetrator Dynamics
Matrix Vector Coupling Systems
Adjoint System
Optimization

The solution to problems in both spatial and time domains using the finite element method can be based on the variational principle employing bilinear functional and adjoint formulation. This principle is extended to matrix vector coupling systems such as in penetration dynamics. The present hyperbolic type partial differential equation of interest has two dependent and two independent variables with the coupling in the spatial domain.
## TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>THE VARIATIONAL PRINCIPLE</td>
<td>1</td>
</tr>
<tr>
<td>INTEGRAL OF BILINEAR EXPRESSION</td>
<td>3</td>
</tr>
<tr>
<td>THE SYMMETRICAL ADJOINT SYSTEM</td>
<td>6</td>
</tr>
<tr>
<td>INITIAL CONDITIONS FOR THE ADJOINT SYSTEM</td>
<td>9</td>
</tr>
<tr>
<td>THE GENERALIZED BOUNDARY CONDITIONS</td>
<td>9</td>
</tr>
<tr>
<td>CONDITIONS FOR THE COUPLING TERMS</td>
<td>10</td>
</tr>
<tr>
<td>THE FIRST VARIATION</td>
<td>11</td>
</tr>
<tr>
<td>CONCLUSIONS</td>
<td>13</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>14</td>
</tr>
<tr>
<td>APPENDIX</td>
<td>15</td>
</tr>
<tr>
<td>APPENDIX REFERENCES</td>
<td>17</td>
</tr>
</tbody>
</table>

## LIST OF ILLUSTRATIONS

1. Image Reflection of the Adjoint System. 7
INTRODUCTION

The elastic-plastic stress-strain relations for a rod have been derived by T. Wright (ref 1), and the differential equations of the rod itself have been cast into variational form. The variational principle using bilinear functional and adjoint formulation has served as a basis to determine numerical solutions by the finite element method. This principle has now been extended to coupling systems such as in the impact dynamics. The present hyperbolic type partial differential equation has two dependent and two independent variables, with coupling in the spatial domain. This new formulation is also ready to be used for the coupled impact problem which is given in the Appendix of this report.

THE VARIATIONAL PRINCIPLE

A dynamical system can be modeled by the matrix vector partial differential equation.

\[ L(\zeta) \ y(\zeta) = -Q(\zeta) \]  \hspace{1cm} (1)

with appropriate boundary and initial conditions. In the above equation, \( L \) is a matrix linear operator in both spatial and temporal domain, \( y \) is a vector dependent variable, \( Q \) is a vector forcing function, and \( \zeta \) represents all independent variables, both spatial and temporal.

The inner product \( \langle Q, y \rangle \) of an adjoint forcing function \( Q \) and the solution \( y(\zeta) \) of Eq. (1) can be used for the purpose of estimation. This inner product is \( \langle Q, y \rangle \).

An accurate estimation can be made by constructing a variational principle (refs 2-5). By using the adjoint variable $y$ as a Lagrange multiplier for Eq. (1) and adding to $\langle Q, y \rangle$, we have

$$J_1[y,y] = \langle Q, y \rangle + \langle y, (Q+Ly) \rangle = \langle Q, y \rangle + \langle y, Q \rangle + \langle y, Ly \rangle \tag{2}$$

To keep the system symmetrical, let us define the adjoint system as

$$L(\xi)y(\xi) = -Q(\xi) \tag{3}$$

By using the original variable $y$ as a Lagrange multiplier for Eq. (3) and adding to $\langle Q, y \rangle$, we have

$$J_2[y,y] = \langle Q, y \rangle + \langle y, (Q+Ly) \rangle = \langle Q, y \rangle + \langle y, Q \rangle + \langle y, Ly \rangle \tag{4}$$

The relationship of the adjoint system to the original system is

$$D = \langle y, Ly \rangle - \langle y, Ly \rangle = D_e \tag{5}$$

where $D$ is the bilinear concomitant (ref 2). Combining Eqs. (2), (4), and (5) one obtains

$$J_1 = J_2 + D_e \tag{6a}$$

In order to keep the functional symmetrical, we have

$$J_1 = \frac{1}{2} [J_1 + J_2 + D_e] \tag{6b}$$

---

which is of the form
\[ J_1 = \langle Q, y \rangle + \langle y, Q \rangle + \frac{1}{2} \langle y, Ly \rangle + \frac{1}{2} \langle y, Ly \rangle - \frac{D_e}{2} \] (6c)

Similarly,
\[ J_2 = \langle Q, y \rangle + \langle y, Q \rangle + \frac{1}{2} \langle y, Ly \rangle + \frac{1}{2} \langle y, Ly \rangle - \frac{D_e}{2} \] (6d)

**INTEGRAL OF BILINEAR EXPRESSION**

The integral of a bilinear expression for a two-dimensional problem having a system of second order partial derivatives in time and in space can be written as
\[ I = \int_{x_0}^{x_b} \int_{t_0}^{t_b} \Omega[y(x,t), y(x,t)] dtdx \] (7)

where \( \Omega[y,y] \) is a given bilinear expression in the form
\[ \Omega[y,y] = y_T^T \beta y_T - y_T^T A y_T - y_T^T P y_T + y_T^T \Gamma y + y_T^T N y_T \] (8)

The subscripts t and x indicate the partial derivatives for the functions y and y. The matrices A, B, and P are diagonal and \( \Gamma \) and N are off-diagonal.

\[
A = \begin{bmatrix} a_1^2 & 0 \\ 0 & a_2^2 \end{bmatrix}, \quad B = \begin{bmatrix} b_1^2 & 0 \\ 0 & b_2^2 \end{bmatrix}, \quad \text{and} \quad P = \begin{bmatrix} p_1^2 & 0 \\ 0 & p_2^2 \end{bmatrix} \] (9)

\[
\Gamma = \begin{bmatrix} 0 & \gamma_1 \\ \gamma_2 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & \eta_1 \\ \eta_2 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} w \\ u \end{bmatrix} \text{ and } y = \begin{bmatrix} w \\ u \end{bmatrix} \] (10)

Equation (8) can be integrated by parts. Two different forms of integration and end conditions are given. The first form of the integral is obtained by integrating by parts on the adjoint variable.
\[ I_a = - \int \left( y^T B y_{tt} - y^T A y_{xx} + y^T p y - y^T p y_x - y^T N y_x \right) dx \\
+ \int y^T B y_{tt} dx - \int y^T A y_{xx} dt + \int y^T p y dt \]  \hspace{1cm} (11)

which gives

\[ I_a = \int \left[ \frac{1}{2} w_{tt}^2 - a_1^2 w_{xx}^2 + P_1^2 w^2 + q_1 u_x - q_1 u_x^2 \right] dx \\
+ \int \left[ \frac{1}{2} u_{tt}^2 - a_2^2 u_{xx}^2 + P_2^2 u^2 + q_2 w_x - q_2 w_x^2 \right] dx dt \\
+ \int_{t_0}^{t_b} \left[ w b_1 + b_2 u_{tt} \right] dx + \int_{x_0}^{x_b} \left[ w b_1 + b_2 u_{tt} \right] dx \\
- \int_{t_0}^{t_b} \left[ a_1^2 w_x^2 + a_2^2 u_x^2 \right] dx + \int_{t_0}^{t_b} \left[ w_1 u + w_2 u \right] dx \\
- \int \left[ w^2 + \frac{1}{2} w x \right] dx - \int \left[ w^2 + \frac{1}{2} w x \right] dt + \int_{t_0}^{t_b} \left[ w^2 + \frac{1}{2} w x \right] dt \\
- \int_{x_0}^{x_b} \left[ w^2 + \frac{1}{2} w x \right] dt \\
+ \int_{t_0}^{t_b} \left[ w^2 + \frac{1}{2} w x \right] dt \]  \hspace{1cm} (12)

On the other hand, we can perform integration on the original variable to give

\[ I_b = -\int \left( y^T B y_{tt} - y^T A y_{xx} + y^T p y - y^T p y_x + y^T N y_x \right) dx \\
+ \int y^T B y_{tt} dx - \int y^T A y_{xx} dt + \int y^T p y dt \]  \hspace{1cm} (14)

which gives

\[ I_b = \int \left[ -w (b_1^2 w_{tt} - a_1^2 w_{xx} + P_1^2 w + q_1 u_x - q_1 u_x^2) \right] dx \\
+ \int \left( -u (b_1^2 u_{tt} - a_2^2 u_{xx} + P_2^2 u - q_2 w_x + q_2 w_x^2) \right) dx dt \\
+ \int_{t_0}^{t_b} \left[ w b_1 + b_2 u_{tt} \right] dx + \int_{x_0}^{x_b} \left[ w b_1 + b_2 u_{tt} \right] dx \\
- \int_{t_0}^{t_b} \left[ a_1^2 w_x^2 + a_2^2 u_x^2 \right] dx + \int_{t_0}^{t_b} \left[ w_1 u + w_2 u \right] dx \\
- \int \left[ w^2 + \frac{1}{2} w x \right] dx - \int \left[ w^2 + \frac{1}{2} w x \right] dt + \int_{t_0}^{t_b} \left[ w^2 + \frac{1}{2} w x \right] dt \\
- \int_{x_0}^{x_b} \left[ w^2 + \frac{1}{2} w x \right] dt + \int_{t_0}^{t_b} \left[ w^2 + \frac{1}{2} w x \right] dt \]  \hspace{1cm} (15)

\[ I_b = -\left< y, L y \right> + \int_{x_0}^{x_b} \left[ w b_1 + b_2 u_{tt} \right] dx \\
- \int_{t_0}^{t_b} \left[ a_1^2 w_x^2 + a_2^2 u_x^2 \right] dx + \int_{t_0}^{t_b} \left[ w_1 u + w_2 u \right] dx \\
- \int \left[ w^2 + \frac{1}{2} w x \right] dx + \int \left[ w^2 + \frac{1}{2} w x \right] dt - \int_{x_0}^{x_b} \left[ w^2 + \frac{1}{2} w x \right] dt \\
+ \int_{t_0}^{t_b} \left[ w^2 + \frac{1}{2} w x \right] dt \]  \hspace{1cm} (16)
To keep the form symmetrical, we take the average of the previous two expressions

\[ I = \frac{1}{2} I_a + \frac{1}{2} I_b = -\int_{x_0}^{t_b} \frac{1}{2} (-y_{LX} + y_{LX}) dt dx + \]
\[ + \int (y_{TBX} + y_{TBX}) dt \]
\[ - \frac{1}{2} \int (y_{TAYX} + y_{TAYX})^{x_b} dt + \int (y_{TPY} + y_{TPY}) dt \]  

which gives

\[ I = -\frac{1}{2} \langle y, Ly \rangle - \frac{1}{2} \langle y, Ly \rangle + \frac{1}{2} \int_{x_0}^{t_b} \left[ b_1^2 (\frac{\partial}{\partial t} v_{xx} + \sigma_{xx}) + b_2^2 (\frac{\partial}{\partial t} u_{xx} + \sigma_{xx}) \right] dx \]
\[ - \frac{1}{2} \int_{t_0}^{t_b} \left[ a_1^2 (\frac{\partial}{\partial t} v_{xx} + \sigma_{xx}) + b_2^2 (\frac{\partial}{\partial t} u_{xx} + \sigma_{xx}) \right] dt \]
\[ + \frac{1}{2} \int_{t_0}^{t_b} \left[ (\gamma_1 + \gamma_1) w + (\gamma_2 + \gamma_2) u \right] dt \]  

where

\[ L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, \quad L = \begin{bmatrix} - & - \\ - & - \end{bmatrix} \]
\[ L_{11} = L_{11} = b_1^2 \frac{\partial^2}{\partial t^2} + b_1^2 \frac{\partial^2}{\partial x^2}, L_{22} = L_{22} = b_2^2 \frac{\partial^2}{\partial t^2} + b_2^2 \frac{\partial^2}{\partial x^2} \]
\[ L_{12} = L_{12} = - \frac{\partial}{\partial x}, L_{21} = L_{21} = \frac{\partial}{\partial x} \]  

It is noted that we have used the following relationship in obtaining Eq. (21)

\[ \eta_1 - \gamma_1 = 1 \quad \text{and} \quad \eta_2 - \gamma_2 = -1 \]  

by comparing Eqs. (10), (21), and the last equation in the Appendix.

For a spatial and temporal partial system, Eq. (5) becomes

\[ D = \int_{x_0}^{t_b} \int_{t_0}^{t_b} y_{LX} dt dx - \int_{x_0}^{t_b} \int_{t_0}^{t_b} y_{LX} dt dx \]
By equating Eqs. (11) and (14) and solving for D in Eq. (23), we are converting the double integral into single integrals in terms of the boundary conditions.

We can express the quantity $D_e$ as the sum of three parts for end conditions $D_1$, $D_2$, and $D_3$ as

$$D_e = D_1 + D_2 + D_3$$

(24)

The terms in $D_1$ involve the initial conditions of $y$ and $y$ as

$$D_1 = \int_{t_0}^{t_b} (y^T \dot{b}_y - y^T \dot{b}_y) \biggr|_{t_0}^{t_b} \, dx$$

$$= \int_{x_0}^{x_b} \left[ b_1^2 (w_{xT} - w_{xT}) + b_2^2 (u_{xT} - u_{xT}) \right] \biggr|_{t_0}^{t_b} \, dx$$

(25)

The terms in $D_2$ involve the boundary conditions from the second partials of $y$ and $y$ as

$$D_2 = -\int_{t_0}^{t_b} (y^T \dot{a}_y - y^T \dot{a}_y) \biggr|_{x_0}^{x_b} \, dt$$

$$= -\int_{t_0}^{t_b} \left[ a_1^2 (w_{xT} - w_{xT}) + a_2^2 (u_{xT} - u_{xT}) \right] \biggr|_{x_0}^{x_b} \, dt$$

(26)

The terms in $D_3$ involve the boundary conditions from the coupling terms.

$$D_3 = \int_{t_0}^{t_b} (y^T \gamma_y - y^T \gamma_y) \, dt = \int_{t_0}^{t_b} \left[ (\gamma_1 - \gamma_1) w + (\gamma_2 - \gamma_2) u \right] \biggr|_{x_0}^{x_b} \, dt$$

(27)

**THE SYMMETRICAL ADJOINT SYSTEM**

The adjoint independent variable $\tau$ in Figure 1 can be expressed as

$$\frac{\tau - t}{t_b - t_0} = \frac{\tau - t}{t_b - t_0}$$

(28)
Figure 1. Image Reflection of the Adjoint System.
which gives
\[ \tau = \tau_b \quad \text{for} \quad t = t_0 \]  
(29a)
and
\[ \tau = \tau_0 \quad \text{for} \quad t = t_b \]  
(29b)

It is noted from Eq. (28) that
\[ \tau_b - \tau_0 = \tau_b - t_0 \]  
(30a)
\[ \tau = \tau_b + t_0 - t \]  
(30b)
\[ d\tau = -dt \]  
(30c)
\[ \frac{d}{d\tau} = \frac{d}{dt} \]  
(30d)
and
\[ y(x,t) = y(x,\tau = \tau_b + t_0 - t) \]  
(30e)

Let us assume that the adjoint system shown in Figure 1 gives
\[ \hat{y}(x,t=t) = y(x,t=t_b+t_0-t) \]  
(31a)
\[ \hat{y}_t(x,t=t) = -y_t(x,t=t_b+t_0-t) \]  
(31b)
\[ \hat{y}_x(x,t=t) = y_x(x,t=t_b+t_0-t) \]  
(31c)
where \( t \) is a dummy variable for \( t \).

We may define the adjoint system as the image reflection in the time domain of the original system. Equation (31) yields the following known initial conditions:
\[ \hat{y}(x,t=t_b) = y(x,t=t_0) \quad \text{(known)} \]  
(32a)
\[ \hat{y}_t(x,t=t_b) = -y_t(x,t=t_0) \quad \text{(known)} \]  
(32b)
The interpretation of the above equations gives the initial conditions of the original system as the far end conditions for the adjoint system, since the adjoint system is a reflected mirror of the original system in time.
INITIAL CONDITIONS FOR THE ADJOINT SYSTEM

We take a symmetrical approach for the initial conditions of the adjoint system as

\begin{align*}
y(x, t=t_b) &= y(x, t-t_0), \quad y_t(x, t-t_b) = -y_t(x, t=t_0) \\
y(x, t=t_0) &= y(x, t-t_b), \quad y_t(x, t=t_0) = -y_t(x, t-t_b)
\end{align*}

(33)

where \( y \) and \( \dot{y} \) are given in Eq. (10). Thus Eq. (25) becomes

\begin{align*}
D_1 &= \int_{x_0}^{x_b} b_1^2 dx \left[ w(x, t=t_0)w_t(x, t=t_b) + w(x, t=t_b)w_t(x, t=t_0) \right] \\
&\quad - \left[ w(x, t=t_b)w_t(x, t=t_0) + w(x, t=t_0)w_t(x, t=t_b) \right] \\
+ \int_{x_0}^{x_b} b_2^2 dx \left[ u(x, t=t_0)u_t(x, t=t_b) + u(x, t=t_b)u_t(x, t=t_0) \right] \\
&\quad - \left[ u(x, t=t_b)u_t(x, t=t_0) + u(x, t=t_0)u_t(x, t=t_b) \right] = 0
\end{align*}

(35)

Since the integrand of Eq. (35) is zero, the above satisfies Eq. (25).

THE GENERALIZED BOUNDARY CONDITIONS

Let us consider the operator \( L \) in Eqs. (19) through (21). It is assumed that elastic springs are installed at the ends such that

\begin{align*}
y_x(x_b, t) &= K_b y(x_b, t), \quad \dot{y}_x(x_b, t) = K_b \dot{y}(x_b, t) \\
y_x(x_0, t) &= -K_0 y(x_0, t), \quad \dot{y}_x(x_0, t) = -K_0 \dot{y}(x_0, t)
\end{align*}

(36a)

where \( K_b, K_0 \) are diagonal matrices. If Eq. (36) is substituted into Eq. (26), we have

\begin{equation}
D_2 = 0
\end{equation}

(37)

Since \( D_1 = D_2 = 0 \), Eq. (24) becomes \( D_3 \) as given in Eq. (27)

\begin{equation}
D_e = D_3
\end{equation}

(38)
CONDITIONS FOR THE COUPLING TERMS

The sum of the functionals \( J_1 + I \) is obtained by adding Eqs. (6c) and (18) as

\[
J_1 + I + \int_{x_0}^{x_b} \int_{t_0}^{t_b} (Qy+yQ)dxdt + T + B + V + \frac{1}{2}D_3
\]

(39)

where

\[
T = \frac{1}{2} \int_{x_0}^{x_b} \left[ b_1^2(w_t w_t + w_t w_t) + b_2 (u_t u_t + u_t u_t) \right] dx
\]

(40)

\[
B = - \frac{1}{2} \int_{t_0}^{t_b} \left[ a_1^2(w_x w_x + w_x w_x) + a_2 (u_x u_x + u_x u_x) \right] dt
\]

(41)

\[
V = \frac{1}{2} \int_{t_0}^{t_b} \left[ (\gamma_1 + \gamma_1)uw + (\gamma_2 \gamma_2)wu \right] dt
\]

(42)

and

\[
D_3 = \frac{1}{2} \int_{t_0}^{t_b} \left[ (\gamma_1 - \gamma_1)uw + (\gamma_2 - \gamma_2)wu \right] dt
\]

(43)

From the last two equations, one obtains

\[
V + \frac{1}{2}D_3 = \int_{t_0}^{t_b} \left[ (\gamma_1 u w + \gamma_2 w u) \right] dt
\]

(44)

We can let Eq. (44) vanish by choosing

\[
\gamma_1 = \gamma_2 = 0
\]

(45)

which gives

\[
V + \frac{1}{2}D_3 = 0
\]

(46)

From Eq. (22) one obtains

\[
\eta_1 = 1 + \gamma_1 = 1
\]

(46a)

\[
\eta_2 = 1 + \gamma_2 = -1
\]

(46b)
Thus, the functional for the original variables and adjoint variations becomes

\[ J_1 = -I + \langle Q, y \rangle + \langle Q, y \rangle + T + B \quad (47) \]

The sum of the two functionals \( J_2 + I \) is obtained by adding Eqs. (6d) and (18) as

\[ J_2 + I = \int_{x_0}^{x_b} \int_{t_0}^{t_b} (Qy + yQ) \, dx \, dt + T + B + V - \frac{1}{2} D_3 \quad (48) \]

where \( T, B, V \), and \( (1/2)D_3 \) are given in Eqs. (40) through (43). By subtracting \( (1/2)D_3 \) from \( V \) we have

\[ V - \frac{1}{2} D_3 = \int_{t_0}^{t_b} [(\eta_1 u + \eta_2 u)] \, dx \quad (49) \]

In this case we let

\[ \eta_1 = \eta_2 = 0 \quad (50) \]

Then from Eq. (22) one obtains

\[ \gamma_1 = -1 + \eta_1 = -1 \quad (51) \]

\[ \gamma_2 = 1 + \eta_2 = 1 \quad (52) \]

Thus, the functional for the adjoint variables and original variations becomes

\[ J_2 = -I + \langle Q, y \rangle + \langle Q, y \rangle + T + \frac{1}{2} \quad (53) \]

which gives the same form as \( J_1 \) shown in Eq. (47).

**THE FIRST VARIATION**

By taking the variations \( \delta y \) and \( \delta y \) separately, we let

\[ \delta J = \delta J_1(\delta y) + \delta J_2(\delta y) = 0 + 0 \quad (54) \]

Then one obtains from Eqs. (40), (41), and (47) that

\[ \delta J_1(\delta y) = -\delta I(\delta y) + \int Q \delta y \, dx \, dt + \delta T(\delta y) + \delta B(\delta y) = 0 \quad (55) \]

where

\[ \delta T(\delta y) = \frac{1}{2} \int_{x_0}^{x_b} [b_1^2 (\omega T \delta \omega + \omega \delta \omega_T) + b_2^2 (u_T \delta u + u \delta u_T)] \, dx \quad (56) \]
\[ \delta B(\delta y) = - \frac{1}{2} \int_{t_0}^{t_b} \left[ a_1^2 (\delta w_x + \delta w_x^*) + a_2^2 (\delta u_x + \delta u_x^*) \right]_{x_0}^{x_b} \, dt \] (57)

and \( \delta I(\delta y) \) can be derived from Eq. (18) with \( \gamma_1 = \gamma_2 = 0 \) and \( \eta_1 = \eta_2 = 1 \)

\[- \delta I(\delta y) = - \int_{x_0}^{x_b} \int_{t_0}^{t_b} \left\{ [b_1^2 \delta w_t + p_1 \delta w - a_1^2 \delta w_x] + [b_2^2 \delta u_t + p_2 \delta u - a_2^2 \delta u_x^*] + [u_x \delta w - u_x^* \delta w] \right\} dt \, dx \] (58)

The second term on the right side of Eq. (54) is

\[ \delta J_2(\delta y) = - \delta I(\delta y) + \int \int Q_0 \delta y \, dx \, dt + \delta T(\delta y) + \delta B(\delta y) = 0 \] (59)

where

\[ \delta T(\delta y) = \frac{1}{2} \int_{x_0}^{x_b} \left[ b_1^2 (\delta w_t + \delta w_x^*) + b_2^2 (\delta u_t + \delta u_x^*) \right]_{x_0}^{x_b} \, dx \] (60)

\[ \delta B(\delta y) = - \frac{1}{2} \int_{t_0}^{t_b} \left[ a_1^2 (\delta w_x + \delta w_x^*) + a_2^2 (\delta u_x + \delta u_x^*) \right]_{x_0}^{x_b} \, dt \] (61)

and \( \delta I(\delta y) \) can be derived from Eq. (18) with \( \eta_1 = \eta_2 = 0 \) and \( - \gamma_1 = \gamma_2 = 1 \).

\[- \delta I(\delta y) = - \int_{x_0}^{x_b} \int_{t_0}^{t_b} \left\{ [b_1^2 \delta w_t + p_1 \delta w - a_1^2 \delta w_x] + [b_2^2 \delta u_t + p_2 \delta u - a_2^2 \delta u_x^*] + [-u_x \delta u + u_x^* \delta w] \right\} dt \, dx \] (62)

The adjoint equation has the same form of the original equation by dropping and adding the bars simultaneously on every variable.

Equations (55) through (58) are the key equations to be used for the finite element method. It is noted that the first variation \( \delta J_1(\delta y) \) is the same as the first variation \( \delta J_2(\delta y) \) by adding or dropping the bar on top of the variables and their variations. We do not need to solve for the adjoint system in Eqs. (39) through (41) since these give exactly the same solutions as the ones of the original system.
CONCLUSIONS

The functional in bilinear matrix vector form is symmetrical about the original variables and the adjoint variables. The Euler-Lagrange equations for the coupling systems are derived using the fundamental lemma of the calculus of variations. By integrating the bilinear matrix vector expression by parts, one can obtain the bilinear concomitant in terms of initial and boundary terms. The adjoint system can be arranged in a manner that it is a reflected mirror of the original system in time. Thus the initial conditions for the bilinear concomitant become zero. Generalized boundary conditions using many types of "springs" relating the various spatial partial derivatives can be defined to satisfy the boundaries of the concomitant. Algorithms are developed for use in the finite element method by taking the first variations of the functional. These algorithms are simplified because the adjoint system gives exactly the same solutions as those of the original system.
REFERENCES


APPENDIX

The wave equation in rods derived by T. W. Wright (ref A-1) is given as

the following system:

\[ w_{\xi\xi} + \frac{2\lambda}{\lambda+2\mu} u_{\xi} = \frac{c^2}{c_1^2} w_{tt} \quad (A-1a) \]

\[ u_{\xi\xi} - \left[ \delta \frac{\lambda+\mu}{\mu} u + 4 \frac{\lambda}{\mu} w_{\xi} \right] = \frac{c^2}{c_2^2} u_{tt} \quad (A-1b) \]

which can be transformed to

\[ \left( \frac{\lambda+2\mu}{2\lambda} w_{tt} - \frac{\lambda+2\mu}{2\lambda} w_{xx} \right) - u_x = 0 \quad (A-2a) \]

\[ w_x + \left( \frac{\mu c^2}{4\lambda c_2^2} u_{tt} - \frac{\mu}{4\lambda} u_{xx} + \frac{2(\lambda+\mu)}{\lambda} u \right) = 0 \quad (A-2b) \]

With appropriate group of parameters, we have the following form:

\[ b_1^2 w_{tt} - a_1^2 w_{xx} + p_1^2 w - u_x = 0 \quad (A-3a) \]

\[ w_x + b_2^2 u_{tt} - a_2^2 u_{xx} + p_2^2 u = 0 \quad (A-3b) \]

The above system of equations can be expressed by a matrix vector form of equations as

\[ \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (A-4) \]

where

\[ L_{11} = b_1^2 \frac{\partial^2}{\partial t^2} - a_1^2 \frac{\partial^2}{\partial x^2} + p_1^2, \quad L_{12} = -\frac{\partial}{\partial x} \quad (A-5) \]

\[ a_1^2 \]

\[ L_{21} = \frac{\partial}{\partial x} \quad \text{and} \quad L_{22} = b_2^2 \frac{\partial^2}{\partial t^2} - a_2^2 \frac{\partial^2}{\partial x^2} + p_2^2 \quad (A-6) \]

which can be written as Eq. (1) in the text.

The notations for the wave equation in rods are:

- \( w \) = axial displacement
- \( u \) = radial strain
- \( \xi = z/a \) = nondimensional axial coordinates
- \( \tau = ct/a \) = nondimensional time
- \( a \) = elastic stored energy per unit length
- \( c_1 = \sqrt{\frac{(\lambda+2\mu)}{\rho}} \) = longitudinal wave speed
- \( c_2 = \sqrt{\frac{\mu}{\rho}} \) = shear wave speed
- \( \lambda \) and \( \mu \) are Lame' constants

The above system of equations was first developed in different form by Mindlin and Herrmann (ref A-2) and can be grouped into a single equation as:

\[ \left( \frac{\partial^2}{\partial \xi^2} - \frac{c_1^2}{c_2^2} \frac{\partial^2}{\partial \tau^2} \right) \left( \frac{\partial^2}{\partial \xi^2} - \frac{c_2^2}{c_1^2} \frac{\partial^2}{\partial \tau^2} \right) - \delta \frac{\lambda+\mu}{\mu} \frac{c_b^2}{c_1^2} \left( \frac{\partial^2}{\partial \xi^2} - \frac{c_1^2}{c_2^2} \frac{\partial^2}{\partial \tau^2} \right) (w \text{ or } u) = 0 \quad (A-7) \]

APPENDIX REFERENCES


## TECHNICAL REPORT INTERNAL DISTRIBUTION LIST

<table>
<thead>
<tr>
<th>Department</th>
<th>Recipients</th>
<th>Copies</th>
</tr>
</thead>
<tbody>
<tr>
<td>CHIEF, DEVELOPMENT ENGINEERING BRANCH</td>
<td>SMCAR-LCB-D</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-DA</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-DP</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-DR</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-DS (SYSTEMS)</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-DS (ICAS GROUP)</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-DC</td>
<td>1</td>
</tr>
<tr>
<td>CHIEF, ENGINEERING SUPPORT BRANCH</td>
<td>SMCAR-LCB-S</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-SE</td>
<td>1</td>
</tr>
<tr>
<td>CHIEF, RESEARCH BRANCH</td>
<td>SMCAR-LCB-R</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>-R (ELLEN FOGARTY)</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-RA</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-RM</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>-RP</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-RT</td>
<td>1</td>
</tr>
<tr>
<td>TECHNICAL LIBRARY</td>
<td>SMCAR-LCB-TL</td>
<td>5</td>
</tr>
<tr>
<td>TECHNICAL PUBLICATIONS &amp; EDITING UNIT</td>
<td>SMCAR-LCB-TL</td>
<td>2</td>
</tr>
<tr>
<td>DIRECTOR, OPERATIONS DIRECTORATE</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>DIRECTOR, PROCUREMENT DIRECTORATE</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>DIRECTOR, PRODUCT ASSURANCE DIRECTORATE</td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

**NOTE:** PLEASE NOTIFY DIRECTOR, BENET WEAPONS LABORATORY, ATTN: SMCAR-LCB-TL, OF ANY ADDRESS CHANGES.
<table>
<thead>
<tr>
<th>NO. OF COPIES</th>
<th>NO. OF COPIES</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>COMMANDER DEFENSE TECHNICAL INFO CENTER ATTN: DTCIC-DDA CAMERON STATION ALEXANDRIA, VA 22314</td>
<td>COMMANDER ROCK ISLAND ARSENAL ATTN: SMCR-ENM (MAT SCI DIV) ROCK ISLAND, IL 61299</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
</tr>
<tr>
<td>DIRECTOR US ARMY INDUSTRIAL BASE ENG ACTV ATTN: DRXIB-M ROCK ISLAND, IL 61299</td>
<td></td>
</tr>
<tr>
<td>COMMANDER US ARMY TANK-AUTMV R&amp;D COMD ATTN: TELCIB-M WARREN, MI 48090</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>COMMANDER US ARMY TANK-AUTMV COMD ATTN: DRSTA-RC WARREN, MI 48090</td>
<td></td>
</tr>
<tr>
<td>COMMANDER US MILITARY ACADEMY ATTN: CHMN, MECH ENGR DEPT WEST POINT, NY 10996</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>DIRECTOR US ARMY MISSILE COMD REDSTONE SCIENTIFIC INFO CTR ATTN: DOCUMENTS SECT, BLDG. 4484 REDSTONE ARSENAL, AL 35898</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>COMMANDER US ARMY FGN SCIENCE &amp; TECH CTR ATTN: DRXST-SD CHARLOTTESVILLE, VA 22901</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>COMMANDER</td>
<td>NO. OF COPIES</td>
</tr>
<tr>
<td>-------------------------------</td>
<td>---------------</td>
</tr>
<tr>
<td>US ARMY MATERIALS &amp; MECHANICS</td>
<td>2</td>
</tr>
<tr>
<td>RESEARCH CENTER</td>
<td></td>
</tr>
<tr>
<td>ATTN: TECH LIB - DRXMR-PL</td>
<td></td>
</tr>
<tr>
<td>WATERTOWN, MA 01272</td>
<td></td>
</tr>
<tr>
<td>COMMANDER</td>
<td>1</td>
</tr>
<tr>
<td>US ARMY RESEARCH OFFICE</td>
<td></td>
</tr>
<tr>
<td>ATTN: CHIEF, IPO</td>
<td></td>
</tr>
<tr>
<td>P.O. BOX 12211</td>
<td></td>
</tr>
<tr>
<td>RESEARCH TRIANGLE PARK, NC 27709</td>
<td></td>
</tr>
<tr>
<td>COMMANDER</td>
<td>1</td>
</tr>
<tr>
<td>US ARMY HARRY DIAMOND LAB</td>
<td></td>
</tr>
<tr>
<td>ATTN: TECH LIB</td>
<td></td>
</tr>
<tr>
<td>2800 POWDER MILL ROAD</td>
<td></td>
</tr>
<tr>
<td>ADELPHIA, MD 20783</td>
<td></td>
</tr>
<tr>
<td>COMMANDER</td>
<td>1</td>
</tr>
<tr>
<td>NAVAL SURFACE WEAPONS CTR</td>
<td></td>
</tr>
<tr>
<td>ATTN: TECHNICAL LIBRARY</td>
<td></td>
</tr>
<tr>
<td>CODE X212</td>
<td></td>
</tr>
<tr>
<td>DAHLGREN, VA 22448</td>
<td></td>
</tr>
</tbody>
</table>

**Note:** Please notify Commander, Armament Research and Development Center, US Army AMCOM, ATTN: BENET WEAPONS LABORATORY, SMCAR-LCB-TL, Watervliet, NY 12189, of any address changes.