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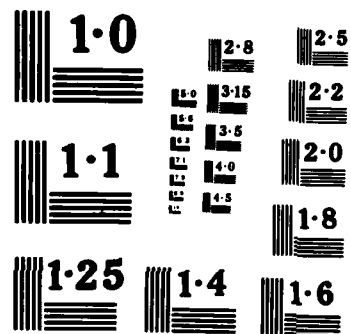UNCLASSIFIED   R/D-4933-EN-09   F/G 8/13   NL

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THEORY OF CONSOLIDATION OF SOFT SEDIMENTS, I

(being the first Interim Report to U.S.A.R.D.S.G. - U.K.
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1.   Introduction

The objectives of the research undertaken are to provide a theory of
consolidation of fine-grained soils of sufficient generality to enable
predictions to be made in engineering situations where the sediment is so soft
that allowance must be made for changes in material properties during the
progress of consolidation, where large strains and displacements must occur,
account being taken of pore water flow in two or three dimensions.

The study is proceeding in two Stages. The first Stage (being undertaken
largely by Prof. R.E. Gibson) involves a critical examination of presently
available theories of consolidation which are modified and extended to meet the
objectives of the research. The aim in Stage I is to develop mathematical
equations governing the consolidation process based on physical assumptions
which accord with the known behaviour of soft sediments.

In Stage II (Principal Investigators: Prof. R.L. Schiffman and Prof. R.E.
Gibson) numerical procedures and associated algorithms will be considered based
on the analysis which would permit prediction of the distribution and time
variation within the medium of quantities of engineering interest, namely: the
pore water pressure, the effective stresses and the void ratio. This may not be
straightforward and analytical modifications to the Stage I equations may be
needed to secure computable results which can be generated economically and
efficiently.

This first Interim Report is devoted to a synoptic account of the progress
achieved in Stage I.

2.   Theories of Consolidation: a brief Survey

The mechanism of the consolidation process in fine-grained soils and its
expression in analytical form were given first by Terzaghi in 1923 [1]. The
theory was restricted to pore water flow in one (Cartesian) dimension but as was
recently pointed out did take some account of large strains [2]. It was aimed
at providing a means for predicting the progress of settlement in uniformly
loaded clay layers. Since that time the theory of one-dimensional consolidation
has been greatly extended to take account of non-linear behaviour, including the
variation during consolidation of the coefficients of permeability and
compressibility, and other effects associated with large strain and deformation
(see, for example, [3], [4], [5]) and this development was made possible by the
intrinsic simplicity of one-dimensional deformation.

The problems of consolidation which face the civil engineer are rarely one-dimensional and a need soon developed for a more general formulation of the theory which would, for example, allow the settlement of structures to be predicted or sand drain installations to be rationally designed: both cases where soil deformation and pore water flow occur in three dimensions. Severe difficulties were soon encountered in this quest and it was found that only by introducing radical simplifying assumptions could any progress be made. This phase of development is associated with Terzaghi and Rendulic - both engineers - and with the mathematician Maurice Biot. The nature of their contributions reflect their difference in outlook and objective. Biot developed a rigorous and self-consistent theory ([6]) based on the assumption that the soil skeleton behaves as a porous perfectly elastic medium, although this is only a crude approximation to the known behaviour of real soils. Terzaghi, on the other hand, tried to avoid commitment to particular constitutive relations between components of strain and effective stress and he succeeded, but only at the cost of implying that during consolidation (under constant loading) all components of total stress remain constant everywhere ([7],[8]). In reviewing these theories in 1943 ([9]) Terzaghi remarks "... the existing theoretical methods for dealing with two- and three-dimensional problems of the consolidation of clay under load are not yet ready for practical application". Owing to the complexity of the problem progress was slow, but by 1969 the consequences of these two differing approaches had been examined in a sufficient number of particular cases for some general conclusions to be drawn by Schiffman and co-workers ([10]).

In all these studies the assumption was invariably made, either tacitly or explicitly, that during consolidation the effective stress-strain relations are linear, the coefficient of permeability of the soil remains constant and that the strains and deformation of the soil are small. However, these factors can have an important influence on the progress of consolidation, in particular upon the magnitude and changing pattern of pore water pressure distribution. More recently engineers have sought to remove those restrictions from the theory, but owing to the complexity of the general case have confined their attention to problems of one-dimensional compression and flow (see, for example, [16],[17],[18]). Contemporaneously Biot's work has been extended to large strains and deformations ([19],[20],[21],[22]), but the "soil" considered remains highly idealized*.

3. The Present Study

The typical problem with which we are concerned here is the consolidation under its own weight of a mound of soft saturated clay which has been placed beneath water. It has been assumed that the strains and deformations that may develop during consolidation are large: that is, account must be taken in the theory of the variation of compressibility and permeability during consolidation.

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*We remark that a theory of large strain consolidation which regards the coefficient of permeability as a soil constant is a contribution exclusively to Applied Mechanics.

Attention has for the present been restricted to plane two-dimensional pore-water flow and one-dimensional soil deformation. The details of the analyses are given in Appendices A (Eulerian description) and B (Lagrangian description). The most promising approach seems at present to lie with a Lagrangian description allied with the void ratio (e) as the dependent variable. The governing equation (B19, B26) which results is non-linear and must be solved numerically using an iterative procedure.

A numerical study of specific cases may reveal that strains and deformations are not unduly large, in which case alternative approximations to those adopted may prove to be more appropriate. This is likely to be helpful if the more general problem of successive accretion of material on an existing mound is to be treated economically, although the present theory can, in fact, be used to cover this case.

4.   Extension to the Present Study

It is intended that the concluding parts of Stage I of the Study should be devoted to an examination of the feasibility of extending the work recorded in Appendix B to cover the cases of axially symmetric and three-dimensional consolidation within the context of the assumptions adopted therein.

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APPENDIX A

In this Appendix we consider the case of pore-water flow and soil skeleton deformation restricted to two dimensions (the x,z-plane) and we seek to derive the equations governing the motion of these phases by adopting the so-called Eulerian scheme of description. We therefore consider the soil grains and water which at any time (t) reside within the rectangular element of sides (δx, δz) located at the point (x,z).

We denote by (w_x, w_z) the velocity components of the solid phase and by (v_x, v_z) those of the pore water. These components will be functions of the three independent variables (x,z,t).

1.   Equations of Continuity

The volume porosity (volume of void space V_v per unit of bulk volume V) we denote by n, so that

$$n = V_v/V,$$

while the area porosity (area of void space A_v per unit of bulk area A) we denote by n_a. If the number of grains occupying the rectangle is in some sense large, it can be shown that

$$n_a = n.$$

The rate of increase of weight of solids within (δx, δz) must equal the net rate of flow of weight across the four faces of the element and this leads us to the equation

$$\frac{\partial}{\partial t}\left[(1-n)\rho_s\right] + \frac{\partial}{\partial x}\left[w_x(1-n)\rho_s\right] + \frac{\partial}{\partial z}\left[w_z(1-n)\rho_s\right] = 0 \qquad (A1)$$

where ρ_s is the unit weight of the solids.

Similarly, for the pore-water

$$\frac{\partial}{\partial t}\left[n\rho_f\right] + \frac{\partial}{\partial x}\left[v_x n\rho_f\right] + \frac{\partial}{\partial z}\left[v_z n\rho_f\right] = 0 \qquad (A2)$$

where ρ_f is the unit weight of the fluid. These are the equations of continuity.

Since, in the range of stress with which we are concerned, the unit weights ρ_s, ρ_f can be regarded as constants, it follows from (A1) and (A2) by addition, that

$$\frac{\partial}{\partial x}\left[nv_x + (1-n)w_x\right] + \frac{\partial}{\partial z}\left[nv_z + (1-n)w_z\right] = 0. \qquad (A3)$$

2. Flow Rule

We shall assume that the pore-water moves through the soil skeleton in accordance with Darcy's law, due account being taken of the fact that it is the relative velocity between the phases which induces a drag on the soil skeleton. Accordingly we write

$$n\left(v_x - w_x\right) = -\frac{k_x}{\rho_f}\frac{\partial u}{\partial x} \qquad (A4)$$

$$n\left(v_z - w_z\right) = -\frac{k_z}{\rho_f}\frac{\partial u}{\partial z} \qquad (A5)$$

where (k_x, k_z) are the coefficients of permeability appropriate to the (x,z) directions and u is an excess pore water pressure which is defined in terms of the pore-water pressure (p) by

$$u = p + \rho_f z + \text{const.} \qquad (A6)$$

where the positive direction of z is against gravity.

3. The Governing Equations

If we set (A4), (A5) in (A3) we find that

$$\frac{\partial}{\partial x}\left[k_x\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial z}\left[k_z\frac{\partial u}{\partial z}\right] = \rho_f\left[\frac{\partial w_x}{\partial x} + \frac{\partial w_z}{\partial z}\right] \qquad (A7)$$

and it can be seen that if the soil skeleton is stationary ($w_x=0$, $w_z=0$) then a familiar equation governing the excess pore-water pressure is obtained, namely

$$\frac{\partial}{\partial x}\left[k_x\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial z}\left[k_z\frac{\partial u}{\partial z}\right] = 0. \qquad (A8)$$

It is worth noting that just after dumping a mound, $w_x = 0$ and $w_z = 0$ almost everywhere and so (A8) holds at this instant; this equation again holds when consolidation is complete.

Now, from (A1)

$$\frac{\partial w_x}{\partial x} + \frac{\partial w_z}{\partial z} = \frac{1}{(1-n)}\left[\frac{\partial n}{\partial t} + w_x\frac{\partial n}{\partial x} + w_z\frac{\partial n}{\partial z}\right] \qquad (A9)$$

and eliminating the rate of soil skeleton dilatation between (A7) and (A9) we find

$$\Box^2 u + \frac{D}{Dt}\left[\ln(1-n)\right] = 0 \qquad (A10)$$

where the operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + w_x \frac{\partial}{\partial x} + w_z \frac{\partial}{\partial z}$$

denotes a differentiation following the motion and

$$\square^2 \equiv \frac{\partial}{\partial x} \left[\frac{k_x}{\rho_f} \frac{\partial}{\partial x} \right] + \frac{\partial}{\partial z} \left[\frac{k_z}{\rho_f} \frac{\partial}{\partial z} \right].$$

In order to proceed further the relations connecting strain increments with the effective stresses and the effective stress increments must be assumed. These relations, together with the equations of equilibrium, would allow a complete theory to be developed. However, in view of the wide variety of soil types and their complicated response to load a completely general approach is at present not feasible.

We shall therefore proceed in the spirit of Terzaghi and Rendulic's early work and seek plausible simplifications which will permit solutions to be obtained to problems which will be sufficiently exact for engineering purposes.

4.   One-dimensional compression

We shall assume here that the development of a mound takes place in a number of stages each consisting of the sudden dumping of soil followed by a period of consolidation during which no further loading occurs.

During the consolidation it will be assumed that:

(a)   lateral displacement of the soil can be ignored.

The mound will therefore consolidate with the development of vertical strains only but the pore water flow we shall not assume to be constrained in any way.

The porosity (or void ratio) at any point in the clay will depend on the initial porosity and state of effective stress there and also upon the subsequent development of the components of effective stress as consolidation proceeds. The detailed effective stress-strain behaviour of soils is very complicated and to obtain an engineering solution to the problem we introduce the following simplifying assumptions:

(b)   The void ratio (e) of a soft saturated clay depends to a good approximation only upon the major principal effective stress σ_1' (the so-called "American Hypothesis"; see, for example, refs. [13-15]).

(c)   The major principal stress is (almost everywhere) equal to the vertical stress σ_{zz} (see ref.[16]) in a mound-like structure.

It follows from these two assumptions that

$$e = e(\sigma_1') = e(\sigma_{zz}'). \tag{A11}$$

For simplicity of exposition we restrict our remarks to the case where $k_x = k_z = k$(const); the more general problem can if required readily be treated later in detail. Equation (A10) can be written in the form:

$$\frac{k}{\rho_f} \nabla^2 u = \frac{1}{(1+e)} \frac{de}{d\sigma_1'} \frac{D\sigma_{zz}'}{Dt} \tag{A12}$$

$$\text{where } \frac{D}{Dt} = \frac{\partial}{\partial t} + w_z \frac{\partial}{\partial z}. \tag{A13}$$

Since

$$\sigma_{zz}' = \sigma_{zz} - p$$

$$c_v \nabla^2 p = \frac{Dp}{Dt} - \frac{D\sigma_{zz}}{Dt} \tag{A14}$$

where

$$c_v = -\frac{k(1+e)}{\rho_f} \frac{d\sigma_1'}{de}$$

which is, apart from the factor $(1+e)$ in place of $(1+e_0)$, Terzaghi's coefficient of one-dimensional consolidation.

5.   One-dimensional consolidation

It is worth noting that in cases of strictly one-dimensional compression and pore water flow, the above assumptions (a), (b) and (c) are satisfied and equation (A14)

$$c_v \frac{\partial^2 p}{\partial z^2} = \frac{Dp}{Dt} - \frac{D\sigma_{zz}}{Dt} \tag{A15}$$

is exact.

This equation governing the pore water pressure has a structure very similar to that encountered in Terzaghi's theory and indeed his equation can be recovered by replacing the differential operator D/Dt by $\partial/\partial t$. From (A13) this can be seen to be tantamount to ignoring, for example, $w_z \partial p/\partial z$ compared with $\partial p/\partial t$.

There is no a priori reason to suppose that this can be justified. Some indication of the error involved can be found from (A3) which reduces to

$$n v_z^f + (1-n) v_z^s = 0 \tag{A16}$$

when there exists a plane on which the condition $v_z^f = v_z^s = 0$ persists. (This is so in the oedometer test: the mid-plane with two-way drainage, or the base if there is no flow from it). It then follows from (A16) and (A5) that

$$w_z = \frac{k}{\rho_f} \frac{\partial u}{\partial z}$$

and so, for example:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{k}{\rho_f} \left(\frac{\partial u}{\partial z}\right)^2. \tag{A17}$$

What then is the equation governing the pore-water pressure which is exact within the context of the assumptions made in the theory, when the Eulerian description is used?

We commence from the form of (A15) which allows for the variation of permeability, namely

$$\frac{1}{m_v} \frac{\partial}{\partial z}\left(\frac{k}{\rho_f} \frac{\partial u}{\partial z}\right) = \frac{Dp}{Dt} - \frac{D\sigma_{zz}}{Dt} \tag{A18}$$

where, from (A6):

$$\frac{Dp}{Dt} = \frac{\partial u}{\partial t} + \frac{k}{\rho_f} \frac{\partial u}{\partial z} \left(\frac{\partial u}{\partial z} - \rho_f\right). \tag{A19}$$

Also

$$\frac{D\sigma_{zz}}{Dt} = \frac{\partial \sigma_{zz}}{\partial t} + \frac{k}{\rho_f} \frac{\partial u}{\partial z} \frac{\partial \sigma_{zz}}{\partial z}, \tag{A20}$$

but vertical equilibrium requires that

$$\frac{\partial \sigma_{zz}}{\partial z} = - n\rho_f - (1-n)\rho_s \tag{A21}$$

Setting (A21) in (A20) and subtracting the resulting equation from (A19), it is found that (A18) becomes

$$\frac{1}{m_v} \frac{\partial}{\partial z} \left[\frac{k}{\rho_f} \frac{\partial u}{\partial z}\right] - \frac{k}{\rho_f}\left(\frac{\partial u}{\partial z}\right)^2 - \frac{k(\rho_s - \rho_f)}{(1+e)\rho_f} \frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} - \frac{\partial \sigma_{zz}}{\partial t}. \tag{A22}$$

Using (A6) a similar equation can be derived for the pore-water pressure (p).

This equation is highly non-linear even in the "thin" layer case ($\rho_s = \rho_f$), and its use in problems of large displacement and strain is limited for the following reasons:

(i)   The parameters k and m_v are related to the void ratio e, but this in turn is not connected to u (or p) in a straightforward way;

(ii) The vertical total stress σ_{zz} and its time derivative $\partial \sigma_{zz}/\partial t$ for a fixed value of z varies with time in a way which is not given as part of the data of the problem and must be discovered as part of the solution;

(iii) The geometry of the mound boundary on which a condition of the type u=0 persists is, again, not known ab initio and must be found during the solution process.

However, one distinct advantage of working in terms of u is that the initial conditions can be specified in terms of the initial state of total stress in the mound. this allows a more complete and exact description than can be achieved when the void ratio or porosity is used as the independent variable (see Appendix B).

A rather more promising governing equation emerges if the porosity (n) is taken as the dependent variable. Commencing from (A10) we can write

$$\frac{1}{\rho_f} \frac{\partial}{\partial z} \left(k \frac{\partial u}{\partial z} \right) = \frac{1}{(1-n)} \left[\frac{\partial n}{\partial t} + v_s \frac{\partial n}{\partial z} \right]. \tag{A23}$$

Now

$$\frac{\partial u}{\partial z} = \frac{\partial p}{\partial z} + \rho_f = \frac{\partial \sigma_{zz}}{\partial z} - \frac{\partial \sigma_{zz}'}{\partial z} + \rho_f \tag{A24}$$

so that using (A21)

$$\frac{\partial u}{\partial z} = -(1-n)(\rho_s - \rho_f) - \frac{d\sigma'}{dn} \frac{\partial n}{\partial z}. \tag{A25}$$

Setting this expression into (A23) we find, after some algebra, that

$$\frac{\partial}{\partial z} \left[c_v \frac{\partial n}{\partial z} \right] = \frac{\partial n}{\partial t} + \left(\frac{\rho_s}{\rho_f} - 1 \right) \frac{d}{dn} \left[k(1-n)^2 \right] \frac{\partial n}{\partial z} \tag{A26}$$

which is essentially equation (23) of ([5]). It is not open to objections (i) and (ii) above. The difficulty (iii) remains but it has been claimed that this can be overcome by using suitable numerical techniques.*

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*Lee and Sills (ref.[5]) use a numerical technique originating from work by Crank and Gupta ([11],[12]) which effectively updates the position of the moving boundary as the solution proceeds. They take the case of a "thin" soil layer where the approximation $\rho_s = \rho_f$ can be justified on physical grounds, and compare their numerical solution with that given by Gibson, England and Hussey (Ref.4). The agreement is apparently excellent, but is not wholly convincing as the coefficient c_F in ref.[4] is taken incorrectly as $c_F = c_v/(1+e)^2$ instead of $c_F = c_v(1+e_0)^2/(1+e)^2$.

APPENDIX B

In this Appendix we consider the physical problem discussed in Appendix A, in particular the case of two-dimensional pore-water flow and one-dimensional compression in a submarine clay mound. However, here we use the Lagrangian scheme of description and examine the motion of an element of soil which contains the same solids throughout its history.

For reasons mentioned in Appendix A we shall not seek to derive an equation governing the pore-water pressure (or an excess pore-water pressure), but rather work in terms of the void ratio (e) or the porosity (n). The same physical assumptions used there will again be adopted here. The argument which ensues follows closely that used in ref.[*] and we shall not repeat the details here.

1.   Equations of Continuity

The deformation of the soil skeleton is described by following the motion of an element of initial area ($\delta a \times \delta b$) lying at t = 0 at (a,b)* which during its subsequent motion always contains the same solids. The location of a material point initially at (a,b) is given subsequently by

$$\xi = \xi(a,b,t) \tag{B1}$$

$$\eta = \eta(a,b,t) \tag{B2}$$

where, by definition,

$$a = \xi(a,b,o) \tag{b3}$$

$$b = \eta(a,b,0). \tag{B4}$$

Since we shall assume that the motion of the soil skeleton is confined to the a direction it follows that (B2) may be replaced by

$$\eta = b. \tag{B2}\underline{bis}$$

The equation of continuity of solids takes the simple form

$$\frac{\partial \xi}{\partial a} = \frac{1-n_0}{1-n} = \frac{1+e}{1+e_0} \tag{B5}$$

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*the co-ordinate a points upward against gravity.

where

$$n_0 = n(a,b,0)$$

$$e_0 = e(a,b,0)$$

are, respectively, the initial (t = 0) porosity and void ratio at (a,b).

As the element translates and deforms during its subsequent motion, pore-water moves into and out from its four faces and the equation of continuity of this (incompressible) phase is found to be

$$\frac{\partial}{\partial a}\left[n(v_a - w_a)\right] + \frac{\partial}{\partial b}\left(nv_b\frac{\partial\xi}{\partial a}\right) + \frac{\partial}{\partial t}\left(n\frac{\partial\xi}{\partial a}\right) = 0. \tag{B6}$$

2. Flow Rule

Darcy's Law takes the form*

$$n(v_a - w_a) = -\frac{k}{\rho_f}\frac{\partial u}{\partial \xi} \tag{B7}$$

$$n(v_b - w_b) = -\frac{k}{\rho_f}\frac{\partial u}{\partial \eta} \tag{B8}$$

with this description, but since

$$\frac{\partial u}{\partial a} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial a}, \tag{B9}$$

$$\frac{\partial u}{\partial b} = \frac{\partial u}{\partial \eta} \tag{B10}$$

and $$w_b = 0 \tag{B11}$$

it follows that

$$n(v_a - w_a)\frac{\partial \xi}{\partial a} = -\frac{k}{\rho_f}\frac{\partial u}{\partial a} \tag{B12}$$

$$nv_b = -\frac{k}{\rho_f}\frac{\partial u}{\partial b}. \tag{B13}$$

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* We have taken $k_a = k_b = k$ here, but the extension to the anisotropic case is trivial.

The excess pore-water pressure ($\underline{u}$) is related to the pore-water pressure (p) by

$$u = p + \rho_f \xi + \text{const.} \tag{B14}$$

3. The Governing Equations

Setting (B12) and (B13) in (B6) we find

$$\frac{\partial}{\partial a}\left[\frac{k}{\rho_f}\frac{\partial u}{\partial a}(\frac{1+e_0}{1+e})\right] + \frac{\partial}{\partial b}\left[(\frac{1+e}{1+e_0})\frac{k}{\rho_f}\frac{\partial u}{\partial b}\right] = \frac{\partial}{\partial t}\left[\frac{e}{1+e_0}\right] \tag{B15}$$

where (B5) has been used. Using now (B5) and (B14), equation (B15) can be expressed in terms of the pore-water pressure:

$$\frac{\partial}{\partial a}\left[\frac{k}{\rho_f}\frac{\partial p}{\partial a}(\frac{1+e_0}{1+e}) + k\right] + \frac{\partial}{\partial b}\left[\frac{k}{\rho_f}\frac{\partial p}{\partial b}(\frac{1+e}{1+e_0})\right] = \frac{1}{(1+e_0)}\frac{\partial e}{\partial t}. \tag{B16}$$

Vertical equilibrium requires that

$$\frac{\partial \sigma}{\partial a} + \left[n\rho_f + (1-n)\rho_s\right]\frac{\partial \xi}{\partial a} = 0, \tag{B17}$$

where σ is the vertical total stress, and this may be used in conjunction with

$$p = \sigma - \sigma' \tag{B18}$$

which defines the vertical effective stress (σ'), to eliminate p from (B16).

The resulting equation governing the void ratio (e) is

$$\frac{\partial}{\partial a}\left[\frac{C_F}{1+e_0}\frac{\partial e}{\partial a}\right] + \frac{\partial}{\partial b}\left[C_F\frac{(1+e)^2}{(1+e_0)^3}\frac{\partial e}{\partial b}\right] - (G_S-1)\frac{d}{de}(\frac{k}{1+e})\frac{\partial e}{\partial a} + \frac{\partial}{\partial b}\left[\frac{k}{\rho_f}(\frac{1+e}{1+e_0})\frac{\partial \sigma}{\partial b}\right]$$

$$= \frac{1}{(1+e_0)}\frac{\partial e}{\partial t}$$

in the general case, which reduces when e_0 is a constant to

$$\frac{\partial}{\partial a}\left[C_F\frac{\partial e}{\partial a}\right] + \frac{\partial}{\partial b}\left[C_F(\frac{1+e}{1+e_0})^2\frac{\partial e}{\partial b}\right] - (G_S-1)\frac{d}{de}\left[k(\frac{1+e_0}{1+e})\right]\frac{\partial e}{\partial a} + \frac{\partial}{\partial b}\left[\frac{k}{\rho_f}(1+e)\frac{\partial \sigma}{\partial b}\right] = \frac{\partial e}{\partial t} \tag{B19}$$

where (as in ref.") the coefficient of finite consolidation

$$C_F = -\frac{k}{\rho_f}\frac{(1+e_0)^2}{(1+e)}\frac{d\sigma'}{de} \tag{B20}$$

and

$$G_S = \rho_s/\rho_f. \tag{B21}$$

4.   The total stress term

The last term on the left-hand side of (B19) is not known a priori and must be found and updated during the solution. In this Section we show how the term $\partial\sigma/\partial b$ can be found at any time from the current void ratio distribution $e(a,b,t)$.

Consider a vertical cylinder of material, of unit cross-sectional area, extending from the base of the mound to the water level above the mound and denote this height by H. It is evident that the weight of material (solids and water) within this cylinder would remain constant if no water flowed across the surfaces (i.e. strict one-dimensional consolidation) or was added (H = constant). Under these conditions $\partial\sigma/\partial b$ on the base depends on b alone. However, on the surface of the mound $a = a_0(b)$ settlement increases the water pressure and hence the total vertical stress there. Thus at the top of the clay $\partial\sigma/\partial b$ depends both on b and t. At any other height a, or when the conditions of simple vertical flow of pore-water no longer obtain or H varies with time, the term $\partial\sigma/\partial b$ can be expected to depend upon (a,b,t). We shall now determine this relation.

Integrating (B17) with respect to a, and using (B5), it is found that

$$\sigma = \rho_f \int_a^{a_0(b)} \frac{e}{1+e_0}da + \rho_s \int_a^{a_0(b)} \frac{da}{1+e_0} + \rho_f\left[H(t) - \xi(a_0(b),b,t)\right] \quad (B22)$$

where the last term represents the water pressure on the surface $a = a_0(b)$ of the mound, which equals the total stress σ in (B22) when $a = a_0(b)$.

From (B5):

$$\xi(a,b,t) = \int_0^a \frac{1+e(a,b,t)}{1+e_0(a,b)} da \quad\quad (B23)$$

since $\xi(o,b,t) = 0$. Substituting (B23) into (B22) and re-arranging, it is found that*

$$\sigma = \rho_f H - \rho_f\int_0^a \left(\frac{1+e}{1+e_0}\right)da + (\rho_s-\rho_f)\int_a^{a_0} \frac{da}{1+e_0}. \quad\quad (B24)$$

It follows from (B24) that

$$\frac{\partial\sigma}{\partial b} = - \rho_f\int_0^a \frac{\partial}{\partial b}\left(\frac{1+e}{1+e_0}\right)da + (\rho_s-\rho_f)\left[\frac{1}{1+e_0(a_0,b,t)} \frac{da_0}{db} + \int_a^{a_0} \frac{\partial}{\partial b}\left(\frac{1}{1+e_0}\right)da\right],$$

$$(B25)$$

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* The first term represents the weight of water if it completely filled the cylinder from the base to the water surface. The second term is the weight of water if it filled the cylinder from the base to the height $\xi(a,b,t)$ at which σ is acting. The last term is the buoyant weight of solids above the plane on which σ acts.

which reduces in the case e_0 = const. to

$$\frac{\partial \sigma}{\partial b} = \frac{(\rho_s-\rho_f)}{1+e_0} \frac{da_0}{db} - \frac{\rho_f}{1+e_0} \int_0^a \frac{\partial e}{\partial b} \, da, \qquad (B26)$$

the first term arising from the initial non-uniform surface profile of the mound, the second being a time-varying correction to this resulting from the settlement.

5.   Some typical boundary conditions

If a mass of saturated clay of more-or-less uniform void ratio e_0 is deposited in a short period of time beneath water and rests on (say) an impervious base, the void ratio distribution in the mound at subsequent times will, on the basis of the assumptions that have been made herein, be governed by (B19) with (B26) and subject to:

(a) the initial condition

$$e(a,b,c) = e_0 (\text{constant}); \qquad (B27)$$

(b) the mound surface boundary condition

$$e(a_0(b),b,t) = e_0 (\text{constant}); \qquad (B28)$$

(c) the impervious base condition

$$w_a(o,b,t) = 0 \qquad (B29)$$

$$v_a(o,b,t) = 0 \qquad (B30)$$

which result in the following condition on the void ratio:

$$\frac{\partial e}{\partial a} + \frac{(\rho_s-\rho_f)}{(1+e_0)} \frac{de}{d\sigma'} = 0 \qquad (B31)$$

on $a = 0$.

END

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