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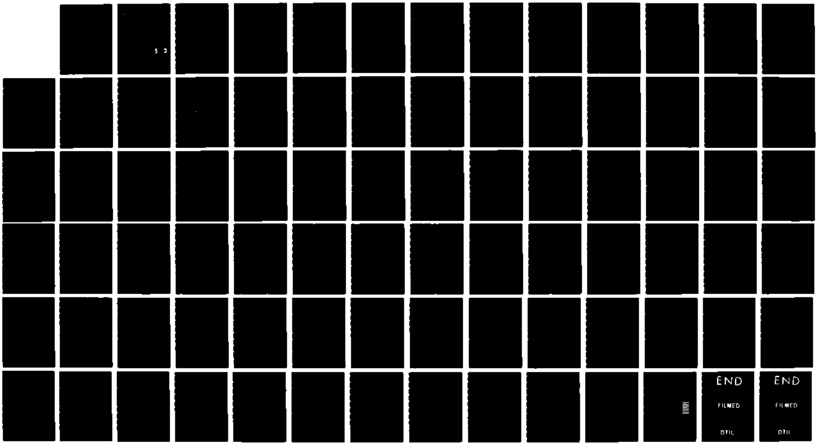

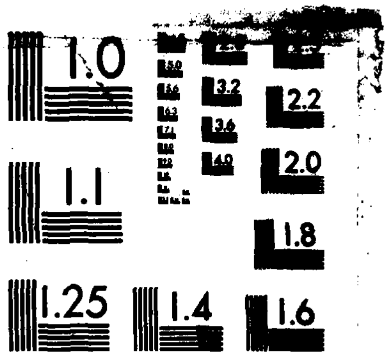
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AD-A157 637

ON THE MOTION OF RODS WITH INITIAL SPACE CURVATURE

Herbert B. Kingsbury

June 1985

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Previous formulations and models of the vibrations of gun tubes have been based on the assumption of an axisymmetric distribution of material, i.e. a perfectly made albeit curved gun tube. This report contains a new formulation for the mathematical problem related to gun tube vibrations with this constraint removed: the formulation admits non-colinear centroidal and bore axes, reflecting the actual case with production gun tubes. These factors may play a significant role in "jump" inherent in direct fire weapons

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including aspects related to "fleet zero" or other zeroing procedures in tank guns.

The general large deformation equations of motion are derived for a linearly elastic rod with initial space curvature. The equations allow examination of such effects as transverse shear deformation, rotatory inertia, initial twist, variable cross section size and shape, and variable curvature and torsion on the motion and deformation of the rod. A linearized engineering theory is developed from the general results and equations for several special cases are formulated~~. The~~ detailed mathematical approach is discussed. Keywords: → to field 19

The study includes only theoretical aspects and derivations related to the formulation. Studies to investigate the importance of the newly-introduced terms on gun tube vibrations and flexure, and the numerical implementation of the formulation have not yet been undertaken. Additional work is planned.

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I. INTRODUCTION

The work described in this report was motivated by the need to explain observed inconsistencies between measured dynamic response data and response predictions of various analytical and finite element models of certain rod-like structures.

It was felt that a sufficiently general mathematical model of this class of structures might illuminate hitherto unnoticed dynamic effects and interactions caused by such phenomena as large displacements and rotations, initial space curvature, variable shape and area of cross section, initial twist of cross section principal coordinates along the rod axis, transverse shear deformation and rotatory inertia.

Although the study of vibrations of rods and bars is one of the most ancient in structural mechanics, a completely general formulation of the problem is difficult to find in the literature.

One of the seminal treatments of the derivation of equations of motion of rods with initial curvature appears in the text on Elasticity by A.E.H. Love.[1] Although restrictive forms of Love's equations are used directly, or are re-derived, by subsequent investigators of small amplitude vibrations of curved rods, there are deficiencies in Love's work which render his strain-displacement equations unsuitable as the starting point for a general examination of the motion of rods with space curvature.

In Love's work, the final state of deformation is assumed to be such that cross sections remain plane and normal to the centerline of the deformed rod. By these assumptions not only is transverse shear deformation excluded but, as will be shown, kinematical inconsistencies are introduced.

Love employs a two stage deformation process consisting of first the imposition of twist and curvature on the initially straight rod centerline followed by imposition of small displacements to points of the cross section. This approach leads to confusion of local coordinate systems and of displacement components which renders suspect his expression for the change in length of line segment connecting adjacent points. The generality of the strain-displacement equations presented is further limited by the assumptions that displacements are small and that dimensions of the rod cross section are small in comparison to the radius of curvature. Finally, strain-displacement equations for an initially curved rod appear explicitly only in a restrictive, plane curvature, form.

Equations governing both static and dynamic deformation of curved rods are employed in such areas of structural mechanics as vibrations of curved beams, large deflection and stability analysis of beams and columns and non-linear dynamics, including stability of oscillations of beams and strings. A bibliography of this literature with emphasis of curved beam vibrations is included with this report.

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1 *A.E.H. Love, A Treatise On The Mathematical Theory of Elasticity, 4th Ed., Dover Publication, New York, 1927.*

In spite of the large number of papers dealing with these subjects, there does not appear to have been a re-examination of Love's strain-displacement equations or a more general independent derivation. The equations of motion employed by the various investigators are either those presented by Love or are derived for each particular application on an "ad-hoc" basis.

In this report a derivation of the equilibrium equations for a rod with general space curvature is presented in Section II. This derivation is similar to that presented by Love, or more recently by Kachanov.[2] although it is somewhat more rigorous than those in its examination of the relative importance of higher order terms in Taylor's series expansions.

A completely general set of small strain but large displacement and rotation strain-displacement equations is then formulated in Section III for a rod with initial twist and arbitrary space curvature. In these equations the displacement components are referred to the local tangent, principal normal and bi-normal space curve coordinates rather than to fixed global coordinates. No assumptions regarding the structural action or mode of deformation of the rod are made so that the functional form of the displacement components is left unspecified.

In Section IV a linearized form of the general strain-displacement equations is first presented. A technical theory of rods with space curvature is then developed based in the usual beam theory assumptions that cross sections remain plane and undeformed. The displacements are then expressed as appropriate linear combinations of three central curve displacement components and three rotation components.

Based upon these displacement functions, the inertia force and moment terms appearing in the equilibrium equations can then be formulated in terms of time derivatives of these six displacement variables.

The linearized strain displacement equations phrased in terms of the central curve displacements and cross section rotations are then used to derive force-moment-strain-curvature equations for a linear elastic rod. The equations are then further simplified by elimination of transverse shear and rotatory inertia effects. When these equations are combined the equilibrium equations, a system of four coupled equilibrium equations in the four displacement and rotation variables results.

The final section of the report is devoted to examination of the displacement equilibrium equations for special cases in order to examine the conditions under which various uncouplings of motions occur.

2 *L.M. Kachanov, "A Brief Course On The Theory of Buckling," T&AM Report No. 441. University of Illinois Urbana-Champaign, 1980.*

II. EQUILIBRIUM EQUATIONS FOR A ROD ELEMENT

A. Force and Moment Equilibrium

A typical element of a rod with space curvature is shown in Figure 1.

The line joining centroids of cross sections is a space curve whose radius (ρ), or curvature $\kappa=1/\rho$, and torsion (λ) * are arbitrary functions of arc length (s). The shape of the cross sections is arbitrary but assumed to be a continuous function of s. At the centroid of any cross section an orthogonal coordinate system (x,y,z) is constructed with corresponding unit vectors **i**, **j**, **k**, such that **k** is the unit tangent vector to the space curve determined by the line of centroids. Vector **k** therefore points in the direction of increasing s. Unit vectors **i** and **j** are the principal normal and bi-normal vectors respectively so that **jxk** = **i**.

The principal directions of the cross section area of the rod with respect to the centroid are, in general, inclined with respect to the x, y coordinates. The angle between these two sets of coordinate axes may vary continuously with s for the case of a pre-twisted rod.

Figure 2 shows a section of curved rod of incremental length δs lying between points 0 and 0' of the space curve of centroids. By the usual definition, the face at 0 is a negative face since its outward normal points in the direction opposite to the local tangent vector while the face at 0' is a positive face. The usual sign convention for stress components in positive and negative faces will be employed.

The force and moment resultants of the stress components (σ_{xz}, σ_{yz}, σ_{zz}) acting on a cross section of current area A are defined as

$$V_x = \int_A \sigma_{xz} dA \ , \qquad (1a)^{**}$$

$$V_y = \int_A \sigma_{yz} dA \ , \qquad (1b)$$

$$V_z = \int_A \sigma_{zz} dA \ , \qquad (1c)$$

$$M_x = \int_A y \, \sigma_{zz} dA \ , \qquad (2a)$$

$$M_y = - \int_A x \, \sigma_{zz} dA \ , \text{ and} \qquad (2b)$$

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* *Appendix A presents a discussion of space curve geometry*

** *Equations are numbered consecutively within each section.*

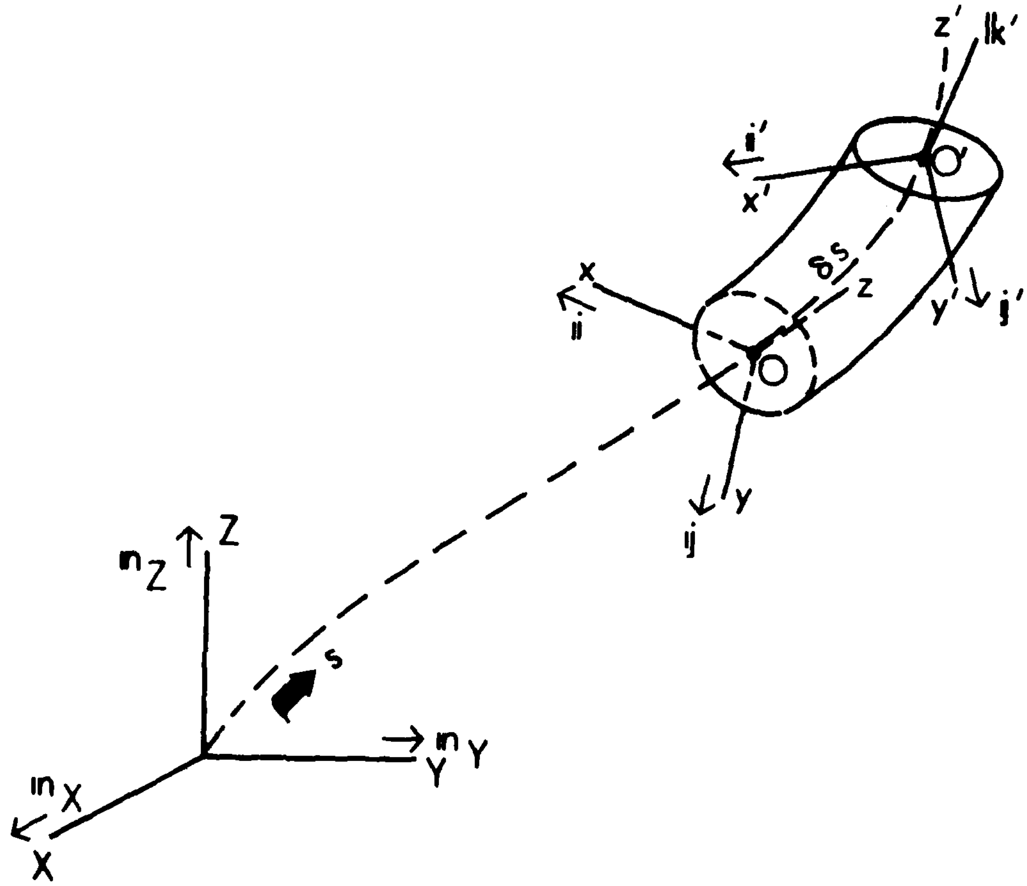

Figure 1. Space Curve Coordinates

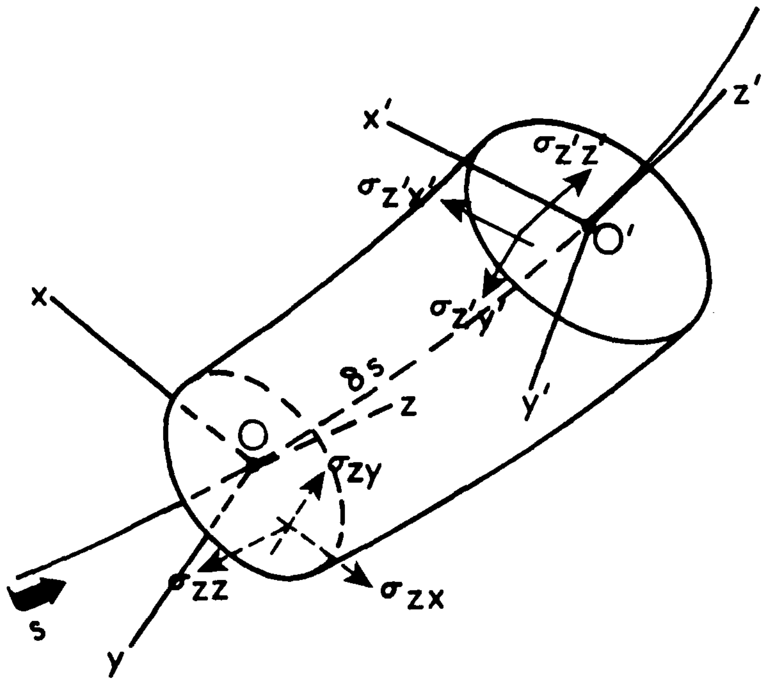

Figure 2. Stress Sign Convention

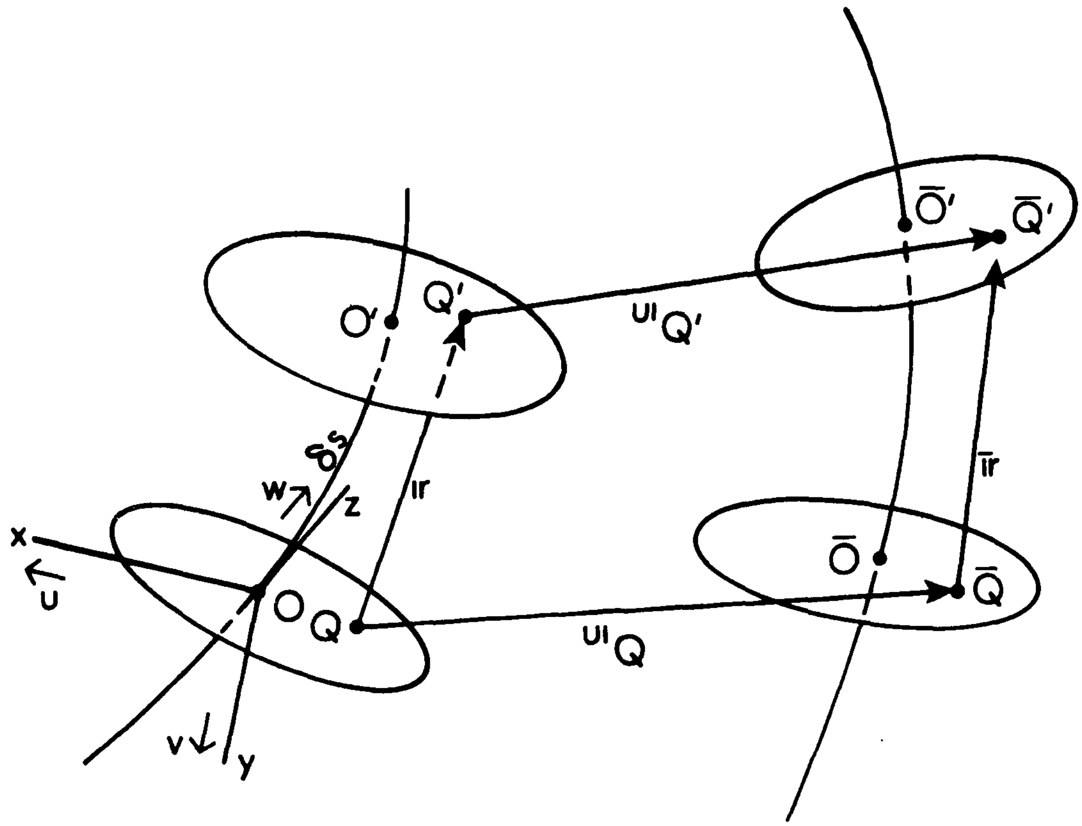

Figure 6. Displacements of Points of Adjacent Cross Sections

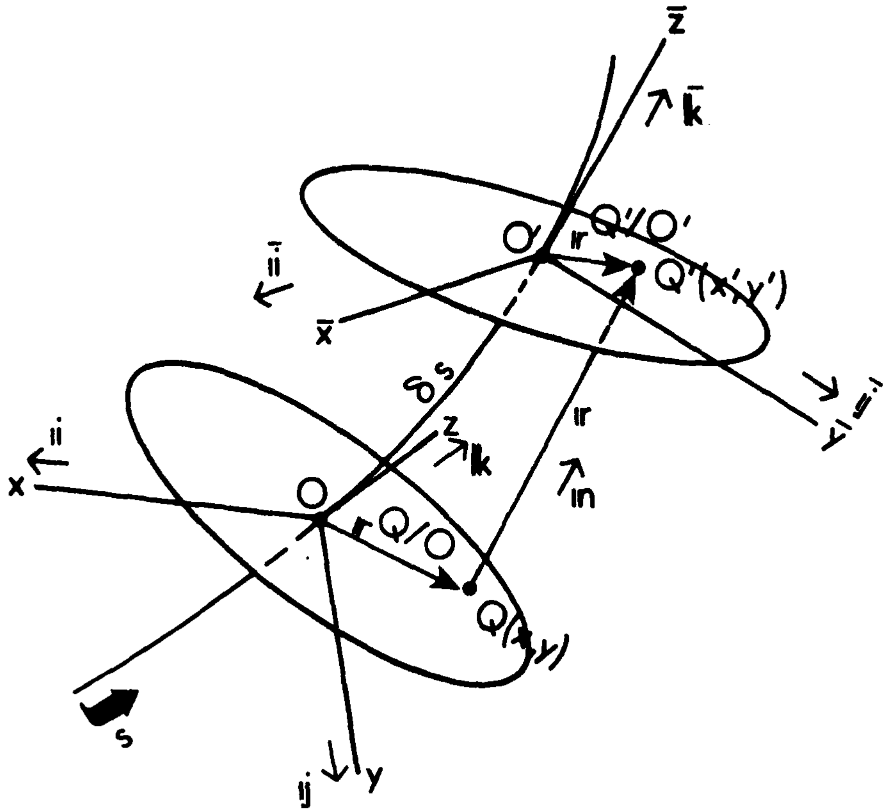

Figure 5. Adjacent Cross Sections Before Deformation

$$m_{Iz} = \int_A \{y\, F_{Ix} - x\, F_{Iy}\}\, dA = \int_A \gamma y \ddot{u}\, dA - \int_A \gamma x \ddot{v}\, dA \,. \quad (26c)$$

III. STRAIN-DISPLACEMENT EQUATIONS

Equations relating the components of strain at a point in the rod to the displacement field are derived by considering the change in length of a line segment connecting to arbitrarily close points of the rod as the rod is deformed.

Figure 5 shows two adjacent cross-sections of the rod before deformation. The centroidal curve has initial curvature κ_o and torsion λ_o.

The centroid of one cross-section is point O and the other, at a small distance δs along the curve, is O'. The vector principal normal, binormal and tangent at O are **i**, **j** and **k** respectively while the corresponding vectors at O' are denoted by $\bar{\mathbf{i}}$, $\bar{\mathbf{j}}$ and $\bar{\mathbf{k}}$. Coordinates x, y, z directed along **i**, **j** and **k** respectively comprise a local coordinate system with origin at O. Corresponding coordinates at O' are $\bar{x}$, $\bar{y}$ and $\bar{z}$.

Point Q lies in the cross-section at O while Q' lies in that at O'. It is assumed that Q and Q' are initially arbitrarily close so that the position vector of Q', **r** can be expressed as

$$\mathbf{r} = \delta x\, \mathbf{i} + \delta y\, \mathbf{j} + \delta z\, \mathbf{k} \quad (1)$$

where δx, δy, and δz are arbitrarily small distances.

If **n** is a unit vector parallel to **r** and directed from Q to Q', and r is the magnitude of **r** . Then

$$\mathbf{r} = r\mathbf{n} = r\,\ell\, \mathbf{i} + rm\, \mathbf{j} + r\, n\, \mathbf{k} \quad (2)$$

when ℓ, m, and n are the direction cosines of **n** relation to **i**, **j** and **k** respectively. Equations (1) and (2) imply the relationships

$$\delta x = r\ell, \quad \delta y = rm, \quad \text{and} \quad \delta z = rn \,. \quad (3)$$

The length of the position vector joining material points initially at Q and Q' as the rod undergoes a deformation which carries these points to $\bar{Q}$ and $\bar{Q}'$ respectively is next determined.

For a discussion of various representations of the angular acceleration and angular velocity vectors the reader is referred to a recent book by Kane, et. al.[4]

If

$$\mathbf{F}_I = F_{Ix}\, \mathbf{i} + F_{Iy}\, \mathbf{j} + F_{Iz}\, \mathbf{k}$$

the scalar inertia forces become

$$F_{Ix} = - \gamma \ddot{u} \,, \tag{24a}$$

$$F_{Iy} = - \gamma \ddot{v} \,, \text{ and} \tag{24b}$$

$$F_{Iz} = - \gamma \ddot{w} \,. \tag{24c}$$

Upon integrating over the area of the cross section, A, the inertia forces per unit length of rod arising from the three body force components are next obtained,

$$T_{Ix} = \int_A F_{Ix} dA = - \int_A \gamma\, \ddot{u}\, dA \,, \tag{25a}$$

$$T_{Iy} = \int_A F_{Iy} dA = - \int \gamma\, \ddot{v}\, dA \,, \text{ and} \tag{25b}$$

$$T_{Iz} = \int_A F_{Iz}\, dA = - \int \gamma\, \ddot{w}\, dA \,. \tag{25c}$$

Finally the inertia force couples are presented in terms of the three acceleration components

$$m_{Ix} = \int_A y\, F_{Iz}\, dA = - \int_A \gamma\, y\, \ddot{w}\, dA \,, \tag{26a}$$

$$m_{Iy} = \int_A x\, F_{Iz}\, dA = \int_A \gamma x \ddot{w}\, dA \,, \text{ and} \tag{26b}$$

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4 *T.R. Kane, P.W. Likins, and D.A. Levinson, "Spacecraft Dynamics," McGraw Hill, New York, 1983.*

The reader is referred to a paper by E. Reissner[3] for corresponding large displacement curved beam equilibrium equations in which the force and moment components are referred to the undeformed coordinates.

B. Inertia Forces and Couples

The inertia body force, $\mathbf{F}_I$, acting at any point, Q, in a cross section is defined by

$$\mathbf{F}_I = - \gamma \ddot{\mathbf{u}}_Q \tag{22}$$

where γ is the mass density of the material of that point, $\mathbf{u}_Q$ is the displacement vector associated with that point and superscript "dot" indicates differentiation with respect to time.

The displacement vector $\mathbf{u}_Q$ is expressed in terms of the local space curve coordinates as

$$\mathbf{u}_Q = u\,\mathbf{i} + v\,\mathbf{j} + w\,\mathbf{k}$$

where u, v, and w are the scalar displacement components in the x, y and z directions respectively.

Since the local coordinate system is moving in the fixed (inertial) reference frame, the acceleration vector may be expressed in the form:

$$\ddot{\mathbf{u}}_Q = \mathbf{a}_Q + \alpha \times \mathbf{u}_Q + \omega \times (\omega \times \mathbf{u}_Q) + 2\,\omega \times \mathbf{v}_Q \tag{23}$$

where

$$\mathbf{v}_Q = \dot{u}\,\mathbf{i} + \dot{v}\,\mathbf{j} + \dot{w}\,\mathbf{k}\,,$$

$$\mathbf{a}_Q = \ddot{u}\,\mathbf{i} + \ddot{v}\,\mathbf{j} + \ddot{w}\,\mathbf{k}\,,$$

and ω and α are the angular velocity and angular acceleration vectors respectively of the local coordinates in the fixed reference frame. For problems involving small rotations and/or small displacements of the rod cross the non-linear terms in Eq. (23) may often be neglected in comparison with $\mathbf{a}_Q$.

Since this report will be concerned primarily with equations governing small motions the non-linear terms in Eq. (23) will henceforth be neglected.

[3] *E. Reissner, "On One-Dimensional Large-Displacement Finite-Strain Beam Theory," Studies in Applied Math, Vol 52, pp 87-95, 1973.*

$$\mathbf{V}(s^+) = V_x\,\mathbf{i} + V_y\,\mathbf{j} + V_z\,\mathbf{k} \text{ and}$$

$$\mathbf{m}(s) = m_x\,\mathbf{i} + m_y\,\mathbf{j} + m_z\,\mathbf{k}\,,$$

Eq. (19) becomes

$$\left(\frac{\partial M_x}{\partial s} - \lambda M_y + \kappa M_z - V_y + m_x\right)\mathbf{i}$$

$$+ \left(\frac{\partial M_y}{\partial s} + \lambda M_x + V_x + m_y\right)\mathbf{j}$$

$$+ \left(\frac{\partial M_z}{\partial s} - \kappa M_x + m_z\right)\mathbf{k} = 0\,.$$

The three scalar equations representing the conditions of moment equilibrium for the rod element become

$$\frac{\partial M_x}{\partial s} - \lambda M_y + \kappa M_z - V_y + m_x = 0\,, \tag{21a}$$

$$\frac{\partial M_y}{\partial s} + \lambda M_x + V_x + m_y = 0\,, \text{ and} \tag{21b}$$

$$\frac{\partial M_z}{\partial s} - \kappa M_x + m_z = 0\,. \tag{21c}$$

Although this manner of derivation of the equations of force and moment equilibrium for a curved rod used above is different from that presented by Love[1] the resulting equations are essentially identical. Differences in form result from differences in force and moment sign conventions and because Love employs local coordinate axes which are the principal directions of the area of the local cross section of the rod rather than the natural space curve coordinates.

In general, the local x, y, z, coordinates are the deformed coordinates of the centroidal curve of the rod. The torsion and curvature terms appearing in Eqs. (12) and (21) are the total values of these quantities consisting of the sums of their initial values and their respective changes due to deformation of the centroidal curve. For small deformation analysis the initial coordinates and initial values of λ and κ are employed.

$$\mathbf{R}' \times \mathbf{V}(s+\delta s) = [\mathbf{k}\,\delta s + \kappa/2\ \mathbf{i}\,(\delta s)^2] \times [\mathbf{V}(s^+) + \frac{\partial \mathbf{V}}{\partial s})\Big|_{s^+}\delta s$$

$$+ \frac{\partial^2 \mathbf{V}}{\partial s^2}\Big|_{s^+} \tfrac{1}{2}(\delta s)^2]$$

$$= \mathbf{k} \times \mathbf{V}(s^+)\,\delta s + [\tfrac{1}{2}\kappa\ \mathbf{i} \times \mathbf{V}(s^+) + \mathbf{k} \times \frac{\partial \mathbf{V}}{\partial s}\Big|_{s^+}]\,(\delta s)^2 \text{ and} \qquad (17)$$

$$\mathbf{M}(s+\delta s) \doteq \mathbf{M}(s^+) + \frac{\partial \mathbf{M}}{\partial s}\Big|_{s^+}\delta s + 1/2\ \frac{\partial^2 \mathbf{M}}{\partial s^2}\Big|_{s^+}(\delta s)^2 , \qquad (18)$$

where again $\mathbf{V}(s^+)$ and $\mathbf{M}(s^+)$ indicate the force and moment resultants acting on the positive face at $\delta s = 0$ so that $\mathbf{M}(s^+) = -\mathbf{M}(s)$.

Finally, substitution of the result given by Eqs. (14) and (16) through (18) into Eq. (13) and division by δs yields the following vector moment equilibrium equations in the limit as $\delta s \to 0$.

$$\frac{\partial \mathbf{M}}{\partial s}\Big|_{s^+} + \mathbf{k} \times \mathbf{V}(s^+) + \mathbf{m}(s) = 0 \qquad (19)$$

To obtain the scalar equations represented by Eq. (19), $\mathbf{M}(s^+)$ is expressed in terms of its vector components and the Serret-Frenet formulae are used to transform the derivatives of the unit vectors

$$\frac{\partial \mathbf{M}}{\partial s}\Big|_{s^+} = \frac{\partial M_x}{\partial s}\mathbf{i} + \frac{\partial M_y}{\partial s}\mathbf{j} + \frac{\partial M_z}{\partial s}\mathbf{k} + M_x\frac{d\mathbf{i}}{ds} + M_y\frac{d\mathbf{j}}{ds} + M_z\frac{d\mathbf{k}}{ds}$$

$$= \frac{\partial M_x}{\partial s}\mathbf{i} + \frac{\partial M_y}{\partial s}\mathbf{j} + \frac{\partial M_z}{\partial s}\mathbf{k} + M_x(\lambda\mathbf{j} - \kappa\mathbf{k}) + M_y(-\lambda)\,\mathbf{i}$$

$$+ \kappa M_z\,\mathbf{i}\ . \qquad (20)$$

Also, since

In Figure 4, $\mathbf{R}_o$, $\mathbf{R}'_o$ and $\mathbf{P}$ are position vectors of points, at positions "s", "s + δs" and "s + ζ" relative to a fixed point Q. If $\mathbf{R}'(s)$ is the position vector of any point in the space curve relative to 0 we may write

$$\mathbf{R}' = \mathbf{R}'_o - \mathbf{R}_o$$

$$= \mathbf{R}_o + d\mathbf{R}/ds \Big|_s \delta s + \frac{1}{2} \frac{d^2\mathbf{R}}{ds^2} \Big|_s (\delta s)^2 + \ldots - \mathbf{R}_o$$

$$= d\mathbf{R}/ds \Big|_s \delta s + \frac{1}{2} \frac{d^2\mathbf{R}}{ds^2} \Big|_s (\delta s)^2 + \ldots$$

$$= \mathbf{k}\, \delta s + \kappa/2\, \mathbf{i}\, (\delta s)^2 , \tag{15a}$$

since by definition $\mathbf{k} = d\mathbf{R}/ds$ and by Eq. (10a) $d\mathbf{k}/ds = \kappa\, \mathbf{i}$.

In a similar way it is found that

$$\overline{\mathbf{R}} = \mathbf{P} - \mathbf{R}_o$$

$$= \mathbf{k}\zeta + \kappa/2\, \mathbf{i}\, \zeta^2 . \tag{15b}$$

Combining Eqs. (7) and (15b) yields

$$\overline{\mathbf{R}} \times \mathbf{T} = \mathbf{k}(s) \times \mathbf{T}(s)\, \zeta + [\kappa/2\, \mathbf{i} \times \mathbf{T} + \mathbf{k} \times \frac{\partial \mathbf{T}}{\partial s} \Big| \zeta^2 + \ldots$$

Therefore,

$$\int_s^{s+\delta s} \overline{\mathbf{R}} \times \mathbf{T}\, ds = \mathbf{k} \times \mathbf{T} \int_o^{\delta s} \zeta d\zeta + [\kappa/2\, \mathbf{i}\times\mathbf{T} + \mathbf{k} \times \partial\mathbf{T}/\partial s] \int_s^{\delta s} \zeta^2 \delta\zeta$$

$$= \mathbf{k}(s) \times \mathbf{T}\, {}^1\!/_2\, (\delta s)^2 . \tag{16}$$

Next,

$$\left(\frac{\partial V_z}{\partial s} - \kappa V_x\right) \mathbf{k} + \left(\kappa V_z + \frac{\partial V_x}{\partial s} - \lambda V_y\right)\mathbf{i}$$

$$+ (\lambda V_x + \partial V_y/\partial s)\mathbf{j} + T_x\mathbf{i} + T_y\mathbf{j} + T_z\mathbf{k} = 0 . \qquad (11)$$

Finally, Eq. (11) may be written as three scalar equations which express the condition of force equilibrium for a beam element

$$\partial V_x/\partial s - \lambda V_y + \kappa V_z + T_x = 0 , \qquad (12a)$$

$$\partial V_y/\partial s + \lambda V_x + T_y = 0 \text{ , and} \qquad (12b)$$

$$\partial V_z/\partial s - \kappa V_x + T_z = 0 . \qquad (12c)$$

Next, the conditions for moment equilibrium are formulated from consideration of the free-body diagram of Figure 4. Summing moments about point 0 yields

$$\mathbf{M}(s) + \mathbf{M}(s + \delta s) + \mathbf{R}'\mathbf{x}\mathbf{V}(s + \delta s)$$

$$+ \int_s^{s+\delta s} \overline{\mathbf{R}}\mathbf{x}\mathbf{T}(s)ds + \int_s^{s+\delta s} \mathbf{m}\, ds = 0 \qquad (13)$$

Again the terms appearing in Eq. (13) must be expressed in terms of their values and those of their derivatives at "s"

$$\mathbf{m}(s + \zeta) = \mathbf{m}(s) + \frac{\partial \mathbf{m}}{\partial s}\Big|_s \zeta + \frac{\partial^2 \mathbf{m}}{\partial s^2}\Big|_s \zeta^2/2 + \ldots \text{ and}$$

$$\int_s^{s+\delta s} \mathbf{m} ds = \mathbf{m}(s) \int_o^{\delta s} d\zeta + \frac{\partial \mathbf{m}}{\partial s}\Big|_s \int_o^{\delta s} \zeta d\zeta + \frac{1}{2} \frac{\partial^2 \mathbf{m}}{\partial s^2}\Big|_s \int_o^{\delta s} \zeta \, d\zeta^2 + \ldots$$

$$= \mathbf{m}(s)\delta s + \frac{\partial \mathbf{m}}{\partial s}\Big|_s \frac{(\delta s)^2}{2} + \ldots \qquad (14)$$

$$\int_{s}^{s+\delta s} \mathbf{T}\, ds = \mathbf{T}(s)\delta s + \left.\frac{\partial \mathbf{T}}{\partial s}\right|_{s} (\delta s)^2/2 + \ldots \tag{8}$$

Returning to Eq. (7), we next express $\mathbf{V}(s + \delta s)$ in terms of its value and those of its derivatives at $\delta s = 0$ using vector Taylor's series.

$$\mathbf{V}(s + \delta s) = \mathbf{V}(s^+) + \left.\frac{\partial \mathbf{V}}{\partial s}\right|_{s^+} \delta s + \ldots \tag{9}$$

The symbol "s^+" is used in Eq. (9) as a reminder that this force acts on a positive face of the element. Therefore,

$$\mathbf{V}(s^+) = V_x\, \mathbf{i} + V_y\, \mathbf{j} + V_z\, \mathbf{k} \text{ and} \tag{9a}$$

$$\left.\frac{\partial \mathbf{V}}{\partial s}\right|_{s^+} = \frac{\partial V_x}{\partial s}\mathbf{i} + \frac{\partial V_y}{\partial s}\mathbf{j} \frac{\partial V_z}{\partial s}\mathbf{k} + V_x \frac{d\mathbf{i}}{ds} + V_y \frac{d\mathbf{j}}{ds} + V_z \frac{d\mathbf{k}}{ds}\,. \tag{9b}$$

Expressions for the rate of change of the unit tangent, principal normal and binormal vectors with respect to arc length are given by the Serret - Frenet formulae (Appendix A) as

$$d\mathbf{k}/ds = 1/\rho\ \mathbf{i} \equiv \kappa\ \mathbf{i}\,, \tag{10a}$$

$$\frac{d\mathbf{j}}{ds} = -\ \lambda \mathbf{i}\ , \text{ and} \tag{10b}$$

$$d\mathbf{i}/ds = \lambda \mathbf{j} - \kappa \mathbf{k} \tag{10c}$$

where $\kappa(= 1/\rho)$ is the local curvature of the line of centroids. Upon substitution of Eq. (8) through Eq. (10) into (6) and division by δs, the following vector equation results in the limit as δs approaches zero,

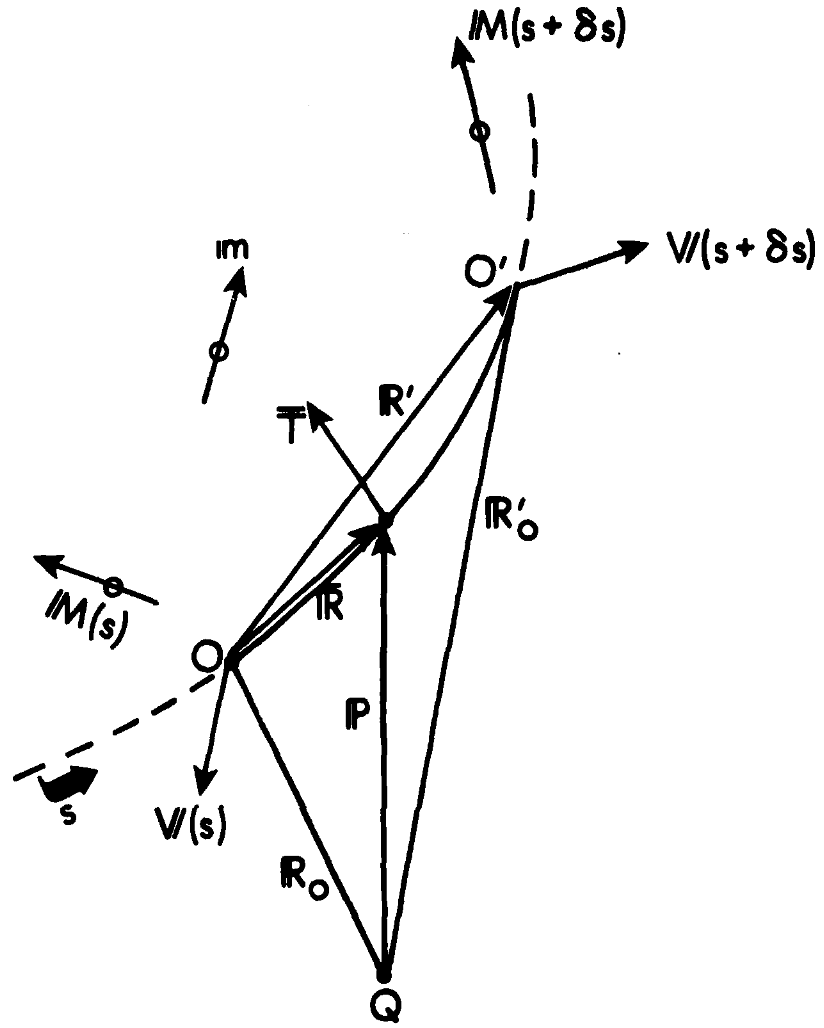

Figure 4. Free Body Diagram of Rod Element

The surface tractions and body forces may also contribute to a moment resultant per unit length about the three coordinate axes. This moment resultant is defined by

$$\mathbf{m} = m_x \mathbf{i} + m_y \mathbf{j} + m_z \mathbf{k} . \tag{5b}$$

For problems involving motion of the rod element, the force **T** will contain terms involving inertia forces while the moment **m** may contain moments of inertia forces about the three axes.

A free-body diagram of a rod element is shown in Figure 4. In Figure 4, points 0 and 0' are again points in the curve of centroids at positions "s" and "s + δs" respectively. The force and moment resultants acting on the rod cross sections of these positions are indicated as well as the surface traction vector and surface traction moment resultant at a typical point $s + \zeta$ where $s < s + \zeta < s + \delta s$.

$\overline{\mathbf{R}}$ is the position vector of the point $x + \zeta$ relative to 0 while $\mathbf{R}'$ is the position vector of $0'$ relative to 0.

The condition of force equilibrium applied to the rod element yields the following equation

$$\mathbf{V}(s) + \mathbf{V}(s + \delta s) + \int_s^{s+\delta s} \mathbf{T}\, ds = 0 . \tag{6}$$

Now,

$$\mathbf{T}(s + \zeta) = \mathbf{T}(s) + \frac{\partial \mathbf{T}}{\partial s}\Big|_s \zeta + \frac{1}{2} \frac{\partial^2 \mathbf{T}}{\partial s^2}\Big|_s \zeta^2 + \ldots \tag{7}$$

where

$$s < (s + \zeta) < (s + \delta s) \text{ and } ds = d\zeta .$$

Therefore,

$$\int_s^{s+\delta s} \mathbf{T}\, ds = \mathbf{T}(s) \int_0^{\delta s} d\zeta + \frac{\partial \mathbf{T}}{\partial s}\Big|_s \int_0^{\delta s} \zeta d\zeta + \frac{1}{2} \frac{\partial^2 \mathbf{T}}{\partial s^2}\Big|_s \int_0^{\delta s} \zeta^2 d\zeta + \ldots \text{ or}$$

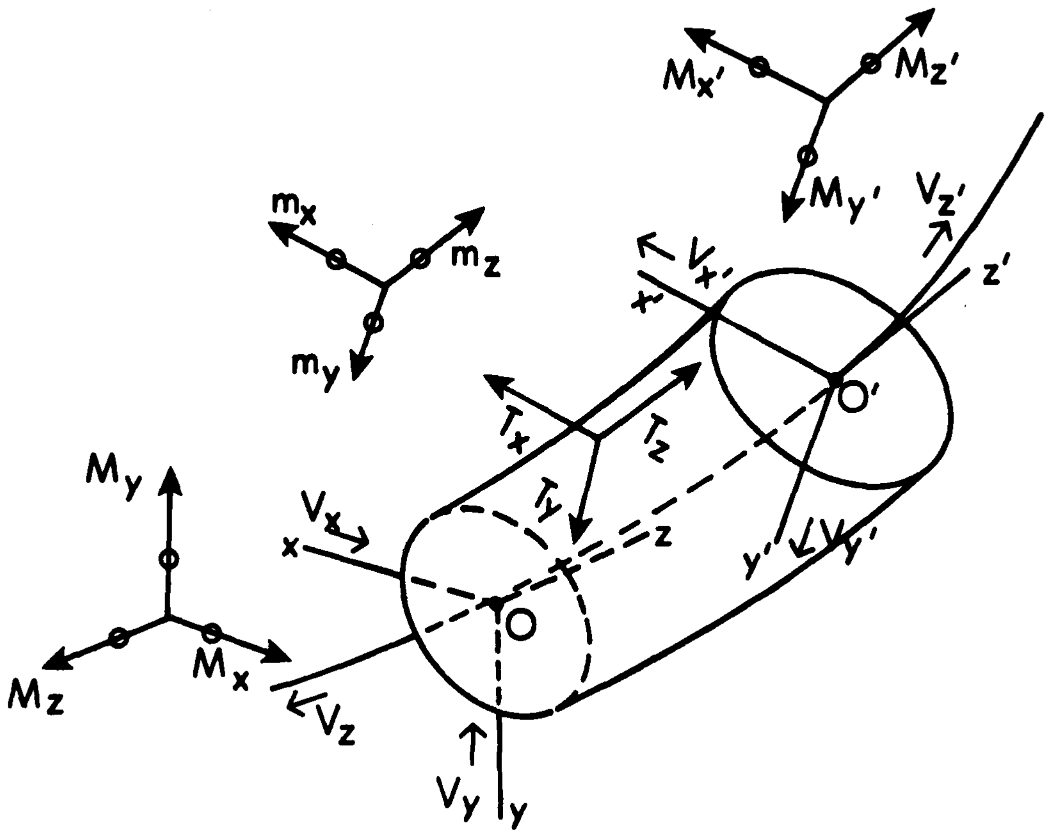

Figure 3. Force and Moment Resultants On Rod Element

$$M_z = \int(x\,\sigma_{yz} - y\,\sigma_{xz})dA \;. \qquad (2c)$$

Based on the above definitions, the vector force and moment resultant of the tractions acting on the face with centroid "O" at point "s" on the space curve of centroids is

$$\mathbf{V}(s) = -\,V_x(s)\mathbf{i}(s) - V_y(s)\mathbf{j}(s) - V_z(s)\mathbf{k}(s) \text{ and} \qquad (3a)$$

$$\mathbf{M}(s) = -\,M_x(s)\mathbf{i}(s) - M_y(s)\mathbf{j}(s) - M_z(s)\mathbf{k}(s) \;. \qquad (3b)$$

Similarily, these vector quantities acting at $s + \delta s$ on the face with centroid $0'$ are

$$\mathbf{V}(s + \delta s) = V_x(s+\delta s)\mathbf{i}(s + \delta s) + V_y(s + \delta s)\mathbf{j}(s + \delta s)$$
$$+ V_z(s + \delta s)\mathbf{k}(s + \delta s) \text{ and} \qquad (4a)$$

$$\mathbf{M}(s + \delta s) = M_x(s + \delta s)\mathbf{i}(s + \delta s) + M_y(s + \delta s)\mathbf{j}(s + \delta s)$$
$$+ M_z(s + \delta s)\mathbf{k}(s + \delta s) \;. \qquad (4b)$$

These quantities are illustrated in Figure 3.

Note that the unit vectors used to describe the directions of the vector forces and moments at s are different from those at $s + \delta s$ since the directions of the space curve unit vectors are constantly changing along the curve.

The resultant per unit length of the tractions acting on the lateral surface and of the body forces acting in the interior of the element is given by

$$\mathbf{T} = T_x\mathbf{i} + T_y\mathbf{j} + T_z\mathbf{k} \qquad (5a)$$

where T_x, T_y, T_z have the dimensions of force per unit length and **i**, **j**, **k** are local unit vectors.

Figure 6 illustrates the initial and final positions of these material points as Q and Q' undergo displacements $\mathbf{u}_Q$ and $\mathbf{u}_{Q'}$. The magnitude of the position vector of $\overline{Q}'$ with respect to $\overline{Q}$, $\overline{\mathbf{r}}$ is found by use of the following vector equation relating the initial and final relative position vectors of these points to their respective displacement vectors,

$$\overline{\mathbf{r}} = \mathbf{r} + \mathbf{u}_{Q'} - \mathbf{u}_Q . \tag{4}$$

The displacement components u, v, and w at point Q are again given by

$$\mathbf{u}_Q = u\,\mathbf{i} + v\,\mathbf{j} + w\,\mathbf{k} . \tag{5}$$

The displacement vector at point Q', adjacent to point Q, can then be expressed as

$$\mathbf{u}_{Q'} = \mathbf{u}_Q + \frac{\partial \mathbf{u}_Q}{\partial x}\,\delta x + \frac{\partial \mathbf{u}_Q}{\partial y}\,\delta y + \frac{\partial \mathbf{u}_Q}{\partial z}\,\delta z \tag{6}$$

where higher order terms in this Taylor's series expansion of the displacement vector about point Q have been dropped. The vector derivatives appearing in Eq. (6) are next presented in terms of their vector components,

$$\frac{\partial \mathbf{u}_Q}{\partial x} = \frac{\partial u}{\partial x}\,\mathbf{i} + \frac{\partial v}{\partial x}\,\mathbf{j} + \frac{\partial w}{\partial x}\,\mathbf{k} , \tag{7a}$$

$$\frac{\partial \mathbf{u}_Q}{\partial y} = \frac{\partial u}{\partial y}\,\mathbf{i} + \frac{\partial v}{\partial y}\,\mathbf{j} + \frac{\partial w}{\partial y}\,\mathbf{k} , \tag{7b}$$

$$\frac{\partial \mathbf{u}_Q}{\partial z} = \frac{\partial u}{\partial z}\,\mathbf{i} + \frac{\partial v}{\partial z}\,\mathbf{j} + \frac{\partial w}{\partial z}\,\mathbf{k} + u\,\frac{\partial \mathbf{i}}{\partial z} + v\,\frac{\partial \mathbf{j}}{\partial z} + w\,\frac{\partial \mathbf{k}}{\partial z}$$

$$= \frac{\partial u}{\partial z}\,\mathbf{i} + \frac{\partial v}{\partial z}\,\mathbf{j} + \frac{\partial w}{\partial z}\,\mathbf{k} + u\,(\lambda_o\,\mathbf{j} - \kappa_o\,\mathbf{k}) - v\lambda_o\,\mathbf{i} + w\,\kappa_o\,\mathbf{i} , \text{ or}$$

$$\frac{\partial u_Q}{\partial z} = \left(\frac{\partial u}{\partial z} - \lambda_o v + \kappa_o w\right) \mathbf{i} + \left(\frac{\partial v}{\partial z} + \lambda_o u\right) \mathbf{j} + \left(\frac{\partial w}{\partial z} - u\kappa_o\right) \mathbf{k} \tag{7c}$$

where the Serret-Frenet formulae have been used to express the rate of change of the local coordinate unit vectors along the curve in terms of the local curvature and torsion.

Substitution of Eqs. (5) through (7) into (4) and using Eqs. (3) yields the following result for the position vector of $\overline{Q}'$ relative to $\overline{Q}$:

$$\begin{aligned} \bar{\mathbf{r}} = {} & r\left[\ell\left(1 + \frac{\partial u}{\partial x}\right) + m\frac{\partial u}{\partial y} + n\left(\frac{\partial u}{\partial z} - \lambda_o v + \kappa_o w\right)\right] \mathbf{i} \\ & + r\left[\ell\frac{\partial v}{\partial x} + m\left(1 + \frac{\partial v}{\partial y}\right) + n\left(\lambda_o u + \frac{\partial v}{\partial z}\right)\right] \mathbf{j} \\ & + r\left[\ell\frac{\partial w}{\partial x} + m\frac{\partial w}{\partial y} + n\left(1 - \kappa_o u + \frac{\partial w}{\partial z}\right)\right] \mathbf{k} \,. \end{aligned} \tag{8}$$

If the magnitudes of $\mathbf{r}$ and $\bar{\mathbf{r}}$ are denoted by r and $\bar{r}$ respectively, then the unit elongation, e, of the original line segment is defined by

$$\bar{r} = r(1 + e) \,. \tag{9}$$

If e is a small quantity, then

$$\bar{r}^{\,2} = r^2(1 + 2e) \tag{10a}$$

or

$$e = \tfrac{1}{2}\left(\frac{\bar{r}^{\,2}}{r^2} - 1\right) . \tag{10b}$$

Upon squaring the magnitudes of the vector $\bar{\mathbf{r}}$ given by Eq. (8), substituting the result into Eq. (10b), and grouping the resulting terms as coefficients of the products of the direction cosines ℓ, m and n, the elongation, e, may be expressed in the following form,

$$e = \varepsilon_{xx}\, \ell^2 + \varepsilon_{yy}\, m^2 + \varepsilon_{zz}\, n^2 + 2\varepsilon_{xy}\, \ell m + 2\varepsilon_{xz}\, \ell n + 2\varepsilon_{yz}\, mn \tag{11}$$

where

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \tfrac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right], \tag{12}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} + \tfrac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right], \tag{12b}$$

$$\begin{aligned} \varepsilon_{zz} = {} & \frac{\partial w}{\partial z} - \kappa_o u + \tfrac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \\ & + \tfrac{1}{2}\left[(\kappa_o^2 + \lambda_o^2)\, u^2 + \lambda_o^2\, v^2 + \kappa_o^2\, w^2 - 2\,\lambda_o \kappa_o\, v\, w\right. \\ & \left. - 2\,\lambda_o v \frac{\partial u}{\partial z} + 2\,\lambda_o\, u \frac{\partial v}{\partial z} - 2\kappa_o\, u \frac{\partial w}{\partial z} + 2\,\kappa_o\, w \frac{\partial u}{\partial z}\right], \end{aligned} \tag{12c}$$

$$2\,\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \tag{12d}$$

$$\begin{aligned} 2\,\varepsilon_{xz} = {} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} - \lambda_o\, v + \kappa_o w + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} \\ & + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z} - \lambda_o v \frac{\partial u}{\partial x} + \kappa_o w \frac{\partial u}{\partial x} + \lambda_o u \frac{\partial v}{\partial x} - \kappa_o u \frac{\partial w}{\partial x}, \end{aligned} \tag{12e}$$

$$\begin{aligned} 2\,\varepsilon_{yz} = {} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \lambda_o u + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z} \\ & + \lambda_o u \frac{\partial v}{\partial y} - \kappa_o u \frac{\partial w}{\partial y} - \lambda_o v \frac{\partial u}{\partial y} + \kappa_o w \frac{\partial u}{\partial y}. \end{aligned} \tag{12f}$$

By definition, the quantities ε_{xx}, ε_{yy}, ε_{zz}, ε_{xy}, ε_{xz}, and ε_{yz} are the normal and shear strains at Q referred to the local coordinates at P. Eqs. (12) therefore constitute the required strain-displacement equations.

Clearly, various approximate forms of Eq. (12) may be formulated depending upon the simplifying assumptions appropriate for a given problem. These formulations may postulate small deformations with large displacements; small deformation with small displacements but large curvatures and/or torsions; small curvatures and torsions; or various other combinations of simplifying assumptions. The reader may verify that the technical theory of curved beams, in which the normal strain is a non-linear function of the transverse coordinate, requires retention of terms involving squares of curvatures and of displacements. It is also noted that consideration of buckling or of stretch-stiffening effects requires retention of non-linear terms in Eq. (12).

The simplest form of strain displacement equations which contains the effect of initial curvature and torsion of the centroidal curve is obtained by dropping all non-linear terms in displacements and displacement gradients. The strain-displacement equations then become

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \tag{13a}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}, \tag{13b}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} - \kappa_o u, \tag{13c}$$

$$\varepsilon_{xy} = \tfrac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{13d}$$

$$\varepsilon_{xz} = \tfrac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} - \lambda_o v + \kappa_o w\right), \text{ and} \tag{13e}$$

$$\varepsilon_{yz} = \tfrac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \lambda_o u\right). \tag{13f}$$

At this point the functional form of the displacement components in terms of the coordinates is completely unspecified. In the next section the strain-displacement equations are put in a form which corresponds to the usual technical beam theory.

IV. A TECHNICAL THEORY OF RODS WITH INITIAL CURVATURE AND TORSION

A. Displacement Functions

A linearized "engineering" theory of curved rods is next developed on the basis of assumptions employed in the strength of material formulations for the bending and torsion of prismatic bars.

The fundamental assumption of both technical beam theory and that of torsion of straight rods is cross sections unchanged in size and shape after the structure is bent or twisted. It is not necessary to initially postulate that cross sections remain normal to the bent centerline.

A typical cross section is therefore assumed to undergo a small rotation about each of the three coordinate axes while its centroid is displaced along each of these axes. These assumptions imply the following form for the displacement functions,

$$w(x,y,z) = w_o(z) - x\,\alpha_y(z) + y\,\alpha_x(z)\ , \tag{1a}$$

$$v(x,y,z) = v_o(z) + x\,\phi(z)\ ,\ \text{and} \tag{1b}$$

$$u(x,y,z) = u_o(z) - y\,\phi(z)\ . \tag{1c}$$

In Eq. (1), u_o, v_o and w_o are the displacement components at a point on the centroidal curve (the origin of the local x,y coordinates) while α_x, α_y and ϕ are small angles of rotation about the x, y and z axes in that order, as shown in Figure 7.

It is noted that theory of torsion of bars with non-circular cross sections requires the addition to Eq. (1) of a term representing warping of the cross section. Introduction of such a term would greatly complicate the ensuing development with very little effect on the state of stress other than near a fixed boundary. For this reason, the displacement functions of Eq. (1) will be employed irrespective of the shape of the cross section of the bar.

Warping of compact cross sections increases as the ratio of the principal second moments of area differs increasingly from unity. The warping phenomenon is thoroughly treated in texts by Timoshenko and Goodier[5] and by Sokolnikoff.[6]

B. Strain-Displacement Equations

The linearized form of the strain-displacement equations derived in Section III are used as the basis for the strain-displacement equations for the technical theory.

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[5] *S. Timoshenko and J.N. Goodier, Theory of Elasticity, 2nd Ed., McGraw-Hill, New York, 1951.*

[6] *I.S. Sokolnikoff, Mathematical Theory of Elasticity, 2nd Ed., McGraw-Hill, New York, 1956.*

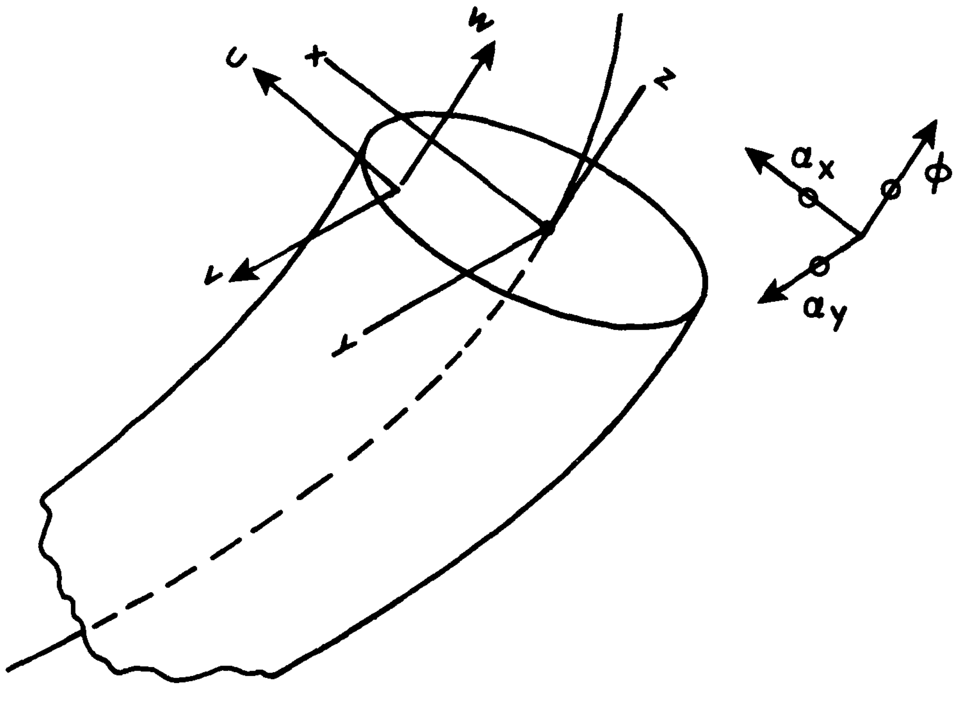

Figure 7. Displacement and Rotation Components

Substitution of Eq. (1) into Eq. (III-13) yields the following expressions for the components of strain,

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{xy} = 0 \ , \tag{2a}$$

$$\varepsilon_{zz} = \frac{\partial w_o}{\partial z} - x\frac{\partial \alpha_y}{\partial z} + y\frac{\partial \alpha_x}{\partial z} - \kappa_o(u_o - y\phi) \ , \tag{2b}$$

$$2\ \varepsilon_{xz} = \frac{\partial u_o}{\partial z} - \alpha_y - \lambda_o v_o + \kappa_o w_o + y\left(\kappa_o \alpha_x - \frac{\partial \phi}{\partial z}\right) - x(\lambda_o \phi + \kappa_o \alpha_y) \ , \text{ and} \tag{2c}$$

$$2\ \varepsilon_{yz} = \frac{\partial v_o}{\partial z} + \lambda_o u_o + \alpha_x + x\frac{\partial \phi}{\partial z} - y\ \lambda_o \phi \ . \tag{2d}$$

Eq. (2a) represents the condition that cross sections do not deform and thereby confirm the satisfaction of the initial displacement assumptions.

Eqs. (2c) and (2d) show that transverse shear deformation cannot be made to vanish for all values of x and y. Introduction of the angle ϕ to represent torsional rotation would lead to this result even for a straight bar. For the curved bar it is noted that the presence of initial torsion and curvature prevent these strains from vanishing everywhere even with the omission of the rotation ϕ.

The condition that cross sections remain normal to the centroidal curve can be invoked locally at x=0, y=0 or, alternatively, in an average sense by integrating Eqs. (2c) and (2d) over the cross section. In either case, the following equations expressing the vanishing of transverse shear deformation (in the above meaning) result:

$$\alpha_y = \frac{\partial u_o}{\partial z} - \lambda_o v_o + \kappa_o w_o \text{ and} \tag{3a}$$

$$\alpha_x = -\frac{\partial v_o}{\partial z} - \lambda_o u_o \ . \tag{3b}$$

Another simplifying assumption often invoked is that the centroidal curve (x=y=0) remains unchanged in length during the motion of the rod. From Eq. (2b), this condition may be expressed as

$$\frac{\partial w_o}{\partial z} - \kappa_o u_o = 0 \ . \tag{3c}$$

C. Inertia Forces and Moments

The inertia force resultants and couples per unit length are obtained by introducing the assumed displacement functions into Eqs. (II-25) and (II-26) and carrying out the indicated integrations. It should be remembered that the origin of the coordinate system is the area centroid of the cross section so that integrals of the form $\int_A x dA$ and $\int_A y dA$ vanish . The resulting expressions, assuming no variation of mass density within a cross section, are presented below.

$$T_{Ix} = -\gamma \ddot{u}_o A \, , \tag{4a}$$

$$T_{Iy} = -\gamma \ddot{v}_o A \, , \tag{4b}$$

$$T_{Iz} = -\gamma \ddot{w}_o A \, , \tag{4c}$$

$$m_{Ix} = -\gamma I_{xx} \ddot{\alpha}_x + \gamma I_{xy} \ddot{\alpha}_y \, , \tag{5a}$$

$$m_{Iy} = \gamma I_{xy} \ddot{\alpha}_x - \gamma I_{yy} \ddot{\alpha}_y \, , \text{ and} \tag{5b}$$

$$m_{Iz} = -\gamma J_p \ddot{\phi} \, , \tag{5c}$$

where

$$I_{xx} = \int_A y^2 \, dA \, , \tag{6a}$$

$$I_{yy} = \int_A x^2 \, dA \, , \tag{6b}$$

$$I_{xy} = \int_A xy \, dA \, , \text{ and} \tag{6c}$$

$$J_p = I_{xx} + I_{yy} \, . \tag{6d}$$

I_{xx} and I_{yy} are the second moments of area of the cross section for the centroid about the x and y axes respectively; I_{xy} is the product of area with respect to the centroid for the x and y directions; and J_p is the polar moment of area for the z direction with respect to the centroid.

The area and shape of the cross section of the rod may vary in an arbitrary way along the rod. The orientation of the principal directions of the cross section area with respect to the plan of curvature also may vary. The quantities defined by Eq. (6) should therefore be considered to be functions of the z coordinate.

D. Moment-Curvature and Force-Displacement Equations

The stress-strain relations which will be used for the derivation of the equations relating the force and moment resultants acting as a cross section of the rod on one hand, to the displacement and rotation components, on the other, are predicated on the assumptions that the rod is constructed from a linear elastic material which is elastically homogeneous in each cross section and that the normal stress components in the directions transverse to the centroidal curve are small in comparison with the normal stress component σ_{zz}.

The applicable stress-strain equations then become

$$\sigma_{zz} = E\,\epsilon_{zz} , \tag{7a}$$

$$\sigma_{xz} = 2G\,\epsilon_{xz} , \text{ and} \tag{7b}$$

$$\sigma_{yz} = 2G\,\epsilon_{yz} . \tag{7c}$$

The required equations are next obtained by combining Eqs. (2) and (3) with the definitions of the various force and moment resultants given by Eq. (II-1),

$$M_x = \int_A y\,\sigma_{zz}\,dA$$

$$= E \int_A y \left[\frac{\partial w_o}{\partial z} - \kappa_o u_o - x\,\frac{\partial \alpha_y}{\partial z} + y\left(\frac{\partial \alpha_x}{\partial z} + \kappa_o \phi\right)\right] dA$$

$$= - E\,I_{xy}\,\frac{\partial \alpha_y}{\partial z} + E\,I_{xx}\left(\frac{\partial \alpha_x}{\partial z} + \kappa_o \phi\right) , \qquad \text{8a)}$$

$$M_y = -\int x\sigma_{zz} dA = E\, I_{yy} \frac{\partial \alpha_y}{\partial z} - E\, I_{xy} \left(\frac{\partial \alpha_x}{\partial z} + \kappa_o \phi\right), \tag{8b}$$

$$\begin{aligned} M_z &= \int_A (x\, \sigma_{yz} - y\, \sigma_{xz})\, dA \\ &= G\, J_p \frac{\partial \phi}{\partial z} - G\, I_{xx} \kappa_o\, \alpha_x + G I_{xy} \kappa_o \alpha_y , \end{aligned} \tag{8c}$$

$$V_x = \int_A \sigma_{xz}\, dA = k_x AG \left(\frac{\partial u_o}{\partial z} - \alpha_y - \lambda_o\, v_o + \kappa_o w_o\right), \tag{8d}$$

$$V_y = \int_A \sigma_{yz}\, dA = k_y\, AG \left(\frac{\partial v_o}{\partial z} + \lambda_o u_o + \alpha_x\right), \text{ and} \tag{8e}$$

$$V_z = \int_A \sigma_{zz}\, dA = E\, A \left(\frac{\partial w_o}{\partial z} - \kappa_o u_o\right). \tag{8f}$$

The coefficients k_x and k_y appearing in Eqs. (8d) and (8e) are called Timoshenko shear coefficients. These are dimensionless quantities, dependent upon the shape of the cross section, which are introduced to compensate for the fact that not only is the shear stress non-uniform over the cross section of the rod but its actual distribution is inconsistent with the function forms describing the shear strain-displacement equations. Values of the Timoshenko shear coefficients are documented for various cross sections as, for example, by Cowper.[7]

The complete system of equations constituting the linear, technical, theory of rods with space curvature is comprised of Eqs. (II-12) and II-21) along with Eqs. (4), (5) and (8) of this section. For convenience of the following exposition, Eqs. (II-12) and (II-21) combined with (4) and (5) are presented below where, in a manner consistent with the employment of linearized strain-displacement equations, the torsion and curvature are taken as those of the undeformed rod.

$$\frac{\partial V_x}{\partial z} - \lambda_o\, V_y + \kappa_o V_z = \gamma A\, \frac{\partial^2 u_o}{\partial t^2} - T_{xs}\, , \tag{9a}$$

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7 *G.R. Cowper, "The Shear Coefficients in Timoshenko Beam Theory," ASME J. Applied Mechanics, Vol 33, Series E, No. 2, pp 335-340, 1966.*

$$\frac{\partial V_y}{\partial z} + \lambda_o V_x = \gamma A \frac{\partial^2 v_o}{\partial t^2} - T_{ys} , \tag{9b}$$

$$\frac{\partial V_z}{\partial z} - \kappa_o V_x = \gamma A \frac{\partial^2 w_o}{\partial t^2} - T_{zs} , \tag{9c}$$

$$\frac{\partial M_x}{\partial z} - \lambda_o M_y + \kappa_o M_z - V_y = \gamma I_{xx} \frac{\partial^2 \alpha_x}{\partial t^2} - \gamma I_{xy} \frac{\partial^2 \alpha_y}{\partial t^2} - m_{xs} , \tag{9d}$$

$$\frac{\partial M_y}{\partial z} + \lambda_o M_x + V_x = - \gamma I_{xy} \frac{\partial^2 \alpha_x}{\partial t^2} + \gamma I_{yy} \frac{\partial^2 \alpha_y}{\partial t^2} - m_{ys} , \text{ and} \tag{9e}$$

$$\frac{\partial M_z}{\partial z} - \kappa_o M_x = \gamma J_p \frac{\partial^2 \phi}{\partial t^2} - m_{zs} . \tag{9f}$$

In Eq. (9), the terms T_{xs}, T_{ys}, T_{zs} represent the portion of the surface and body forces acting which are not acceleration dependent while m_{xs}, m_{ys} and m_{zs} are the corresponding parts of the couples of these forces.

To determine the motion of a rod with space curvature using this linearized formulation, the system of twelve coupled equations consisting of Eqs. (8) and (9) above must be solved for the six force and moment resultants and the six displacement and rotation components subject to appropriate initial and boundary conditions. This formulation retains the effects of transverse shear deformation, rotatory inertia, and extension of the centroidal curve. Since some or all of these effects can safely be neglected for problems involving thin rods and low frequencies, more simplified formulations will next be considered.

It is noted that no restrictions have yet been imposed on the shape of the cross section, the orientation of the principal axes of the cross section with respect to the local plane of curvature, or the change in these quantities along the rod. The cross section area, the torsion and curvature, and the tensile and shear moduli may also have arbitrary variation with z.

E. Displacement Equations Of Equilibrium Neglecting Transverse Shear Deformation And Rotatory Inertia

1. General Form

In this section the equilibrium equations for the rod are phrased in terms of displacement functions. The effects of torsion, curvature and orientation of the principal axes of the cross section areas as well as the variation of these quantities along the rod are then examined.

In the following formulation transverse shear strains are neglected in the sense defined by Eq. (3). Since rotatory inertia generally has less effect upon predictions of frequency and motion than has transverse shear deformation, it is consistent to set terms involving α_x and α_y equal to zero in Eq. (9) although rotational inertia effects about the z axis are retained. Equations (8d) and (8c) for V_x and V_y respectively become inconsistent because, although ε_{xz} and ε_{yz} are set equal to zero on the average, equilibrium considerations require non-zero transverse shear forces. These forces are eliminated from the equilibrium equations by solving for V_y and V_x in Eqs. (9d) and (9e) and substituting these results into the remaining four equations. The equilibrium equations then become

$$M_y'' - \lambda_o^2 M_y + 2\lambda_o M_x' + \lambda_o' M_x + \lambda_o \kappa_o M_z$$

$$- \kappa_o V_z = - \gamma A \ddot{u}_o + T_{xs} - m'_{ys} + \lambda_o m_{xs} \,, \tag{10a}$$

$$M_x'' - \lambda_o^2 M_x - 2\lambda_o M_y' - \lambda_o' M_y + (\kappa_o M_z)' = \gamma A \ddot{v}_o - T_{ys} + m'_{xs} + \lambda_o m_{ys} \,, \tag{10b}$$

$$\kappa_o M_y' + \kappa_o \lambda_o M_x + V_z' = \gamma A \ddot{w}_o - T_{zs} - \kappa_o m_{ys} \,, \text{ and} \tag{10c}$$

$$M_z' - \kappa_o M_x = \gamma J_p \ddot{\phi} - m_{zs} \,. \tag{10d}$$

In Eq. (10) superscript "prime" indicates differentiation with respect to z. Eq. (9f) is repeated as Eq. (10d).

To obtain the required displacement form of the equilibrium equations, the functions α_x and α_y are first eliminated from Eq. (8) by use of Eq. (3). The resulting expressions for the moments and V_z in terms of the four displacement and rotation functions are then substituted into Eq. (10). The resulting equations can be presented in the following form:

$$A_{12}\, u_o'' + A_{11}\, u_o' + A_{10}\, u_o + B_{12}\, v_o'' + B_{11}\, v_o' + B_{10}\, v_o$$

$$+ C_{11}\, w_o' + C_{10}\, w_o + D_{12}\, \phi'' + D_{11}\, \phi' + D_{10}\, \phi = \gamma J_p \ddot{\phi} - m_{zs} \,, \tag{11a}$$

$$A_{24}\, u_o^{IV} + A_{23} u_o''' + A_{22} u_o'' + A_{21} u_o' + A_{20} u_o$$

$$+ B_{24} v_o^{IV} + B_{23} v_o''' + B_{22} v_o'' + B_{21} v_o' + B_{20} v_o$$

$$+ C_{23} w_o''' + C_{22} w_o'' + C_{21} w_o' + C_{20} w_o$$

$$+ D_{22}\phi'' + D_{21}\phi' + D_{20}\phi = -\gamma A \ddot{u}_o + T_{xs} - m'_{ys} + \lambda_o m_{xs} \,, \tag{11b}$$

$$A_{34} u_o^{IV} + A_{33} u_o''' + A_{32} u_o'' + A_{31} u_o' + A_{30} u_o$$

$$+ B_{34} v_o^{IV} + B_{33} v_o''' + B_{32} v_o'' + B_{31} v_o' + B_{30} v_o$$

$$+ C_{33} w_o''' + C_{32} w_o'' + C_{31} w_o' + C_{30} w_o$$

$$+ D_{32}\phi'' + D_{31}\phi' + D_{30}\phi = \gamma A \ddot{v}_o - T_{ys} - m'_{xs} - \lambda_o m_{ys} \,, \text{ and} \tag{11c}$$

$$A_{43}u_o''' + A_{42}u_o'' + A_{41}u_o' \; A_{40}u_o$$

$$+ B_{43}v_o''' + B_{42}v_o'' + B_{41}v_o' + B_{40}v_o$$

$$+ C_{42}w_o'' + C_{41}w_o' + C_{40}w_o$$

$$+ D_{41}\phi' + D_{40}\phi = \gamma A\ddot{w}_o - T_{zs} + \kappa_o m_{ys} \; . \tag{11d}$$

The fifty-eight coefficients appearing in Eq. (11) are presented in Appendix B. There it is seen that these coefficients depend upon the geometric properties of the space curve described by the centroids of the cross sections, the geometry of the cross section and its relationship to the local vector principal normal and binormal, the elastic moduli of the material, and the variations of these quantities along the rod.

It is noted that in the general case described by Eq. (11), all motions are fully coupled since all variables appear in each equation. This means, for instance, that any exciting force will cause motion involving all four displacement and rotation variables. Also, solution of the free vibration problem will yield mode shapes which will each have four corresponding natural frequencies.

To aid in the understanding of how various geometric and stiffness quantities interact to effect the rod motion, several more specialized cases will next be examined.

2. <u>Straight Rod With Variable Section Properties.</u>

If λ_o and k_o and their derivatives are set equal to zero, the following equations are obtained:

$$(GJ_p\phi')' = \gamma J_p\ddot{\phi} - m_{zs} \; , \tag{12a}$$

$$(EI_{yy}u_o'')'' + (EI_{xy}v_o'')'' = -\gamma A\ddot{u}_o + T_{xs} - m'_{ys} \; , \tag{12b}$$

$$\boldsymbol{\beta} = \frac{\mathbf{P}' \times \mathbf{P}''}{|\mathbf{P}' \times \mathbf{P}''|} \tag{A-5}$$

where superscript prime indicates differentiation of vector function **P** with respect to z.

Proof:

$$\mathbf{C}_1 = \mathbf{P}_1 - \mathbf{P} = h\,\mathbf{P}' + \frac{h^2}{2!}\mathbf{P}'' + \ldots ,$$

$$\mathbf{C}_2 = \mathbf{P}_2 - \mathbf{P} = -h\,\mathbf{P}' + \frac{h^2}{2!}\mathbf{P}'' - \ldots ,$$

$$\mathbf{C}_1 \times \mathbf{C}_2 = h^3\mathbf{P}'\mathrm{x}\mathbf{P}'' + \ldots , \text{ and}$$

$$|\mathbf{C}_1 \mathrm{x} \mathbf{C}_2| = |h^3|\;|\mathbf{P}'\mathrm{x}\mathbf{P}''| + \ldots$$

Note that $\boldsymbol{\tau}$ and $\boldsymbol{\beta}$ are perpendicular to each other since $\mathbf{P}' \cdot (\mathbf{P}'\mathrm{x}\mathbf{P}'') = 0$

**If z is the arc length s from a fixed point P_o of C to point P, $\boldsymbol{\beta}$ is then given by

$$\boldsymbol{\beta} = \frac{\dfrac{d\mathbf{P}}{ds} \times \dfrac{d^2\mathbf{P}}{ds^2}}{\left|\dfrac{d^2\mathbf{P}}{ds^2}\right|} . \tag{A-6}$$

Proof: Since

$$\left|\frac{d\mathbf{P}}{ds}\right| = 1,$$

$$\frac{d\mathbf{P}}{ds} \cdot \frac{d^2\mathbf{P}}{ds^2} = \tfrac{1}{2}\,\frac{d}{ds}\left(\frac{d\mathbf{P}}{ds}\right)^2 = 0 .$$

Therefore $\mathbf{P}'(s)$ is perpendicular to $\mathbf{P}''(s)$ so that

$$|\mathbf{P}'(s) \times \mathbf{P}''(s)| = |\mathbf{P}'|\;|\mathbf{P}''| = |\mathbf{P}''(s)| .$$

**The plane passing through P and normal to $\boldsymbol{\beta}$ is called the <u>plane of curvature</u> or <u>oscillating plane</u> of C at P.

Proof: If h is the length of arc between P and P_1; Then

$$\lim_{h\to 0} \frac{|\mathbf{c}|}{h} = 1 = \left|\frac{d\mathbf{P}}{ds}\right| .$$

The plane passing through P normal to $\boldsymbol{\tau}$ is called the normal plane of c at P.

2. Vector Binormal Of A Space Curve (β)

Referring to Figure A-2, P_1 and P_2 are points on C either side of point P such that z increases by an amount h from P to P_1 and decreases by the same amount from P to P_2. $\mathbf{C}_1$ and $\mathbf{C}_2$ are position vectors of P_1 and P_2 with respect to P respectively

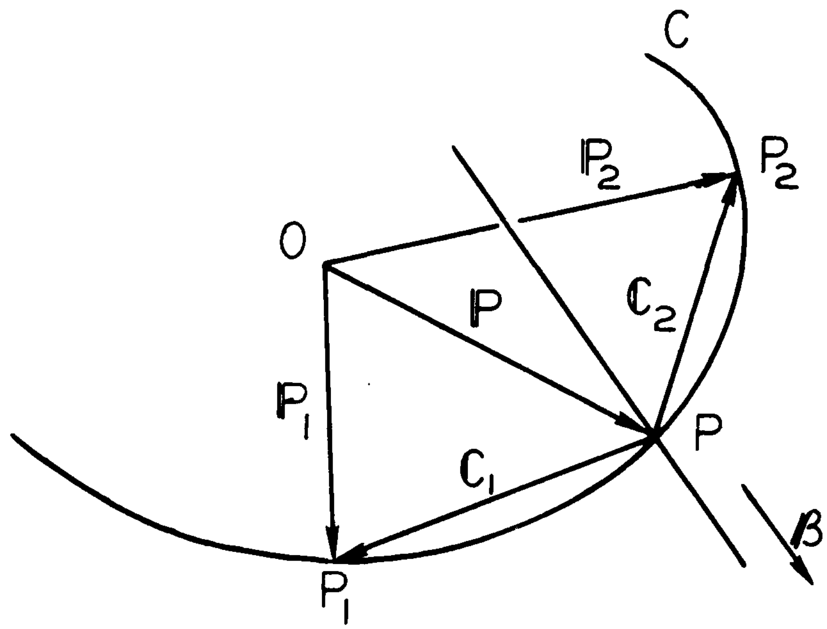

Figure A-2.

A vector binormal, $\boldsymbol{\beta}$, of C at P is defined as

$$\boldsymbol{\beta} = \lim_{h\to 0} \frac{\mathbf{C}_1 \times \mathbf{C}_2}{|\mathbf{C}_1 \times \mathbf{C}_2|} . \qquad \text{(A-4)}$$

Again, the sense of $\boldsymbol{\beta}$ depends upon the choice of scalar variable z used to describe the position of P.

If $\mathbf{P}$ is the position vector of a point P on C relative to a fixed point O, the vector binormal can be expressed as

A vector tangent, $\boldsymbol{\tau}$, to C at point P is defined by

$$\boldsymbol{\tau} = \lim_{h \to 0} \frac{\mathbf{c}}{|\mathbf{c}|} \tag{A-1}$$

where h is the increase in z from point P to P. Note that the sense of $\boldsymbol{\tau}$ is not unique since it depends on the choice of scalar variable z. It points in the direction in which z increases along the curve.

**The vector tangent can be expressed as:

$$\boldsymbol{\tau} = \frac{\mathbf{P}'}{|\mathbf{P}'|} \tag{A-2}$$

where $\mathbf{P}'$ is the derivative of $\mathbf{P}$ with respect to z

Proof:

$$\mathbf{c} = \mathbf{P}_1 - \mathbf{P}$$

where

$$\mathbf{P}_1 \equiv \mathbf{P}(z+h)$$

By Taylor's series for vector functions,

$$\mathbf{P}(z+h) = \mathbf{P} + \mathbf{P}'h + \mathbf{P}''h^2/2! + \ldots$$

so that

$$\frac{\mathbf{c}}{|\mathbf{c}|} = \frac{\mathbf{P}' + \tfrac{1}{2}h\,\mathbf{P}'' + \ldots}{[\mathbf{P}'^2 + h\,\mathbf{P}'\cdot\mathbf{P}'' + \ldots]^{1/2}} \text{ and}$$

$$\boldsymbol{\tau} = \lim_{h \to 0} \frac{\mathbf{c}}{|\mathbf{c}|} .$$

**If the scalar variable governing the position of P on C is the scalar arc-length displacement of P relative to a fixed point P_o of C then

$$\boldsymbol{\tau} = \frac{d\mathbf{P}}{ds} . \tag{A-3}$$

APPENDIX A

SPACE CURVE GEOMETRY

This appendix defines and presents relationships among various vector and scalar quantities used to describe a line curved in three dimensional space. These quantities include the vector tangent, vector principal normal, vector binormal, the radius of curvature, and the curvature and the torsion of a curve at a point.

Although the ensuing material may be found in many sources, this particular exposition is abstracted from the book by T.R. Kane.* Rigorous proofs, which have been omitted from this presentation in favor of intuitive arguments, can be found in that book.

1. Vector Tangents Of A Space Curve (τ)

In Figure A-1, C is a curve in space, O is a point fixed in space while P and P_1 are points on C whose position along C depends on a variable z. Vectors **P** and $\mathbf{P}_1$, are position vectors of P and P_1 respectively, with respect to O while **C** is the position vector of P_1 with respect to P.

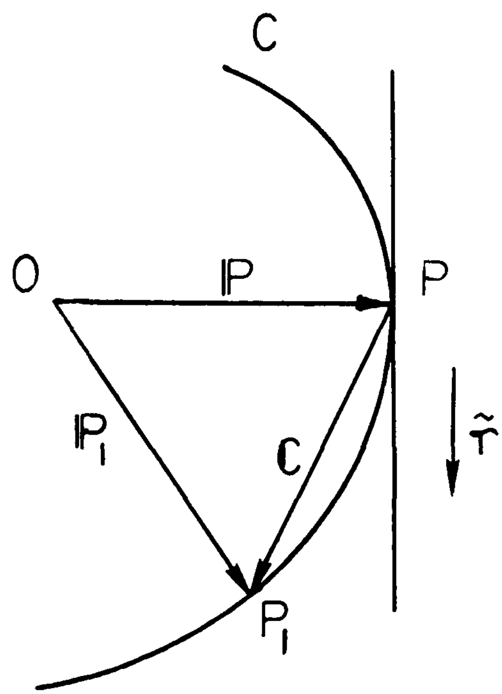

Figure A-1.

* *T.R. Kane, Analytical Elements of Mechanics: Vol. 2 Dynamics, Academic Press, New York, 1981.*

APPENDIX A

SPACE CURVE GEOMETRY

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V. CONCLUSIONS

This work was motivated by a requirement for a general analytical model of a gun tube which could be used to examine the effects on tube motion of centroidal axis curvature and twist, cross section eccentricity and variation in cross section shape along the axis.

The model which has been formulated is based upon first principles of mechanics and is sufficiently general to permit study of the above effects and also of motions involving large displacements and rotations of the tube axis.

The model can provide useful information in its own right; both by illustrating the manner in which the governing equations are coupled when terms representing various affects are added or deleted, and by the free and forced response solutions which may be obtained for various special cases. It can also be used to provide guidance for the development and use of more approximate analytical techniques, such as finite element models, by establishing the potential error associated with various approximations to the actual structure. Since the derived equations show the coupling among the displacement and rotation variables caused by axis curvature and cross section non-uniformity effects, they can aid in the interpretation of physical measurements of gun tube motion.

ACKNOWLEDGMENT

The author wishes to thank Mr. Alexander Stowell Elder for suggesting that this problem was worthy of examination and for his patient endurance of the sometimes-frustrated author's monologues.

$$- \lambda_o [EI_{xx} + 2EI_{yy}] u_o''' - \lambda_o[2(EI_{xx})' + 2(EI_{yy})'] u_o''$$

$$- \lambda_o [(EI_{xx})'' + \lambda_o^2 EI_{xx} + \kappa_o^2 GI_{xx}] u_o' + \kappa_o^2 \lambda_o (GI_{xx})' u_o - (EI_{xx} v_o'')''$$

$$+ \kappa_o^2 (GI_{xx})' v_o' - 2 \lambda_o \kappa_o EI_{yy} w_o'' - 2\lambda_o \kappa_o (EI_{yy})' w_o'$$

$$+ \kappa_o (EI_{xx} + GJ_p)\phi'' + \kappa_o[2 (EI_{xx})' + (GJ_p)']\phi'$$

$$+ [\kappa_o (EI_{xx})'' - \lambda_o^2 \kappa_o EI_{xx}] \phi = + \gamma A \ddot{v}_o - T_{ys} + m'_{xs} + \lambda_o m_{ys} \text{ , and} \tag{15c}$$

$$\kappa_o EI_{yy} u_o''' + \kappa_o (EI_{yy})' u_o'' - \kappa_o[\lambda_o^2 EI_{xx} + EA] u_o'$$

$$- \kappa_o (EA)' u_o - \lambda_o \kappa_o EJ_p v_o'' - \lambda_o \kappa_o (EI_{yy})' v_o'$$

$$+ (EA + \kappa_o^2 EI_{yy}) w_o'' + [\kappa_o^2 (EI_{yy})' + (EA)'] w_o'$$

$$+ \kappa_o^2 \lambda_o EI_{xx} \phi = \gamma A \ddot{w}_o - T_{zs} + \kappa_o m_{ys} . \tag{15d}$$

Introduction of torsion again causes coupling among all of the equations even for a rod with symmetric cross section. It is also noted that although the equations describing motion of a rod with constant cross sectional properties would have somewhat fewer and simpler coefficients, they would retain the coupling among the motions described by the four variables.

It is seen that while inclusion of effects of variable curvature somewhat complicates the coefficients and increases their number, the equation with I_{xy} equal to zero still describes two separate pairs of coupled motion involving v_o and ϕ together and u_o and w_o.

5. Rod With Constant Curvature And Torsion And With Variable Cross Section

The final special case of Eq. (11) which will be considered is for a rod with constant torsion and curvature and variable but axisymmetric cross section so that I_{xy} remains zero. The displacement equilibrium equations for this case are given below:

$$\lambda_o\kappa_o(G+E)I_{xx}u_o' + \kappa_o\lambda_o(GI_{xx})'\,u_o + \kappa_o(G+E)I_{xx}\,v_o'' + (\kappa_o GI_{xx})'v_o'$$

$$+ GJ_p\,\phi'' + (GJ_p)'\phi' - \kappa_o^2 EI_{xx}\,\phi = \gamma J_p\ddot{\phi} - m_{zs}\,, \qquad (15a)$$

$$EI_{yy}\,u_o^{IV} + 2(EI_{yy})'\,u_o''' + [(EI_{yy})''-\lambda_o^2E(I_{yy}+2I_{xx})u_o'' - 2\,\lambda_o^2\,(EI_{xx})'\,u_o'$$

$$+ [\kappa_o^2EA + \kappa_o^2\lambda_o^2GI_{xx}\,]u_o - \lambda_o[E\,(I_{yy} + 2I_{xx})]\,v_o'''$$

$$+ 2\,\lambda_o[(EI_{yy})'+(EI_{xx})']v_o''+[\lambda_o(EI_{yy})''+\lambda_o^3EI_{yy} +\kappa_o^2\lambda_o GI_{xx}]v_o'+\kappa_o EI_{yy}\,w_o'''$$

$$+ 2\,\kappa_o(EI_{yy})'\,w_o'' + [\kappa_o(EI_{yy})'' - \lambda_o^2\kappa_o\,EI_{yy} - \kappa_o\,EA]w_o'$$

$$+ \lambda_o\kappa_o\,[2EI_{xx} + GJ_p]\phi' + 2\kappa_o\lambda_o(EI_{xx})'\phi$$

$$= -\,\gamma A\,\ddot{u}_o + T_{xs} - m'_{ys} + \lambda_o\,m_{xs}\,, \qquad (15b)$$

$$\kappa_o (EI_{yy}u_o'')' - \kappa_o (EAu_o)' + \kappa_o (EI_{xy}v_o'')' + [(EA + \kappa_o^2 EI_{yy})w_o']'$$

$$- \kappa_o^2 (EI_{xy}\phi)' = \gamma A\ddot{w}_o - T_{zs} - \kappa_o m_{ys} . \qquad (13d)$$

Eqs. (13) are seen to be fully coupled in the variables. If however, I_{xy} is zero as, for example, for a circular cross section, then two pair of equations result; one pair involving v_o and ϕ only and the other u_o and w_o.

4. Rod Curved In A Plane With Variable Curvature But Constant Section Properties.

The next special case is that of a rod for which λ_o is zero but the curvature completely variable. The principal normal and binormal directions are assumed to be principal directions for the cross section. Eqs. (11) become

$$\kappa_o(G+E)I_{xx}v_o'' + \kappa_o' GI_{xx}v_o' + GJ_p\phi'' - \kappa_o^2 EI_{xx}\phi = \gamma J_p\ddot{\phi} - m_{zs} , \qquad (14a)$$

$$EI_{yy}u_o^{IV} + \kappa_o^2 EAu_o + EI_{yy}(\kappa_o w_o)'' - \kappa_o EA w_o' =$$

$$- \gamma A\ddot{u}_o + T_{xs} - m'_{ys} , \qquad (14b)$$

$$EI_{xx}v_o^{IV} + GI_{xx}(\kappa_o^2 v_o')' + \kappa_o(EI_{xx} + GJ_p)\phi'' + \kappa_o'(2EI_{xx} + GJ_p)\phi'$$

$$+ \kappa_o'' EI_{xx}\phi = \gamma A\ddot{v}_o - T_{ys} - m'_{xs} , \text{ and} \qquad (14c)$$

$$\kappa_o EI_{yy}u_o''' - EA (\kappa_o u_o)' + (EA + \kappa_o^2 EI_{yy})w_o'' + 2 \kappa_o \kappa_o' EI_{yy} w_o'$$

$$+ \kappa_o \kappa_o'' EI_{yy}w_o = \gamma A\ddot{w}_o - T_{zs} - \kappa_o m_{ys} . \qquad (14d)$$

$$(EI_{xy}u_o'')'' + (EI_{xx}v_o'')'' = -\gamma A\ddot{v}_o + T_{ys} + m'_{xs}\ , \text{ and} \tag{12c}$$

$$(EAw_o')' = \gamma A\ddot{w}_o - T_{zs}\ . \tag{12d}$$

Eqs. (12) are seen to be the usual equations of motion of shafts, beams and bars. Equations involving u_o and v_o are uncoupled if the x and y axes are the principal axes of the cross section.

3. Rod Curved In A Plane With Constant Curvature And Variable Section Properties.

In this case, the rod is initially in the form of an arc of a circle so that κ_o is constant and λ_o is zero. The governing equations thus become

$$\kappa_o\, EI_{xy}u_o'' + \kappa_o\,(GI_{xy}u_o')' + \kappa_o\, EI_{xx}v_o'' + \kappa_o\,(GI_{xx}v_o')' +$$

$$\kappa_o^2\, EI_{xy}w_o' + \kappa_o^2\,(GI_{xy}w_o)' + (GJ_p\phi')' - \kappa_o^2\, EI_{xx}\phi = \gamma J_p\ddot{\phi} - m_{zs}\ , \tag{13a}$$

$$(EI_{yy}u_o'')'' + \kappa_o^2\, EAu_o + (EI_{xy}v_o'')'' + \kappa_o(EI_{yy}w_o')'' - \kappa_o EAw_o' - \kappa_o(EI_{xy}\phi)''$$

$$= -\gamma A\ddot{u}_o + T_{xs} - m'_{ys}\ , \tag{13b}$$

$$(EI_{xy}u_o'')'' - \kappa_o^2\,(GI_{xy}u_o')' + (EI_{xx}v_o'')'' - \kappa_o^2(\,GI_{xx}v_o')' + \kappa_o(EI_{xy}w_o')''$$

$$-\kappa_o^3\,(GI_{xy}w_o)' - \kappa_o(EI_{xx}\phi)'' - \kappa_o(GJ_p\phi')' = -\gamma A\ddot{v}_o + T_{ys} - m'_{xs}\ , \tag{13c}$$

3. The Vector Principal Normal Of A Space Curve (ν)

The unit vector given by

$$\boldsymbol{\nu} = \boldsymbol{\beta} \times \boldsymbol{\tau} \tag{A-7}$$

is called the vector principal normal of C of P. Contrary to the cases of $\boldsymbol{\beta}$ and $\boldsymbol{\tau}$, the sense of ν is unique.

An alternative expression for $\boldsymbol{\nu}$ in terms of derivatives of position vector **P** with respect to coordinate *z* is

$$\nu = \frac{(\mathbf{P}'\times\mathbf{P}'') \times \mathbf{P}'}{|(\mathbf{P}' \times \mathbf{P}'') \times \mathbf{P}'|} . \tag{A-8}$$

In terms of derivatives of **P** with respect to s. $\boldsymbol{\nu}$ is given by

$$\nu = \frac{\mathbf{P}''(s)}{|\mathbf{P}''(s)|} . \tag{A-9}$$

This follows from the fact that $\mathbf{P}''(s)$ is perpendicular to $\mathbf{P}'(s)$ and $\mathbf{P}'(s)$ is a unit vector. The plane passing through P and normal to $\boldsymbol{\nu}$ is called the *rectifying plane* of C at P.

4. The Vector Radius Of Curvature Of A Space Curve (ρ)

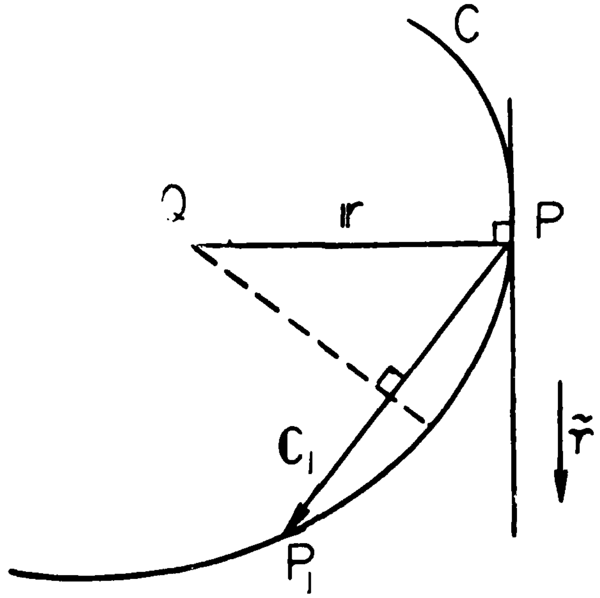

Figure A-3.

Referring to Figure A-3, P is a point on C and P_1 is the point of C with which P coincides when scalar variable z is increased by an amount h. C_1 is the position vector of P_1 relative to P. By definition, the center of curvature of C at P is the point approached, as P_1 approaches P by the center Q, of a circle which is tangent to the tangent line of C at P and passes through P_1. If r is the position vector of Q relative to P, the vector radius of curvature, ρ , is defined by

$$\rho = \lim_{h \to 0} r \, . \tag{A-10}$$

**In terms of the derivatives with respect to coordinate z of the position vector, P, of P relative to a point 0 fixed in space, the vector radius of curvature is given by:

$$\rho = \frac{(P')^2}{(P'xP'')^2} \; (P'xP'') \times P' \, . \tag{A-11}$$

Proof: r(by construction) is perpendicular to both τ and $\tau \times C_1$. Hence, there exists some scalar, say δ such that

$$r = \delta\tau \times (\tau \times c_1) \, . \tag{A}$$

On the other hand, as Q lies in the perpendicular bisector of line $\overline{P\,P_1}$, r may be regarded as the sum of $\frac{1}{2}$ c_1 and a vector perpendicular to both c_1 and $\tau x c_1$. Hence, there exists some scalar, say μ , such that

$$r = \frac{c_1}{2} + \mu c_1 x(\tau \times c_1) \, . \tag{B}$$

Dot multiply each of Eqs. (A) and (B) with c_1 (to eliminate δ), equate the resulting expressions for $r \cdot c_1$, solve for δ and substitute into Eq. (A) yields

$$r = \frac{c_1^{\,2}\tau \times (\tau \times c_1)}{2\, c_1 \cdot \tau \times (\tau \times c_1)} \, . \tag{C}$$

Next, use the relationships

$$\tau = \frac{\mathbf{P}'}{|\mathbf{P}'|} \; ; \quad \mathbf{c} = h\,\mathbf{P}' + h^2/2 \;\; \mathbf{P}'' + \ldots$$

and substitute Eq. C into Eq. A-10.

5. The Radius Of Curvature.

The vector radius of curvature $\boldsymbol{\rho}$ can be expressed as

$$\boldsymbol{\rho} = \rho\,\boldsymbol{\nu} \qquad \text{(A-12)}$$

where ρ is an intrinsically positive quantity called the radius of curvature of C at P which is given by

$$\rho = \frac{|\mathbf{P}'|^3}{|\mathbf{P}' \mathrm{x} \mathbf{P}''|} \, . \qquad \text{(A-13)}$$

Proof: Multiply the numerator and denominator of the expressions for P of Eq. (A-11) with $|\mathbf{P}'|$, giving

$$\boldsymbol{\rho} = \frac{|\mathbf{P}'|^3}{|\mathbf{P}' \mathrm{x} \mathbf{P}''|} \; \frac{(\mathbf{P}' \mathrm{x} \mathbf{P}'') \;\mathrm{x}\; \mathbf{P}'}{|\mathbf{P}' \mathrm{x} \mathbf{P}''| \; |\mathbf{P}'|} \, .$$

By Eq. (A-8), Eq. (A-12) is an identity if ρ is defined by Eq. (A-13).

**In terms of derivatives of P with respect to s, $\boldsymbol{\rho}$ and ρ are respectively given by

$$\boldsymbol{\rho} = \frac{\mathbf{P}''(s)}{|\mathbf{P}''(s)|^2} \quad \text{and}$$

$$\rho = \frac{1}{|\mathbf{P}''(s)|} \qquad \text{(A-15)}$$

Proof: $|\mathbf{P}'(s)|^2 = 1$ and

$$[\mathbf{P}'(s) \times \mathbf{P}''(s)]^2 = |\mathbf{P}'(s) \times \mathbf{P}''(s)|^2 = |\mathbf{P}''(s)|^2 \ .$$

Furthermore,

$$(\mathbf{P}'(s) \times \mathbf{P}''(s)) \times \mathbf{P}'(s) = (\mathbf{P}')^2\mathbf{P}'' - \mathbf{P}''\cdot\mathbf{P}' \ \mathbf{P}' = \mathbf{P}''(s) \ .$$

6. The Serret-Frenet Formulae

The derivatives of the vectors τ, β and ν with respect to s are given by

$$\frac{d\tau}{ds} = \nu/\rho \ , \qquad \text{(A-16)}$$

$$\frac{d\beta}{ds} = -\lambda \ \nu \ , \text{ and} \qquad \text{(A-17)}$$

$$\frac{d\nu}{ds} = \lambda\beta - \tau/\rho \qquad \text{(A-18)}$$

where ρ is the radius of curvature and λ, called the torsion of the curve c at point P is given by

$$\lambda = \rho^2[\mathbf{P}'(s) \cdot \mathbf{P}''(s) \times \mathbf{P}'''(s)] \ . \qquad \text{(A-19)}$$

Eqs. (16), (17) and (18) are called the Serret-Frenet formulae.

Proofs:

$$\tau = \mathbf{P}'(s)$$

$$\tau'(s) = \mathbf{P}'' = \mathbf{P}'' = \left|\mathbf{P}''\right| \ \nu = \boldsymbol{\nu}/\rho \ , \qquad \text{(A-9)}$$

$$\beta = \mathbf{P}'(s) \times \mathbf{P}''(s)[(\mathbf{P}'')^2]^{-1/2} \text{, and}$$

$$\beta'(s) = \mathbf{P}'\mathrm{x}\mathbf{P}'''[(\mathbf{P}'')^2]^{-1/2}$$

$$- \mathbf{P}'\mathrm{x}\mathbf{P}''[(\mathbf{P}'')^2]^{-3/2} \ \mathbf{P}''\bullet\mathbf{P}''' \ .$$

Eliminate the first and second derivatives of P by using

$$\mathbf{P}' = \tau \text{ and } P'' = \nu/\rho \ .$$

Then

$$\beta' = \rho \ \tau \times (\mathbf{P}''' - \boldsymbol{\nu} \ \boldsymbol{\nu}\bullet\mathbf{P}''')$$

$$= \rho \ \tau \times [\boldsymbol{\nu} \times (\mathbf{P}'''\mathrm{x}\boldsymbol{\nu})]$$

$$= \rho \ \tau \bullet(\mathbf{P}'''\mathrm{x}\nu) \ \boldsymbol{\nu} - \rho \ \tau \bullet \boldsymbol{\nu} \ (P'''\mathrm{x}\nu) \ .$$

Since,

$$\boldsymbol{\nu}\bullet\boldsymbol{\tau} = 0,$$

$$\beta' = \rho\tau \bullet (\mathbf{P}''' \times \nu) \ \boldsymbol{\nu}$$

$$= - \lambda \ \nu \ ,$$

where

$$\lambda = - \rho \ \tau \bullet (\mathbf{P}'''\mathrm{x}\nu)$$

$$= \rho^2 \ [\mathbf{P}' \bullet \mathbf{P}''\mathrm{x}\mathbf{P}'''] \ .$$

Finally,

$$\boldsymbol{\nu} = \boldsymbol{\beta} \times \boldsymbol{\tau} \text{ , and}$$

$$\boldsymbol{\nu}' = \boldsymbol{\beta}' \times \boldsymbol{\tau} + \boldsymbol{\beta} \times \boldsymbol{\tau}'$$

$$= -\lambda\, \boldsymbol{\nu} \times \boldsymbol{\tau} + \boldsymbol{\beta} \times \boldsymbol{\nu}/\rho$$

But,

$$\boldsymbol{\nu} \times \boldsymbol{\tau} = (\boldsymbol{\beta} \times \boldsymbol{\tau}) \times \boldsymbol{\tau} = -\boldsymbol{\beta} \text{ and}$$

$$\boldsymbol{\beta} \times \boldsymbol{\nu} = \boldsymbol{\beta} \times (\boldsymbol{\beta} \times \boldsymbol{\tau}) = -\boldsymbol{\tau} \text{ .}$$

Hence,

$$\boldsymbol{\nu}' = \lambda \boldsymbol{\beta} - \boldsymbol{\tau}/\rho \text{ .}$$

**The torsion, λ, expressed in terms of derivatives of P with respect to a variable z other than s is given by:

$$\lambda = \rho^2 \, [\mathbf{P}'(z) \cdot \mathbf{P}''(z) \times \mathbf{P}'''(z)] \left|\mathbf{P}'(z)\right|^{-6} \qquad \text{(A-20)}$$

Proof:

$$\mathbf{P}'(s) = \mathbf{P}'(z) \frac{dz}{ds} \text{ ,}$$

$$\mathbf{P}''(s) = \mathbf{P}''(z) \left(\frac{dz}{ds}\right)^2 + \mathbf{P}'(z) \frac{d^2 z}{ds^2} \text{ , and}$$

$$P'''(s) = P'''(z)\left(\frac{dz}{ds}\right)^3 + 3\,P''(z)\,\frac{dz}{ds}\,\frac{d^2z}{ds^2} + P'\,\frac{d^3z}{ds^3}\ .$$

Then,

$$P'(s) \cdot P''(s) x P'''(s) = P'(z) \cdot P''(z) x P'''(z)\left(\frac{dz}{ds}\right)^6\ ,$$

also

$$\left(\frac{dz}{ds}\right)^2 P'(z)^2 = P'(s)^2 = 1\ .$$

Therefore,

$$\left(\frac{dz}{ds}\right)^6 = [P'(z)^2]^{-3} = |P'(z)|^{-6}\ .$$

7. **Example:** The use of the Serret-Frenet formulas is illustrated by use of the following example.

A straight line AC is drawn on a rectangular sheet of paper ABCD having the dimensions shown in Figure A-4. The paper is then folded to form a right cylinder of radius a/π, the line AC thereby being transformed into the circular helix H shown.

a. Determine the cosine of the angle ψ between a unit vector **k** parallel to the axis of the cylinder and the vector binormal of H at point P.

Solution: Let P' be a point on the cylinder base (Line AB) such that PP' is parallel to **k**. Let z be the distance between P and P', 0 the intersection of the axis of the cylinder with its base, **n** a unit vector parallel to line OP', θ the radian measure of angle AOP', and P the position vector of P relative to 0.

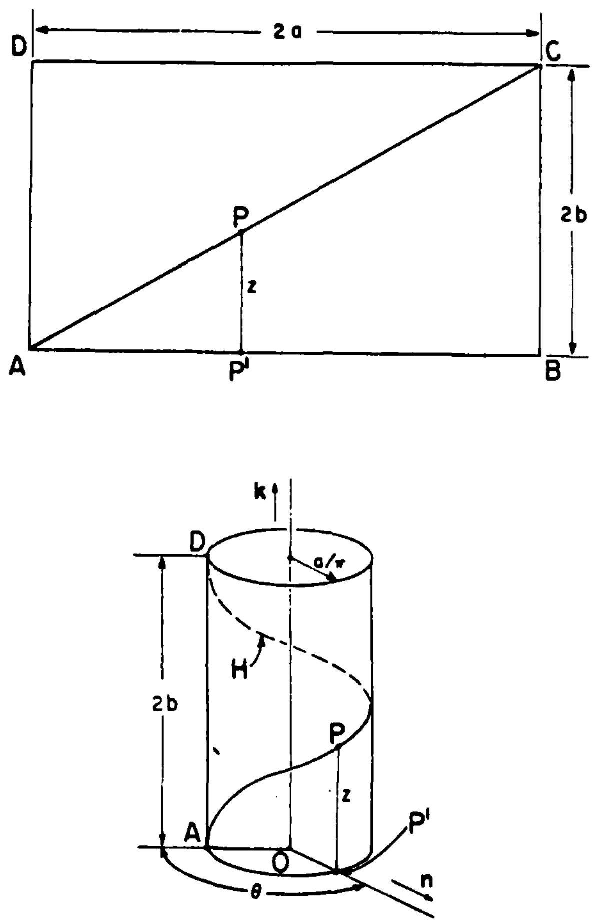

Figure A-4.

$$\cos\psi = \mathbf{k} \cdot \boldsymbol{\beta} ,$$

$$\boldsymbol{\beta} = \frac{\mathbf{P}'(z) \times \mathbf{P}''(z)}{|\mathbf{P}' \times \mathbf{P}''|} ,$$

$$\mathbf{P} = a/\pi \, \mathbf{n} + z \, \mathbf{k} ,$$

$$\mathbf{P}' = a/\pi\ \mathbf{k} \times \mathbf{n}\,\frac{d\theta}{dz} + \mathbf{k}\ ,$$

$$\theta = \pi\,\frac{z}{b}\quad (\text{Since arc } AP' = z\,\frac{a}{b}\ ,\ \text{and}$$

$$\frac{d\theta}{dz} = \frac{\pi}{b}\ .$$

Therefore,

$$\mathbf{P}'(z) = \frac{a}{b}\,\mathbf{k}\mathrm{x}\mathbf{n} + \mathbf{k}\ ,$$

$$\mathbf{P}''(z) = \frac{a}{b}\,\mathbf{k}\mathrm{x}(\mathbf{k}\mathrm{x}\mathbf{n})\,\frac{d\theta}{dz}$$

$$= -\,\pi\,\frac{a}{b^2}\ \mathbf{n}\quad (\text{Since } \mathbf{k}\mathrm{x}(\mathbf{k}\mathrm{x}\mathbf{n}) = -\,\mathrm{n})\ .$$

$$\mathbf{P}'\mathrm{x}\mathbf{P}'' = \frac{\pi a}{b^2}\ \left(\frac{a}{b}\,\mathbf{k} + \mathbf{n} \times \mathbf{k}\right)\ \text{and}$$

$$\left|\mathbf{P}'\mathrm{x}\mathbf{P}''\right| = \pi\,\frac{a}{b^2}\left(1 + \frac{a^2}{b^2}\right)^{1/2}\ .$$

Thus ,

$$\boldsymbol{\beta} = \left(\frac{a}{b}\,\mathbf{k} + \mathbf{n} \times \mathbf{k}\right)\left(1 + \frac{a^2}{b^2}\right)^{-1/2}\ .$$

Finally,

$$\cos\,\psi = \mathbf{k}\cdot\boldsymbol{\beta} = \left(1 + \frac{b^2}{a^2}\right)^{-1/2} .$$

b. Determine The Radius Of Curvature At P Of The Helix H.

$$\rho = \frac{|\mathbf{P}'|^3}{|\mathbf{P}'\mathrm{x}\mathbf{P}''|} ,$$

$$\mathbf{P}' = \frac{a}{b}\,\mathbf{k} \times \mathbf{n} + \mathbf{k} ,$$

$$|\mathbf{P}'\mathrm{x}\mathbf{P}''| = \pi \frac{a}{b^2} \left(1 + \frac{a^2}{b^2}\right)^{1/2} , \text{ and}$$

$$\rho = \frac{\left[1 + \frac{a^2}{b^2}\right]^{3/2}}{\pi \frac{a}{b^2} \left[1 + \frac{a^2}{b^2}\right]^{1/2}} = \frac{a}{\pi}\left(1 + \frac{b^2}{a^2}\right) .$$

c. Determine the torsion λ of the curve H.

$$\lambda = \rho^2 [\mathbf{P}'\cdot\mathbf{P}''\mathrm{x}\mathbf{P}'''] [\mathbf{P}']^{-6} ,$$

$$\mathbf{P}' = \frac{a}{b}\,\mathbf{k} \times \mathbf{n} + \mathbf{k} ,$$

$$\mathbf{P}'' = -\frac{\pi a}{b^2}\,\mathbf{n} ,$$

$$\mathbf{P}''' = -\pi\frac{a}{b^2}\,\frac{d\mathbf{n}}{dz} = -\pi\,\frac{a^2}{b^3}\,\mathbf{k}\mathrm{x}\mathbf{n} ,$$

$$|\mathbf{P}'| = \left(1 + \frac{a^2}{b^2}\right)^{1/2} \text{, and}$$

$$\mathbf{P}' \cdot (\mathbf{P}'' \mathrm{x} \mathbf{P}''') = \pi^3 \frac{a^2}{b^5} \ .$$

Then,

$$\lambda = \frac{\pi b}{a^2+b^2} \ .$$

8. Angular Velocity Of Space Curve Coordinates In A Fixed Reference Frame.

Let P be a point moving along a space curve c with speed $\frac{ds}{dt}$ where s is the distance to P along c from some point of c. If R is a reference frame in which the tangent (τ), principal normal (ν) and binormal (β) unit vectors to c at P are fixed then the angular velocity of R in the reference frame R' in which c is fixed is given by

$$R' \, {}_{\omega} \, R = (\lambda \ \tau + 1/\rho \ \beta) \frac{ds}{dt} \ . \tag{A-21}$$

Proof: Given two reference frames R and R' and any two non-parallel vectors **a** and **b** fixed in R. The angular velocity of R in R' is defined by Kane as

$$R' \, {}_{\boldsymbol{\omega}} \, R = \frac{\frac{d\mathbf{a}}{dt} \mathrm{x} \frac{d\mathbf{b}}{dt}}{\frac{d\mathbf{a}}{dt} \cdot \mathbf{b}} \ . \tag{A-22}$$

If **a** and **b** are taken as the vector principal normal, $\boldsymbol{\nu}$, and the vector binormal, $\boldsymbol{\beta}$, respectively at P then Eq. (22) may be written as

$$R' \, {}_{\boldsymbol{\omega}} \, R = \frac{\frac{d\boldsymbol{\nu}}{ds} \mathrm{x} \frac{d\boldsymbol{\beta}}{ds}}{\frac{d\nu}{ds} \cdot \boldsymbol{\beta}} \ \frac{ds}{dt} \ . \tag{A-23}$$

Next, the Serret-Frenet formulas give

$$\frac{d\boldsymbol{\nu}}{ds} = \lambda\boldsymbol{\beta} - 1/\rho\ \tilde{\tau} \quad \text{and}$$

$$\frac{d\boldsymbol{\beta}}{ds} = -\lambda\ \nu\ .$$

Substitution of the above into Eq. (A-22) gives the desired result.

APPENDIX B

APPENDIX B. COEFFICIENTS APPEARING IN EQUATIONS (IV-11)

APPENDIX B

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$A_{12} = \kappa_o(G+E)I_{xy}$

$A_{11} = \lambda_o\kappa_o(G+E)I_{xx} + (\kappa_o GI_{xy})'$

$A_{10} = \kappa_o\lambda_o(GI_{xx})'+(\kappa_o'\lambda_o+\kappa_o\lambda_o')GI_{xx}+\kappa_o\lambda_o' \; EI_{xy}$

$B_{12} = \kappa_o(G+E)I_{xx}$

$B_{11} = (\kappa_o GI_{xx})' - \kappa_o\lambda_o(G+E)I_{xy}$

$B_{10} = - \kappa_o\lambda_o(GI_{xy})' - \kappa_o'\lambda_o GI_{xy}-\kappa_o\lambda_o'(G+E)I_{xy}$

$C_{11} = \kappa_o^2(G+E)I_{xy}$

$C_{10} = \kappa_o^2(GI_{xy})' + \kappa_o\kappa_o'(2G+E)I_{xy}$

$D_{12} = GJ_P$

$D_{11} = (GJ_P)'$

$D_{10} = - \kappa_o^2 \; EI_{xx}$

$A_{24} = EI_{yy}$

$A_{23} = 2(EI_{yy})' - \lambda_o \; EI_{xy}$

$A_{22} = (EI_{yy})'' + 2\lambda_o' \; EI_{xy} - \lambda_o^2 \; E(2I_{xx}+I_{yy})$

$$A_{21} = \lambda_o(EI_{xy})'' + 4\lambda_o'(EI_{xy})' + (3\lambda_o''-\lambda_o^3)\, EI_{xy}$$

$$+ \kappa_o^2\lambda_o GI_{xy} - 2\lambda_o^2(EI_{xx})' - 5\lambda_o\lambda_o'\, EI_{xx}$$

$$A_{20} = \kappa_o^2 EA - 2\lambda_o\lambda_o'(EI_{xx})' - (2\lambda_o\lambda_o''+\lambda_o'^2)EI_{xx}$$

$$+ \kappa_o^2\lambda_o^2 GI_{xx} + \lambda_o'(EI_{xy})'' + 2\lambda_o''(EI_{xy})'$$

$$+ (\lambda_o''' - \lambda_o'\lambda_o^2)EI_{xy}$$

$$B_{24} = EI_{xy}$$

$$B_{23} = 2(EI_{xy})' - 2\lambda_o EI_{xx} - \lambda_o\, EI_{yy}$$

$$B_{22} = -2\lambda_o(EI_{yy})' - 3\lambda_o'\, EI_{yy} - 2\lambda_o(EI_{xx})'$$

$$- \lambda_o'\, EI_{xx} + (EI_{xy})'' + \lambda_o^2\, EI_{xy}$$

$$B_{21} = -\lambda_o(EI_{yy})'' - 4\lambda_o'(EI_{yy})' + (\lambda_o^3-3\lambda_o'')EI_{yy}$$

$$+ \kappa_o^2\lambda_o GI_{xx} + 2\lambda_o^2(EI_{xy})' + 5\lambda_o\lambda_o' EI_{xy}$$

$$B_{20} = -\lambda_o'(EI_{yy})''-2\lambda_o''(EI_{yy})' + (\lambda_o^2\lambda_o'-\lambda_o''')EI_{yy}$$

$$+ 2\lambda_o\lambda_o'(EI_{xy})' + (2\lambda_o\lambda_o''+\lambda_o'^2)EI_{xy}-\kappa_o^2\lambda_o^2 GI_{xy}$$

$$C_{23} = \kappa_o\, EI_{yy}$$

$$C_{22} = 2\kappa_o(EI_{yy})' + 3\kappa_o' EI_{yy} - 2\lambda_o\kappa_o EI_{xy}$$

$$C_{21} = \kappa_o(EI_{yy})'' + 4\kappa_o'(EI_{yy})' + (3\kappa_o'' - \lambda_o^2\kappa_o)EI_{yy}$$

$$- 2\kappa_o\lambda_o(EI_{xy})' - (4\lambda_o\kappa_o' + \lambda_o'\kappa_o)EI_{xy} - \kappa_o EA$$

$$C_{20} = \kappa_o'(EI_{yy})'' + 2\kappa_o''(EI_{yy})' + (\kappa_o''' - \kappa_o'\lambda_o^2)EI_{yy}$$

$$- 2\lambda_o\kappa_o'(EI_{xy})' - (2\lambda_o\kappa_o'' + \lambda_o'\kappa_o')EI_{xy}$$

$$+ \kappa_o^3\,\lambda_o GI_{xy}$$

$$D_{22} = -\kappa_o EI_{xy}$$

$$D_{21} = 2\lambda_o\kappa_o\, EI_{xx} + \kappa_o\lambda_o\, GJ_p - 2\kappa_o(EI_{xy})'$$

$$- 2\,\kappa_o'\, EI_{xy}$$

$$D_{20} = 2\,\kappa_o\lambda_o(EI_{xx})' + (\lambda_o'\kappa_o + 2\lambda_o\kappa_o')EI_{xx}$$

$$- \kappa_o(EI_{xy})'' - 2\,\kappa_o'(EI_{xy})' + (\lambda_o^2\kappa_o - \kappa_o'')EI_{xy}$$

$$A_{34} = -EI_{xy}$$

$$A_{33} = -\lambda_o\, EI_{xx} - 2\,\lambda_o\, EI_{yy} - 2(EI_{xy})'$$

$$A_{32} = -2\lambda_o(EI_{xx})' - 3\lambda_o'\, EI_{xx} - 2\lambda_o(EI_{yy})' - \lambda_o'\, EI_{yy}$$

$$- (EI_{xy})'' - \lambda_o^{2} EI_{xy} + \kappa_o^{2}\, GI_{xy}$$

$$A_{31} = -\lambda_o(EI_{xx})'' - 4\,\lambda_o'(EI_{xx})' + (\lambda_o^{3} - 3\lambda_o'')EI_{xx}$$

$$+ \kappa_o^{2}\lambda_o\, GI_{xx} - 2\lambda_o^{2}(EI_{xy})' - 5\,\lambda_o\lambda_o'\, EI_{xy}$$

$$+ \kappa_o^{2}(GI_{xy})' + 2\,\kappa_o\kappa_o'\, GI_{xy}$$

$$A_{30} = -\lambda_o'(EI_{xx})'' - 2\lambda_o''(EI_{xx})' + (\lambda_o'\lambda_o^{2} - \lambda_o''')\, EI_{xx}$$

$$+ \kappa_o^{2}\,\lambda_o(GI_{xx})' + \kappa_o\,(2\,\kappa_o'\lambda_o + \kappa_o\lambda_o\,')GI_{xx}$$

$$- 2\,\lambda_o\lambda_o'(EI_{xy})' - (2\,\lambda_o\lambda_o'' + \lambda_o'^{2})\, EI_{xy}$$

$$B_{34} = -\,EI_{xx}$$

$$B_{33} = -\,2(EI_{xx})' - \lambda_o\, EI_{xy}$$

$$B_{32} = -\,(EI_{xx})'' + \lambda_o^{2}\, EI_{xx} + \kappa_o^{2}\, GI_{xx} + 2\,\lambda_o^{2} EI_{yy}$$

$$+ 2\,\lambda_o'\,(EI_{xy})$$

$$B_{31} = 2\,\kappa_o\kappa_o'\, GI_{xx} + \kappa_o^{2}(GI_{xx})' + 5\,\lambda_o\lambda_o'\, EI_{yy}$$

$$+ (3\lambda_o'' - \lambda_o^3)\, EI_{xy} + \kappa_o^2\lambda_o\, GI_{xy}$$

$$+ 4\lambda_o'(EI_{xy})' + \lambda_o(EI_{xy})''$$

$$B_{30} = (2\,\lambda_o'^2 + 2\,\lambda_o\lambda_o'')EI_{yy} + 2\,\lambda_o^2(EI_{yy})'$$

$$= (\lambda_o''' - \lambda_o^2\lambda_o')EI_{xy} - (2\,\kappa_o\kappa_o'\lambda_o + \kappa_o^2\lambda_o')GI_{xy}$$

$$+ 2\,\lambda_o''\,(EI_{xy})' - \lambda_o\,\kappa_o^2(GI_{xy})' + \lambda_o'(EI_{xy})''$$

$$C_{33} = -\,\kappa_o\, EI_{xy}$$

$$C_{32} = -\,2\,\lambda_o\kappa_o EI_{yy} - \kappa_o'\, EI_{xy} - 2\,\kappa_o(EI_{xy})'$$

$$C_{31} = -\,(2\lambda_o\kappa_o' + \kappa_o\lambda_o')EI_{yy} - 2\lambda_o\kappa_o(EI_{yy})'$$

$$+ (\lambda_o^2\kappa_o - 3\kappa_o'')EI_{xy} + \kappa_o^3\, GI_{xy} - 4\kappa_o'(EI_{xy})'$$

$$- \kappa_o(EI_{xy})''$$

$$C_{30} = -\,(2\lambda_o\kappa_o'' + \lambda_o'\,\kappa_o')EI_{yy} - 2\,\lambda_o\kappa_o'\,(EI_{yy})'$$

$$+ (\lambda_o^2\kappa_o' - \kappa_o''')EI_{xy} + 4\,\kappa_o^2\kappa_o'\, GI_{xy}$$

$$- 2\,\kappa_o''(EI_{xy})' + \kappa_o^3(GI_{xy})' - \kappa_o'\,(EI_{xy})''$$

$$D_{32} = \kappa_o(EI_{xx} + GJ_p)$$

$$D_{31} = \kappa_o' \, (2EI_{xx} + GJ_P) + 2\,\kappa_o\lambda_o \, EI_{xy}$$

$$+ 2\,\kappa_o \, (EI_{xx})' + \kappa_o(GJ_P)'$$

$$D_{30} = (\kappa_o''-\lambda_o^2\kappa_o) \, EI_{xx} + (2\lambda_o\kappa_o' + \kappa_o\lambda_o') \, EI_{xy}$$

$$+ 2\kappa_o'(EI_{xx})' + \kappa_o(EI_{xx})'' + 2\kappa_o \, \lambda_o(EI_{xy})'$$

$$A_{43} = \kappa_o \, EI_{yy}$$

$$A_{42} = \kappa_o(EI_{yy})'$$

$$A_{41} = -\,\kappa_o\lambda_o^2 \, EI_{xx} - \kappa_o EA + 2\,\lambda_o'\kappa_o \, EI_{xy} + \lambda_o\kappa_o(EI_{xy})'$$

$$A_{40} = -\,\kappa_o\lambda_o\lambda_o' \, EI_{xx} - \kappa_o' \, EA + \kappa_o\lambda_o'' \, EI_{xy}$$

$$-\,\kappa_o(EA)' + \kappa_o \, \lambda_o' \, (EI_{xy})'$$

$$B_{43} = \kappa_o \, EI_{xy}$$

$$B_{42} = -\,\lambda_o\kappa_o \, EJ_P + \kappa_o \, (EI_{xy})'$$

$$B_{41} = -\,2\,\kappa_o \, \lambda_o' \, EI_{yy} - \lambda_o \, \kappa_o \, (EI_{yy})' + \kappa_o \, \lambda_o^2 \, EI_{xy}$$

$$B_{40} = -\,\lambda_o''\kappa_o \, EI_{yy} - \lambda_o'\kappa_o(EI_{yy})' + \kappa_o\lambda_o\lambda_o' \, EI_{xy}$$

$$C_{42} = EA + \kappa_o^2 \, EI_{yy}$$

$$C_{41} = 2\,\kappa_o\kappa_o'\ EI_{yy} + \kappa_o^{\,2}(EI_{yy})' + (EA)'$$

$$-\ \kappa_o^{\,2}\lambda_o\ EI_{xy}$$

$$C_{40} = \kappa_o\kappa_o''\ EI_{yy} + \kappa_o\kappa_o'\ (EI_{yy})' - \kappa_o\kappa_o'\lambda_o\ EI_{xy}$$

$$D_{41} = -\ \kappa_o^{\,2}\ EI_{xy}$$

$$D_{40} = \kappa_o^{\,2}\lambda_o\ EI_{xx} - \kappa_o\kappa_o'\ EI_{xy} - \kappa_o^{\,2}\ (EI_{xy})'$$

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