A TRANSFORMATION METHOD FOR THE EVALUATION OF THE RESPONSE MOMENTS OF ELASTIC STRUCTURES WITH RANDOM PROPERTIES

BY

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B.S. Civil Engineering

University of Virginia, 1978

M.E. Civil Engineering

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DISSERTATION

Submitted in Partial Fulfillment of the
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RESPONSE MOMENTS OF ELASTIC STRUCTURES WITH RANDOM PROPERTIES

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In this investigation a new method for computing the first and second moments of the displacement response of linear elastic structures with random stiffness and damping is developed.

The method accurately computes the mean and variance of the displacement of a dynamically loaded structure as functions of time. The probability that the displacement will be greater than a specified value is also computed.

The results of the transformation method are compared to exact solutions for a number of cases.

The influence of the randomness in the underlying parameters on the randomness of the response is discussed.

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CHAPTER I

INTRODUCTION

1.1 OBJECTIVES AND SCOPE

The objective of this dissertation is to develop a method for determining the first and second moments of the dynamic response of structural systems with random parameters.

The randomness in the response of structural systems is due to two primary factors. The first of these is the randomness in the loading which has been discussed extensively in the past. The other is the randomness associated with the structure and has usually been ignored in computing the moments of the response.

The dissertation will develop a new method for including the structural uncertainties directly in the computation of the response moments.

The systems under study are limited to those with linear, elastic properties. The random structural properties considered are the stiffness and damping. The response parameters of interest are the deflections at key points on the structure. The desired results are the mean and covariances of the deflections as functions of time.

The method developed will not require knowledge of the distribution of the structural parameter other than their means and variances. Similarly the distribution of the response will not be computed. For the computation of first passage probabilities the response will be assumed Gaussian.

1.2 MOTIVATION

The randomness in structural parameters, such as stiffness, strength and damping, has long been realized by designers. For typical structures which are designed to resist normal fluctuations in environmental loads, such uncertainties may be safely accounted for by using conservative values for the stiffness or strength rather than the mean values. However, for structures which must resist extreme loads such as earthquakes, accidental blast, or weapons effects, such a conservative approach is often economically impractical. A preferable approach establishes a probability of failure that is deemed acceptable, and the structure is designed to insure such a probability is not exceeded.

For the engineer to design a structure with a specific probability of failure three things are required. The first of these is a probabilistic description of the loading; secondly, the probabilistic characterization of the structure is required. Third, and the most difficult, is a method for computing the moments of the response and, hence, the probability of failure.

The probability of failure is the probability that a response parameter will exceed an allowable value which may or may not be a random variable. When the allowable value is deterministic the moments of the response plus the assumption or computation of its distribution function will be sufficient to compute the probability or failure. When the allowable value is a random variable then the joint distribution of the allowable value and the response values is required.

Typical response parameters of interest in computing structural failure include strains, stresses, displacements, velocities, accelerations, number of stress reversals and ductility ratios. Strains, stresses and ductility ratios are usually of interest when examining the possibility of overload resulting in breaking of a member. Stress reversals may be of interest in examining fatigue failure. Displacement, velocity and acceleration are of interest when examining stability, service or shock isolation type failures. For example when the deflection of a shock isolated piece of equipment exceeds available rattlespace the system may be assumed to have failed. Similarly a tall building may suffer a service failure if the displacement of the upper floors in high winds results in the occupants feeling uncomfortable. This type of failure may occur when the stresses and strains are still well below a dangerous level.

While much work has been done in the area of load modeling for earthquakes, wind and blast loads, and in the calculation of the response of deterministic systems to random loads, little has been done in establishing techniques for computing the response moments in the presence of randomness in the structural parameters.

Recent studies described in the literature review which follows have concentrated on quantifying the structural uncertainties, but not on their effects on response uncertainties.

1.3 LITERATURE REVIEW

This review deals with studies on the theory and calculation of random dynamic structural response. Books of general interest will be discussed first followed by papers dealing with the quantification of structural uncertainties. Papers involving the computation of response characteristics in the presence of structural uncertainties will then be discussed. Finally, some papers dealing with the first passage and peak values of structural response will be reviewed.

The theory of random vibrations of linear elastic structures with deterministic mass, damping and stiffness properties is well developed.

The equation of motion for such systems when they are mass excited is,

$$[M]\{\ddot{X}(t)\} + [C]\{\dot{X}(t)\} + [K]\{X(t)\} = \{P(t)\} \qquad (1\text{-}1)$$

where $[M]$ = System Mass Matrix

$[C]$ = System Damping Matrix

$[K]$ = System Stiffness Matrix

$\{X\}$ = System Displacement Vector

$\{P(t)\}$ = System Forcing Vector

When $\{P(t)\}$ is deterministic $\{X(t)\}$ will be as well.

Lin (1), Crandall (2), and Papoulis (3) all address the stationary and transient response of such linear systems for both stationary and nonstationary loadings. Single-degree-of-freedom, (SDOF), systems may be solved using the system's impulse response function. Multiple-degree-of-freedom, (MDOF), systems are analyzed via their normal modes. Expressions for the autocorrelation functions and first passage probabilities of the response are developed as well as the expressions for the mean responses. In a more recent work Vanmarke (4) addresses the extension of this work to include first crossing probabilities for both stationary and nonstationary envelopes about the process. Elishakoff (5) provides an excellent text for quantifying the response statistics for both static and dynamic loads as well as a discussion of the Monte Carlo method of analysis. Soong (6) expresses the solution of differential equations with random coefficients as a function of the coefficients. The mean of the solution at any time may then be obtained by integrating the product of the solution and the joint probability density function of the coefficients over the range of the coefficients. The variance may be obtained in a similar manner. However, the resulting integrals may have to be evaluated numerically at each time step.

Several studies have addressed the quantification of structural uncertainties. MacGregor, Mirza and Ellingwood (7) compiled information on the means and variances for concrete strength, steel yield strength, error in steel placement and total resistance. Ellingwood (8) has also examined the statistical properties of reinforced concrete beam columns. These studies were based on tests of existing structures and laboratory specimens.

An analytical study was performed by Grant, Mirza and MacGregor (9) to examine the strength of concrete columns based on the underlying random variables. The formula for the strength of the columns was evaluated repeatedly with a Monte Carlo procedure to determine the mean and variance. Ramsey, Mirza and MacGregor (10) applied a similar technique to the computation of deflections of concrete beams. Mlakar (11) used a Taylor series expansion about the mean to evaluate the mean and variance of the resistance of a beam column. Thielen (12) used a similar technique to examine the behavior of concrete beams near the limit state.

The consideration of structural uncertainties in response analysis is fairly recent. The building codes have recognized strength variation through the specification of strength reduction factors and working stresses well below the average values. The field of reliability and safety analysis was among the first to handle the uncertainties explicitly.

Studies within this field have concentrated on the proper design loads and strengths to use for structures in routine circumstances. Ang and Cornell (13) investigated the influence random properties such as yield, bond and shear strength have on structural reliability and proper design levels. Ellingwood and Ang (14) proposed procedures for risk based design as did Ravionda, Lind and Siu (15). Moses (16) examined the reliability of structural systems.

There have been few studies concerned with the calculation of the moments of the dynamic response of a system with random stiffness and damping. Collins and Thompson (17) addressed the problem of random eigenvalues and eignevectors for a system such as Equation (1-1) with a random stiffness matrix. Hasselman and Hart (18) developed expressions for the modal response of MDOF systems based on random eigenvectors.

In both of these papers the expressions for the eigenvalue means were taken to be the same values that would result from the deterministic expression and the variances were evaluated by Taylor series expansions about the means. Hisada and Nakgiri (19) used a Taylor series expansion about the means to quantify the effects of node location uncertainty on the stiffness matrix and resulting stresses and strains for the static case. Handa and Anderson (20) developed an approximation for solving the static problem by separating the random variables into a deterministic variable equal to the mean and a random variable with mean zero but a variance equal to the variance of the original random variable. However, the cross-correlation between the stiffness and deflection is ignored. This results in the mean response being equal to the response of the system when the system parameters are treated as being deterministic and equal to their means. This is known to be incorrect.

There are many papers related to first passage probabilities and peak values for stochastic processes. The topic is also well covered in many of the basic texts listed earlier.

In a classic paper Rice (21) developed the expressions for first passage times of a white noise process. Karlin and Taylor (22) developed expressions for first passage probabilities of a random process represented by Brownian motion with a drift in the mean. Corotis, Vanmarke and Cornell (23) developed expressions for the first passage of nonstationary random processes. Crandall, Chandiramani and Cook (24) used a numerical simulation technique to examine first passage probabilities of the response of a SDOF system. The system was linear, elastic and the system properties were deterministic. The loading was repeatedly applied in a Monte Carlo style and the statistics of the response including the expected time to first passage beyond a barrier computed. A number of types of barriers were investigated as were the zero and stationary start conditions. These authors also developed a model where the sample space is divided into cells and the probability mass is repeatedly redistributed in accordance with theoretically derived transition probabilities. Krenk, Madsen and Madsen (25) developed a method for modifying stationary crossing frequencies for use in transient response envelopes. Ang (26) developed a method for computing the response moments of a linear system, and then developed first passage probabilities based on these. Simple nonlinear elastic structures were examined.

CHAPTER 2

RESPONSE OF SDOF SYSTEMS WITH RANDOM STIFFNESS AND DAMPING

2.1 INTRODUCTION

In this chapter a new method is developed for analyzing linear, elastic, SDOF systems with random stiffness and damping. The system may be excited by either a deterministic signal or random process. In either case the response of the system will be a stochastic process. Such processes may frequently be satisfactorily characterized by their first two moments, the mean and variance. For the processes considered here these will vary as functions of time.

The method developed involves the replacement of the random variables appearing in the system's equation of motion with two new variables. The first of the new variables is deterministic and equal to the mean of the original random variable. The second variable is random with mean zero and has a variance equal to that of the original random variable.

This substitution has been used by Handa and Anderson (20) for analyzing the static loading of structures but this is its first application to the dynamic response of structures. Handa and Anderson also neglect the correlation between the randomness in the system parameters and the randomness in the response introducing an error in the mean which they acknowledge. The correlation is included here and hence the mean generated is the true mean.

The results of the method are the mean and variance of the system's displacement and velocity as functions of time. The correlation between the displacement and velocity may be readily computed as well.

The details of the transformation of the random variables appearing in the equation of motion are explained in Section 2.2. The resulting transformed equation of motion is solved using forward differences, the details of this operation are explained in Section 2.3. A computer code was developed to perform the computations and its basic numerical algorithm is provided in Section 2.4. The effects of some higher order terms which have been neglected in the computation of the variance are examined in Section 2.5. Several numerical examples varifying the method are provided in Section 2.6. The method will be referred to as the transformation method.

The following assumptions are made concerning the system's parameters and response characteristics,

a) The mass of the system is deterministic.

b) The system response is twice mean square differentiable with respect to time.

c) The system's damping and stiffness are random variables whose values are not functions of time. The two may be correlated or uncorrelated.

d) When the forcing function is stochastic it has an autocorrelation function with one factor which is a delta function.

e) The forcing function is independent of the system parameters.

2.2 TRANSFORMATION OF THE EQUATIONS OF MOTION

The equation of motion of a mass excited SDOF system is,

$$M\ddot{X} + C\dot{X} + KX = F(t) \tag{2-1}$$

where,

M = System Mass

C = Damping Resistance

K = Stiffness

$X(t)$ = System Displacement

$F(t)$ = Forcing Function

When K and C are deterministic Equation (2-1) may be solved exactly for either deterministic or stochastic forcing functions. When K and C are random variables the response, $X(t)$, is a stochastic process, the mean and variance of which may be evaluted by the method of Soong (6). However, this method will in general require numerical integration at each time of interest.

To develop expressions for the mean and mean square values of $X(t)$ a substitution is made in Equation (2-1) for the random variables. The substitution replaces the original variables with the sum of two other variables, one of which is deterministic and equal to the mean of the original variable; the second variable is random with mean zero and variance equal to the variances of the original variable. The substitution is made for K, C, $X(t)$, $\dot{X}(t)$, $\ddot{X}(t)$ and $F(t)$.

The deterministic portion of the response $X(t)$ is represented by $\mu(t)$ and the random portion by $\delta(t)$. Hence the substitution for $X(t)$ is,

$$X(t) = \mu(t) + \delta(t) \tag{2-2}$$

Taking the expected value of Equation (2-2) yields,

$$E(X(t)) = E(\mu(t)) + E(\delta(t)) \tag{2-3a}$$

$$= \mu(t) + 0 \quad . \tag{2-3b}$$

To develop an expression for the mean square value of the response one squares Equation (2-2) which yields,

$$X(t)^2 = \mu(t)^2 + 2\mu(t)\delta(t)+\delta(t)^2 \quad . \tag{2-4}$$

Taking the expected value of Equation (2-4) yields an expression for the mean square response of X(t)

$$E\left(X(T)^2\right) = \mu(t)^2 + 2\mu(t)\cdot 0 + E\left(\delta(t)^2\right) \tag{2-5a}$$

$$E\left(X(t)^2\right) = \mu(t)^2 + E\left(\sigma(t)^2\right) \tag{2-5b}$$

The variance of X(t) is given by,

$$\mathrm{Var}(X(t)) = E\left(X(t)^2\right) - E(X(t))^2 \tag{2-6}$$

Substituting the expressions in Equation (2-5b) for $E\left(X(t)^2\right)$ and Equation (2-3b) for $E(X(t))$ yields,

$$\mathrm{Var}(X(t)) = E\left(\delta(t)^2\right) \tag{2-7}$$

Similar substitutions may be developed for the derivatives of X(t) by differentiating Equation (2-2). This yields for the velocity,

$$\dot{X}(t) = \dot{\mu}(t) + \dot{\delta}(t) \tag{2-8}$$

And for the acceleration,

$$\ddot{X}(t) = \ddot{\mu}(t) + \ddot{\delta}(t) \tag{2-9}$$

Taking the expected value of Equation (2-8) yields an expression for the mean of the velocity,

$$E(\dot{X}(t)) = E(\dot{\mu}(t)) + E(\dot{\delta}(t)) \tag{2-10}$$

Since $\mu(t)$ is deterministic its derivatives are as well, hence,

$$E(\dot{\mu}(t)) = \dot{\mu}(t) \tag{2-11}$$

Since X(t) and thus $\delta(t)$ have been assumed mean square differentiable the expectation and differentiation operations may be interchanged. And since $E(\delta(t))$ is zero for all times the expected value of the change of $\delta(t)$ with time must also be zero, yielding,

$$E(\dot{\delta}(t)) = 0 \quad . \tag{2-12}$$

When the relation in Equations (2-11) and (2-12) are substituted into Equation (2-10) the resulting expression for the mean of the velocity is,

$$E(\dot{X}(t)) = \dot{\mu}(t) \quad . \tag{2-13}$$

Squaring Equation (2-8) and taking the expected value provides the expression for the mean square value of the velocity,

$$E(\dot{X}(t)^2) = \dot{\mu}(t)^2 + E(\dot{\delta}(t)^2) \quad . \tag{2-14}$$

Similar expressions for the mean and mean square values of the acceleration may be developed from Equation (2-9). The expected value of the acceleration is,

$$E(\ddot{X}(t)) = \ddot{\mu}(t) \quad . \tag{2-15}$$

The mean square value of the acceleration is given by,

$$E(\ddot{X}(t)^2) = \ddot{\mu}(t)^2 + E(\ddot{\delta}(t)^2) \quad . \tag{2-16}$$

Representing the stiffness K as the sum of K_D and K_R with K_D deterministic and K_R a random variable with mean zero yields,

$$K = K_D + K_R \quad . \tag{2-17}$$

Taking the expected value of Equation (2-17) yields,

$$E(K) = K_D + E(K_R) \quad . \tag{2-18}$$

And since K_R is mean zero,

$$E(K) = K_D \quad . \tag{2-19}$$

Squaring Equation (2-17) to develop an expression for the square of K yields,

$$K^2 = K_D^2 + 2K_R K_D + K_P^2 \quad . \tag{2-20}$$

Taking the expected value of Equation (2-20) yields

$$E(K^2) = K_D^2 + E(K_R^2) \quad . \tag{2-21}$$

Upon rearranging one has the expression for the variance of K.

$$Var(K) = E(K^2) - E(K)^2 \tag{2-22a}$$

$$= E(K_R^2) \quad . \tag{2-22b}$$

For the damping, C, let C_D and C_R be the deterministic and random portions, respectively. Then following the development used for the stiffness, the expressions for C, the mean of C, and the variance of C are,

$$C = C_D + C_R \tag{2-23}$$

$$E(C) = C_D \tag{2-24}$$

$$Var(C) = E(C_R^2) \quad . \tag{2-25}$$

The substitution for the forcing function, F(t), consists of the deterministic portion P(t) and a random portion q(t). Making this substitution and following the development above results in the following expressions for the forcing function, F(t), the mean forcing function, P(t), and the variance of the forcing function,

$$F(t) = P(t)+q(t) \tag{2-26}$$

$$E(F(t)) = P(t) \tag{2-27}$$

$$Var(F(t)) = E(q^2(t)) \quad . \tag{2-28}$$

Substituting the results of Equations (2-2), (2-8), (2-9), (2-17), (2-23), and (2-26) into the equation of motion, Equation (2-1), results in,

$$M(\ddot{\mu}+\ddot{\delta}) + (C_D+C_R)(\dot{\mu}+\dot{\delta}) + (K_D+K_R)(\mu+\delta) = P(t) + q(t) \quad . \tag{2-29}$$

When the system parameters and response are considered correlated the expected value of Equation (2-29) is,

$$M\ddot{\mu} + C_D\dot{\mu} + K_D\mu + E(K_R\delta) + E(C_R\dot{\delta}) = P(t) \quad . \tag{2-30}$$

A solution for $\mu(t)$, the mean response of the system, may be sought based on this expression.

To develop an expression for $\delta(t)$ one may subtract Equation (2-30) from Equation (2-29) yielding the result,

$$M\ddot{\delta} + C_D\dot{\delta} + K_D\delta + CRT + KRT = q(t) - K_R\mu - C_R\dot{\mu} \tag{2-31a}$$

where

$$CRT = C_R\dot{\delta}-E(C_R\dot{\delta}) \tag{2-31b}$$

$$KRT = K_R\delta-E(K_R\delta) \quad . \tag{2-31c}$$

The term CRT represents the fluctuation of the random portion of the damping force about its mean. The fluctuation of the random portion of the stiffness force is represented by KRT.

When the random portions of the restoring forces are considered small in comparison to the deterministic portions, $C_D\dot{\mu}$ and $K_D\mu$, the fluctuations may be considered negligible and set to zero. This assumption is made in the development of the technique. The implications of the assumption are examined in Section 2.5 and the examples in Section 2.6.

When the CRT and KRT terms are neglected Equation (2-31a) becomes,

$$M\ddot{\delta} + C_D\dot{\delta} + K_D\delta = q(t) - K_R\mu - C_R\dot{\mu} \tag{2-32}$$

The response characteristic of interest in this analysis is $E(\delta(t)^2)$. Furthermore, the equations for $\mu(t)$ and $\delta(t)$ are coupled in that $\mu(t)$ appears in the expression for $\delta(t)$ and vice versa. The separate the equations and permit the computation of $\mu(t)$ and $E(\delta(t)^2)$ the forward difference technique will be applied as explained in the following section.

2.3 APPLICATION OF FINITE DIFFERENCES

Many structural mechanics problems involve differential equations. They arise in both static and dynamic analysis and may be solved directly for a relatively small number of cases. For those cases which may not be solved directly, various numerical integration schemes have been developed. A common method of performing the integration is the application of finite differences. This technique may be applied to a differential equation provided the function and its derivatives as they appear in the equation are continuous.

Consider the definition of a derivative:

$$\frac{df(y)}{dy} = \lim_{\Delta y \to o} \frac{f(y+\Delta y) - f(y)}{\Delta y} \quad . \tag{2-33}$$

When Δy approaches a small, fixed value, s, rather than zero an approximation to the derivative of f(y) is:

$$\frac{df(Y)}{dy} \simeq \frac{f(y+s) - f(y)}{s} \quad . \tag{2-34}$$

This expression is the basis for the forward difference technique. If one considers a time, τ, to equal $n\Delta t$, where n is a positive integer and Δt is a small, positive time constant, the expressions for $f(\tau)$ and its derivatives are,

$$f(\tau) = f_n \tag{2-35}$$

$$\dot{f}(\tau) \simeq \frac{f_{n+1} - f_n}{\Delta t} \tag{2-36}$$

$$\ddot{f}(\tau) \simeq \frac{f_{n+2} - 2f_{n+1} + 1f_n}{\Delta t^2} \quad . \tag{2-37}$$

For further information on the finite difference approach and its applications Wang (31) provides a good introductory discussion. Bathe (32) addresses numerical integration schemes in considerable detail as well.

To develop the relations required to evaluate $\mu(t)$, Equations (2-35), (2-36), and (2-37) are applied to Equation (2-30) in place of $\mu(t)$ and its derivatives resulting in,

$$\frac{M}{\Delta t^2}(\mu_{n+2}-2\mu_{n+1}\mu_n) + \frac{C_D}{\Delta t}(\mu_{n+1}-\mu_n) + K_D(\mu_n) =$$

$$= P_n - E(K_R\delta)_n - E(C_R\dot{\delta})_n \quad . \qquad (2\text{-}38)$$

Combining terms and solving for μ_{n+2} yields,

$$\mu_{n+2} = \frac{1}{\overline{M}}\left(P_n - E(K_R\delta)_n - E(C_R\dot{\delta})_n + A_1\mu_{n+1} + A_2\mu_n\right) \quad . \qquad (2\text{-}39a)$$

where

$$\overline{M} = M/\Delta t^2 \qquad (2\text{-}39b)$$

$$\overline{C}_D = C_D/\Delta t \qquad (2\text{-}39c)$$

$$A_1 = 2\overline{M}-\overline{C}_D \qquad (2\text{-}39d)$$

$$A_2 = \overline{C}_D-\overline{M}-K_D \qquad (2\text{-}39e)$$

Equation (2-39a) provides an expression by which $\mu(t)$ may be calculated in a step-by-step manner, based on information calculated at previous time steps, provided that information can be developed. The main difficulty lies in the calculation of $E(K_R\delta)_n$ and $E(C_R\dot{\delta})_n$. These are the terms which have been ignored or assumed zero in many previous studies. Their inclusion here represents an improvement over previous approximations. The means for calculating these terms will be discussed shortly.

Applying the finite difference relations of Equations (2-35), (2-36), and (2-37) to $\delta(t)$ and its derivatives in Equation (2-32) yields,

$$\overline{M}(\delta_{n+2}-2\delta_{n+1}+\delta_n) + \overline{C}_D(\delta_{n+1}-\delta_n) + K_D(\delta_n) = q_n - K_R\mu_n - C_R\dot{\mu}_n \quad . \quad (2\text{-}40)$$

Rearranging and solving for δ_{n+2},

$$\delta_{n+2} = \frac{1}{\overline{M}}\ (q_n - K_R\mu_n - C_R\dot{\mu}_n + A_1\delta_{n+1} + A_2\delta_n) \quad . \qquad (2\text{-}41)$$

Since an expression for $\delta(t)^2$ is sought, Equation (2-41) is squared resulting in,

$$\begin{aligned}\delta_{n+2}^2 = \frac{1}{\overline{M}^2}\ (q_n^2 &- 2q_nK_R\mu_n - 2q_nC_R\dot{\mu}_n + 2q_nA_1\delta_{n+1} \\ &+ 2q_nA_2\delta_n + K_R^2\mu_n^2 + 2K_RC_R\mu_n\dot{\mu}_n \\ &- 2\mu_nK_RA_1\delta_{n+1} - 2\mu_nK_RA_2\delta_n + C_R^2\dot{\mu}_n^2 \\ &- 2C_R\dot{\mu}_nA_1\delta_{n+1} - 2C_R\dot{\mu}_nA_2\delta_n \\ &+ A_1^2\ \delta_{n+1}^2 + 2A_1A_2\delta_{n+1}\delta_n + A_2^2\ \delta_n^2\ ) \qquad (2\text{-}42)\end{aligned}$$

When expected values are taken in the above expression some terms on the right side will be zero. Due to the assumption outlined at the end of Section 2.1, terms involving a product of K_R and qn or C_R and qn will be zero, for the randomness in the loading is independent of the system parameters. Hence, when the expected value of Equation (2-42) is taken the result is,

$$E(\delta_{n+2}^{2}) = \frac{1}{\bar{M}^2}\Big[\mathrm{Var}(F_n) + \mu_n^2\,\mathrm{Var}(K) + \dot{\mu}_n^2\,\mathrm{Var}C$$

$$+ 2\mu_n\dot{\mu}_n\,\mathrm{Cov}(K,C) - 2\mu_n A_1 E(K_R\delta_{n+1})$$

$$- 2\mu_n A_2 E(K_R\delta_n) - 2\dot{\mu}_n A_1 E(C_R\delta_{n+1})$$

$$-2\dot{\mu}_n A_2 E(C_R\delta_n) + A_1^2 E(\delta_{n+1}^{2})$$

$$+ A_2^2 E(\delta_n^2) + 2A_1A_2E(\delta_{n+1}\delta_n) + 2A_1E(q_n\delta_{n+1})$$

$$+ 2A_2E(q_n\delta_n)\Big] \qquad (2\text{-}43)$$

Note in Equation (2-43) that all terms which involve the randomness of the system parameters are multiplied by the mean response or its derivatives. Hence, when the system has a zero mean response the variance and mean square values are functions of the randomness of the excitation only. This is due to the assumptions regarding the terms designated CRT and KRT in Equations (2-31a), (2-31b), and (2-31c). These terms are discussed further in Section 2.5.

The terms Var(K), Var(C), Cov(K,C) and $E(q_n^2)$ must be specified prior to commencing the solution. The terms involving $E(\delta_{n+1}{}^2)$ and $E(\delta_n{}^2)$ on the right hand side are the results of Equation (2-43) at previous time steps. The formulation of the remaining terms, $E(C_R\dot{\delta}_n)$, $E(K_R\delta_n)$, $E(\delta_{n+1}\delta_n)$, and $E(q_n\delta_n)$ proceeds as follows. Consider Equation (2-41) with the index shifted by one. This provides an expression for δ_{n+1}.

$$\delta_{n+1} = \frac{1}{\bar{M}} \left(q_{n-1} - K_R\mu_{n-1} - C_R\dot{\mu}_{n-1} + A_1\delta_n + A_2\delta_{n-1}\right) \quad . \qquad (2\text{-}44)$$

Multiplying this expression by K_R yields

$$K_R\delta_{n+1} = \frac{1}{\bar{M}} \left(K_Rq_{n-1} - K_R^2\mu_{n-1} - K_RC_R\dot{\mu}_{n-1} + A_1K_R\delta_n + A_2K_R\delta_{n-1}\right) \quad . \qquad (2\text{-}45)$$

Taking the expected value of both sides results in,

$$E(K_R\delta_{n+1}) = \frac{1}{\bar{M}} \left(-Var(K)\mu_{n-1} - \dot{\mu}_{n-1}Cov(K,C) + A_1E(K_R\delta_n) + A_2E(K_R\delta_{n-1})\right) \quad . \qquad (2\text{-}46)$$

This expression may then be readily evaluated in a step-by-step manner from the zero start condition. The expression for $E(C_R\delta_{n+1})$ is developed similarly and results in,

$$E(C_R\delta_{n+1}) = \frac{1}{\bar{M}} \left(-\mu_{n-1}Cov(K,C) - \dot{\mu}_{n-1}Var(C) + A_1E(C_R\delta_n) + A_2E(C_R\delta_{n-1})\right) \qquad (2\text{-}47)$$

The expression for $E(\delta_{n+1}\delta_n)$ is developed in a similar manner. It is,

$$E(\delta_{n+1}\delta_n) = \frac{1}{\bar{M}} \left(-\mu_{n-1}E(K_R\delta_n) - \dot{\mu}_{n-1}E(C_R\delta_n) + A_1E(\delta_n^2) + A_2E(\delta_n\delta_{n-1})\right) \qquad (2\text{-}48)$$

The expression for $E(q_n \delta_{n+1})$ is

$$E(q_n \delta_{n+1}) = \frac{1}{M} \big(E(q_n q_{n-1}) - E(q_n K_R)\mu_{n-1}$$

$$- E(q_n C_R)\dot{\mu}_{n-1} + A_1 E(q_n \delta_n)$$

$$+ A_2 E(q_n \delta_{n-1})\big) \qquad (2\text{-}49)$$

However, since the loading is independent of the system parameters and uncorrelated with itself for any time shift, all terms in the above expression are zero. Hence,

$$E(q_n \delta_{n+1}) = E(q_n \delta_n) = 0 \qquad (2\text{-}50)$$

Since these terms are zero the randomness in the response due to the loading and that due to the system parameters are independent. Thus they may be computed separately and combined merely by adding the two independent variances.

The expressions for the variance of the velocity may be developed from Equations (2-48), (2-43) and the finite different approximation to the velocity.

$$\dot{\delta}_n \simeq \frac{\delta_{n+1} - \delta_n}{\Delta t} \qquad (2\text{-}37)$$

Squaring and taking the expected value yields,

$$E(\dot{\delta}_n^2) = \frac{E(\delta_{n+1}^2) - 2E(\delta_{n+1}\delta_n) + E(\delta_n^2)}{\Delta t^2} \qquad (2\text{-}51)$$

Hence the method of transforming the random variables by a shift about the mean provides expressions which enable the analyst to evaluate the means, variances and covariance of displacement and velocity. Note these expressions are independent of the distribution of the system parameters and the loading function. This is due to the linearization of the solution to the equation of motion at each time step.

An algorithm for computing the mean and variance of the displacement response of SDOF systems is outlined in the following section.

2.4 THE COMPUTATION ALGORITHM

The following algorithm was developed to compute the mean and variance of the displacement response of an SDOF system. The system stiffness and damping are considered random. They are described in terms of their means and variances, which are input at the beginning of the calculation as is their correlation coefficient. The randomness in the loading is described in terms of its mean and variance which are defined at the beginning of the calculation as well. These may vary with time provided the function describing the variation is defined.

The algorithm is based on a zero start condition and proceeds as follows,

1. Read input parameters;
   K, Var(K), C, Var(C), Cov(K,C)
   M, Δt, and problem duration.

2. Initialize all variables for zero start condition.

3. Set, NT, the total number of time steps equal to the problem duration/Δt.

4. Do for each step N=1,NT
   a) Compute the mean and variance of the loading function based on predetermined functions.

   b) Compute $E(\delta_{n+1}\delta_n)$ from Equation (2-48).

   c) Compute μ_{n+2} from Equation (2-39a).

   d) Compute $E(\delta_{n+2}^2)$ from Equation (2-43).

   e) Compute $E(K_R\delta_{n+2})$ from Equation (2-46).

   f) compute $E(C_R\delta_{n+2})$ from Equation (2-47).

   g) Write desired results.

   h) Time = Time + Δt.

   i) Go to step 4a until Time = Problem duration.

This computational algorithm is implemented in the computer program RSDOF, a listing of which is included in Appendix A. Numerical examples in Section 2.6 use the code to varify the adequacy of the method as well as to investigate the influence the random system parameters have on the response moments.

2.5 THE EFFECT OF HIGHER ORDER TERMS

This section discusses the effects of the assumptions made in Section 2.2 regarding the fluctuations in the random portion of the system's stiffness and damping restoring forces.

Equation (2-31a) may be rewritten as,

$$M\ddot{\delta} + C_D\dot{\delta} + K_D\delta = q(t) - K_R\mu - C_R\dot{\mu} - CRT - KPT \qquad (2\text{-}52)$$

where

$$CRT = C_R\dot{\delta} - E(C_R\dot{\delta}) \qquad (2\text{-}31b)$$

$$KRT = K_R\delta - E(K_R\delta) \qquad (2\text{-}31c)$$

In Section 2.2 KRT and CRT were assumed to be zero. Hence, they do not appear in the development of the finite different expressions for δ_{n+2}, Equation (2-41) and $E(\delta_{n+2}^2)$, Equation (2-43) in Section 2.3. In the following development no assumptions regarding the values of CRT and KRT are made and they are included in the development of new expressions for δ_{n+2} and $E(\delta_{n+2}^2)$.

When the difference expressions are substituted for $\delta(t)$ and its derivatives on the left hand side of Equation (2-52) the result is,

$$\overline{M}(\delta_{n+2} - 2\delta_{n+1} + \delta_n) + \overline{C}_D(\delta_{n+1} - \delta_n) + K_D\delta_n$$

$$= q_n - K_R\mu_n - C_R\dot{\mu}_n - CRT - KRT \qquad (2\text{-}53)$$

Solving Equation (2-53) for δ_{n+2} yields,

$$\delta_{n+2} = \frac{1}{\overline{M}}(q_n - K_R\mu_n - C_R\dot{\mu}_n - CRT - KRT + A_1\delta_{n+1} + A_2\delta_n) \qquad (2\text{-}54)$$

Note the difference expressions could have been applied to the $\delta(t)$ terms in CRT and KRT as well, but this would only complicate the algebra without changing the results.

To develop an expression for the variance of the response one first squares Equation (2-54) resulting in,

$$\delta_{n+2}^2 = \frac{1}{M^2}\left(q_n^2 - 2q_nK_R\mu_n - 2q_nC_R\dot{\mu}_n + 2q_nA_1\delta_{n+1}\right.$$

$$+ 2q_nA_2\delta_n + K_R^2\mu_n^2 + 2K_RC_R\mu_n\dot{\mu}_n$$

$$- 2A_1K_R\delta_{n+1} - 2A_2K_R\delta_n\mu_n + C_R^2\dot{\mu}_n^2 - 2A_1C_R\delta_{n+1}\dot{\mu}_n$$

$$- 2A_2C_R\delta_n\dot{\mu}_n + A_1^2\delta_{n+1}^2 + 2A_1A_2\delta_{n+1}\delta_n + A_2^2\delta_n^2$$

$$+ \{CRT^2 + 2CRTKRT + KRT^2 - 2q_nCRT$$

$$+ 2\mu_nK_RCRT + 2\dot{\mu}_nC_RCRT - 2A_1\delta_{n+1}CRT$$

$$- 2A_2\delta_nCRT - 2q_nKRT + 2K_R\mu_nKRT$$

$$\left. + 2\dot{\mu}_nC_RKRT - 2A_1\delta_{n+1}KRT - 2A_2\delta_nKRT\}\right) \qquad (2\text{-}55)$$

Equation (2-55) is identical to Equation (2-42) developed in Section 2.3 with the exception of the terms involving CRT and KRT contained within the brackets, { }, above. Hence when one takes the expected value of Equation (2-55) to determine the variance of the response the result is Equation (2-43) plus the expected value of the terms within the brackets. Thus the expected value of the terms within the brackets provides an estimate of the error in the variance of the response due to assuming CRT and KRT are zero. Designating this term REM one finds,

$$REM_n = \frac{1}{\overline{M}^2} E\{CRT_n^2 + 2CRT_nKRT_n + KRT_n^2$$

$$+ 2(K_R\mu_n + C_R\dot{\mu}_n - q_n - A_1\delta_{n+1} - A_2\delta_n)\ CRT_n$$

$$+ 2(K_R\mu_n + C_R\dot{\mu}_n - q_n - A_1\delta_{n+1} - A_2\delta_n)KRT_n\} \quad (2\text{-}56)$$

Evaluating the right hand side of Equation (2-56) term by term yields,

$$E(CRT^2) = E[(C_R\dot{\delta} - E(C_R\dot{\delta}))^2]$$

$$= Var(C_R\dot{\delta})$$

Likewise,

$$E(KRT^2) = Var(K_R\delta) \quad (2\text{-}58)$$

And,

$$E(KRTCRT) = E[(C_R\dot{\delta} - E(C_R\dot{\delta}))(K_R\delta - E(K_R\delta))]$$

$$= Cov(K_R\delta, C_R\dot{\delta}) \quad (2\text{-}59)$$

For the remaining terms on the right hand side of Equation (2-56) involving CRT one gets,

$$E((K_R\mu_n + C_R\dot{\mu}_n - q_n - A_1\delta_{n+1} - A_2\delta_n)CRT_n) =$$

$$= E((K_R\mu_n + C_R\dot{\mu}_n - q_n - A_1\delta_{n+1} - A_2\delta_n)(C_R\dot{\delta})_n)$$

$$- E((K_R\mu_n + C_R\mu_n - q_n - A_1\delta_{n+1} - A_2\delta_n)E(C_R\dot{\delta})_n) \quad (2\text{-}60)$$

Since $E(K_R)$, $E(C_R)$, $E(\delta_{n+1})$, and $E(\delta_n)$ are all zero, and the $E(C_R\delta)_n$ is deterministic the result is,

$$E((K_R\mu_n + C_R\dot{\mu}_n - q_n - A_1\delta_{n+1} - A_2\delta_n)CRT_n) =$$

$$= \mu_n E(K_R C_R \dot{\delta}_n) + \dot{\mu}_n E(C_R^2 \dot{\delta}_n) - E(q_n C_R \dot{\delta}_n)$$

$$- A_1 E(\delta_{n+1}\dot{\delta}_n C_R) - A_2 E(\delta_n \dot{\delta}_n C_R) \qquad (2\text{-}61)$$

The corresponding expression for the terms involving KRT in Equation (2-56) is,

$$E((K_R\mu_n + C_R\dot{\mu}_n - q_n - A_1\delta_{n+1} - A_2\delta_n)KRT_n) =$$

$$= \mu_n E(K_R^2 \delta_n) + \dot{\mu}_n E(K_R C_R \delta_n) - E(q_n K_R \delta_n)$$

$$- A_1 E(\delta_{n+1}\delta_n K_R) - A_2 E(\delta_n \delta_n K_R) \qquad (2\text{-}62)$$

Combining Equations (2-57), (2-58), (2-59), (2-61) and (2-62) the resulting expression for the remainder terms, REM is,

$$REM = \frac{1}{\overline{M}^2}\left(Var(K_R\delta)_n + Var(C_R\dot{\delta})_n + 2Cov(K_R\delta, C_R\dot{\delta})_n\right)$$

$$+ 2\{\mu_n[E(K_R C_R \delta_n) + E(K_R^2 \delta_n)] + \dot{\mu}_n[E(K_R C_R \delta_n) + E(C_R^2 \dot{\delta}_n)]$$

$$- E(q_n C_R \dot{\delta}_n) - E(q_n K_R \delta_n) - A_1[E(\delta_{n+1}\dot{\delta}_n C_R)$$

$$+ E(\delta_{n+1}\delta_n K_R)] - A_2[E(\delta_n^2 K_R) + E(\delta_n \dot{\delta}_n C_R)]\} \qquad (2\text{-}63)$$

This expression may be further reduced since q_n is independent of the system parameters C_R and K_R and is independent of δ_{n+1} and δ_n as well.

Hence,

$$E(q_n C_R \dot{\delta}_n) = E(q_n)E(C_R \delta_n) \quad (2\text{-}64)$$

And,

$$E(q_n K_R \delta) = E(q_n)E(K_R \delta_n) \quad (2\text{-}65)$$

However, since q_n is mean zero,

$$E(q_n C_R \dot{\delta}_n) = 0 \quad (2\text{-}66)$$

And,

$$E(q_n K_R \delta_n) = 0 \quad (2\text{-}67)$$

Thus Equation (2-63) is reduced to,

$$\begin{aligned} REM = \frac{1}{\bar{M}^2} \Big(& Var(K_R \delta)_n + Var(C_R \dot{\delta})_n + 2Cov(K_R \delta, C_R \dot{\delta})_n \\ & + 2\{\mu_n[E(K_R C_R \dot{\delta}_n) + E(K_R^2 \delta_n)] \\ & + \dot{\mu}_n[E(K_R C_R \delta_n) + E(C_R^2 \dot{\delta}_n)] \\ & - A_1[E(\delta_{n+1} \dot{\delta}_n C_R) + E(\delta_{n+1} \delta_n K_R)] \\ & - A_2[E(\delta_n^2 K_R) + E(\delta_n \dot{\delta}_n C_R)] \} \Big) \end{aligned} \quad (2\text{-}68)$$

Examination of the terms on the right hand side of Equation (2-68) reveals that these terms provide a means for the uncertainties in the system parameters entering the computation of the response's variance when the response is mean zero.

When the variation of the response is small compared to its mean then one would expect that the value of $Var(K_R\delta)$ would be small compared to that of $E(K_R^2)\mu^2$. Similarly one would expect that the $Var(C_R\delta)$ would be small compared to $E(C_R^2)\mu^2$ and $Cov(K_R\delta, C_R\dot{\delta})_n$ would be small in comparison to $E(K_R C_R)\dot{\mu}_n\mu_n$. Hence one would expect that these terms have little influence on the response. Similar arguments may be made for discounting the rest of the terms in Equation (2-68) as they all have corresponding terms which one would expect to be much larger in Equation (2-43).

Several of the following examples verify this assumption by comparing the results of the computer code RSDOF to analytical methods which do not ignore the terms included in Equation (2-68).

2.6 NUMERICAL EXAMPLES

This section includes examples of the transformation technique developed earlier in this chapter applied to the solution of specific problems. The following problems are covered.

1) Response of deterministic random systems to a white noise loading.
2) Response of a random system to a deterministic constant loading.
3) Response of a random system to a deterministic blast loading.

The first two problems were used as a method of checking the results of the technique. These examples were also used to quantify the effects the random system parameters have on the response. The third demonstrates the use of the technique in a practical application.

EXAMPLE 1 - White Noise Excitation of An SDOF System

Consider the system diagramed in Figure 2-1. All system parameters are unitless. When F(t) is white noise with spectral density ϕ the mean square response of the system with deterministic parameters, is given by Lin (1) as,

$$E(X(t)^2) = \frac{\pi\phi}{2\varepsilon\omega_o{}^3 m^2} \left\{1 - \frac{\exp(-2\varepsilon\omega_o t)}{\omega_D^2} \left[\omega_D^2 + 2(\varepsilon\omega_o \sin\omega_D t)^2 + \varepsilon\omega_o \sin 2\omega_D t\right]\right\} \tag{2-69}$$

where
$$\varepsilon = \frac{C}{2\sqrt{KM}} \tag{2-70}$$

$$\omega_o = \sqrt{\frac{K}{M}} \tag{2-71}$$

$$\omega_D = \omega_o\sqrt{1-\varepsilon^2} \tag{2-72}$$

As the response reaches its stationary value Equation (2-69) reduces to,

$$E_s(X(t)^2) = \frac{\pi\phi}{2\varepsilon\omega_o^3 m^2} \tag{2-73}$$

To model the response of the system one first has to develop a suitable model for the white noise excitation. A white noise random process with spectral density ϕ has an autocorrelation function, $R_{FF}(\tau)$ of,

$$R_{FF}(\tau) = 2\pi\phi\delta(\tau) \tag{2-74}$$

where $\delta(\tau)$ is the dirac delta function.

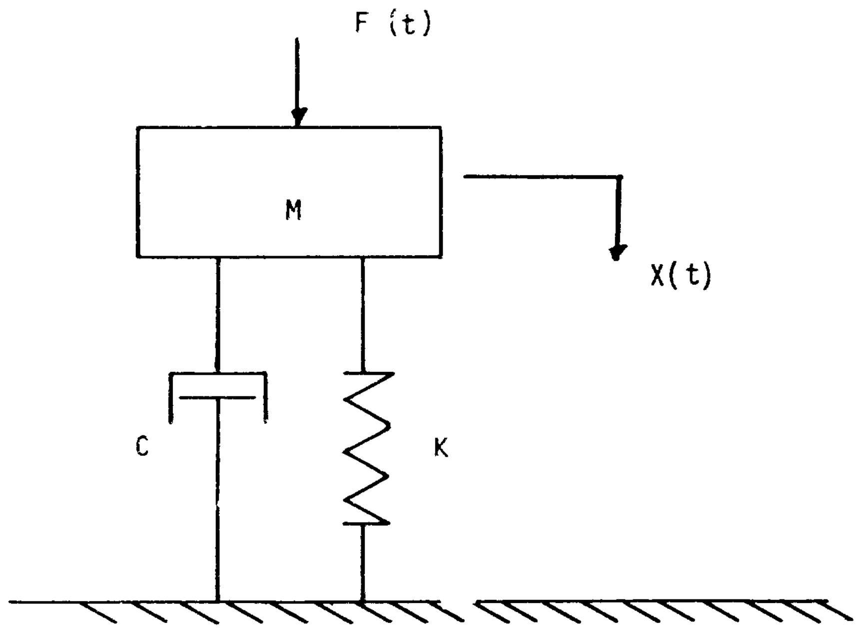

Figure 2-1 Mass Excited SDOF System

When one wishes to excite a structure using a white noise input and perform response computations in a discrete time framework he can model the input as a frequency of independent, identically distributed random variables applied to the structure at Δt time intervals. This is an IID (independent identically distributed) random process (sometimes referred to as a band-limited white noise random process). Let $\{F_j = F(j\Delta t), j = 0, 1, 2, \ldots\}$ be an IID random process. The random process has mean zero and autocorrelation function,

$$R_{FF}((j-k)\sigma t) = \begin{cases} \sigma^2 & j = k \\ 0 & j \neq k \end{cases} \tag{2-75}$$

$$= \sigma^2 \; \delta_{j-k} \tag{2-76}$$

where δ_{j-k} is the Kronecker delta (equal to zero when the subscript is nonzero, and equal to one when the subscript is zero). The spectral density of this random process is,

$$S_{FF}(\omega) = \begin{cases} \dfrac{\Delta t \sigma^2}{2\pi} & \dfrac{-\pi}{\Delta t} \leq \omega \leq \dfrac{\pi}{\Delta t} \\ 0 & \text{elsewhere} \end{cases} \tag{2-77}$$

In view of this, the IID random process is equivalent to the white noise random process if σ^2 is chosen as follows,

$$\sigma^2 = \frac{2\pi}{\Delta t} \phi \tag{2-78}$$

This definition for σ^2 insures equivalence between the white noise and IID random process in terms of their spectral densities. However, equivalence in terms of the structural response these inputs excite is only guaranteed for certain values of Δt.

Specifically, the cutoff frequency of the IID random process, $\pi/\Delta t$, must be much greater than the natural frequency of the structure being excited in order for the IID random process to appear similar to the white noise input, as far as structural response is concerned. This establishes the requirement,

$$\Delta t << \frac{\omega_n}{\pi} \tag{2-79}$$

where ω_n is the natural frequency of the structure being excited.

To address the issue of how small a time step, Δt, is required to insure an adequate approximation of the structural response characteristics a series of calculations with decreasing values of Δt were performed. The ratio of the natural period to the time step is termed N. The computations were run for the case of C = 20, (ε = 0.2), until the response became stationary. With $\sigma^2 = 100/\Delta t$, K = 500, and M = 5, the exact solution for the mean square value of the response, $E_S(X(t)^2)$, as defined in Equation (2-73) is 0.005. The values for $E_S(X(t)^2)$ computed for successive values of N are plotted vs ln(N) in Figure 2-2.

As Δt becomes small (that is as N becomes large), the values obtained from the computations very closely approximate the exact solution, being in error by less than 1% for values of N greater than 1500.

To examine the variation of the error with time the results of computations for different values of the damping parameter C, were compared to the exact solution as given by Equation (2-69). the values of C chosen for the comparison correspond to values of ε, the critical damping ratio, of 0.05, 0.10, and 0.2. The computed values of $E(X(t)^2)$ are

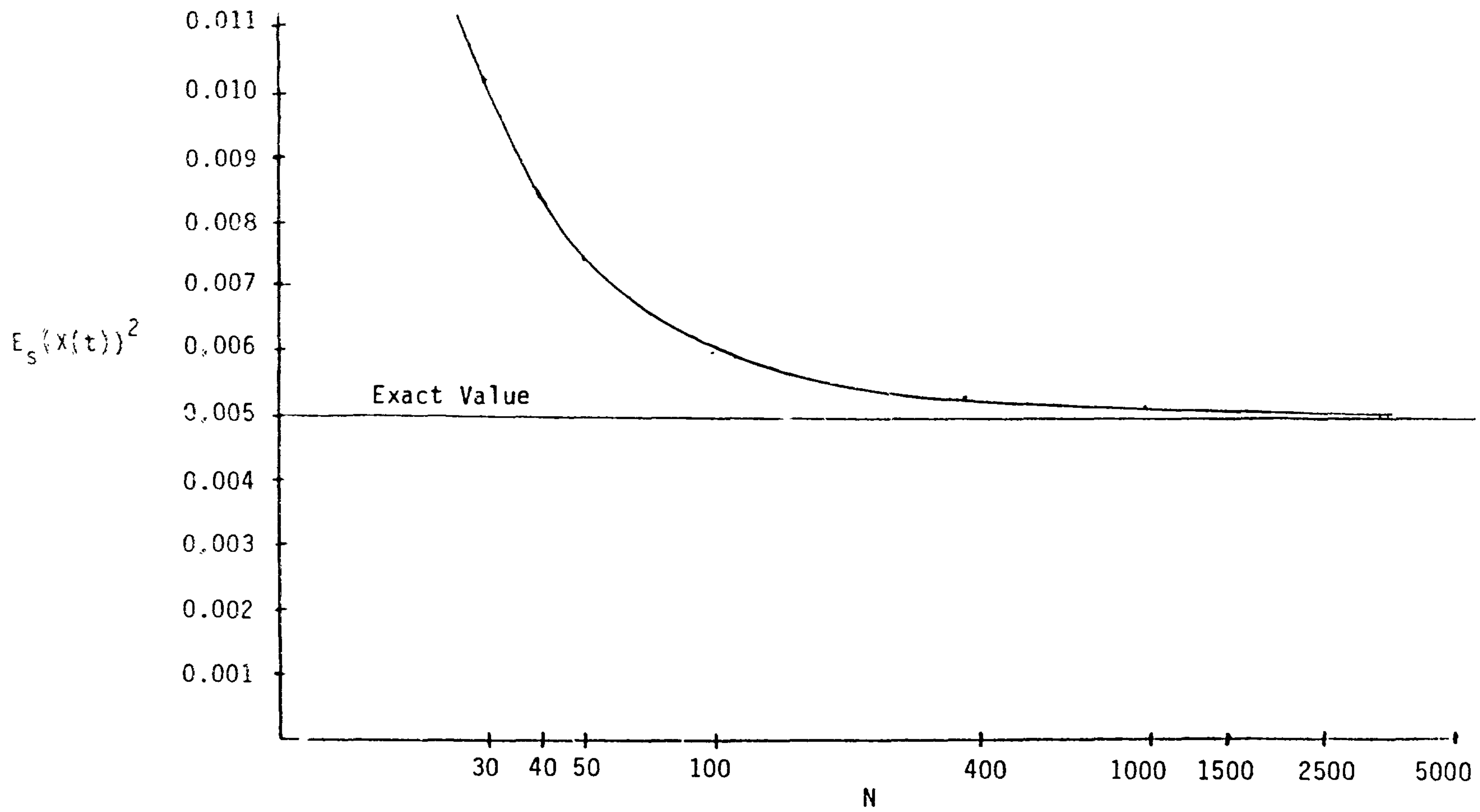

Figure 2-2 Values of $E_s(X(t)^2)$ for Various Values of N, where N = Period/Δt

plotted versus time in Figure 2-3. The relative error, E, was computed ? - , time step by,

$$E = \frac{E\left(X(t)^2\right)_{comp} - E\left(X(t)^2\right)_{exac}}{E\left(X(t)^2\right)_{exac}} \times 100 \tag{2-80}$$

where $E\left(X(t)^2\right)_{comp}$ is the value of the mean square response as computed and $E\left(X(t)^2\right)_{exac}$ is the value for the mean square response given by Equation (2-69). The error was found to be monotonically increasing with time for all three cases, reaching its maximum value when the system reached the point of stationary response.

The values for E_{max}, the maximum error for the three cases are listed in Table (2-1). The maximum error was found to increase with decreasing values of ε. However, since the mean square response goes to infinity as ε goes to zero this is not surprising. A smaller value of Δt would result in smaller errors for the very lightly damped system.

If one now considers the system diagramed in Figure 2-1 to have both random stiffness K and damping C the values obtained from Equation (2-42) for the mean square response will be the same as for when the system is deterministic. This is due to the assumption made in Section 2.1 regarding the variations of the random components of the restoring forces. Specifically it was assumed that,

$$E(C_R\dot{\delta}) = C_R\dot{\delta}$$

and (2-81)

$$E(K_R\delta) = K_R\delta$$

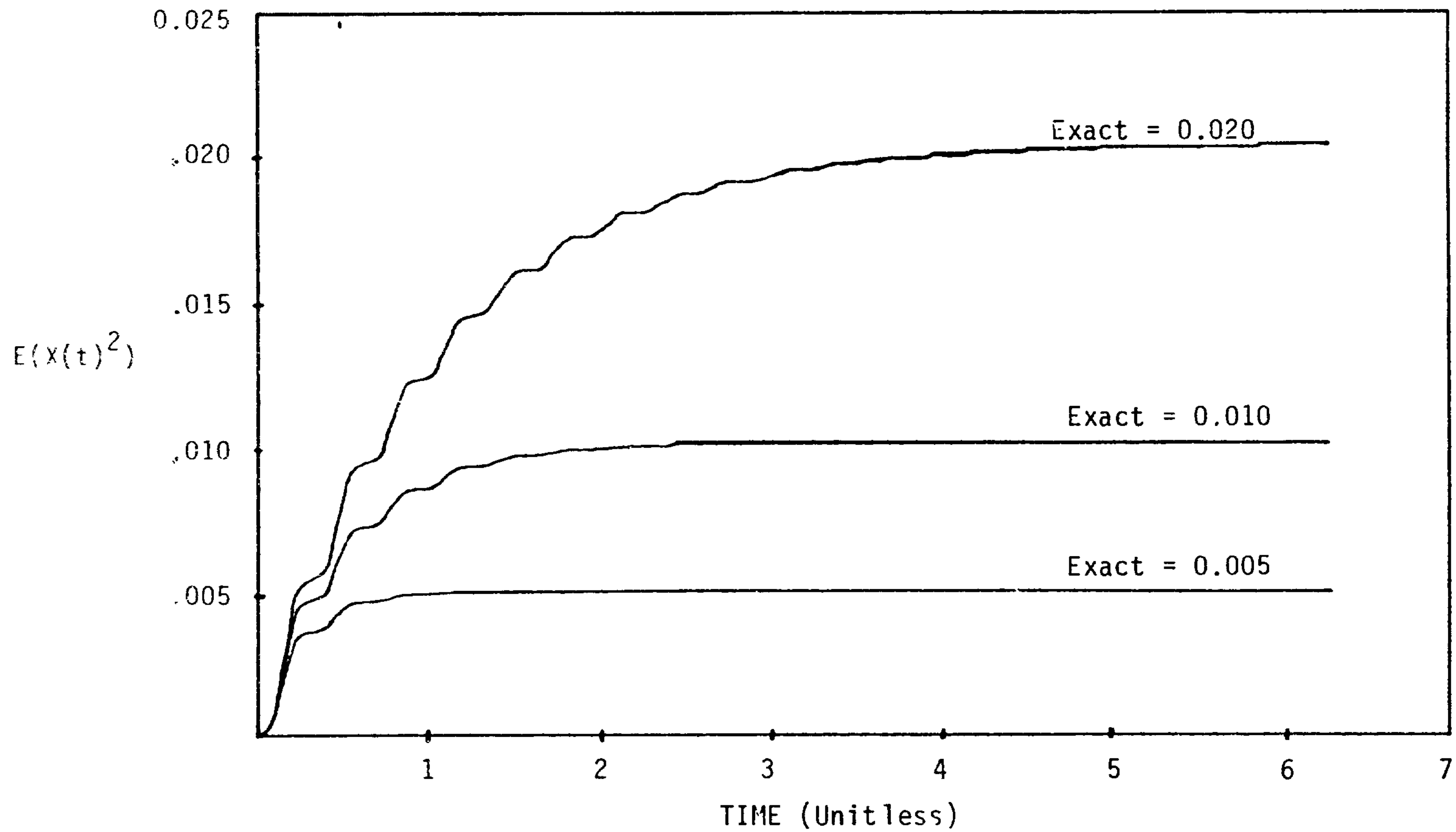

Figure 2-3 Mean Square Respnse for White Noise Loading

TABLE 2-1

MAX ERROR IN THE COMPUTED VALUE OF $E(X(t)^2)$ FOR VARIOUS VALUES OF ϵ

ϵ	$E(X(t)^2)_{comp}$	Max error
0.05	0.020472	2.55%
0.10	0.010128	1.28%
0.20	0.005032	0.64%

N = 2500 for all cases.

These assumptions have no effect on the mean response of the system but do result in the contribution of the randomness in the stiffness and damping to the randomness of the response being multiplied by a factor containing the mean response. Hence when the mean response is zero the effects of the randomness in the system parameters is lost.

To answer the question as to how significant this error is one may write the expression for the mean square response of the stationary system as a function of K and C. Rearranging Equation (2-73) results in,

$$E_s(X(t)^2) = \frac{\pi\phi}{2KC} \tag{2-82}$$

For purposes of illustration one may consider K random and C deterministic, resulting in Equation (2-82) being a function of the random variable K. Taking the expected value of Equation (2-82) results in,

$$E(E_s(X(t)^2)) = \frac{\pi\phi}{2C} E(\frac{1}{K}) \tag{2-83}$$

And since the expectation of an expected value is the expected value the left hand side may be rewritten and Equation (2-83) becomes,

$$E_s(X(t)^2) = \frac{\pi\phi}{2C} E(\frac{1}{K}) \tag{2-84}$$

The $E(\frac{1}{K})$ depends on the distribution of K. When the mean stiffness K_D, is equal to 500 and the coefficient of variation δ_K is 0.15, and K is uniformly distributed with upper bound K_U and lower band K_L, $E(\frac{1}{K})$ is given by,

$$E\left(\frac{1}{K}\right) = \int_{K_L}^{K_U} \frac{1}{K} \frac{1}{SPAN} dK \tag{2-85}$$

where SPAN = $K_U - K_L$ and is given by

$$\text{Var } K = \frac{1}{K} (SPAN)^2 \tag{2-86}$$

And since,

$$\left.\begin{aligned} \text{Var } K &= (0.15)^2 K_D{}^2 \\ \text{Var } K &= 5625 \end{aligned}\right\} \tag{2-87}$$

The span of the distribution is,

$$SPAN = 259.8 \tag{2-88}$$

Yielding values of K_u and K_L the upper and lower limits of the distribution of,

$$\left.\begin{aligned} K_u &= 629.9 \\ K_L &= 370.1 \end{aligned}\right\} \tag{2-89}$$

The resulting value for $E(\frac{1}{K})$ is thus,

$$E\left(\frac{1}{K}\right) = 0.002046 = \frac{1}{488.7} \tag{2-90}$$

this differs from $\frac{1}{K_D}$ by only 2.35%. Thus the true value for the station any mean square response of a white noise excited system with random stiffness is underpredicted by only about 2.35% for the case illustrated. A suitable rule of thumb would be

$$E_s(X(t)^2) = \frac{\phi\pi}{2C\ K_D}\ \left(1 + \delta_K^2\right) \tag{2-91}$$

where δ_K is the coefficient of variation for K and K_D is the mean value. This example provides considerable reassurance that the assumptions made in Section 2.1 will produce only a small error.

EXAMPLE 2 - Comparison of the Results of the Transformation Method to the Exact Solution for the Response of an SDOF System with Random Parameters

The objective of this example is the verification of the transformation method by comparing its results with those obtained by numerically integrating an exact solution for the mean and variance of the system response.

The system under consideration has the following unitless mean values,

$$K_D = 500$$

$$\varepsilon = 0.1$$

$$M = 5$$

and zero initial conditions.

The loading function used for the comparison was deterministic. The load was instantaneously applied at time 0 and remained in place indefinitely. The loading function is diagramed in Figure 2-4.

The system response to such a loading is,

$$X(t) = \frac{F}{K}\left(1 - \exp(-\varepsilon\omega_0 t)\left(\frac{\varepsilon\omega_0}{\omega_D}\sin\omega_D t + \cos\omega_D t\right)\right) \tag{2-92}$$

where

$$\omega_0 = \sqrt{\frac{K}{M}} \tag{2-93}$$

$$\omega_D = \omega_0\sqrt{1-\varepsilon^2} \tag{2-94}$$

The first comparison consists of evaluating the response of the SDOF system using the code RSDOF with the randomness in the system parameters set to zero. The results of this computation were plotted with values obtain from Equation (2-92) in Figure 2-5. As can be seen in the figure t plots are indistinguishable.

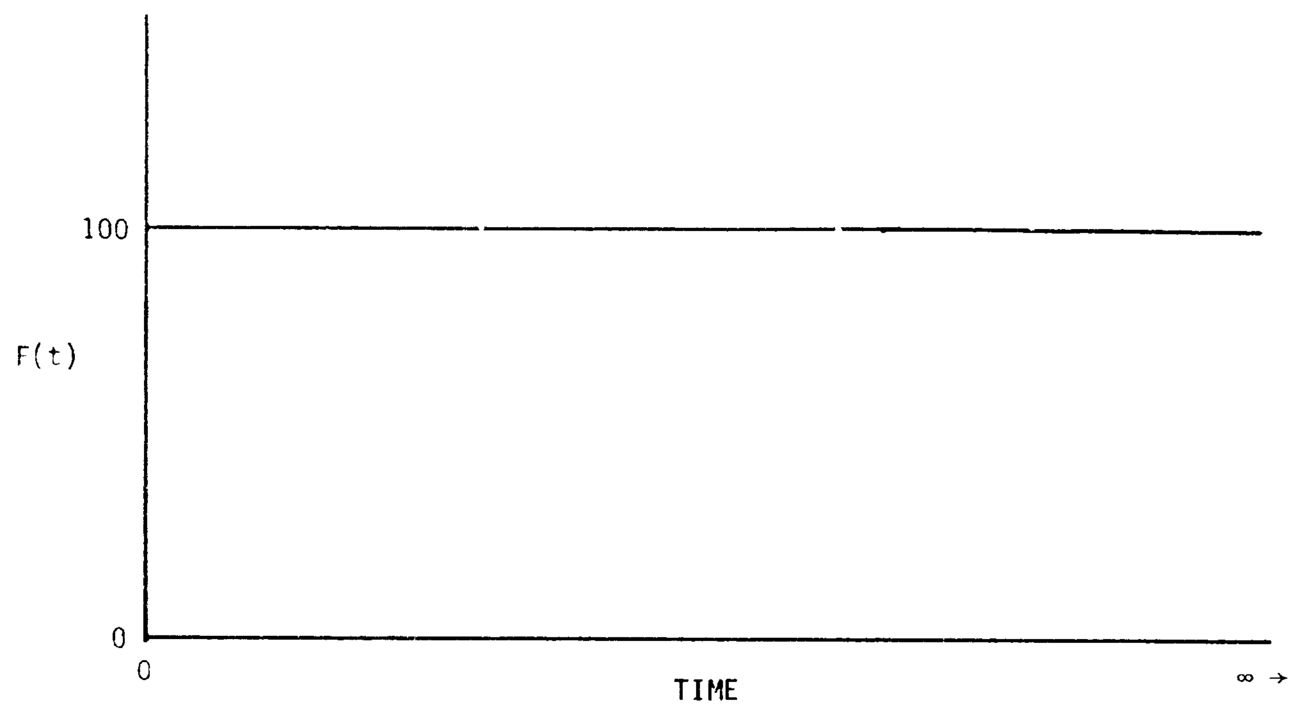

Figure 2-4 Dimensionless Load vs Time for Example 2

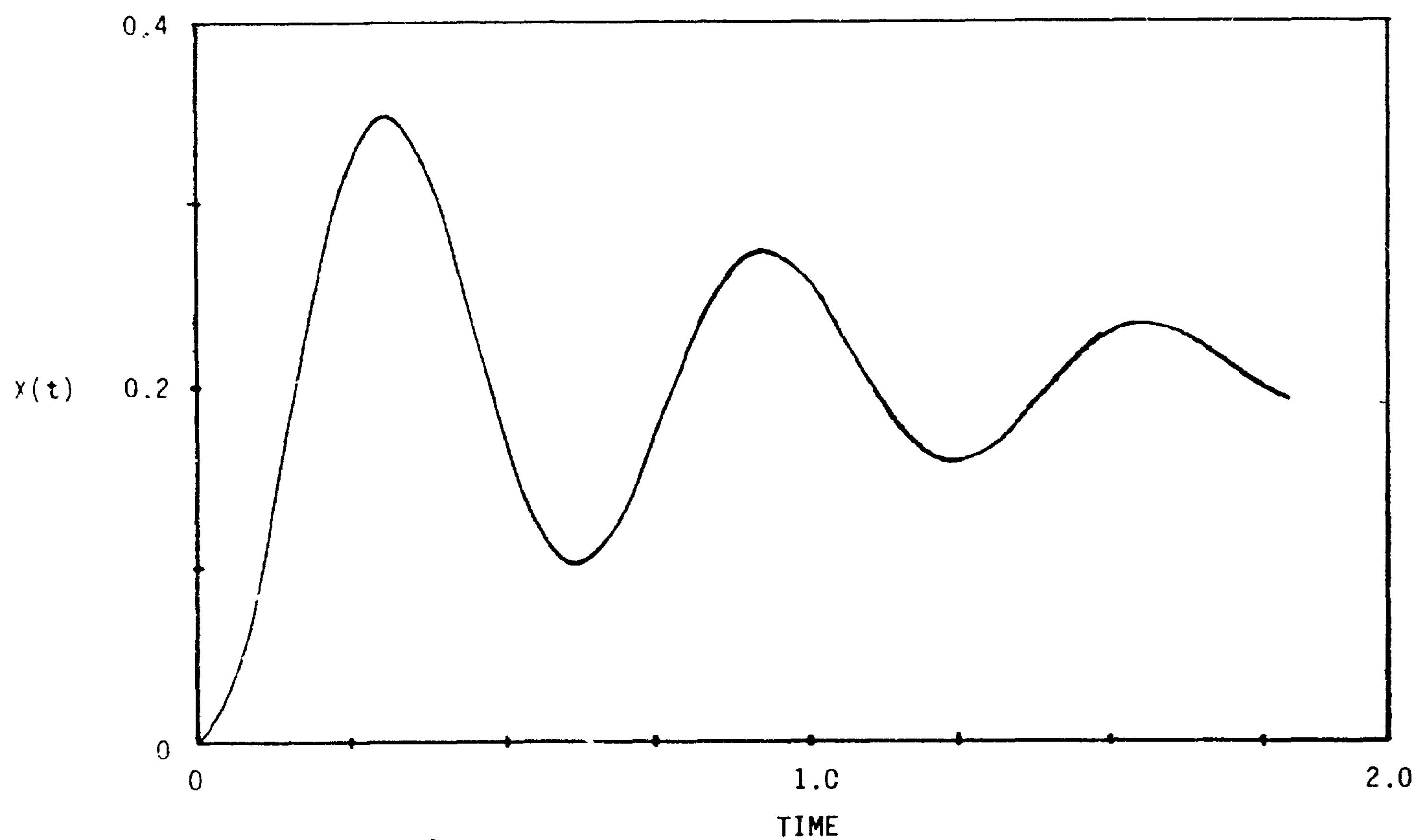

Figure 2-5 Exact and Numerical Solutions fo. the Deterministic Case, Example 2

For the second comparison the stiffness was treated as a random variable with a coefficient of variation of 0.1. To generate the exact solution for the mean and variance of X(t) Equation (2-92) was treated as a function of K at any time t. Thus,

$$X(t) = G(K)$$

And

$$E(X(t)) = E(G(K)) = \int_{-\infty}^{\infty} G(K)P_K\, dK \tag{2-95}$$

And

$$E(X^2(t)) = E(G^2(K)) = \int_{-\infty}^{\infty} G^2(K)P_K\, dK \tag{2-96}$$

$$Var(X(t)) = E(G^2(K)) - E(G(k))^2 \tag{2-97}$$

where P_K is the probability density function for K.

One notes that the exact values for E(X(t)) and Var(X(t)) are functions of the probability distribution function of K. This information is not included in the equations of the transformation method since the method replaces the derivatives with linear approximations. The means and variances of the resulting linear equations are independent of the distributions.

The integrations of Equations (2-95) and (2-96) were performed for two different distributions of the random variable K. The first of these was the uniform distribution. The probability density function for K is then,

$$\left.\begin{aligned} P_K &= \frac{1}{K_U - K_L} \qquad K_L \le K \le K_U \\ &= 0 \qquad \text{elsewhere} \end{aligned}\right\} \tag{2-98}$$

where K_U is the upper limit of integration and K_L the lower. Values of K_K and K_U were 413.4 and 586.6 respectively. These yield a mean value of K of 500 and a coefficient of variation of .1.

The numerical integration was performed using Newton-Quotes algorithm available via the QUANC8 subroutine available on the University of New Mexico's VSPC computer system.

The resulting means and standard deviations for approximately 2.5 cycles of response are plotted in Figures 2-6 and 2-7 respectively. The means are indistinguishable. The transformation method appears to seriously undershoot the troughs of the standard deviation curve. However, since the response at or near the peaks is of primary concern to the engineer this is not considered a serious shortcoming.

For a second comparison the stiffness K was assumed to have a truncated normal distribution. The probability density function for K is then,

$$P_K = \frac{1.00067}{\sigma\sqrt{2\pi}} \exp\left[\frac{-1}{2\sigma^2}(K-\mu_K)^2\right] \qquad \text{for } K_L < K < K_U \qquad (2\text{-}99)$$

$$= 0 \qquad \text{elsewhere}$$

The means are once again indistinguishable, and are plotted in Figure 2-8. The standard deviations are plotted in Figure 2-9. Once again the transformation method undershoots the troughs of the standard deviation vs time curve.

Based on the above information the transformation method provides an adequate approximation for the mean and standard deviation, or variance, of an SDOF system with random parameters.

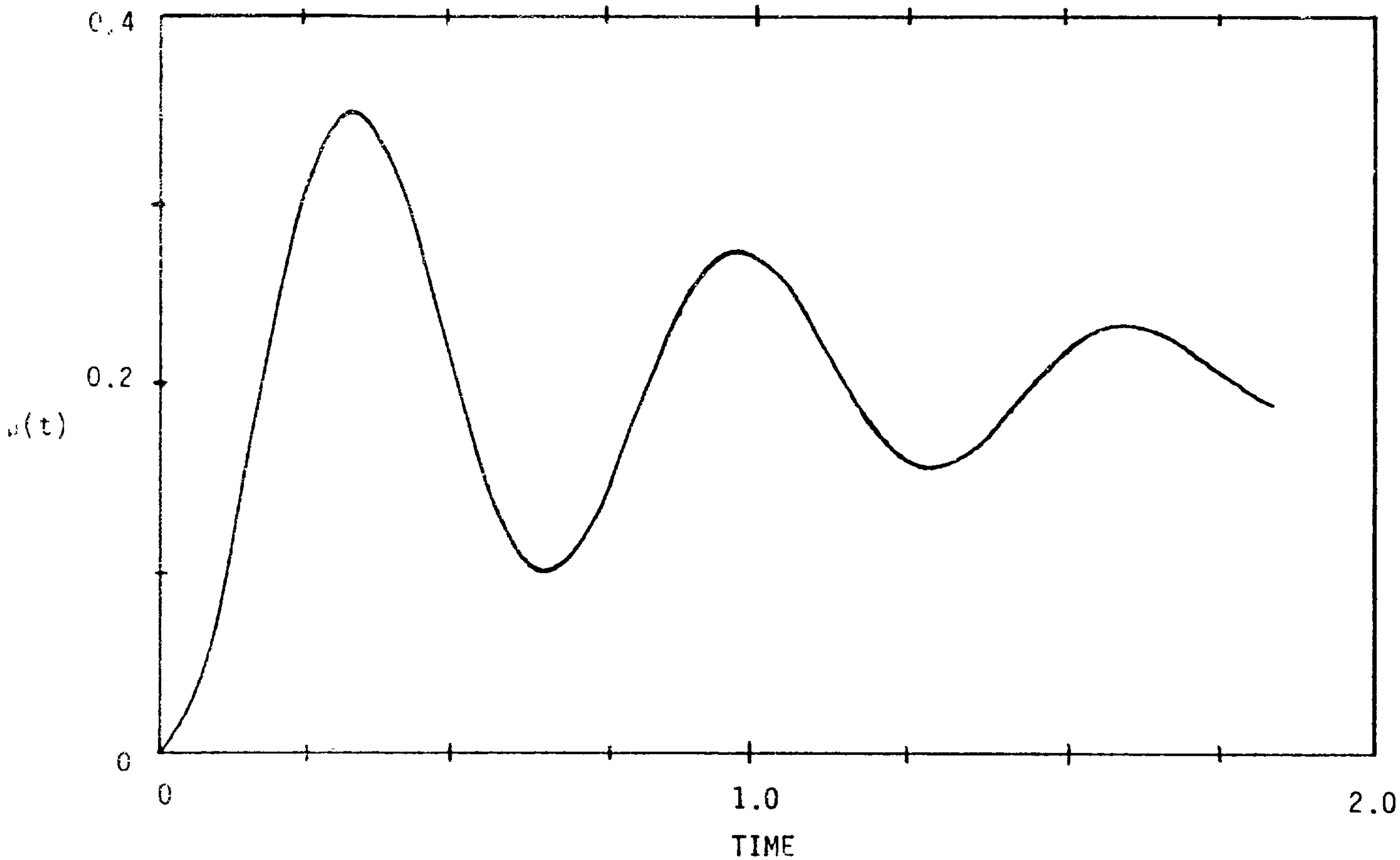

Figure 2-6 Transformation Approximation and Exact Solutions for $\mu(t)$ Uniformly Distributed Stiffness

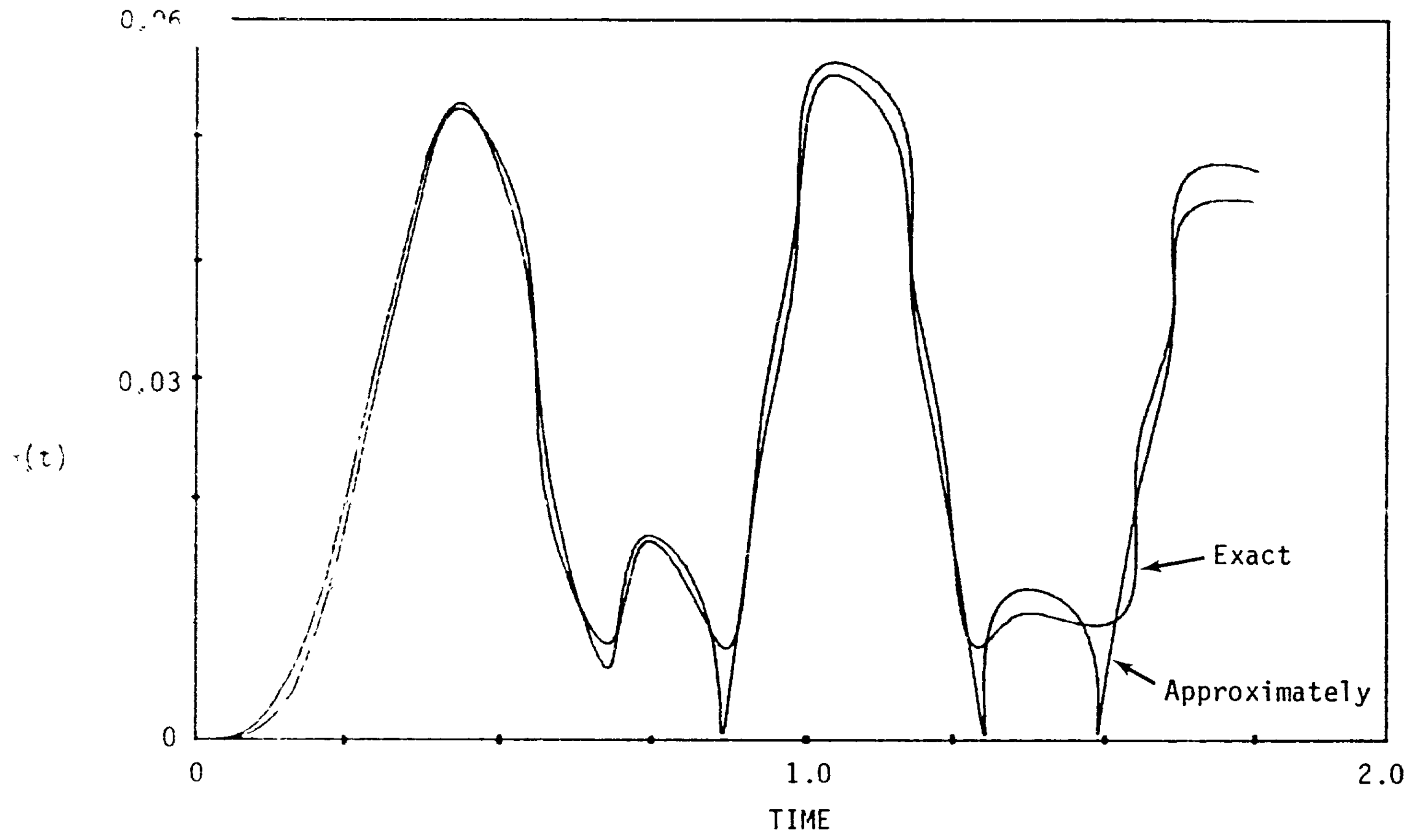

Figure 2-7 Exact and Transformation Approximation for $\sigma(t)$, Uniformly Distributed Stiffness

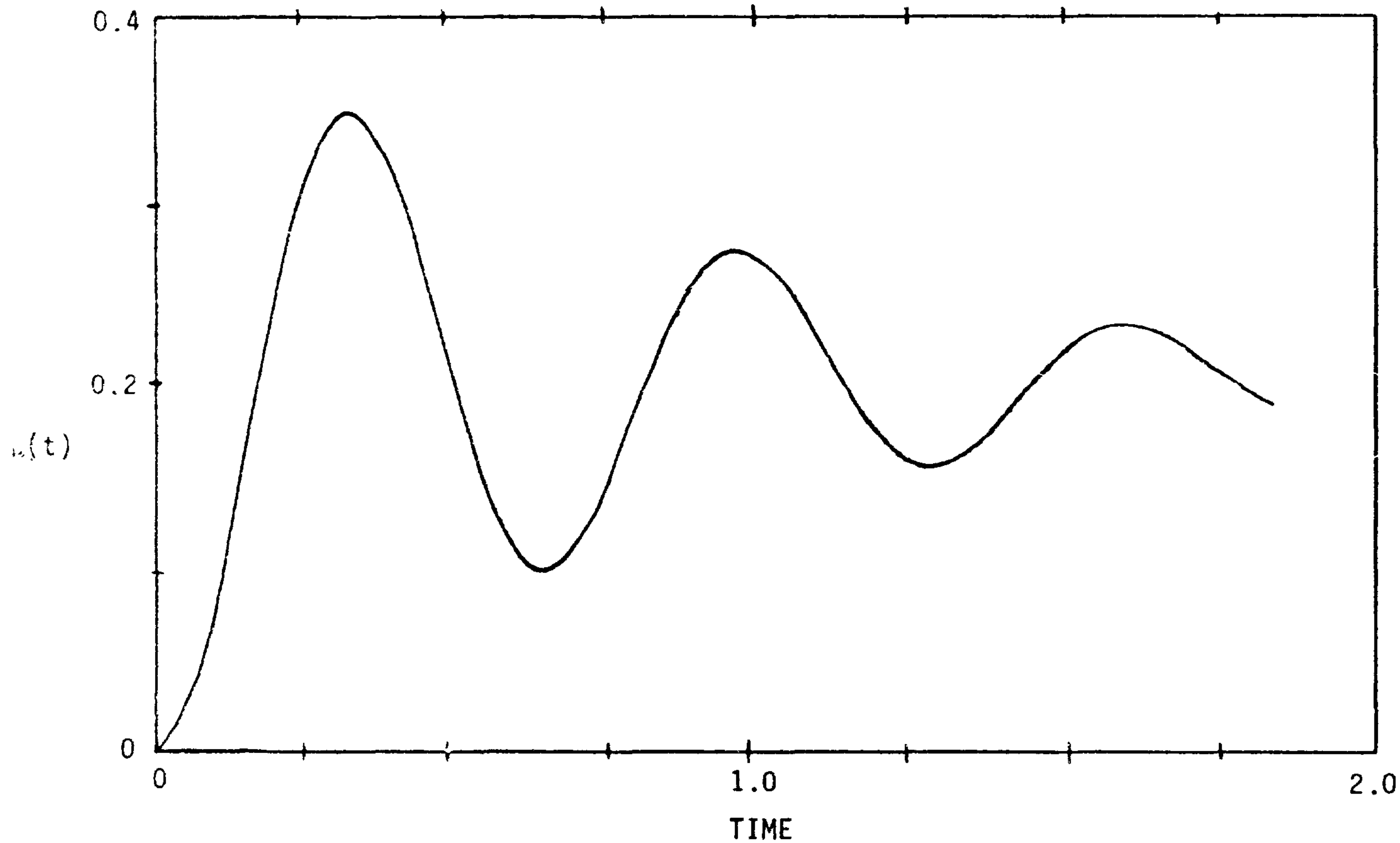

Figure 2-8 Exact and Transformation Approximation for $\mu(t)$, Normally Distributed Stiffness

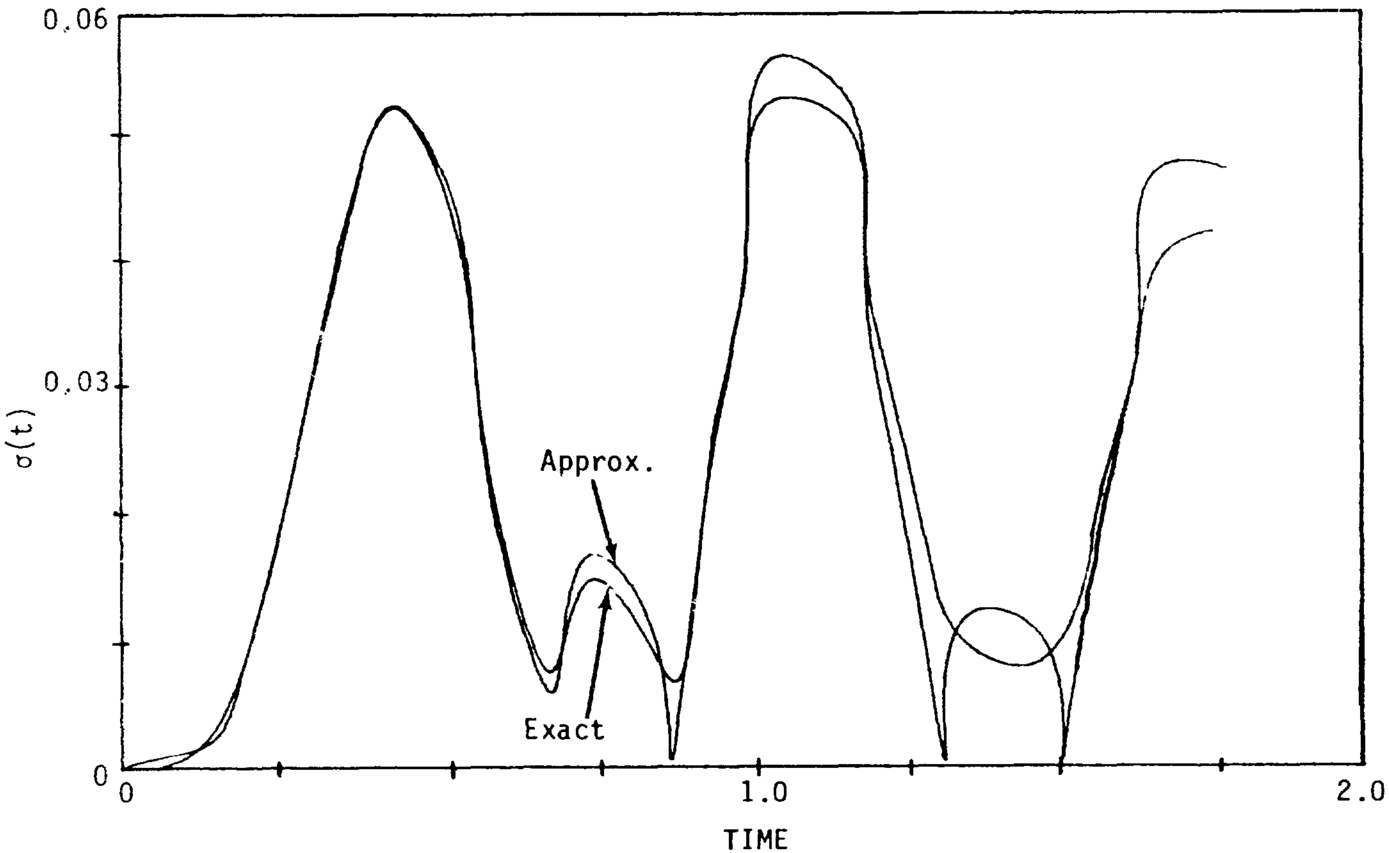

Figure 2-9 Exact and Transformation Approximation for $\sigma(t)$, Normally Distributed Stiffness

EXAMPLE 3 - Evaluation of the Influence of Random Stiffness and Damping on the Response Statistics for an SDOF System

This example consists of a set of computations of the mean and standard deviation for an SDOF system. The system's mean stiffness and mass were held constant. The loading function was the same as in Example 2 and is shown in Figure 2-4. The mean damping ratio, the standard deviations for the stiffness and damping, and the correlation coefficient between the stiffness and damping were varied in individual cases. The values used for these parameters in each case are shown in Table 2-2.

The results of cases 1 and 2 and the deterministic case were used to examine the influence of the randomness in the stiffness on the randomness of the response. The resulting values for the mean, $\mu(t)$, and the standard deviation, $\sigma(t)$, for each case are plotted in Figures 2-10, and 2-11. The deterministic case was discussed in Example 2 and its mean response is plotted in Figure 2-5. One notes that the value for the peak response increases as the standard deviation of the stiffness increases. Furthermore the standard deviation curve lags behind the response curve though they are of similar shape. The trend of increasing peak response is evident by comparing the values for the peak response obtained for cases 1 and 2 and the deterministic value. However, the variation between these cases is smaller than it would be for the static case, hence the variation in the stiffness has a smaller effect on the peak dynamic response than it does on the static. Furthermore the values obtained from the code are at preselected times based on the period of the response as determined by its mean parameters, when the

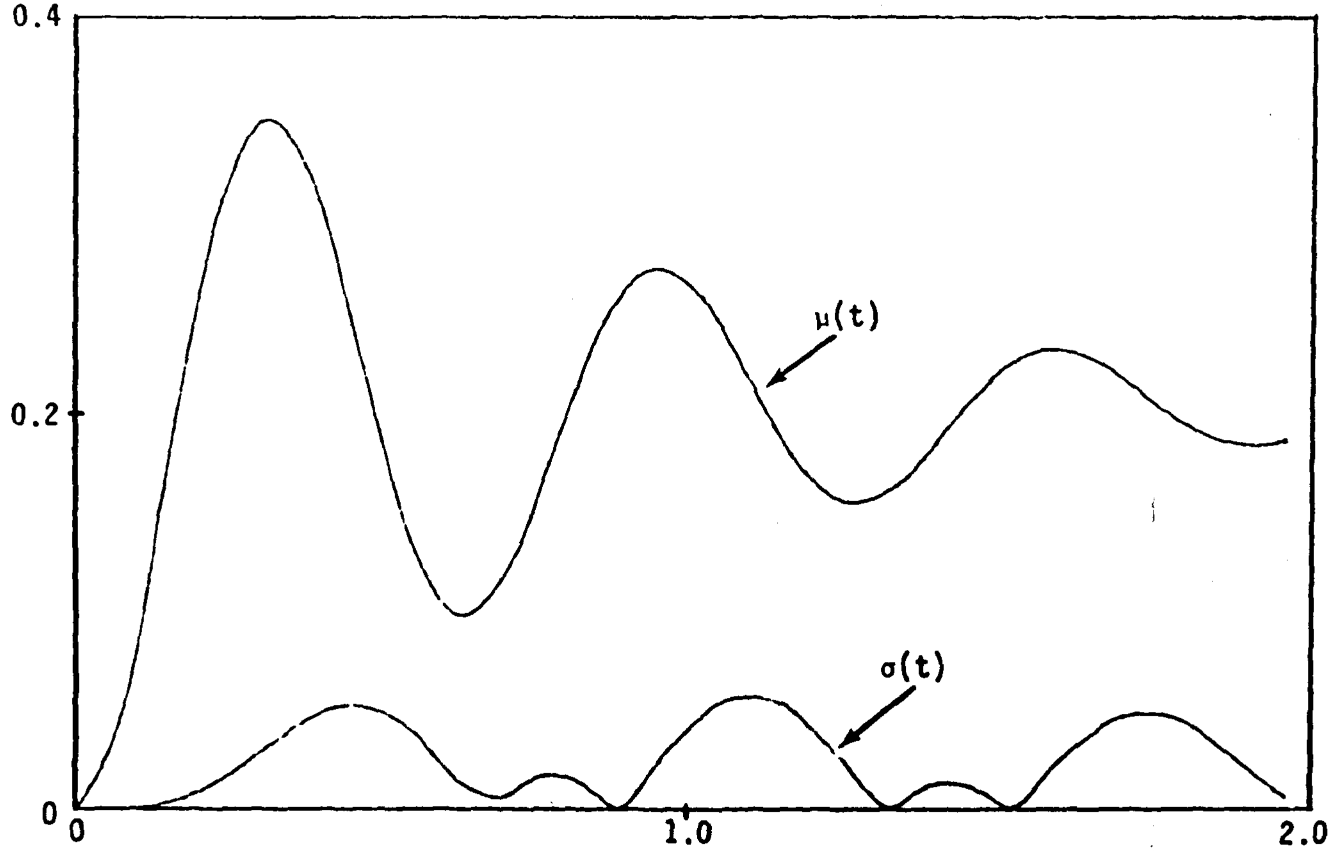

Figure 2-10 $\mu(t)$ and $\sigma(t)$ Case 1, Example 2

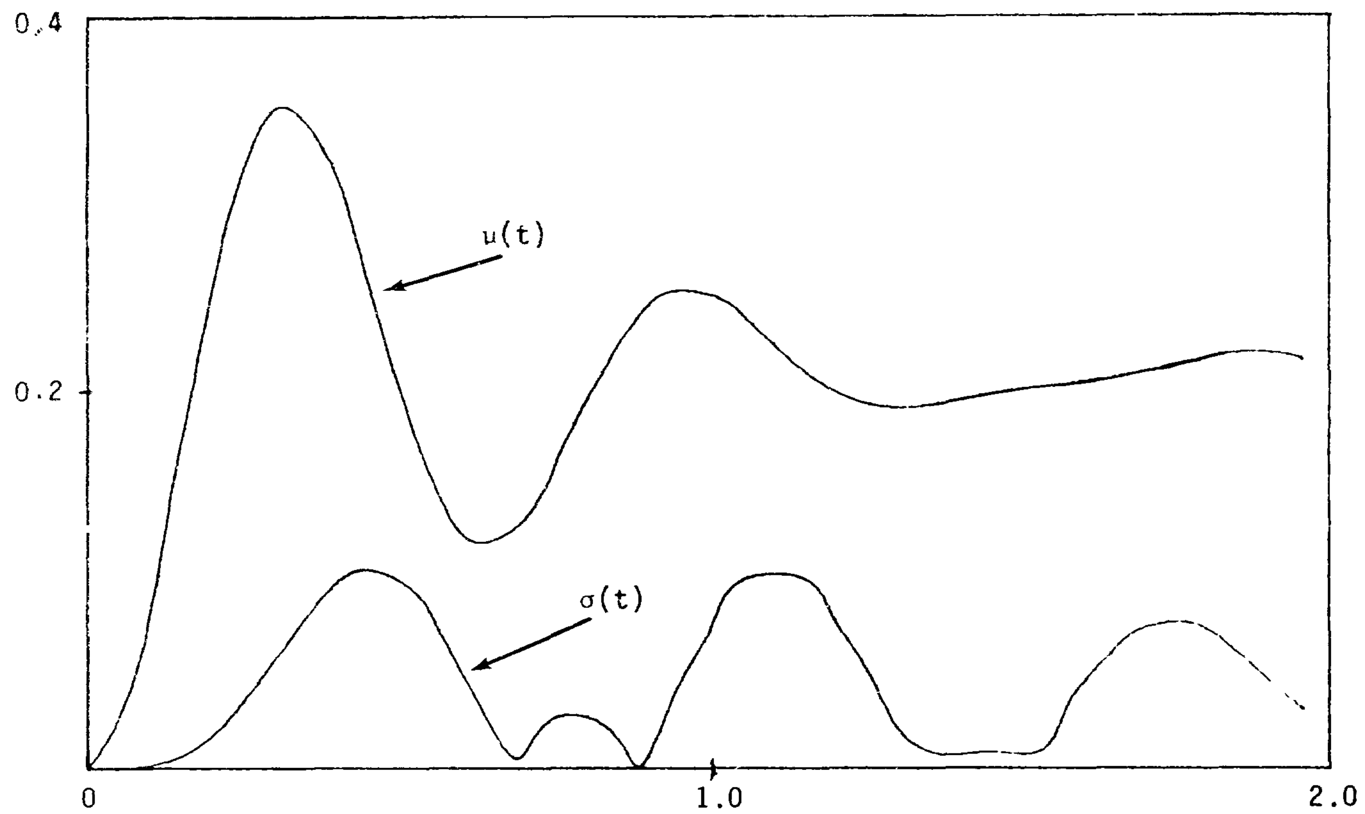

Figure 2-11 $\mu(t)$ and $\sigma(t)$ Case 2, Example 2

TABLE 2-2

PARAMETERS USED IN CASE STUDIES

Case	ϵ	σ_ϵ	σ_K	P_{KC}
1	0.1	0	50	0
2	0.1	0	100	0
3	0.5	0	100	0
4	0.1	0.01	0	0
5	0.1	0.02	0	0
6	0.5	0.02	0	0
7	0.1	0.01	50	0
8	0.1	0.01	50	1.0
9	0.1	0.01	50	-1.0

system has random parameters there can be a slight phase shift resulting in the peak values printed being off by one or two time steps from the true peaks.

The peak response for each of the cases, σ_k = 0, 50, 100 are listed in Table 2-3.

The results of cases 4, 5, and 6 allow one to examine the influence of the randomness in the damping on the randomness of the response. The means and standard deviations for these cases are plotted in Figures 2-12, 2-13, and 2-14 respectively.

By comparing the results of cases 1 and 4 one notes that the values obtained for $\sigma_X(t)$ resulting from a random stiffness are an order of magnitude larger than those obtained for random damping. Cases 2 and 5 show the same effect for larger variations in the stiffness and damping. However, one must note that these results are for a system with damping which is only 10% of the critical value. When the damping is increased to 50% of critical as in cases 3 and 6, plotted in Figures 2-15 and 2-14 respectively, the resulting values for $\sigma_X(t)$ resulting from randomness in the stiffness and damping are still not of the same magnitude.

From these results one concludes that for lightly damped systems which do not have a zero mean, the effects on response of randomness in the damping are minimal compared to those resulting from a randomness in the stiffness. For the case of white noise response the effects of randomness in K and randomness in C will be the same when the coefficients of variation of K and C are equal.

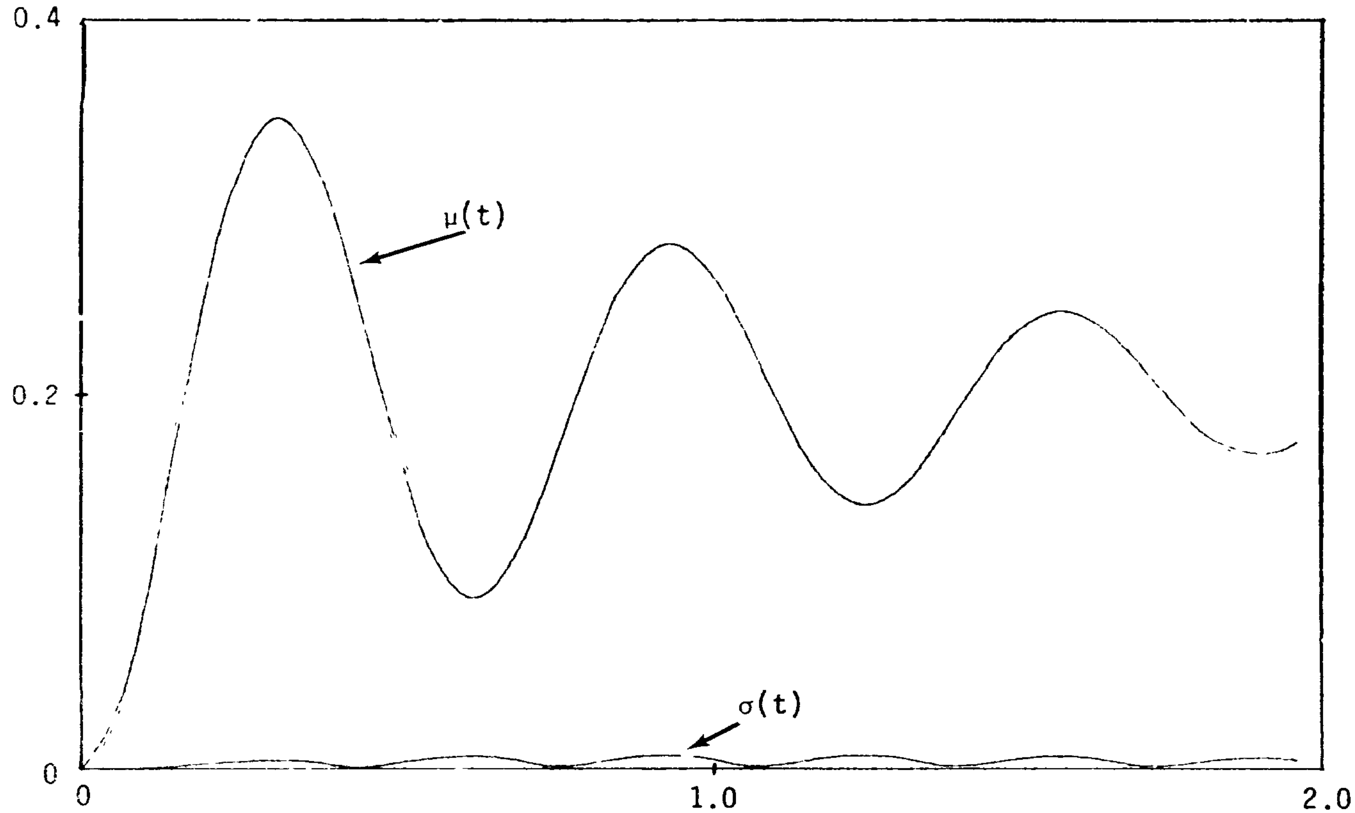

Figure 2-12 $\mu(t)$ and $\sigma(t)$ Case 4, Example 2

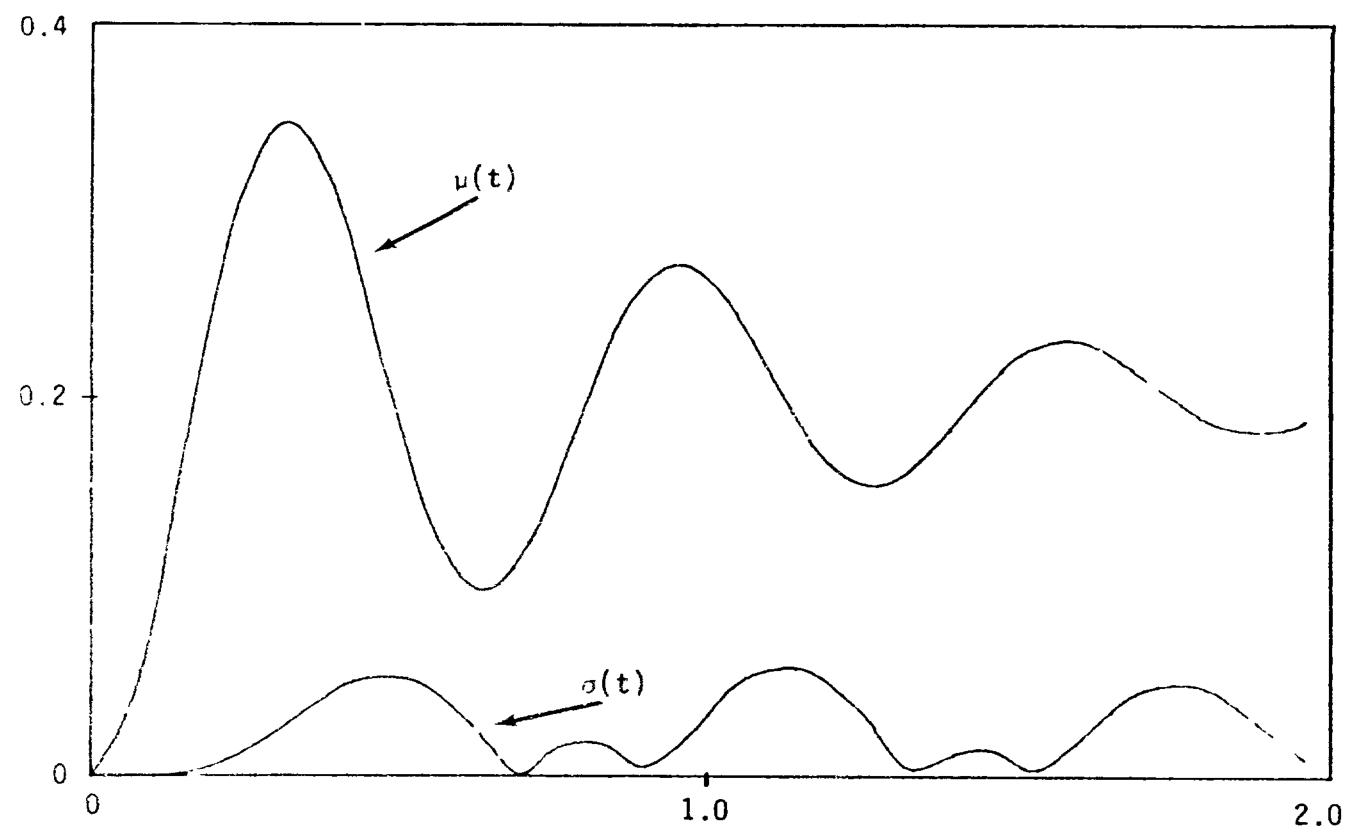

Figure 2-13 $\mu(t)$ and $\sigma(t)$ Case 5, Example 2

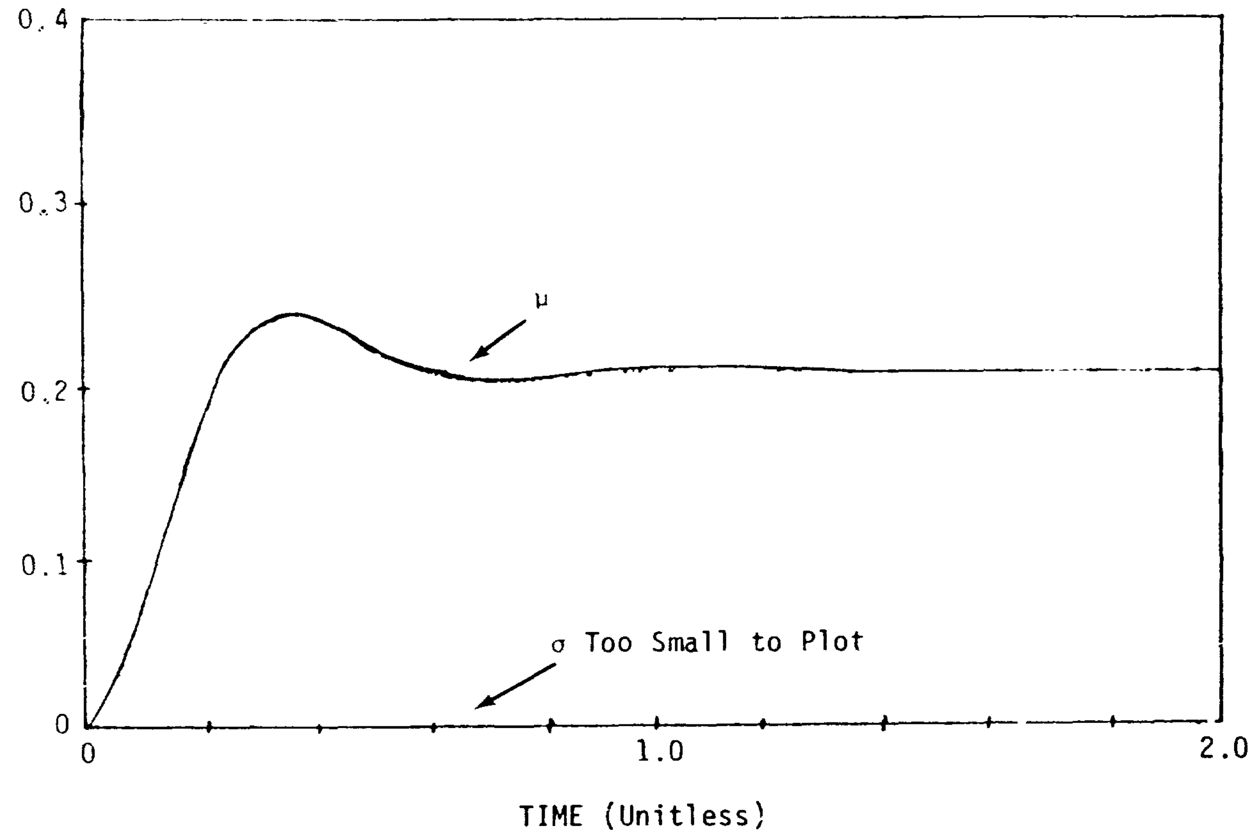

Figure 2-14 μ and σ for Case 6, Example 3

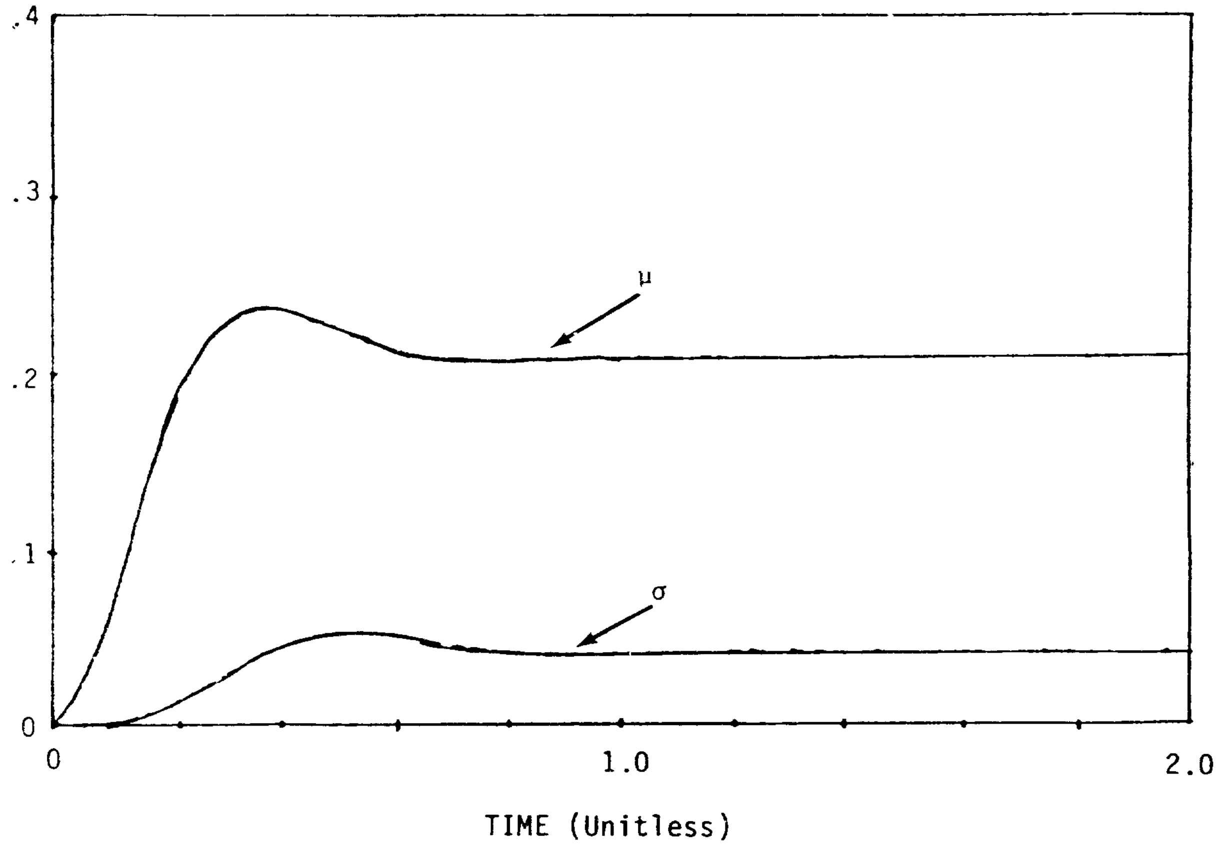

Figure 2-15 μ and σ for Case 3, Example 3

TABLE 2-3

STANDARD DEVIATION OF K AND
CORRESPONDING PEAK RESPONSE VALUES

σ_K	Peak Response
0	0.3473
50	0.3485
100	0.3521

Furthermore while the randomness in the stiffness effects the response for its entire duration, the effects of the random damping dissipate as the system converges to its static deflection which is independent of the damping. However, when the loading is transitory the effects of the random stiffness will also dissipate as the system returns to its initial state.

The results of cases 7, 8, and 9 allow one to assess the impact of correlation between the stiffness and damping. In these cases the correlation coefficient between the stiffness and damping, P_{KD}, was given the values of 0, 1, and -1, respectively.

The difference between these cases were too small to be effectively plotted, however, case 8, $P_{KC} = 1$, produced the largest values of both $\mu_X(t)$ and $\sigma_X(t)$. Case 9, $P_{KC} = -1$, produced the smallest values of $\mu(t)$ and $\sigma_X(t)$.

Based on these results one concludes that $\sigma_X(t)$ at time of peak response increases with increasing correlation between the stiffness and damping. The values for the mean and standard deviation at the time of peak response are provided in Table 2-4 as an illustration.

TABLE 2-4

MEAN AND STANDARD DEVIATION OF
X(t) AT TIME OF PEAK RESPONSE

P_{KC}	$\mu(t)$	$\sigma_K(t)$
-1	0.3479	0.276
0	0.3486	0.326
1	0.3493	0.370

EXAMPLE 4 - Blast Loading of a Simply Supported Beam

This example illustrates the application of the methodology developed in the previous sections to a problem of practical interest.

The structure considered is a simply supported reinforced concrete beam. The loading is a simplified approximation to a blast loading. The structure and its loading time history are shown in Figure 2-16.

A deterministic solution using standard analysis procedures is provided in Biggs (33). The peak deflection resulting from the deterministic and stochastic solutions are compared.

When modeling such a structure with an SDOF system the stiffness is,

$$K_D = \frac{384EI}{5L^3} \tag{2-100}$$

for the present case this yields a value of K_D of 2.08×10^6 lb/in. The stiffness is assumed to have a coefficient of variation of 0.15. This results in the variance of K,$E(K_R{}^2)$, being 9.734×10^{10} lb^2/in^2.

The damping was assumed deterministic and equal to 8% of critical.

The results of the stochastic computation, the mean of the midspan deflection and its standard deviation are plotted in Figure 2-17. The maximum value obtained for the mean deflection was 0.2450". Biggs estimated an upper bound of 0.2726" for the undamped structure.

The key question of interest for the engineer is, "what is the probability of the peak response exceeding a desired level, b?". This question is addressed in Example 2 of Chapter 4 where the passage probabilities for this problem are computed.

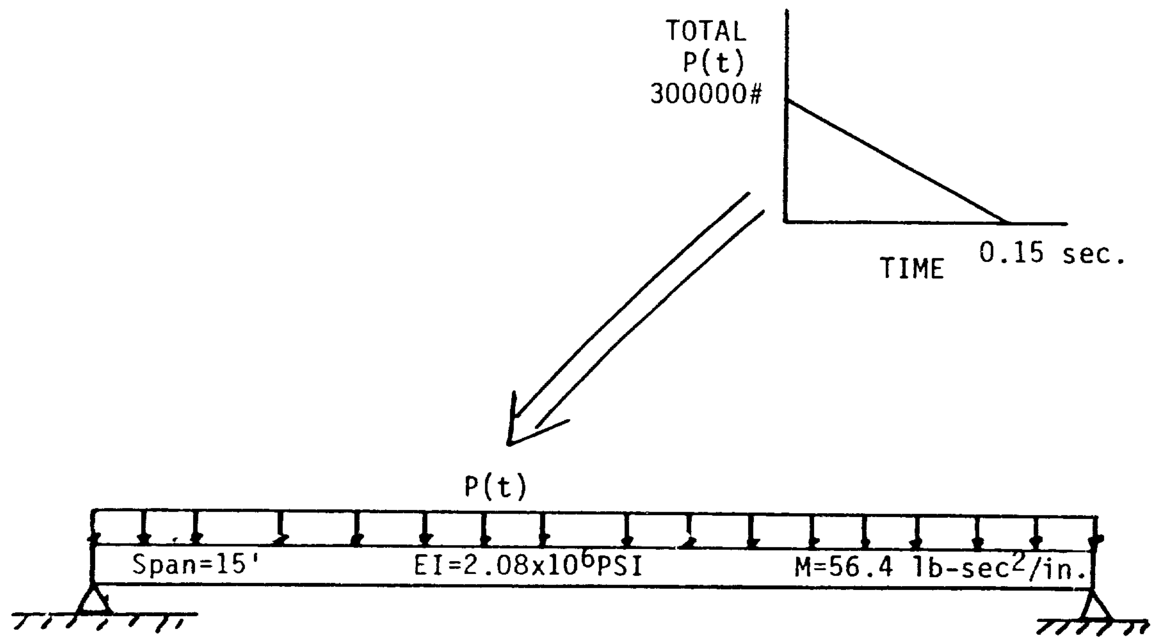

Figure 2-16 Simply Supported, Blast Excited Beam of Example 4

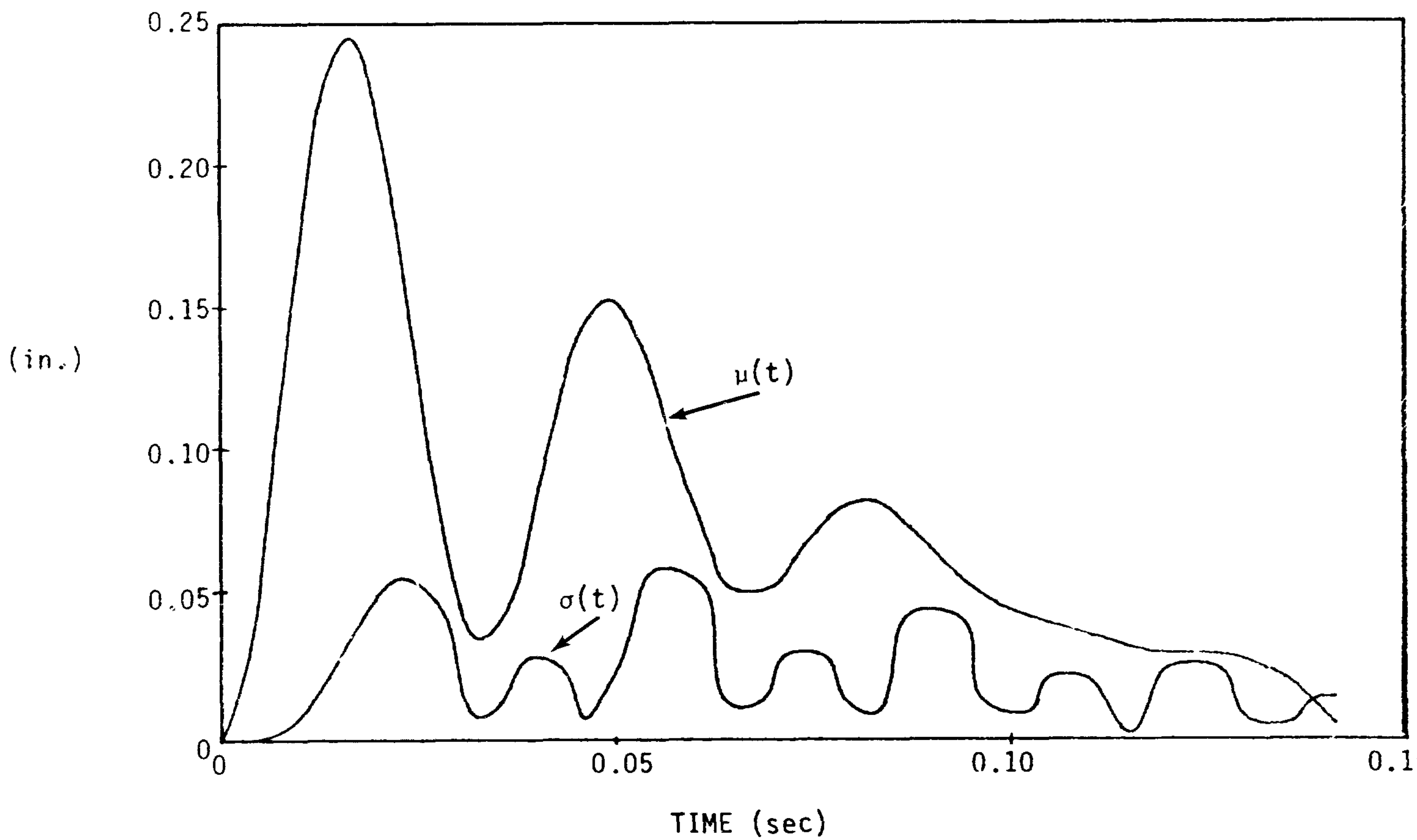

Figure 2-17 $\mu(t)$ and $\sigma(t)$ of Midspan Deflection, Example 4

CHAPTER 3

RESPONSE OF MDOF SYSTEMS WITH RANDOM STIFFNESS AND DAMPING

3.1 INTRODUCTION

In this chapter the transformation method developed in Chapter 2 is expanded to MDOF sytems. This introduces a number of complications involving the correlation between the random variables describing the system and the response parameters. Expressions for the general case where the correlation between system parameters may be user specified and for a simpler case of individual terms of the system matrices being uncorrelated are developed. The latter case is used in the comptuer code developed simply to reduce computation times.

The assumptions regarding the stochastic nature of the displacement response, $X(t)$, and of the loading, $F(t)$, for the SDOF system of Chapter 2 to apply to $\{X(t)\}$, the vector displacement response, and $\{F(t)\}$, the forcing function vector as well.

The desired results of the computation are the mean response vector, $\{\mu(t)\}$, and the Covariance Matrix, $\mathrm{Cov}(\{X(t)\},\{X(t)\}^T)$ as functions of time.

No attempt is made to model the distribution of the response vector or to detemine the correlation between values at different times.

3.2 TRANSFORMATION OF THE EQUATIONS OF MOTION

The equation of motion for a mass excited MDOF system is,

$$[M]\{\ddot{X}(t)\} + [C]\{\dot{X}(t)\} + [K]\{X(t)\} = \{F(t)\} \tag{3-1}$$

where $[M]$ = System Mass Matrix

$[C]$ = System Damping Matrix

$[K]$ = System Stiffness Matrix

$\{X(t)\}$ = System Displacement Vector

$\{F(t)\}$ = Forcing Function Vector

When $[K]$ and $[C]$ are deterministic the response of the system may be obtained by several methods. For a deterministic loading $\{F(t)\}$, Equation (3-1) may be solved by a variety of numerical schemes including modal decomposition, explicit integration, and implicit integration.

When $\{F(t)\}$ is a stochastic process modal decomposition is the method most frequently used. However when $[K]$ and $[C]$ are matrices of random variables the solution of Equation (3-1) is not readily available.

The transformation method developed in Chapter 2 provides a new method for addressing this problem.

Following the development of Section 2.2 results in,

$$\{X(t)\} = \{\mu(t)\} + \{\delta(t)\} \tag{3-2}$$

Taking the expected value of Equation (3-2) yields,

$$E\{X(t)\} = E\{\mu(t)\} + E\{\delta(t)\} \tag{3-3a}$$

$$= \{\mu(t)\} \tag{3-3b}$$

To develop an expresion for the mean square value of the response one multiplies Equation (3-2) by its transpose yielding,

$$\{X(t)\}\{X(t)\}^T = \{\mu(t)\}\{\mu(t)\}^T + \{\mu(t)\}\{\delta(t)\}^T$$

$$+ \{\delta(t)\}\{\mu(t)\}^T + \{\delta(t)\}\{\delta(t)\}^T \tag{3-4}$$

Taking the expected value of Equation (3-4) yields,

$$E[\{X(t)\}\{X(t)\}^T] = E[\{\mu(t)\}\{\mu(t)\}^T] + E[\{\delta(t)\}\{\delta(t)\}^T] \quad (3\text{-}5)$$

The covariance matrix of the response is given by,

$$\mathrm{Cov}[\{X(t)\},\{X(t)\}^T] = E[\{X(t)\}\{X(t)\}^T] - E\{X(t)\} \cdot E\{X(t)\}^T \quad (3\text{-}6a)$$

$$\mathrm{Cov}[\{X(t)\},\{X(t)\}^T] = E[\{\delta(t)\}\{\delta(t)\}^T] \quad (3\text{-}6b)$$

One notes that the expressions in Equation (3-6b) are nxn matrices for a system with n degrees of freedom. The diagonal terms in the matrices are the variances of the displacement responses at the degrees of freedom, and the off diagonal terms are the covariances between displacement responses at the degrees of freedom.

A similar substitution results in the expression for the mean and covariance terms for the velocity vector, $\{\dot{X}(t)\}$, which are,

$$\{\dot{X}(t)\} = \{\dot{\mu}(t)\} + \{\dot{\delta}(t)\} \quad (3\text{-}7)$$

$$E\{\dot{X}(t)\} = \{\dot{\mu}(t)\} \quad (3\text{-}8)$$

$$\mathrm{Cov}(\{\dot{X}(t)\},\{\dot{X}(t)\}^T) = E[\{\dot{\delta}(t)\}\{\dot{\delta}(t)\}^T] \quad (3\text{-}9)$$

The expressions for the mean and covariance of the acceleration vector, $\{X(t)\}$ are similarly derived and are,

$$\{\ddot{X}(t)\} = \{\ddot{\mu}(t)\} + \{\ddot{\delta}(t)\} \quad (3\text{-}10)$$

$$E\{\ddot{X}(t)\} = \{\ddot{\mu}(t)\} \quad (3\text{-}11)$$

$$\mathrm{Cov}(\{\ddot{X}(t)\},\{\ddot{X}(t)\}^T) = E(\{\ddot{\delta}(t)\}\{\ddot{\delta}(t)\}^T) \quad (3\text{-}12)$$

Representing the stiffness matrix, $[K]$, as the sum of $[K_D]$ and $[K_R]$ with $[K_D]$ deterministic and $[K_R]$ a matrix of random variables with mean zero yields,

$$[K] = [K_D] + [K_R] \tag{3-13}$$

Taking the expected value of Equation (3-13) yields

$$E[K] = E[K_D] + E[K_R] \tag{3-14}$$

And since $[K_R]$ is mean zero,

$$E[K] = [K_D] \tag{3-15}$$

To develop covariance terms one must consider individual elements in the matrix $[K]$. Let $(K)_{ij}$ denote the element in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column of $[K]$. Then,

$$(K)_{ij} = (K_D)_{ij} + (K_R)_{ij} \tag{3-16}$$

The product between this and the arbitrary element $(K)_{\ell m}$ is,

$$(K)_{ij}(K)_{\ell m} = (K_D)_{ij}(K_D)_{\ell m} + (K_D)_{ij}(K_R)_{\ell m} + (K_R)_{ij}(K_D)_{\ell m} + (K_R)_{ij}(K_R)_{\ell m} \tag{3-17}$$

Taking the expected value of Equation (3-17) yields,

$$E[(K)_{ij}(K)_{\ell m}] = (K_D)_{ij}(K_D)_{\ell m} + E[(K_R)_{ij}(K_R)_{\ell m}] \tag{3-18}$$

because the random components are mean zero. This quantity is the correlation between the random variables $(K)_{ij}$ and $(K)_{\ell m}$. The covariance between random variable pairs is its correlation minus the product of means. In the present case this is,

$$\mathrm{Cov}[(K)_{ij},(K)_{\ell m}] = E[(K_R)_{ij}(K_R)_{\ell m}] \tag{3-19}$$

Similar expressions are developed for the damping matrix, $[C]$, with $[C_D]$ and $[C_R]$ representing the deterministic and random, mean zero terms respectively,

$$[C] = [C_D] + [C_R] \tag{3-20}$$

The resulting expression for the mean damping matrix is,

$$E[C] = [C_D] \tag{3-21}$$

The covariance between terms of $[C]$ is,

$$\mathrm{Cov}[(C)_{ij}(C)_{\ell m}] = E[(C_R)_{ij}(C_R)_{\ell m}] \tag{3-22}$$

The substitution for the vector of forcing functions, $\{F(t)\}$, consists of the sum of deterministic mean portion $\{P(t)\}$ and the random portion $\{q(t)\}$. Making this substitution and following the development above results in the following expressions for $\{F(t)\}$ its mean and covariance.

$$\{F(t)\} = \{P(T)\} + \{q(t)\} \tag{3-23}$$

$$E\{F(t)\} = \{P(T)\} \tag{3-24}$$

$$\mathrm{Cov}(\{F(t)\}\{F(t)\}^T) = E(\{q(t)\}\{q(t)\}^T) \tag{3-25}$$

Substituting the results of Equations (3-2), (3-7), (3-10), (3-13), (3-20), and (3-23) into Equation (3-1) yields,

$$[M]\{\{\ddot{\mu}\}+\{\ddot{\delta}\}\} + [[C_D]+[C_R]]\{\{\dot{\mu}\}+\{\dot{\delta}\}\} + [[K_D]+[K_R]]\{\{\mu\}+\{\delta\}\} =$$

$$= \{P(t)\} + \{q(t)\} \tag{3-26}$$

When the system parameters and response are considered correlated the expected value of Equation (3-26) is,

$$[M]\{\ddot{\mu}\} + [C_D]\{\dot{\mu}\} + [K_C]\{\mu\} + E([K_R]\{\delta\}) + E([C_R]\{\dot{\delta}\})$$

$$= \{P(t)\} \tag{3-27}$$

A solution for $\{\mu(t)\}$, the mean response vector may be sought based on this expression.

To develop an expression for $\{\delta(t)\}$ one may subtract Equation (3-27) from Equation (3-26) resulting in,

$$[M]\{\ddot{\delta}\} + [C_D]\{\dot{\delta}\} + [K_D]\{\delta\} + \{CRT\} + \{KRT\} =$$

$$= \{q(t)\} - [K_R]\{\mu\} - [C_R]\{\dot{\mu}\} \tag{3-28a}$$

where

$$\{CRT\} = [C_R]\{\dot{\delta}\} - E([C_R]\{\dot{\delta}\}) \tag{3-28b}$$

$$\{KRT\} = [K_R]\{\delta\} - E([K_R]\{\delta\}) \tag{3-28c}$$

The terms in Equations (3-28b) and (3-28c) may be considered very small and may be neglected under the arguments presented in Sections 2.2, and 2.5, resulting in,

$$[M]\{\ddot{\delta}\} + [C_D]\{\dot{\delta}\} + [K_D]\{\delta\} = \{q(t)\} - [K_R]\{\mu\} - [C_R]\{\dot{\mu}\} \quad (3\text{-}29)$$

The forward difference method is used to solve Equation (3-27) for the mean response of the system. The expression for the variance of the response is developed by applying the forward difference equations to Equation (3-29), yielding an expression for $\{\delta(t)\}$ in terms of the underlying random variables, and $\{\delta(t)\}$ at earlier times. This expression is then multiplied by its transpose, and the expected value of the result is equal to the variance at time t.

The details of this development are provided in Section 3.4 after some comments on the development of the system matrices in Section 3.3.

3.3 FORMULATION OF THE SYSTEM MATRICES

The finite element method was used to develop the system's stiffness, damping and mass matrices. The structure of interest is represented as the combination of a number of place frame elements. Each element has the six degrees of freedom shown in Figure 3-1.

Figure 3-1 Plane Frame Element Degrees of Freedom

The stiffness matrix, [K], for such an element is,

$$[k] = \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^2 & -6EI/L^2 \\ 0 & 6EI/l^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix} \quad (3\text{-}30)$$

where AE = Extensional Rigidity of the Element

EI = Flextural Rigidity of the Element

L = Length of the Element

When EI and AE are random variables each term of [k] consists of a random component with mean zero and a deterministic component equal to the values obtained from Equation (3-30) evaluated at the mean values of EI and AE. Thus,

$$[k] = [k_R] + [k_D] \quad (3\text{-}31)$$

Taking the expected value of Equation (3-30) yields,

$$E[k] = E[k_R] + E[k_D] \quad (3\text{-}32)$$

And since $[k_R]$ is mean zero and $[k_D]$ is deterministic,

$$E[k] = [k_D] \quad (3\text{-}33)$$

When individual realizations of EI and AE appearing in Equation (3-30) are assumed uncorrelated then the elements of $[k_R]$ are independent of one another. This assumption is made for ease of computation only. In fact the correlation functions between the terms of the system stiffness matrix will seldom be known in practice. The method developed in the following sections may be applied to cases where the terms in the stiffness matrix are correlated, provided the correlation functions are determined and stored for use at each time step. This becomes apparent in the formulation of the finite difference expressions and is discussed in detail in Section 3.4.

The system stiffness matrix, [K], is formed by summing the contributions of each element for each degree of freedom. When the orientation of the system degrees of freedom differ by an angle γ from the element degrees of freedom the matrix defined by Equation (3-30) must be rotated to the proper coordinates. This rotation is accomplished by pre- and post-multiplying [k] by a transformation matrix [T]. When one considers an element oriented as in Figure 3-2 the stiffness matrix expresses the relations between forces and displacements in the X and Y direction.

The transformation matrix [T], used to develop the rotated stiffness matrix, [k'], which expresses the relations between forces and displacements in the X' and Y' directions is,

$$[T] = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 & 0 & 0 & 0 \\ -\sin\gamma & \cos\gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\gamma & \sin\gamma & 0 \\ 0 & 0 & 0 & -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3\text{-}34)$$

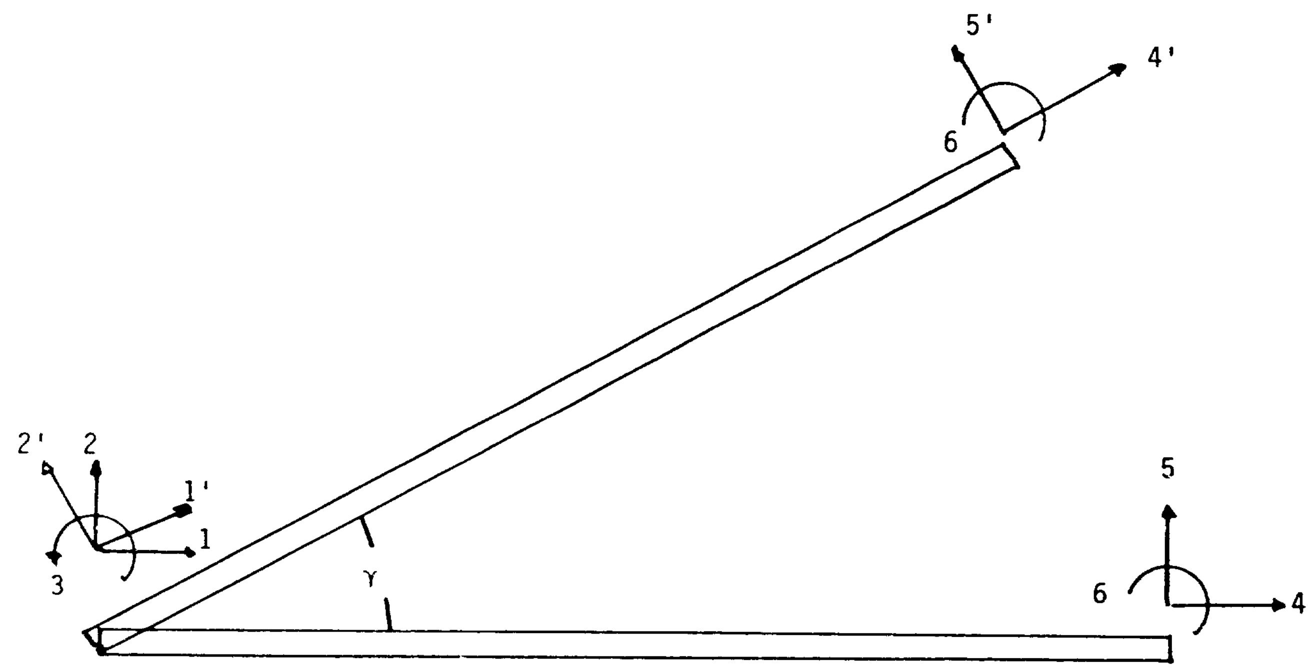

Figure 3-2 Rotation of Plane Frame Element Through an Angle γ

And the rotated stiffness matrix is,

$$[k'] = [T]^T[k][T] \tag{3-35}$$

Since [k] has a deterministic and random components [k'] will as well. Thus [k'] may be expressed as,

$$[k'] = [k_R'] + [k_D'] \tag{3-36}$$

with the random component $[k_R']$ given by,

$$[k_R'] = [T]^T[k_R][T] \tag{3-37}$$

and the deterministic component $[K_D']$ given by

$$[K_D'] = [T]^T[k_D][T] \tag{3-38}$$

The expected value of [K'] is

$$E[k'] = [k_D'] = [T]^\Gamma\{k_D][T] \tag{3-39}$$

The covariance of $[k'][k']^T$ is given by,

$$Cov[k'][k']^T = E[k_R'][k_R']^T \tag{3-40}$$

Replacing $[k_R']$ with the Equation (3-27) yields

$$E[k_R'][k_R']^T = E[T]^T[k_R][T][T]^T[k_R]^T[T] \tag{3-41}$$

However, [T] is orthogonal to $[T]^T$ so $[T][T]^T = [I]$ resulting in,

$$E[k_R'][k_R']^T = E[T]^T[k_R][k_R]^T[T] \tag{3-42}$$

And since [t] is deterministic

$$E[k_R'][k_R']^T = [T]^T E([k_R][k_R]^T)[T] \tag{3-43}$$

Resulting in

$$Cov[K'][k']^T = [T]^T(Cov[k][k]^T)[T] \tag{3-44}$$

However due to the manner in which $[K_R]$ appears in the expressions developed in the following section the properties of interest are the variance of each term of [K] and hence each term of [k']. The $ij^{\underline{th}}$ component of [k'] may be written using indicial notation as

$$k'_{ij} = T^T_{i\ell}\, j_{\ell p}\, T_{pj} \tag{3-45}$$

Hence the $Var(k_{ij})$ may be expressed as

$$Var(k'_{ij}) = (T^T_{i\ell})^2\, Var(k_{\ell p})(T_{pj})^2 \tag{3-46}$$

when the terms of k are independent as assumed earlier. When this is not the case the values of the covariances of $(k_{ij}k_{\ell p})$ must be known. When the correlation between AE and EI is known these may be readily computed from Equation (3-30). However, there will be 6^4 of them. The final assembly of the global matrices would result in n^4 terms of the form $Cov(K_{ij}K_{\ell p})$.

The system stiffness matrix is assembled by summing the components of the element stiffness matrices at each degree of freedom. Thus [K] may be written as,

$$[K] = \sum_{i=1}^{\ell} [k']_i \tag{3-47}$$

where $[k']_i$ is the stiffness matrix for the $i^{\underline{th}}$ element, and ℓ is the total number of elements.

Similar expressions for the random, $[K_R]$ and deterministic, $[K_D]$, components of [K] are,

$$[K_R] = \sum_{i=1}^{\ell} [k_R']_i \tag{3-48}$$

And

$$[K_D] = \sum_{i=1}^{\ell} [k_D']_i \tag{3-49}$$

The variance of each term K_{ij} is the sum of the variances of k'_{mp} for each element where element degrees of freedoms m and p correspond to system degrees of freedom i and j respectively.

The mass matrix for a frame element may take one of several forms. The one chosen for use here is presented by Cook (33). The matrix is diagonal with no zero entries which makes it trivial to invert as well as offering several other advantages in forming the finite difference expressions. The terms of the mass matrix m are given by,

$$(m) = \frac{M}{420} \left(210 \quad 210 \quad L^2 \quad 210 \quad 210 \quad L^2\right) \tag{3-50}$$

where M is the total mass of the element. When required (M) may be rotated as was [k] in Equation (3-35). This results in,

$$(m') = [T]^T (m) [T] \tag{3-51}$$

The system mass matrix, (M) will also be diagonal and is formed by summing the element matrices at appropriate degrees of freedom. Thus (M) may be expressed as

$$(M) = \sum_{i=1}^{\ell} (m')_i \tag{3-52}$$

Where $(m)_i$ is the mass matrix of the $i^{\underline{th}}$ element and ℓ is the total number of elements.

The system damping matrix is a linear combination of [K] and [M]. This formulation is called Rayleigh damping and results in,

$$[C] = \alpha[K] + \beta(M) \tag{3-53}$$

where α and β are deterministic constants. Substituting Equations (3-20) and (3-13) for [C] and [K] above yields.

$$[C]_D + [C_R] = \alpha[K_D] + \alpha[K_R] + \beta(M) \tag{3-54}$$

Taking the expected value of Equation (3-54) yields

$$[C]_D = \alpha[K_D] + \beta(M) \tag{3-55}$$

and hence the random component of [C] is

$$[C_R] = \alpha[K_R] \tag{3-56}$$

Alpha and Beta may be chosen such that the damping of the system is a predetermined portion of crilical at two separate frequencies. Considering the frequencies ω_1 and ω_2 with corresponding percentages of critical damping ε_1 and ε_1, β and α are given by,

$$\alpha = \frac{2(\omega_2\varepsilon_2 - \omega_1\varepsilon_1)}{\omega_2^{\,2} - \omega_1^{\,2}} \tag{3-57}$$

$$\beta = 2\omega_1\varepsilon_1 - \alpha\omega_1^{\,2} \tag{3-58}$$

Once the system matrices, $[K_D]$, $[K_R]$, $[C_D]$, $[C_R]$, and (M) have been obtained the forward difference method is used to compute the mean $\{\mu(t)\}$, and covariance $\{X(t)\}\{X(t)\}^T$ of the response. The details of this computation are explained in the following section.

3.4 APPLICATION OF FINITE DIFFERENCES

The forward difference technique is used to express the equations of motion generated in Section 3.2. The resulting expressions are thus evaluated in a step-by-step manner to yield values for the mean and variance of the response over times

Let $t_n = n\Delta t$, $r = 0, 1, \ldots$, Δt a "small value", be a discretized time variable. The vector forms of the difference expressions developed in Section 2.2 are,

$$\{f(t_n)\} = \{f_n\} \tag{3-59}$$

$$\{\dot{f}(t_n)\} = \frac{1}{\Delta t}\left(\{f_{n+1}\} - \{f_n\}\right) \tag{3-60}$$

$$\{\ddot{f}(t_n)\} = \frac{1}{\Delta t^2}\left(\{f_{n+1}\} - 2\{f_{n+1}\} + \{f_n\}\right) \tag{3-61}$$

To develop the relations required to evaluate $\{\mu(t_n)\}$ and its derivatives in Equation (3-27). This yields,

$$(\overline{M})\ (\{\mu_{n+2}\} - 2\{\mu_{n+1}\} + \{\mu_n\}) + [C_D]\ (\{\mu_{n+1}\}-\{\mu_n\})$$

$$+ [K_D]\{\mu_n\} = \{P_n\} - E([K_R]\{\delta_n\}) - E([C_R]\{\dot{\delta}_n\}) \qquad (3\text{-}62a)$$

where
$$(\overline{M}) = \frac{1}{\Delta t^2}\ (M) \qquad (3\text{-}62b)$$

$$[C_D] = \frac{1}{\Delta t}\ [C_D] \qquad (3\text{-}62c)$$

If the relationship between $[K_R]$ and $[C_R]$ as defined in Equation (3-56) is now imposed, Equation (3-62a) reduces to,

$$(\overline{M})\ (\{\mu_{n+2} - 2\mu_{n+1} + \mu_n\}) + [\overline{C}_D](\{\mu_{n+1} - \mu_n\}) + [K_D]\{\mu_n\} =$$

$$= \{P_n\} - E([K_R]\{\eta_n\}) \qquad (3\text{-}63a)$$

where
$$(\eta_n) = \{\delta_n\} + \alpha\{\dot{\delta}_n\} \qquad (3\text{-}63b)$$

Solving Equation (3-63a) for $\{\mu_{n+2}\}$ yields,

$$\{\mu_{n+1}\} = (\overline{M})^{-1}\ (\{P_n\} - E([K_R]\{\eta_n\}) + [A_1]\{\mu_{n+1}\} + [A_2]\{\mu_n\}) \qquad (3\text{-}64a)$$

where
$$[A_1] = 2\ (\overline{M}) - [\overline{C}_D] \qquad (3\text{-}64b)$$

$$[A_2] = [C_D] - (\overline{M}) - [K_D] \qquad (3\text{-}64c)$$

Replacing $[C_D]$ with Equation (3-55) yields,

$$[A_1] = (2-\Delta t\beta) \quad (\overline{M}) - (\frac{\alpha}{\Delta t}) \; [K_D] \qquad (3\text{-}65a)$$

$$[A_2] = (\beta\Delta t-1) \quad (\overline{M}) + (\frac{\alpha}{\Delta t}) \; [K_D] \qquad (3\text{-}65b)$$

Equation (3-64a) provides an expression by which $\{\mu_{n+2}\}$ may be evaluated in a step-by-step manner similar to that used for the SDOF system discussed in Chapter 2. The values for $\{\mu_{n+1}\}$ and $\{\mu_n\}$ are available from previous time steps.

The expected value of $[K_R]\{n_n\}$ may be developed from the expressions for $\{\delta_n\}$ which now follows.

The process $\{\delta(t)\}$ is implicitly defined by Equation (3-29). Applying the forward difference relations to $\{\delta(t)\}$ and its derivatives results in,

$$(\overline{M}) \;\; (\{\delta_{n+2}\} - 2\{\delta_{n+1}\} + \{\delta_n\}) + [\overline{C}_D](\{\delta_{n+1}\} - \{\delta_n\}) + [K_D]\{\delta_n\}$$

$$= \{q_n\} - [K_R]\{Z_n\} \qquad (3\text{-}66a)$$

where

$$\{Z_n\} = \{\mu_n\} + \alpha\{\dot{\mu}_n\} \qquad (3\text{-}66b)$$

Solving Equation (3-65a) for $\{\delta_{n+2}\}$ yields,

$$\{\delta_{m+2}\} = (\overline{M})^{-1} \;\; (\{q_n\} - [K_R]\{Z_n\} + [A_1]\{\delta_{n+1}\} + [A_2]\{\delta_n\}) \qquad (3\text{-}67)$$

To evaluate the expected value of the product $(K_R)_{ij}(\delta_{n+2})_k$ one writes the expression for $(\delta_{n+2})_k$ by evaluating Equation (3-67) for its $k_{\underline{th}}$ element and multiplying by $(K_R)_{ij}$ and taking the expected value. The equation for the $k_{\underline{th}}$ element of $\{\delta_{n+2}\}$ is,

$$(\delta_{n+2})_k = (\overline{M})^{-1}_{kk}\ ((q_n)_k - (K_R)_{k\ell}(Z_n)_\ell + (A_1)_{kr}(\delta_{n+1})_r$$

$$+ (A_2)_{kp}(\delta_n)_p) \qquad (3\text{-}68)$$

Premultiplying both sides by $(K_R)_{ij}$ yields,

$$(K_R)_{ij}(\delta_{n+2})_k = (\overline{M})^{-1}_{kk}\ ((K_R)_{ij}(q_n)_k - (K_R)_{ij}(K_R)_{k\ell}(Z_n)_\ell$$

$$+ (K_R)_{ij}(A_1)_{kr}(\delta_{n+1})_r + (K_R)_{ij}(A_2)_{kp}(\delta_n)_p) \qquad (3\text{-}69)$$

Taking the expected value of Equation (3-68) yields,

$$E((K_R)_{ij}(\delta_{n+1})_k) = (\overline{M})^{-1}_{kk}\ (E((K_R)_{ij}(q_n)_k) - E((K_R)_{ij}(K_R)_{kk})(Z_n)_\ell$$

$$+ (A_1)_{kr}E((K_R)_{ij}(\delta_{n+1})_r(+ (A_2)_{kp}E((K_R)_{ij}(\delta_n)_p)) \qquad (3\text{-}70)$$

And since $[K_R]$ is independent of $\{q_n\}$ Equation (3-70) reduces to,

$$E((K_R)_{ij}(\delta_{n+2})k) = (\overline{M})^{-1}_{kk}(-E((K_R)_{ij}(K_R)_{k\ell})(Z_n)_\ell + (A_1)_{kr}E((K_R)_{ij}(\delta_{n+1})_r)$$

$$+ (A_2)_{kp}E((K_R)_{ij}(\delta_n)_p)) \qquad (3\text{-}71)$$

Thus, the n^3 terms of the $E((K_R)_{ij}(\delta_{n+2})_k)$ may be evaluated in a step-by-step manner. However, the evaluation will require n^2 calculations for each of the n^3 terms. To reduce the number of computations a simplifying assumption may be imposed. The assumption is that the deflection at degree-of-freedom, (DOF), k is independent of the entries of the random stiffness matrix at dof's other than k.

This may be written

$$E((K_R)_{ij}(\delta_{n+2})_k) = 0 \qquad \text{when } k \neq i \quad \text{or} \quad k \neq j \tag{3-72}$$

When $k = i$ the $E((K_R)_{ij}(\delta_{n+2})_k)$ is equal to $E((K_R)_{kj}(\delta_{n+2})_k)$ and is,

$$E((K_R)_{kj}(\delta_{n+2})_k) = (\bar{M})^{-1}_{kk}[-E((K_R)_{kj}(K_R)_{k\ell})(Z_n)_\ell + (A_1)_{kr}E((K_R)_{kj}(\delta_{n+1})_r) + (A_2)_{kp}E((K_R)_{kj}(\delta_n)_p)] \tag{3-73}$$

(no sum on k or j)

Due to the independence of the elements of $[K_R]$ the expected value of $(K_R)_{kj}(K_R)_{k\ell}$ is zero unless $\ell = j$ then

$$E((K_R)_{kj}(K_R)_{1\ell}) = E(K_R)^2_{kj} \tag{3-74}$$

Furthermore, by taking advantage of the assumption expressed in Equation (3-71) for the terms $(K_R)_{ki}(\delta_{n+1})_r$ and $K_R)_{kj}(\delta_n)_p$,

Equation (3-73) may be rewritten as,

$$E((K_R)_{kj}(\delta_{n+2})_k) = (\bar{M})^{-1}_{kk}\ [-E(K_R)^2_{kj}(Z_n)_j + (A_1)_{kk}E((K_R)_{kj}(\delta_{n+1})_k) + (A_1)_{kj}E((K_R)_{kj}(\delta_{n+1})_j) + (A_2)_{kk}E((K_R)_{kj}(\delta_n)_k) + (A_2)_{kj}E((K_R)_{kj}(\delta_n)_j)] \tag{3-75}$$

(no sum on k or j)

When we consider the case of k=j Equation (3-71) may be written,

$$E((K_R)_{ij}(\delta_n)_j) = (\overline{M})^{-1}_{jj}\,[-E((K_R)_{ij}(K_R)_{j\ell})(Z_n)_\ell$$

$$+ (A_1)_{jr}E((K_R)_{ij}(\delta_{n+1})_r) + (A_2)_{jp}E((K_R)_{ij}(\delta_n)_p)] \quad (3\text{-}76)$$

Once again taking advantage of the independence of the elements of $[K_R]$ and the assumption expressed in Equation (3-72), Equation (3-76) is reduced to

$$E((K_R)_{ij}(\delta_{n+2})_j) = (\overline{M})^{-1}_{ij}\,[-E((K_R)_{ij}(K_R)_{k\ell})(Z_n)_\ell + (A_1)_{ij}E((K_R)_{ij}(\delta_{n+1})_j)$$

$$+ (A_1)_{ji}E((K_R)_{ij}(\delta_{n+1})_i) + (A_2)_{jj}E((K_R)_{ij}(\delta_n)_i) \quad (3\text{-}77)$$

no sum on j or i

However, the expressions in Equations (3-75) and (3-77) allow the $2n^2$ terms of $(K_R)_{ij}(\delta_n)_k$ for k=j and k=i to be evaluated at each point in time. When i=j, Equation (3-71) with the assumption of Equation (3-71) yields,

$$E((K_R)_{ij}(\delta_{n+2})_i) = (\overline{M})^{-1}_{ii}\,[-E(K_R)^2_{ii}(Z_n)_i + (A_1)_{ii}E((K_R)_{ii}(\delta_{n+1})_i)$$

$$+ (A_2)_{ii}E((K_R)_{ii}(\delta_{n+1})_i] \quad (3\text{-}78)$$

However, if one considers a slightly more restrictive assumption than that of Equation (3-72) the calculations may be reduced still further.

Consider the product $(K_R)_{ij}(\delta_n)_j$; this is the component of the random portion of the static restoring force at DOF i due to a deflection at DOFj. If one considers the product $(K_R)_{ij}(\delta_n)_i$ there is no such clear physical meaning. One may therefore say that the terms involving $(K_R)_{ij}(\delta_n)_j$ will be representative of the uncertainties in actual forces and are the terms of primary interest. Since the terms involving $(K_R)_{ij}(\delta_n)_i$ have no such clear physical meaning it is assumed they will not have a significant effect on the solution of the problem. The assumption is made for computational convenience. The more liberal assumption of $E[(K_R)_{ij}(\delta_n)_i]$ and $E[(K_R)_{ij}(\delta_n)_j]$ are both not equal zero would be more accurate provided computation and storage capabilities exist to execute it. The means for doing so are provided by Equations (3-75) and (3-77). The full set of n^3 terms, $E[(K_R)_{ij}(\delta_n)_k]$ may also be assembled at each time step by Equation (3-71), provided once again that sufficient storage and computation capabilities are available.

Evaluating Equation (3-71) under the constraint,

$$E((K_R)_{ij}(\delta_n)_j) = 0 \qquad k \neq j \tag{3-79}$$

yields,

$$E[(K_R)_{ij}(\delta_{n+2})_j] = (\overline{M})_{jj} \, [-E(K_R)_{ij}(K_R)_{j\ell}](Z_n)_\ell$$

$$+ (A_1)_{jr}E[(K_R)_{ij}(\delta_{n+1})_r] + (A_2)_{jp}E[(K_R)_{ij}(\delta_n)_p] \tag{3-80}$$

Applying Equation (3-79) to the $K_R\delta$ terms on the right hand side of Equation (3-80) yields,

$$E((K_R)_{ij}(\delta_{n+2})_j) = (\overline{M})^{-1}_{jj} \, [-E((K_{Rij})(K_{Rj\ell}))(Z_n)_\ell$$

$$+ (A_1)_{jj}E((K_R)_{ij}(\delta_{n+1})_j) + (A_2)_{jj}E((K_R)_{ij}(\delta_n)_j)] \tag{3-81}$$

and due to the independence of the terms of (K_R) when $j \neq i$ this yields,

$$E((K_R)_{ij}(\delta_{n+2})_j) = (\overline{M})^{-1}_{jj} \; [(A_1)_{jj}E((K_R)_{ij}(\delta_{n+1})_j)$$

$$+ (A_1)_{jj}E((K_R)_{ij}(\delta_n)_j)] \qquad (3\text{-}82)$$

And so when $E(K_R)_{ij}(\delta_n)_j)_o = 0$, i.e. the case of zero start, Equation (3-82) results in,

$$E((K_R)_{ij}(\delta_{n+2})_j) = 0 \quad \text{all} \quad n. \qquad (3\text{-}83)$$

then i=j Equation (3-81) becomes,

$$E((K_R)_{ii}(\delta_{n+2})_i) = (\overline{M})^{-1}_{ii} \; [-E(K_{Rii})^2(Z_n)_i + (A_1)_{ii}E(K_{Rii})(\delta_{n+1})_i)$$

$$+ (A_2)_{ii}E((K_R)_{ii}(\delta_{n+1})_i)] \qquad (3\text{-}84)$$

Hence through the assumption of independent elements in the stiffness matrix and the constraint of Equation (3-79) the expression of Equation (3-71) involving n^3 terms requiring a total of n^5 (n^2 per term), computations per time step is reduced to n terms involving a total of n computations per time step.

Equation (3-84) now allows one to evaluate Equation (3-64a) for the mean response since $E[K_R]\{\varsigma_n\}$ is a vector whose $i^{\underline{th}}$ term is given by

$$(E[K_R]\{\eta_n\})_i = \frac{\alpha E(K_R)_{ii}(\delta_{n+1})_i}{\Delta t} + E(K_R)_{ii}(\delta_n)_i(1 - \frac{\alpha}{\Delta t}) \qquad (3\text{-}85)$$

To develop an expression for the variance of the response one first multiplies Equation (3-67), the expression for $\{\delta_{n+2}\}$ by its transpose and then takes the expected value of the resulting expression. This results in,

$$E(\{\delta_{n+2}\}\{\delta_{n+2}\}^T) = (\bar{M})^{-1} (E\{q_n)\}\{q_n\}^T - [E_1]^T - [E_1] + [E_2]$$

$$+ [E_2]^T + [E_3] + [E_3]^T + E[K_R]\{Z_n\}\{Z_n\}^T[K_R]^T$$

$$- [E_4] - [E_4]^T - [E_5] - [E_5]^T + E[A_1]\{\delta_{n+1}\}\{\delta_{n+1}\}^T[A_1]^T + [E_6]^T + [E_6]$$

$$+ E[A_2]\{\delta_n\}\{\delta_n\}^T[A_2]) (\bar{M})^{-1} \qquad (3\text{-}86a)$$

where

$$[E_1] = E(\{q_n\}\{Z_n\}^T[K_R]^T \qquad (3\text{-}86b)$$

$$[E_2] = E(\{q_n\}\{\delta_n\}^T[A_2]^T) \qquad (3\text{-}86c)$$

$$[E_3] = E(\{q_n\}\{\delta_{n+1}\}^T[A_1]^T) \qquad (3\text{-}86d)$$

$$[E_4] = E([K_R]\{Z_n\}\{\delta_{n+1}\}^T[A_1]^T) \qquad (3\text{-}86e)$$

$$[E_5] = E([K_R]\{Z_n\}\{\delta_n\}^T[A_2]^T) \qquad (3\text{-}86f)$$

$$[E_6] = E([A_1]\{\delta_{n+1}\}\{\delta_n\}^T[A_2]^T) \qquad (3\text{-}86g)$$

Since the loading and stiffness are independent $[E_1]$ may be written,

$$[E_1] = E\{q_n\}\{Z_n\}^T E[K_R]^T \tag{3-87}$$

And since $\{q_n\}$ and $[K_R]$ are mean zero,

$$[E_1] = [0] \tag{3-88}$$

Upon examination of Equation (3-67) one notes that $\{\delta\}_n$ is independent of $\{q_{n-1}\}$ and $\{q_n\}$. Hence $[E_2]$ and $[E_3]$ may be written as

$$[E_2] = E\{q_n\}E\{\delta_n\}^T[A_2]^T \tag{3-89}$$

$$[E_3] = E\{q_n\}E\{\delta_{n+1}\}^T[A_1]^T \tag{3-90}$$

And since $\{q_n\}$, $\{\delta_n\}$ and $\{\delta_{n+1}\}^T$ are all mean zero,

$$[E_2] = [0] \tag{3-91}$$

and

$$[E_3] = [0] \tag{3-92}$$

To evaluate $[E_4]$ one may write out the expression for its in $ij^{\underline{th}}$ term,

$$(E_4)_{ij} = E((K_R)_{i\ell}(Z_n)_\ell(\delta_{n+1})_p(A_1{}^T)_{pj}) \tag{3-93}$$

However, from Equation (3-84) this reduces to,

$$(E_4)_{ij} = E((K_R)_{ii}(\delta_{n+1}))Z_{ni}A_1 ij \qquad (3\text{-}94)$$

Similarly,

$$(E_5)_{ij} = E((K_R)_{ii}(\delta_{n+i})Z_{ni}A_2^T ij \qquad (3\text{-}95)$$

The matrix $[E_6]$ may be written as

$$[E_6] = [A_1]E(\{\delta_{n+1}\}\{\delta_n\}^T)[A_2]^T \qquad (3\text{-}96)$$

Note that $\{\delta_{n+1}\}$ may be written

$$\{\delta_{n+1}\} = (\overline{M})^{-1}(\{q_{n-1}\} - [K_R]\{Z_{n-1}\} + [A_1]\{\delta_n\} + [A_2]\{\delta_{n-1}\}) \qquad (3\text{-}97)$$

Postmultiplying the above by $\{\delta_n\}^T$ yields,

$$\{\delta_{n+1}\}\{\delta_n\}^T = (\overline{M})^{-1}(\{q_{n-1}\}\{\delta_n\}^T - [K_R]\{Z_{n-1}\}\{\delta_n\}^T + [A_2]\{\delta_n\}\{\delta_n\}^T$$

$$+ [A_2]\{\delta_{n-1}\}\{\delta_n\}^T) \qquad (3\text{-}98)$$

Taking the expected value yields,

$$E\{\delta_{n+1}\}\{\delta_n\}^T = (\overline{M})^{-1}[0 - E[K_R]\{Z_{n-1}\}\{\delta_n\}^T + [A_1]E\{\delta_n\}\{\delta_n\}^T$$

$$+ A_2(E\{\delta_n\}\{\delta_{n-1}\}^T)^T] \qquad (3\text{-}99)$$

The expression $(E\{\delta_n\}\{\delta_{n-1}\}^T)^T$ is merely the transpose of the left hand side evaluated at the previous time step. The $E([K_R]\{Z_{n-1}\}\{\delta_n\}^T)$ is a diagonal matrix whose entries are,

$$(E[K_R]\{Z_{n-1}\}\{\delta_n\}^T)_{ii} = (Z_{n-1})_i E((K_R)(\delta_n)_i) \qquad (3\text{-}100)$$

The expression $E[K_R]\{Z_n\}\{Z_n\}^T[K_R]^T$ may be written in tensor notation as

$$E((K_R)_{i\ell}(Z_n)_\ell(Z_n)_p(K_R^T)_{pj}] = (Z_n)_\ell(Z_n)_p E(K_{Ri\ell}K^T_{Rpj}) \qquad (3\text{-}101)$$

And since $E(K_{Ri\ell}K^T_{Rpj}) = 0$ unless j-i, p=ℓ, Equation (3-101) reduces to a diagonal matrix with entries given by

$$E[K_R]\{Z_n\}\{Z_n\}^T[K_R]^T_i = E(K_{Rij})^2 Z_{nj}{}^2$$

Equation (3-86c) may now be evaluated at each time step yielding an expression for the $Cov(\{X(t)\}\{X(t)\}^T)$ at each time step.

The numerical algorithm for performing the computation is presented in Section 3.5. The computation provides the expected value of the displacement response at each degree of freedom and the covariance between the responses at various degrees of freedom.

3.5 THE MULTIPLE DEGREE OF FREEDOM COMPUTATIONAL ALGORITHM

The following algorithm is used to evaluate the mean response, $\{\mu(t)\}$, and the covariance of $\{X(t)\}$ using the forward difference equations developed in Section 3.4. The algorithm is implemented in the code RMDOF. A listing of the code is provided in Appendix B.

The algorithm is based on a zero start condition and proceeds as follows,

1. Read the problem description parameters, the number of nodes, the number of elements and the number of restraints.

2. Read the nodal locations.

3. For each element, read the nodes to which it is connected, the material properties, and their uncertainties. Formulate the element stiffness matrix and compute the variance of each term in the stiffness matrix. Sum these into the global matrices. Formulate the element mass matrix and sum the results into the global mass matrix.

4. Read the damping parameters, ω_1, ε_1, ω_2, ε_2 and calculate α and β from Equations (3-57) and (3-58).

5. Formulate the problem solution matrices $[A_1]$ and $[A_2]$ defined in Equations (3-65a) and (3-65b).

6. Impose the constraint conditions.

7. For each time step do the following:

   a) Update time and the index on the arrays $\{\mu_{n+1}\}$, $\{\mu_n\}$, $E[\{\delta_{n+1}\}\{\delta_{n+1}\}^T]$, $E[\{\delta_n\}\{\delta_n\}^T]$, $E[\{\delta_{n+1}\},\{\delta_n\}^T]$, $E[[K_R]\{\delta_n\}]$, $E[[K_R]\{\delta_{n+1}\}]$.

   b) Form $E([K_R]\{\delta_{n+2}\})$ from Equation (3-86).

   c) Compute $E[\{\delta_{n+1}\}\{\delta_n\}^T]$ from Equation (3-93).

d) Compute $\{\mu_{n+2}\}$ from Equation (3-64a).

e) Compute $E[\{\delta_{n+2}\}\{\delta_{n+2}\}^T]$ from Equation (3-86a).

f) Write appropriate results.

8. Repeat 7 until the problem is completed.

3.6 NUMERICAL EXAMPLES

This section presents the results of two example problems which illustrate the analysis of MDOF systems with random stiffness and damping. In both examples the random portion of any element of the global stiffness matrix, [K], was assumed to be perfectly correlated with the random portion of the corresponding element in the damping matrix.

This assumption of independent terms has a serious shortcoming in that a rigid body displacement of any element will contribute to the variances of the displacements.

The random properties of each element of the stiffness and damping matrices were assumed independent of all other entries of these matrices.

These assumptions were made for reasons of computational covenience only. When one is modeling simple structures such as homogeneous bars and beams where the random properties are assumed highly correlated along their length, these assumptions may not be valid.

However, in the examples below the elements are considered to be condensed representations of many beams, columns and girders. For example, if one were to model a large building the axial and flextural properties of several bays and stories may be combined and represented by a single frame element. Thus a large structure may be modeled in an economical manner. Such subassembly modeling is common for large structures including aircraft, ships, and dams as well as buildings.

EXAMPLE 1 - Deflection of a Non-Prismatic Cantilever Beam.

The structure considered here is shown in Figure 3-3. All dimensions are unitless. The structure possesses six degrees of freedom. However, the discussion here will concern the midpoint and tip deflection. The mean and standard deviation of the tip deflection are plotted in Figure 3-4. The mean and standard deviation of the midpoint deflection are plotted in Figure 3-5.
The correlation coefficient between the two is plotted in Figure 3-6.

The mean and standard deviations for this problem follow the same basic trend as that observed for the SDOF examples, namely that the standard deviation curve has the same basic shape as the mean curve. However, the standard deviation curves do not lag behind the mean curves as in the SDOF cases. Furthermore the standard deviation does not display the secondary peaks which were indicated in the SDOF case.

The ratio between the midpoint and tip deflections appear to be about a factor of 1/8. The standard deviation at the tip is approximately 1/3 the value of the mean. The standard deviation at the midpoint on the other hand is approximately 1/2 the mean. The coefficient of variation of the stiffness and damping were 0.15. Since more elements contributed to the stiffness at the midpoint, and the stiffness of the elements is independent, it is not surprising that the midpoint has a larger coefficient of variation.

The coefficient of variation at the tip, however, is still approximately double, (1/3 vs 0.15) what would be expected based on the results of the SDOF computations. However the assumption of independence of terms within the stiffness matrix, means only positive terms, specifically those of Equation (3-102) enter the computations. If the terms

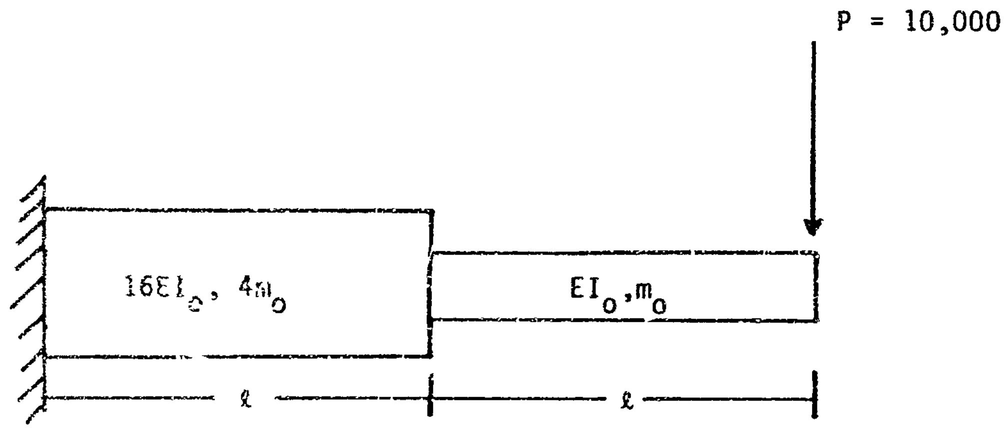

Figure 3-3 Cantilever Beam of Example 1

$\ell = 20 \quad EI_o = 3.867\times10^7 \quad m_o = 2.938\times10^{-3}$

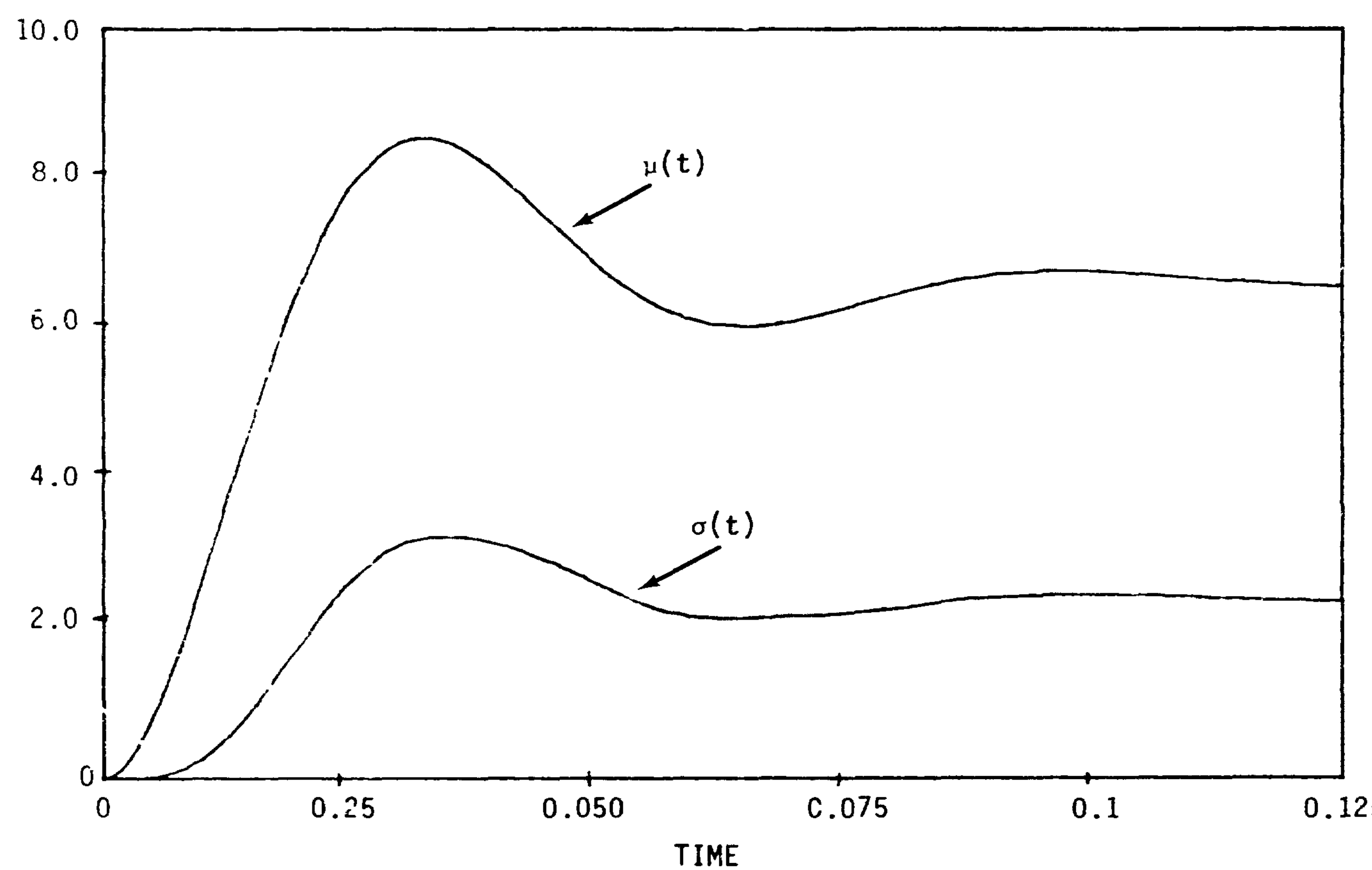

Figure 3-4 $\mu(t)$ and $\sigma(t)$ for Tip Deflection, Example 1

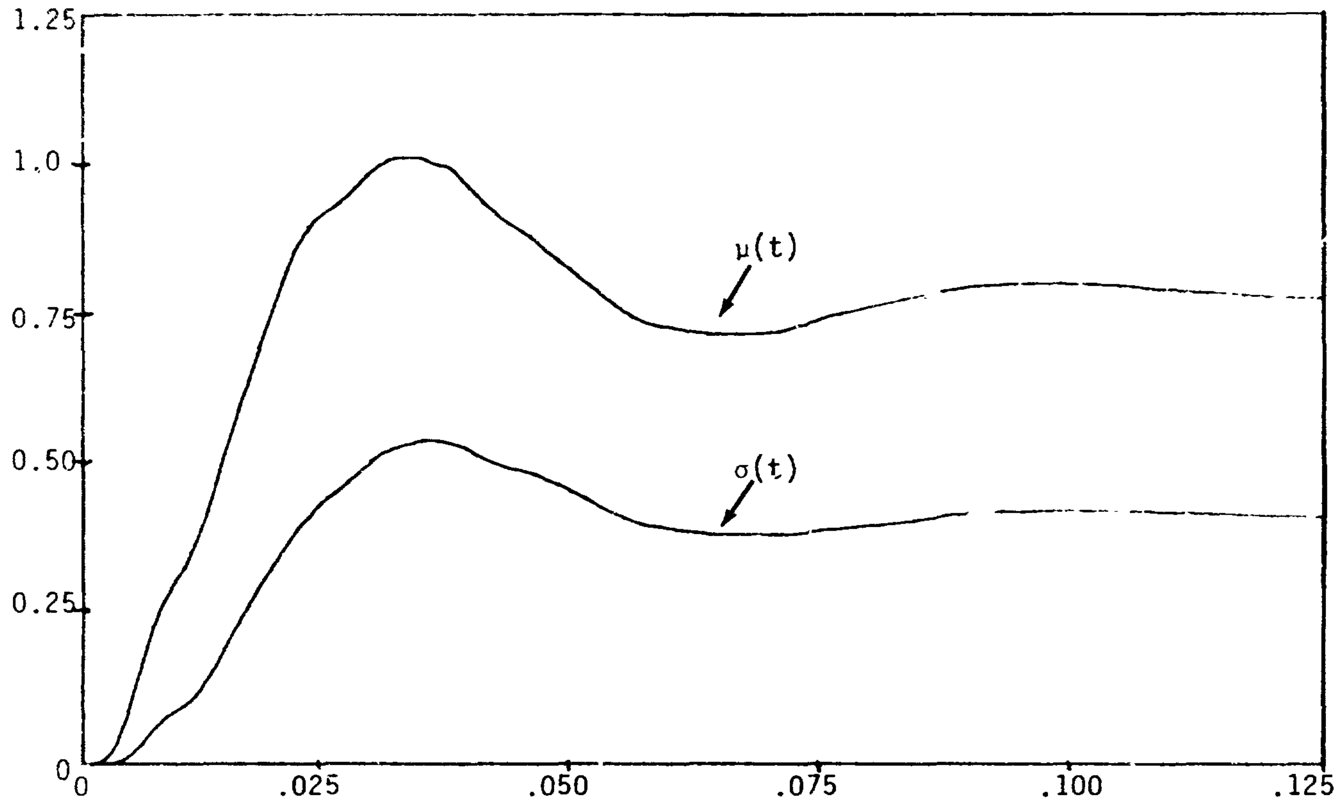

Figure 3-5 $\mu(t)$ and σ (t) for Midpoint Deflection, Example 1

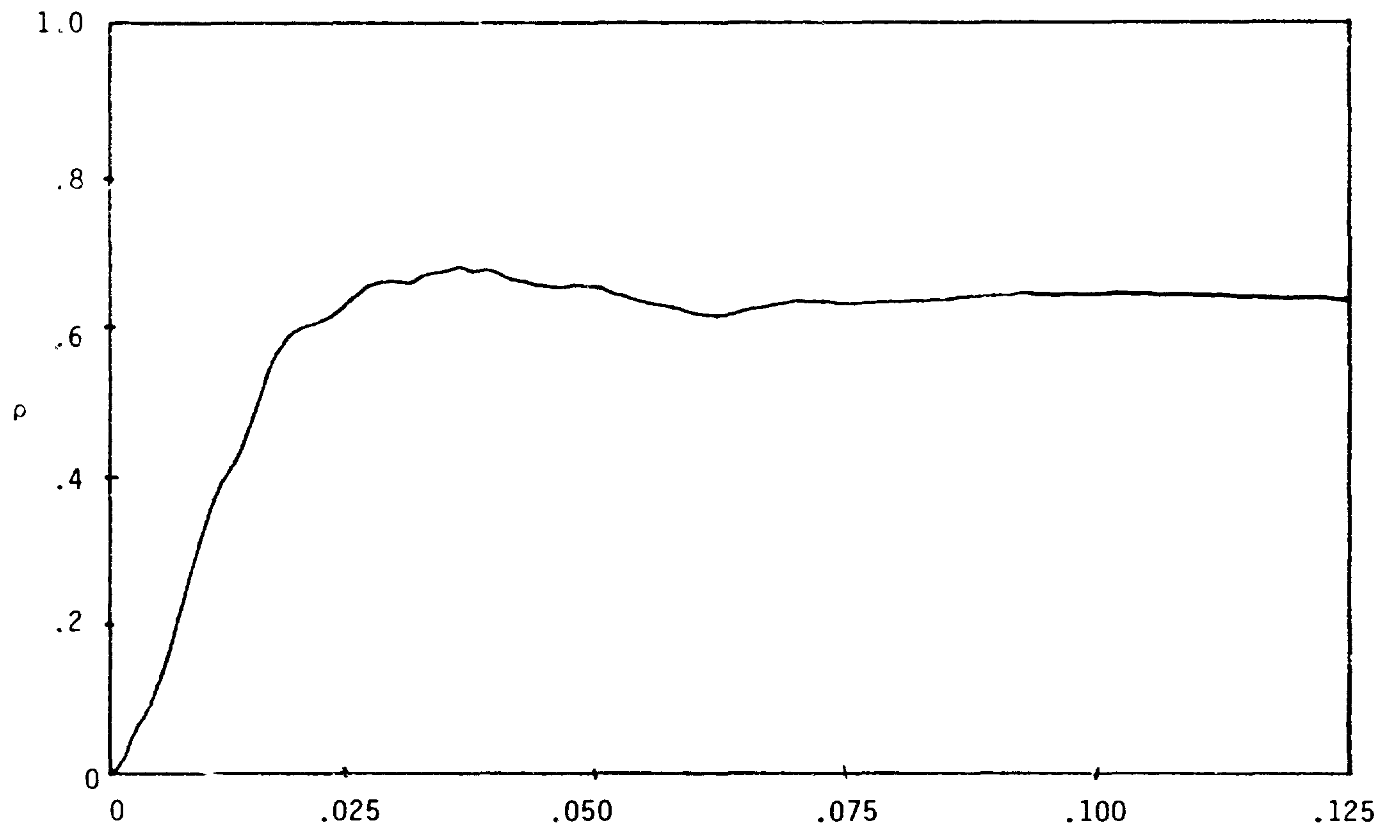

Figure 3-6 Correlation Coefficient Between Deflections at Tip and Midpoint

of the stiffness matrix were to be considered perfectly correlated then negative terms would enter the computation as well, reducing the overall variance. The SDOF approximation to this same problem would in effect assume perfect correlation between the random variables, thus the results would not be comparable.

The correlation between the tip and midpint deflections approaches a constant value of 0.615. The correlation remains positive since the deflections always are in phase, and tied to one another in that the deflected shape of the structure is well defined for this particular problem.

EXAMPLE 2 - Axial Response of a Prismatic Bar.

The structure of interest for this problem is shown in Figure 3-7. All dimensions are unitless. The response parameters of interest were the midpoint and tip deflections. Two loading conditions were investigated, the first being a loading applied suddenly to the midpoint and the second being a loading applied suddenly to the end. The loading in both cases had a value of .1.

For the case of the loading at the midpoint the mean and standard deviation for the deflection at the tip are plotted in Figure 3-8. The mean and standard deviation for the midpoint are plotted in Figure 3-9, and the correlation between them is plotted in Figure 3-10. The same information for loading at the tip is plotted in Figures 3-11, 3-12, and 3-13.

Examination of Figures 3-8 and 3-9 indicate that the standard deviation of the tip deflection is significantly higher than that at the midpoint even though the deflection are nearing their final values. The increase in the tip deflection's standard deviation is due to the assumption of independent entries of the stiffness matrix. Hence while the end element has undergone only a rigid body deformation once the oscillations are damped out this has still contributed to the final standard deviations.

The resulting means are oscillating about a point which is approximately 7% greater than their deterministic static deflection. This is due to the impact of the uncertainties on the mean response.

The correlation coefficient for both cases is approaching a constant value of 0.5. The standard deviations for the case of the end

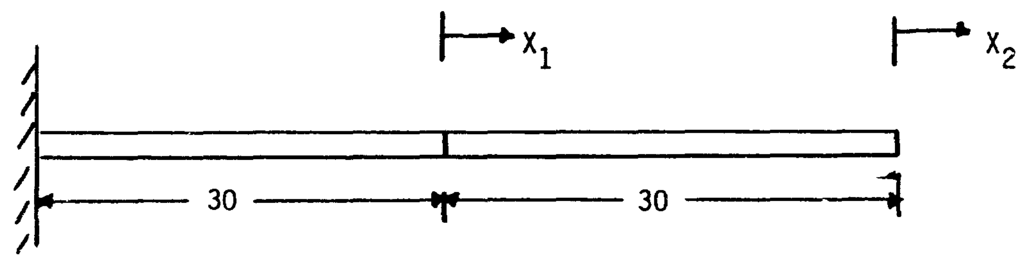

Figure 3-7   Homogeneous Bar of Examples 2 and 3
AE = 60

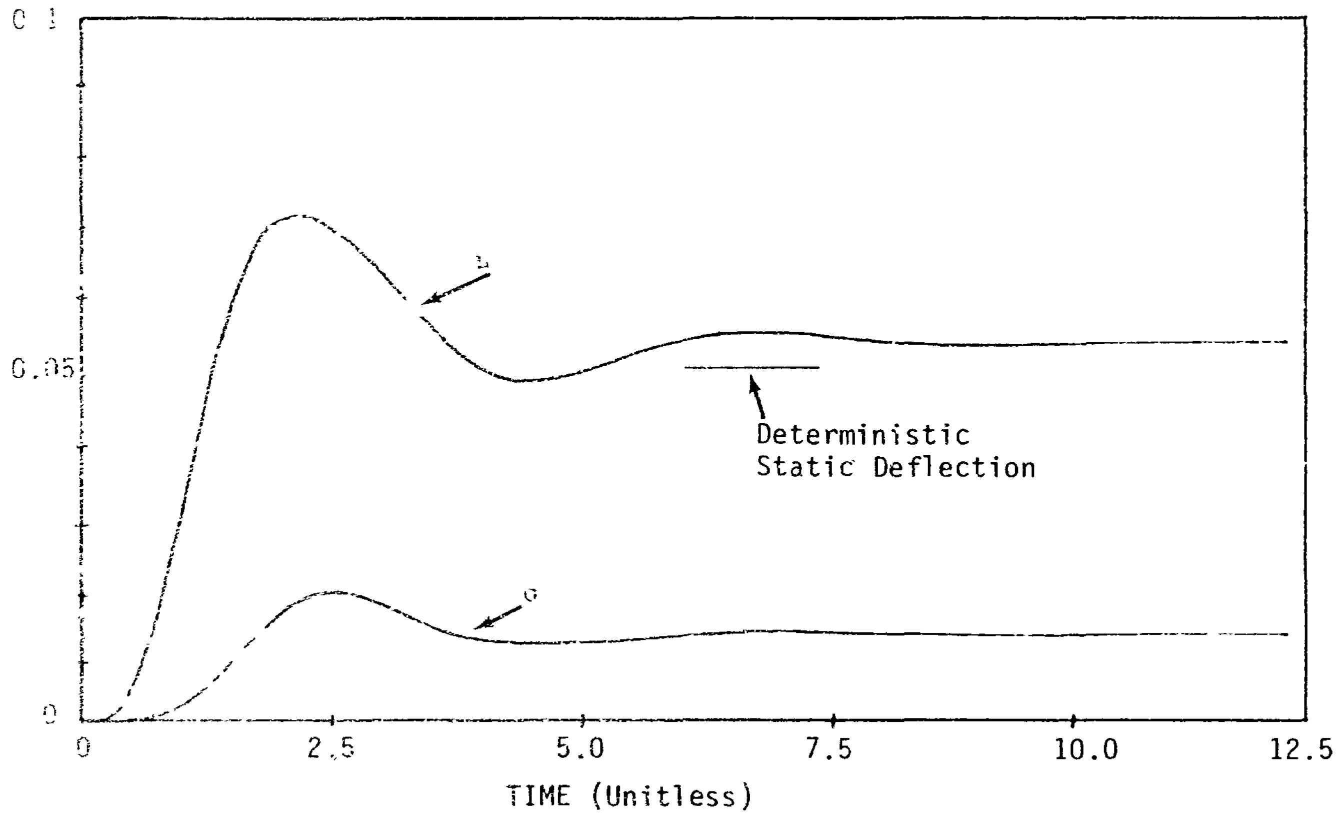

Figure 3-8 Tip Deflection for Bar with Loading Applied at Midpoint

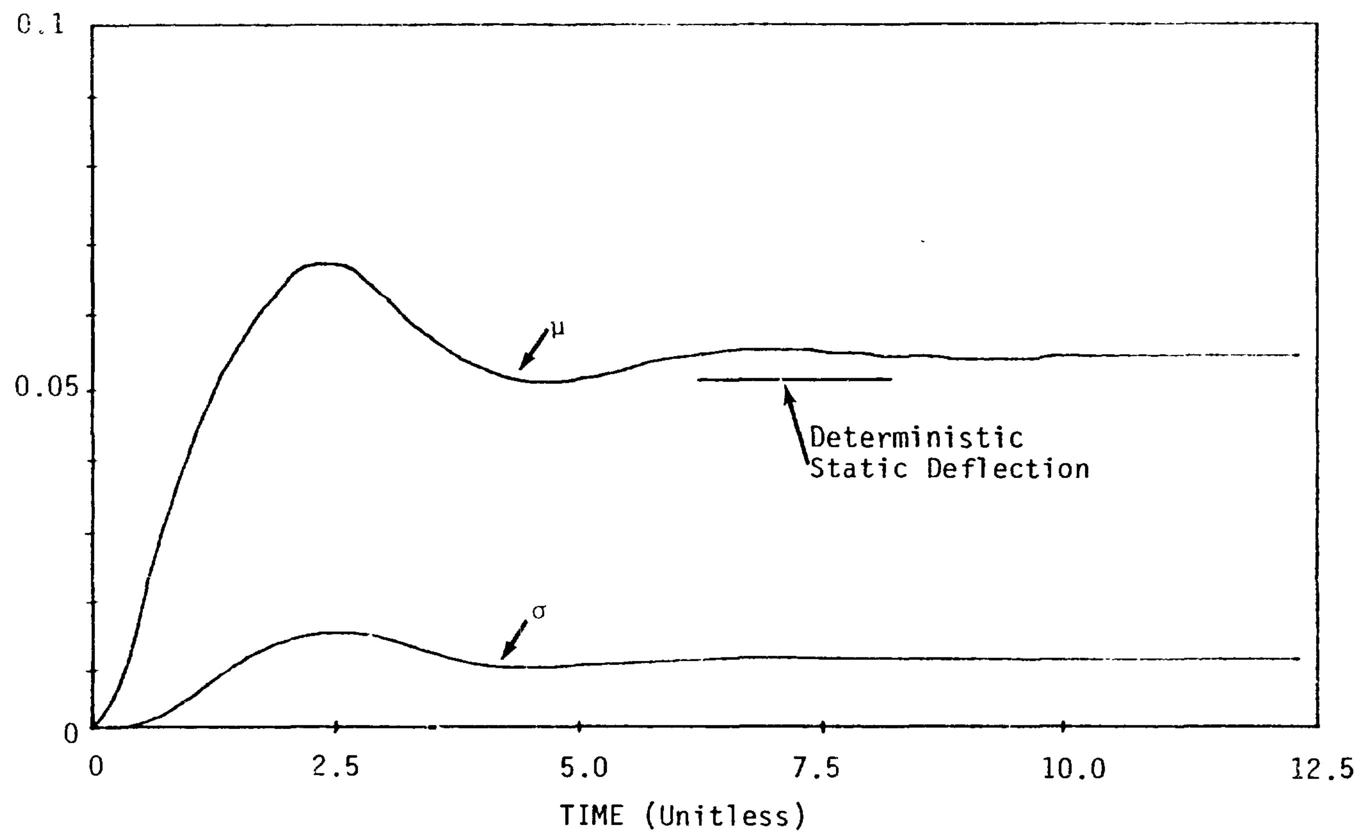

Figure 3-9 μ and σ for Midpoint Deflection of Bar with Loading Applied at Midpoint

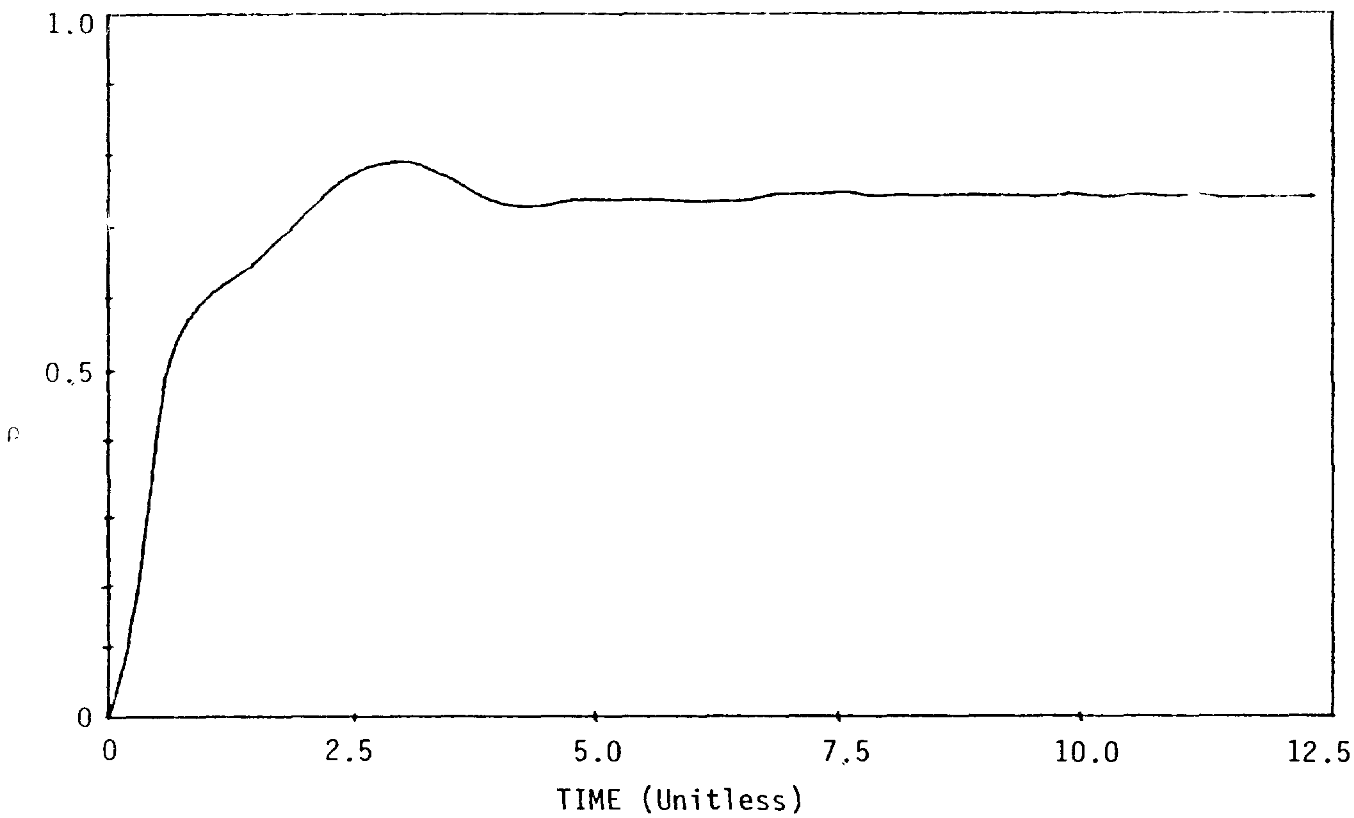

Figure 3-10 Correlation Between the Midpoint and Tip Deflections for a Bar With Loading Applied at the Midpoint

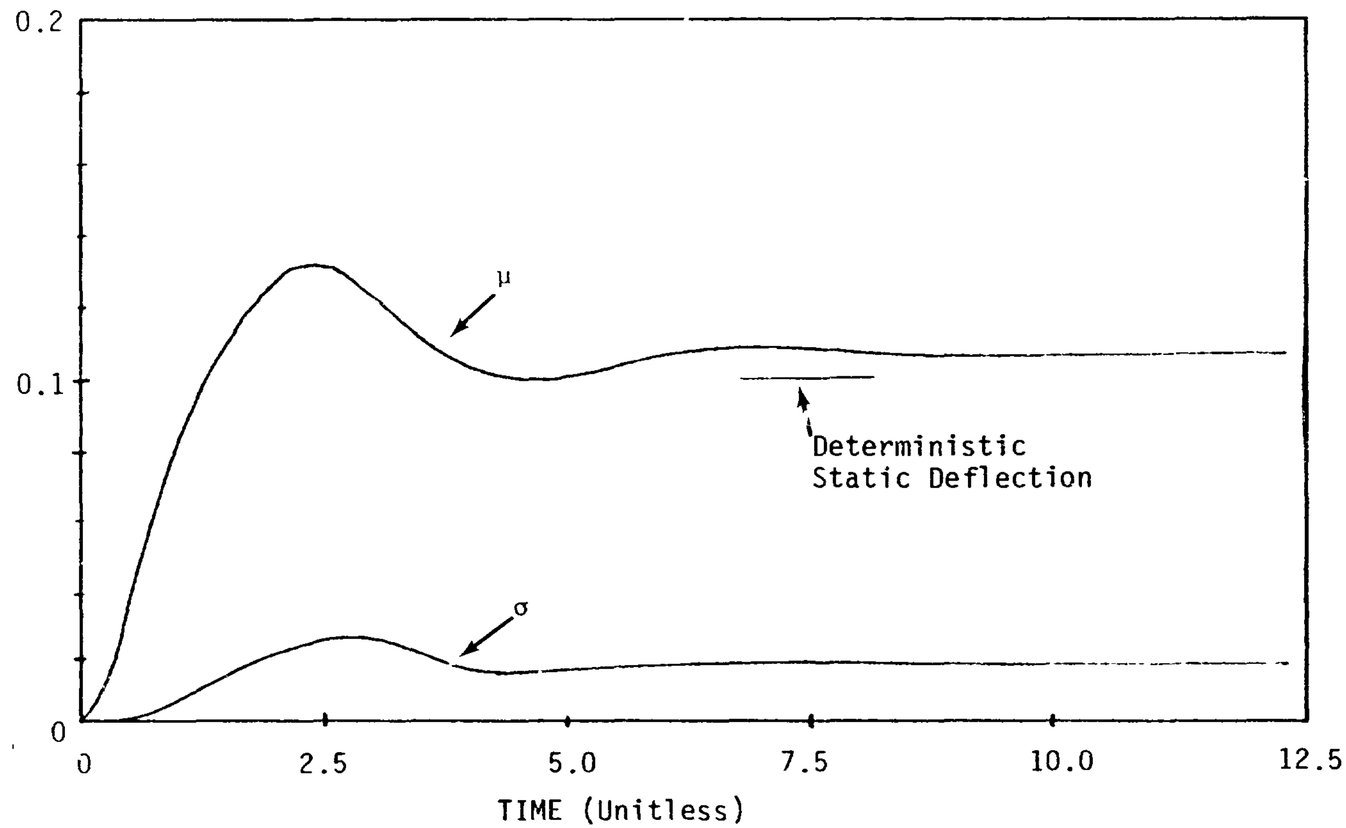

Figure 3-11 μ and σ of Tip Deflection of Bar With Loading Applied at the End

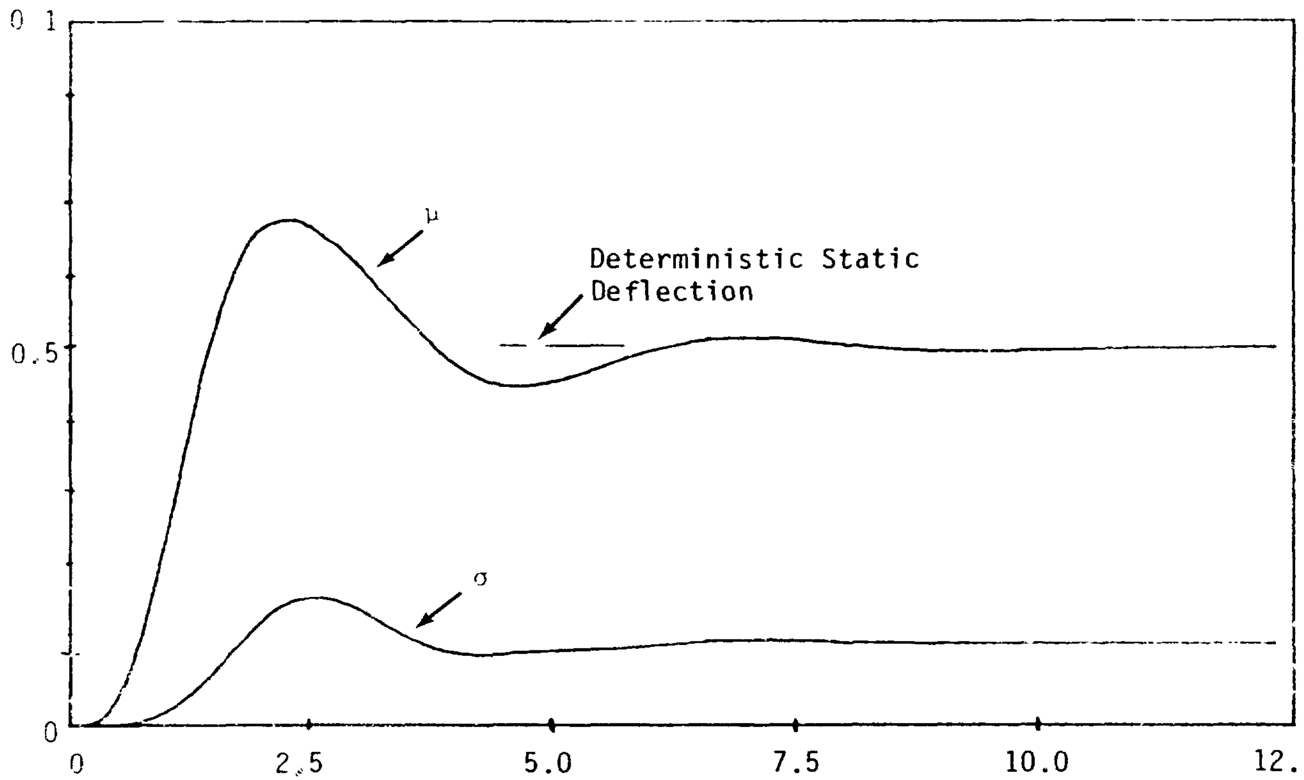

Figure 3-12 μ and σ for Midpoint Deflection of Bar With Loading Applied at the End

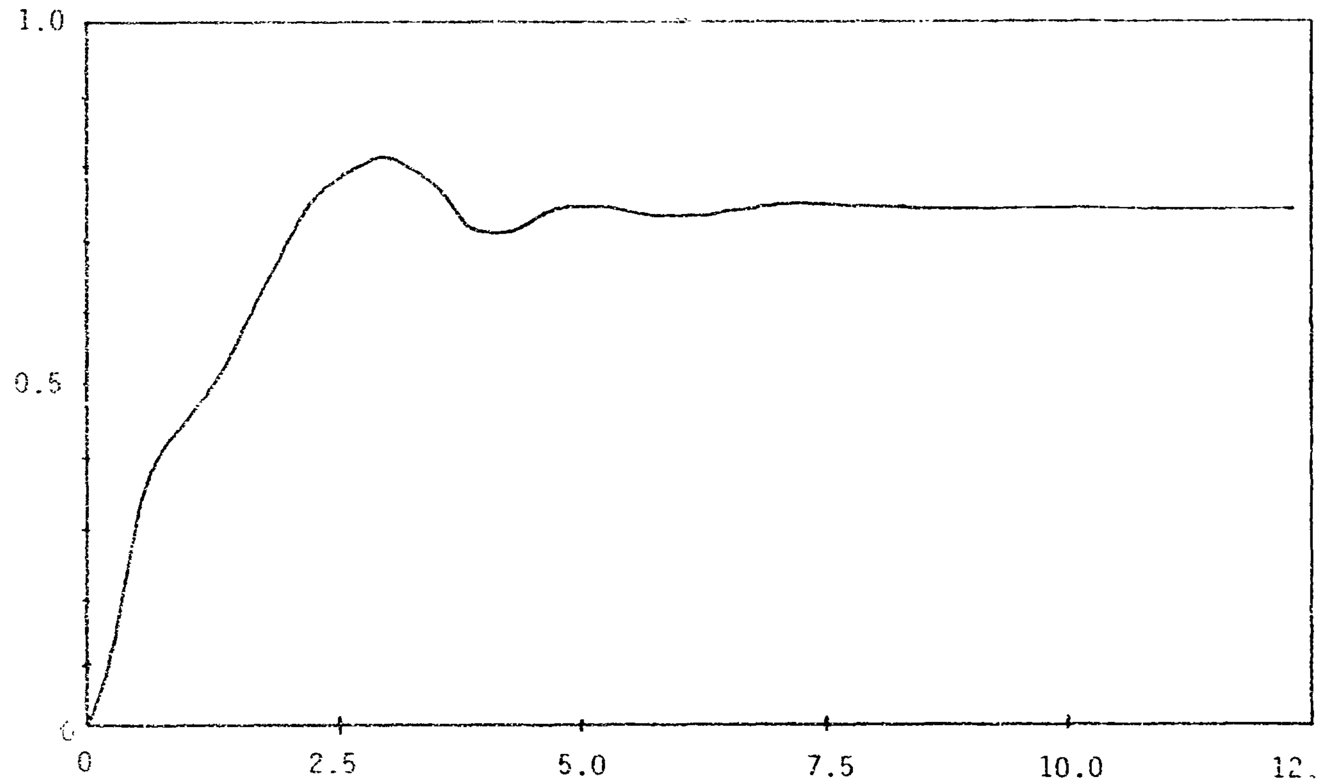

Figure 3-13 Correlation Between the Midpoint and End Deflections for a Bar With the Loading Applied at the End

loading are higher than those for the midpoint loading. This is due to larger deflections on the second half of the bar for the case of an end loading. The mean value of the midpoint deflection is also greater for the case of an end loading than the case of a midpoint loading, but by only a few percent. This is due once again to the effect of the randomness of the system parameters on the mean values.

CHAPTER 4

PASSAGE PROBABILITIES FOR NONSTATIONARY RANDOM PROCESSES

4.1 INTRODUCTION

This chapter presents a new method for estimating the probability of passage beyond a barrier level, b, by the nonstationary random process X(t).

When the barrier b represents a predetermined failure level then the probability of passing beyond this barrier is identical to the probability of failure.

The method of computing the passage probabilities for the nonstationary random process, X(t), presented in this chapter is a new extension of existing methods used for stationary processes. The extension involves the application of conditional probabilities and the assumption that the process X(t) appears stationary during short time intervals.

The calculation of passage probabilities for stationary processes is discussed in Section 4.2. The extension of this to nonstationary processes is presented in Section 4.3 with the computational algorithm for implementing the new method presented in Section 4.4. Numerical examples of the application of the technique are provided in Section 4.5.

4.2 PASSAGE PROBABILITIES FOR STATIONARY RANDOM PROCESSES

The methodology for approximating the passage probabilities for stationary random processes has been well developed by Rice (21) and others. The method developed by Rice is summarized below.

If one considers the random process $X(t)$ to be weakly stationary then its mean, $\mu_X(t)$, and variance, $\sigma_X^2(t)$, are constant values for all values of t. Furthermore the process defining its derivative with regard to t, $\dot{X}(t)$, is independent of $X(t)$.

The characteristic of interest of $X(t)$ is the collection of its random peaks which fall above a predetermined level, b, where $b>\mu_X$. Let $N(b,\tau)$ be a counting process associated with the crossing of the barrier level b. Thus every crossing of b by $X(t)$ is counted by $N(b,t)$. Specifically it is assumed that $N(b,t)$ is a Poisson counting process.

The events counted by a Poisson counting process, in this case the crossings of b by $X(t)$, are assumed to have the following properties:

a) The events occur independently, hence knowing an event has just occurred provides no information regarding the probability of when the next event will occur,

b) The probability of one event occurring in a time interval dt is $\lambda_b(t)dt$ where $dt \ll 1$, and $\lambda(t)_b$ is the mean crossing rate. When $X(t)$ is stationary $\lambda_b(t) = \lambda_b$ a constant.

c) The probability of more than one event occurring in a time interval dt is negligible.

Assumption (a) above is arbitrary for structural response since the realizations of the process $X(t)$ are not independent. When $X(t)$ is at or near a barrier level then $X(t+dt)$ will be as well for $dt \ll 1$. This effect is called "clumping" and is addressed by Vanmarke (4) and others. However, this effect was neglected by Rice and will be neglected in the following sections as well.

The Poisson process, $N(b,t)$, has the following probability function,

$$P_N(j,t) = \exp(-\lambda_b t) \frac{(\lambda_b t)^j}{j!} \tag{4-1}$$

where j is the number of events counted, that is the number of times $X(t)$ crosses the barrier level b.

The probability of not crossing the barrier in the interval (o,t) is provided by Equation (4-1) with $j=o$. This yields,

$$P_N(o,t) = \exp(-\lambda_b t) \tag{4-2}$$

Let $P_N(F,t)$ be the probability of crossing the barrier level at least once in the increment (o,t), then,

$$P_N(F,t) = 1 - \exp(-\lambda_b t) \tag{4-3}$$

All that now remains is to compute the mean crossing rate, λ_b. The mean crossing rate at any time (t), $\lambda_b(t)$, if given by,

$$\lambda_b(t) = \int_{-\infty}^{\infty} |\dot{x}| \, p_{X\dot{X}}(b,\dot{x},t)d\dot{x} \tag{4-4}$$

where $p_{X\dot{X}}(b,x,t)$ is the joint probability density function of $X(t)$ and $\dot{X}(t)$ evaluated at $X(t) = b$. Since the random processes $X(t)$ and $\dot{X}(t)$ have been assumed stationary, $\lambda_b(t) = \lambda_b$, and Equation (4-4) may be reduced to,

$$\lambda_b = \int_{-\infty}^{\infty} |\dot{x}| \, p_{X\dot{X}}(b,\dot{x})d\dot{x}. \tag{4-5}$$

When $X(t)$, $\dot{X}(t)$ are independent, mean zero, normal random processes their joint density is given by,

$$P_{s_{X\dot{X}}}(x,\dot{x}) = \frac{1}{2\pi\ \sigma_X \sigma_{\dot{X}}} \exp\left(\frac{-x^2}{2\sigma_X{}^2} - \frac{\dot{x}^2}{2\sigma_{\dot{X}}{}^2}\right) \tag{4-6}$$

Substituting this expression into Equation (4-5) yields,

$$\lambda_b = \frac{1}{\pi}\ \frac{\sigma_{\dot{X}}}{\sigma_X} \exp\left(\frac{-b^2}{2\sigma_X{}^2}\right) \tag{4-7}$$

Equation (4-7) now defines the mean crossing rate of a barrier b by the stationary mean zero random process $X(t)$. However, this rate reflects crossing the barrier from both above and below and since the engineer is only concerned with crossings from below the crossing rate must be modified. Fortunately this is straightforward since for a stationary random process all crossings from below must be paired with a crossing from above. Hence the mean crossing rate from below, λ_b^+ is merely half the total mean crossing rate,

$$\lambda_b^+ = \frac{1}{2\pi}\ \frac{\sigma_{\dot{X}}}{\sigma_X} \exp\left(\frac{-b^2}{2\sigma_X{}^2}\right) \tag{4-8}$$

The resulting probability of crossing the barrier level b from below within the increment (o,t) is then

$$P_N^+(F,t) = 1 - \exp\left(-\lambda_b^+ t\right) \tag{4-9}$$

These results may now be expanded to include the present case of interest, the nonstationary response of a structure with random stiffness and damping.

4.3 THE NONSTATIONARY CASE

A new method for computing the probability of having crossed a barrier, b, at least once in the interval (o,t) for a nonstationary process will now be presented. The barrier b is greater than the largest values of $\mu_X(t)$. When $\mu_X(t)$ is greater than b, the probability of passage is assumed to be 1.

The method is formulated under the following assumptions,

a) The nonstationary random process $X(t)$ and $\dot{X}(t)$ are continuous. Hence their values may not instantaneously jump at any time. This is completely appropriate for processes defining structural displacement and velocity response under most loadings of interest, but excludes processes such as Brownian motion and quite possibly buckling responses.

b) The means and variances of $X(t)$ and $\dot{X}(t)$ vary only slightly over increments of time, Δt, which are small in comparison to the response time of the structure.

c) $X(t)$ and $\dot{X}(t)$ are correlated normal random processes whose means, $\mu_X(t)$, $\mu_{\dot{X}}(t)$, standard deviations, $\sigma_X(t)$, $\sigma_{\dot{X}}(t)$ and correlation coefficient $r(t)$ are known.

Such a function obviously does not allow the direct use of Equation (4-9) to calculate the probability of passage beyond a barrier, b. Equation (4-4) will still provide the mean crossing rate at any time t. However, consider the behavior of $X(t)$ over a short period of time. For example the interval $(\tau,\tau+\Delta t)$. Then as Δt becomes small,

$$\mu_X(\tau) \approx \mu_X(\tau+\Delta t) \tag{4-10}$$

and

$$\sigma_X(\tau) \approx \sigma_X(\tau+\Delta t) \tag{4-11}$$

Considering the approximations of Equations (4-10) and (4-11) the mean and standard deviation of X(t) within the increment $(\tau,\tau+\Delta t)$ may be approximated by,

$$\mu_X(t) \simeq \mu_X(\tau) \qquad \tau \leq t \leq \tau+\Delta t \tag{4-12}$$

$$\sigma_X(t) \simeq \sigma_X(\tau) \qquad \tau \leq t \leq \tau+\Delta t \quad . \tag{4-13}$$

The values for $\mu_X(\tau+\Delta t)$ and $\sigma_X(\tau+\Delta t)$ could have been used for the right hand sides of Equations (4-12) and (4-13). However, this would only result in the final results being shifted to the left on the time axis by an amount Δt.

The similar expressions for the process $\dot{X}(t)$ are,

$$\mu_{\dot{X}}(t) \simeq \mu_{\dot{X}}(\tau) \qquad \tau \leq t \leq \tau+\Delta t \tag{4-14}$$

and

$$\sigma_{\dot{X}}(t) \simeq \sigma_{\dot{X}}(\tau) \qquad \tau \leq t \leq \tau+\Delta t \tag{4-15}$$

The approximations of Equations (4-12) through (4-15) will have the largest error when the response begins, that is as the structure moves away from the deterministic zero start condition. However, since the probability of exceeding a barrier in a time Δt when starting from a zero position is approximately zero, the errors occurring at early times will in effect be "laundered out" by the computations.

Consider the random processes X(t) and $\dot{X}(t)$, with the properties outlined above. The mean upcrossing rate of a barrier b at an instant of time, τ, is then

$$\lambda_b^+(\tau) = \int_{-\infty}^{\infty} \dot{x}\, p_{X\dot{X},\tau}(b,\dot{x})\,d\dot{x} \tag{4-16}$$

where $P_{X\dot{X},\tau}(b,\dot{x})dx$ is,

$$P_{X\dot{X},\tau}(b,x) = \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}\rho} \exp\left\{- \frac{1}{2\rho^2}\left[\left(\frac{b-\mu_X}{\sigma_X}\right)^2 - 2r\left(\frac{b-\mu_X}{\sigma_X}\right)\left(\frac{\dot{x}-\mu_{\dot{X}}}{\sigma_{\dot{X}}}\right) + \left(\frac{x-\mu_{\dot{X}}}{\sigma_{\dot{X}}}\right)^2\right]\right\} \quad (4\text{-}17a)$$

and

$$\mu_X = \mu_X(\tau) \quad (4\text{-}17b)$$

$$\mu_{\dot{X}} = \mu_{\dot{X}}(\tau) \quad (4\text{-}17c)$$

$$\sigma_X = \sigma_X(\tau) \quad (4\text{-}17d)$$

$$\sigma_{\dot{X}} = \sigma_{\dot{X}}(\tau) \quad (4\text{-}17e)$$

$$r = r(\tau) \quad (4\text{-}17f)$$

$$\rho = \sqrt{1-r^2} \quad (4\text{-}17g)$$

Letting,

$$b' = \frac{b-\mu_X}{\rho\sigma_X} \quad (4\text{-}18)$$

and,

$$Z = \frac{\dot{x}-\mu_{\dot{X}}}{\sigma_X} \quad (4\text{-}19)$$

Equation (4-16) may be written,

$$\lambda_b^+(\tau) = \frac{1}{2\pi\sigma_X} \int_{\frac{-\mu_{\dot{X}}}{\pi\sigma_{\dot{X}}}}^{\infty} (\rho\sigma_X Z+\mu_{\dot{X}}) \exp\left\{- \frac{1}{2}\left[Z^2-2rb'Z+b'^2\right]\right\}dZ \quad (4\text{-}20)$$

Completing the square of the exponential term in the integral yields,

$$\lambda_b^+(\tau) = \frac{\exp\left\{-\frac{1}{2}\left(\rho^2 b'^2\right)\right\}}{2\pi\sigma_X} \int_{\frac{-\mu_{\dot{X}}}{\rho\sigma_{\dot{X}}}}^{\infty} (\rho\sigma_{\dot{X}} Z + \mu_{\dot{X}}) \exp\left\{-\frac{1}{2}(Z-rb')^2\right\} dz \quad (4\text{-}21)$$

Now letting,

$$Z - rb' = S \quad (4\text{-}22)$$

and,

$$\frac{-\mu_{\dot{X}}}{\rho\sigma_{\dot{X}}} - rb' = S_L \quad (4\text{-}23)$$

where S_L is the new lower bound of integration yields,

$$\lambda_b^+(\tau) = \frac{\exp\left\{-\frac{1}{2}\left(\rho^2 b'^2\right)\right\}}{2\pi\sigma_X} \int_{S_L}^{\infty} \left(\rho\sigma_{\dot{X}}(rb+s)+\mu_X \exp\left\{-\frac{1}{2}(S)^2\right\} dS \quad (4\text{-}24)$$

Expanding this expression to two integrals,

$$\lambda_b^+(\tau) = \frac{\exp\left\{-\frac{1}{2}\rho^2 b'^2\right\}}{2\pi\sigma_X} \left[\left(\rho\sigma_{\dot{X}}\rho b + \mu_{\dot{X}}\right) \int_{S_L}^{\infty} \exp\left\{-\frac{1}{2}S^2\right\} dS\right.$$

$$+ \rho\sigma_{\dot{X}} \int_{S_L}^{\infty} S \exp\left\{-\frac{1}{2}S^2\right\} \quad (4\text{-}25)$$

Upon solution this yields,

$$\lambda_b^+(\tau) = \frac{\exp\left\{-\frac{1}{2}\rho^2 b'^2\right\}}{2\pi\sigma_X} \left[\rho\sigma_{\dot{X}} \exp\left\{-\frac{1}{2}S_L{}^2\right\} - S_L\rho\sigma_{\dot{X}}\right.$$

$$\left. - S_L\rho\sigma_{\dot{X}} \int_{S_L}^{\infty} \exp\left\{-\frac{1}{2}S^2\right\} dS\right] \quad (4\text{-}26)$$

The integral on the right hand side of Equation (4-24) may be evaluated in terms of the error function, erf(x) where,

$$\operatorname{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_{o}^{x} \exp\left(-\frac{1}{2}y^2\right)dy \qquad (4\text{-}27)$$

and,

$$\operatorname{erf}(\infty) = \frac{1}{2} \qquad (4\text{-}28)$$

Hence, Equation (4-24) becomes,

$$\lambda_b(\tau) = \frac{\rho\sigma_{\dot{X}} \exp\left\{-\frac{1}{2}\ \rho^2 b'^2\right\}}{2\pi\sigma_X} \left[\exp\left\{-\frac{1}{2}S_L{}^2\right\} - S_L\sqrt{2\pi}\ G\ \right] \qquad (4\text{-}29a)$$

where,

$$G = \frac{1}{2} - \operatorname{erf}(S_L) \qquad o < S_L \qquad (4\text{-}29b)$$

or,

$$G = \frac{1}{2} + \operatorname{erf}(S_L) \qquad S_L < o \qquad (4.29c)$$

Now replacing b' with its definition, Equation (4-18), Equation (4-29a) becomes

$$\lambda_b(\tau) = \frac{\rho\sigma_{\dot{X}}\exp}{2\pi\sigma_X} \left\{-\frac{1}{2}\left(\frac{x-b}{\sigma_X}\right)^2\right\} \left[\exp\left\{-\frac{1}{2}S_L{}^2\right\} - S_L\sqrt{2\pi}\ G\ \right] \qquad (4\text{-}30)$$

where S_L is defined by Equation (4-23) and G is defined by Equations (4-29b) and (4-29c). The value of erf(x) is available as a computer generated function.

Then the probability of crossing the barrier b at least once during the interval $(\tau,\tau+\Delta t)$, $P(b,\tau,\Delta t)$, is

$$P(b,\tau,\Delta t) = 1 - \lambda_b^+(\tau)\Delta t \quad . \tag{4-31}$$

Having the probability of at least one crossing occurring in the time (o,t) may now be developed. Let $t_n=n\Delta t$, where n is a positive integer, and Δt is a small positive time. Let the event α be the crossing of the barrier at lest once in the interval (o,t_n). Let the event β be the crossing of the barrier at least once in the interval (o,t_{n-1}).

The probability of α, $P(\alpha)$ may be written as a the sum of probabilities conditioned on β. This results in,

$$P(\alpha) = P(\alpha/\beta)P(\beta) + P(\alpha/\beta^c)P(\beta^c) \tag{4-32}$$

where $P(\alpha/\beta)$ is the probability that α occurs given that β has occurred and $P(\alpha/\beta^c)$ is the probability that α occurs given β did not occur.

Since $P(\alpha/\beta)$ is the probability of at least one crossing in the interval (o,t_n) given at least one crossing in (o,t_{n-1}),

$$P(\alpha/\beta) = 1 \tag{4-33}$$

And Equation (4-32) becomes,

$$P(\alpha) = P(\beta) + P(\alpha/\beta^c)P(\beta^c) \tag{4-34}$$

However since the only way for a passage to occur in the interval (o,t_n) given that none occurred in the interval (o,t_{n-1}) is for the passage to occur in the interval (t_{n-1},t_n). This is given by Equation (4-31) evaluated for $\tau=t_{n-1}$. Hence

$$P(\alpha) = P(\beta) + \left(1-\lambda_b^+(t_{n-1})\Delta t\right)P(\beta^c) \tag{4-35}$$

Letting $P_{Xb}(j)$ be the probability of X(t) achieving at least one crossing of the barrier b by the end of the $j^{\underline{th}}$ time step yields,

$$P(\alpha) = P_{Xb}(n) \tag{4-36a}$$

$$P(\beta) = P_{Xb}(n-1) \tag{4-36b}$$

$$P(\beta^c) = 1-P(\beta) = 1-P_{Xb}(n-1) \tag{4-36c}$$

Substituting the above into Equation (4-35) yields the recurrence relation,

$$P_{Xb}(n) = P_{Xb}(n-1) + \left(1+\lambda_b^+(t_{n-1})\Delta t\right)\left(1-P_{Xb}(n-1)\right) \tag{4-37}$$

This expression may now be evaluated in a step-by-step manner to yield the probability of X(t) crossing a barrier b by the time t_n.

The computational algorithm for implementing Equation (4-37) is provided in Section 4.4 and some numerical examples are provided and discussed in Section 4.4.

4.4 THE COMPUTATIONAL ALGORITHM

The following algorithm was developed to evaluate Equation (4-37) to compute the probability of passage beyond a barrier b. The algorithm is implemented in the code RSDOF. A listing of RSDOF is included in Appendix A.

1) Let $n=o$, set barrier level of interest

2) Let $n=n+1$, $t_n=n\Delta t$

3) Get the values for $\sigma_X(t_n)$, $\sigma_{\dot{X}}(t_n)$, $\mu_X(t_n)$, $\mu_{\dot{X}}(t_n)$. $r(t_n)$ as computed in Chapter 2.

4) Evaluate B of Equation (4-20c) to determine which expression for λ_b^+ to use

5) If $B < o$ use $\lambda_b^+(t_n)$ as defined in Equation (4-30)

6) If $B > o$ use $\lambda_b^+(t_n)$ as defined in Equation (4-29)

7) Compute the probability of crossing a barrier b during this time step by Equation (4-31).

8) Compute the probability of crossing the barrier in the interval (o,t_n) from Equation (4-37)

9) Go to Step 2 until problem complete.

The results obtained by implementing the above algorithm are discussed in the followng sections.

4.5 NUMERICAL EXAMPLES

This section presents the results of five calculations using the method developed in the previous sections of this chapter.

The first example consisted of three calculations. The probability of having achieved at least one crossing of a barrier was computed for Cases 7, 8, and 9 of Example 2 in Chapter 2. These cases had the parameters shown in Table 2-2. The barrier level was 0.36, which is 90% of the peak response for the undamped system. The only variation between cases was the correlation between the random stiffness and damping.

The resulting passage probabilities are plotted in Figure 4-1. The probabilities rapidly reach a value that is close to their final value as the response reaches its first peak. The probabilities increase slightly with each additional peak but are assymt-tically approaching a final value.

The probability of passage for the case of a negative correlation between the stiffness and damping is the largest of the three cases followed by the case of independent stiffness and damping, with positively correlated stiffness and damping yielding the smallest probability of passage.

The second example is the probability of the system defined in Example 3 of Chapter 2 exceeding the peak level obtained by the deterministic calculation.

The resulting passage probability is plotted in Figure 4-2. These values are for the system with 8% damping and a coefficient of variation for the stiffness of 0.15, the values used in Chapter 2.

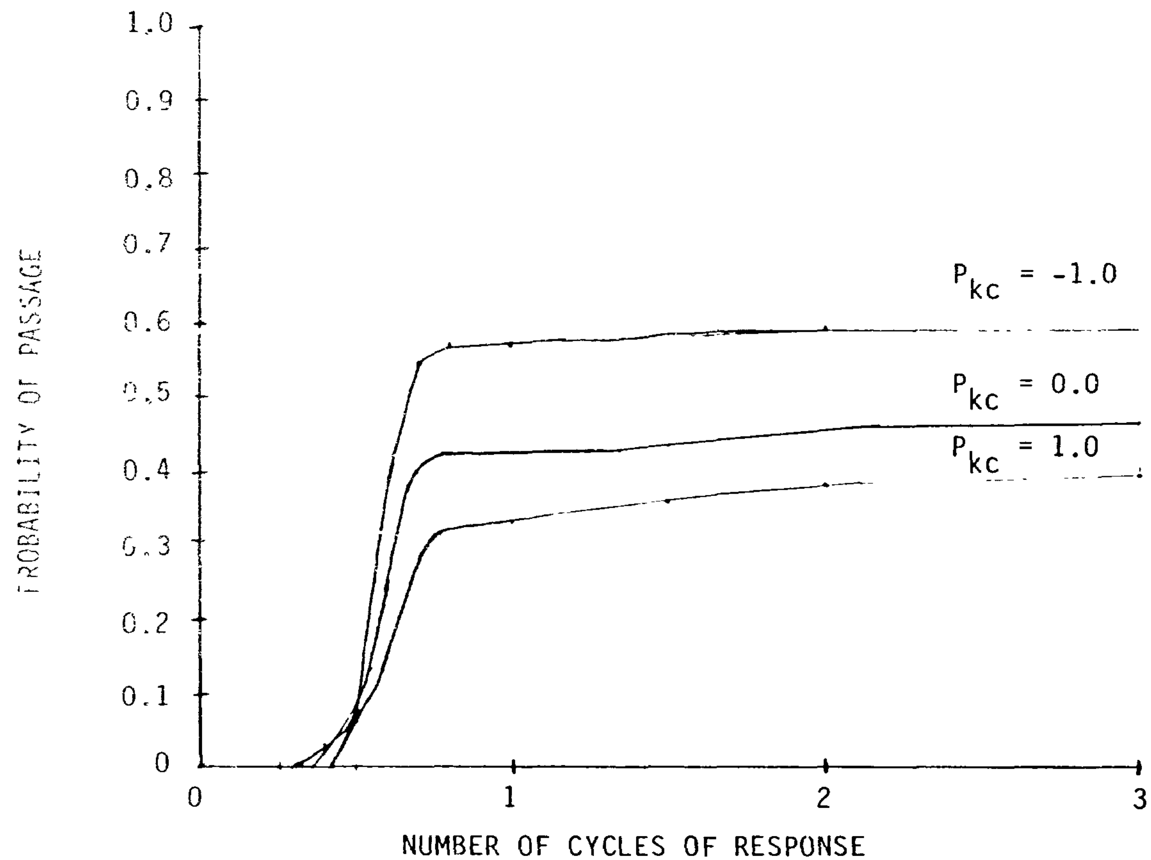

Figure 4-1 Probability of Passage for 3 Systems With Rnadom Stiffness and Damping

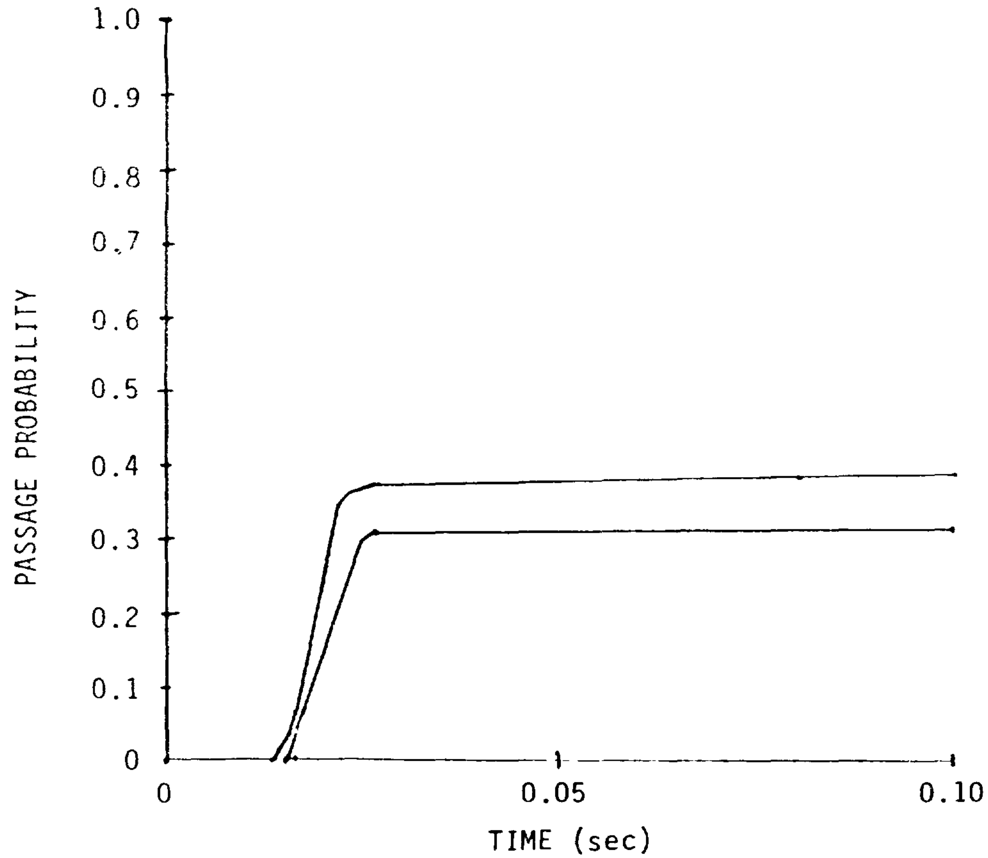

Figure 4-2 Passage Probabilities for Blast Excited Beam

Hence there is almost a 40% chance of this structure failing when failure is defined as exceeding the peak determinist response. The second calculation computed the probability of exceeding the value of 0.2884", which is the deflection resulting from twice the peak load when it is applied as a static load,

$$0.2884'' = \frac{2 \times 300{,}000}{2.08 \times 10^6} \tag{4-38}$$

The results of this computation are plotted in Figure 4-3. The shape of the curve is similar to that of Figure 4-2; however, the resulting probability of passage is now only about .32. Hence an increase in the barrier level of less than 6% decreased the passage probability by 20%.

CHAPTER 5

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.1 SUMMARY

A technique to explicitly account for the randomness of structural stiffness and damping in the computation of the structural response was developed. The technique is suitable for use in cases of both single and multiple degrees of freedom.

The technique was applied to several problems in Chapters 2 and 3 to demonstrate the influence of these underlying uncertainties on the response of typical structures.

The results of some of these computations were used to compute the probability of the response exceeding a given value using a new method developed in Chapter 4.

The methods of Chapters 2 and 3 are limited to linear elastic structures, however they may be readily extended to cases of nonlinearity or plastic response.

The method for computing the passage probabilities developed in Chapter 4 may be applied to any continuous nonstationary random process which satisfies the assumptions made in Chapter 4.

5.2 CONCLUSIONS

The randomness of a systems stiffness and damping will effect the mean and standard deviation of that systems response. The uncertainties will have only minimal influence when the system response is mean zero. This is to be expected since the systems stiffness and damping only enter the problem when the system is displaced.

The relative influence of the system stiffness appears constant for all non-zero deflections. The mean at any time may be approximated by,

$$\mu_X = X_D \cdot \left(1 + Cov_K^2\right) \qquad (5\text{-}1)$$

where μ_X is the mean of X, X_D is the deterministic value and cov_K is the coefficient of variation for the stiffness.

The relative influence of the damping depends on the level of damping, and has no effect on the static deflections. For lightly damped systems, $\varepsilon \leq 0.15$, the influence of the randomness of the damping may probably be ignored for most problems.

The results of Chapter 4 indicate that the randomness in the stiffness must be accounted for when a high degree of survivability is required. A blast excited slab had a significant probability of exceeding the theoretical maximum value obtained for the deterministic system.

5.3 RECOMMENDATIONS

The problem of MDOF system response needs more work, specifically an efficient method of accounting for the correlation of the random variables within the stiffness and damping matrices. This may be easier in a plane strain or plane stress computation where the system is a continuous medium.

A method for accounting for sytem uncertainties in cases where the response is not mean zero should also be developed. This includes such important cases as the response to earthquake, and turbulence.

APPENDIX A

PROGRAM RSDOF

```
C*****************************************************
C*                                                   *
C*   PROGRAM RSDOF                                   *
C*   THIS PROGRAM COMPUTES THE FIRST AND SECOND      *
C*   MOMENTS OF THE RESPONSE OF A LINEAR ELASTIC     *
C*   SDOF SYSTEM WITH RANDOM STIFFNESS AND DAMPING   *
C*                                                   *
C*   THE LOADING MAY BE DETERMINISTIC OR RANDOM      *
C*                                                   *
C*****************************************************
      DOUBLE PRECISION KM,M,PSI,KV,TD,RHO
      DOUBLE PRECISION UN,UN1,UN2,CVN2,EDN,EDN1,EDN2
      DOUBLE PRECISION ZN,EPDN1,C,W1,W2,TP,DELT,ALPHA
      DOUBLE PRECISION MB,MB2,ALPB,T,CK,A1,A2,A12,A22
      DOUBLE PRECISION EKDN,EKDN1,EKDN2,DUM,QN,EDNN1,U21,V
      DOUBLE PRECISION P,VP,VPN
      DOUBLE PRECISION CV,VK,VC,KRCR,ECDN,ECDN1,ECDN2,VN,VM1
      DOUBLE PRECISION W3,W4,W5,W6,W7,EXDNF,W8,T1,ERR
      DOUBLE PRECISION SIGV,SIG,MUA,PFP,PFPT,B,BAR,EX
       DOUBLE PRECISION R,R2,BP,BZ,TEST,SIGV2
      DOUBLE PRECISION ZZ
      DOUBLE PRECISION SL,SL2,T2,B2
C
C  READ AND WRITE PROBLEM PARAMETERS
   10 CONTINUE
       READ(5,*) KM,M,PSI,KV,CV,RHO,TD,N,IN
      READ(5,*) BAR
      WRITE(6,110) KM,M,PSI,KV,CV,RHO,N
      WRITE(6,115) BAR
  110 FORMAT(1H ,'K=',F15.4,/,1H ,'M=',F10.4,/,1H ,'PSI=',F5.3,/,
     11H ,'KV=',F5.3,/,1H ,'CV=',F5.3,/,'RHO=',F5.3,/,1H ,
     2'N=',I4)
  115 FORMAT(1X,'BARRIER LEVEL IS=',F10.6)
```

```
C
C INITIALIZE VARIABLES
      UN=0.0D0
      UN1=0.0D0
      UN2=0.0D0
      VN=0.D0
      EDN=0.D0
      EDN1=0.D0
      EDN2=0.D0
      ECDN=0.D0
      ECDN1=0.D0
      ECDN2=0.D0
      EKDN=0.D0
      EKDN1=0.D0
      EKDN2=0.D0
      P=0.D0
      VP=0.D0
      EDNN1=0.D0
      PFP=0.D0
      PFPT=0.D0
       MUA=0.D0
C
C  PERFORM ONE TIME CALCULATIONS
      C=KM*M
      C=DSQRT(C)
      C=2.D0*PSI*C
      VC=CV*CV*C*C
      VK=KM*KM*KV*KV
      KRCR=RHO*KM*C*CV*KV
      W1=KM/M
      W1=DSQRT(W1)
      W2=1.0D0-PSI*PSI
      W2=DSQRT(W2)
      W2=W2*W1
```

```
      TP=6.28318D0/W2
       TD=TD*TP
      DELT=TP/N
      NT=TD/DELT+1
      PT=TP/10.
       N1=PT/DELT+1
      N2=N1
C
C  SCALE FOR TIME EFFECTS
C
      MB=M/(DELT*DELT)
      T=0.D0
      C=C/DELT
      A1=2.D0*MB-C
      A2=C-MB-KM
      A12=A1*A1
      A22=A2*A2
      MB2=MB*MB
      MB2=1.0D0/MB2
C
C  START CALCULATION AND WRITE TITLES
C
      WRITE(6,120) DELT
  120 FORMAT(1H ,'TIMESTEP= ',F10.8,/,
     11X,'TIME',11X,'DISPL',12X,'VEL',10X,'SIG',10X,'PFP')
      DO 20 I=1,NT
        UM1=UN
        UN=UN1
        UN1=UN2
        UN2=0.D0
        VM1=VN
        VN=(UN1-UN)/DELT
        EDN=EDN1
        EDN1=EDN2
        EDN2=0.D0
```

```
      EKDN=EKDN1
      EKDN1=EKDN2
      EKDN2=0.DO
      ECDN=ECDN1
      ECDN1=ECDN2
      ECDN2=0.DO
      EDNN1=(-UM1*EKDN-VM1*ECDN+A1*EDN+A2*EDNN1)/MB
      CALL LOAD(P,T,VP,DELT)
      UN2=(P-((ECDN1-ECDN)/DELT)-EKDN+A1*UN1+A2*UN)/MB
      EDN2=MB2*(VP+VK*UN*UN+VC*VN*VN+2.DO*UN*VN*KRCR
     1          -2.DO*UN*A1*EKDN1-2.DO*UN*A2*EKDN-2.DO*VN*A1*ECDN1
     2          -2.DO*VN*A2*ECDN+A12*EDN1+2.DO*A1*A2*EDNN1+A22*EDN)
      EKDN2=(-VK*UN-KRCR*VN+A1*EKDN1+A2*EKDN)/MB
C*****************************************************
C*                                                   *
C*  THIS BLOCK OF CODE COMPUTES THE PASSAGE          *
C*  PROBABILITY OF A BARRIER LEVEL SPECIFIED BY THE *
C*  USER.                                            *
C*****************************************************
     IF(EDN.LE.0.0) GOTO 11
     SIGV2=(EDN+EDN1-2*EDNN1)/(DELT*DELT)
     IF(SIGV2.LT.0.0) SIGV2=0.0
     SIGV=DSQRT(SIGV2)
     SIG=DSQRT(EDN)
     IF (SIGV .EQ. 0.0 .OR. SIG .EQ. 0.0) GOTO 11
     R=(EDNN1-EDN)/(DELT*SIG*SIGV)
     IF ( R.GT. 1.) R=1.
     IF (R.LT.-1.) R=-1.
     IF(R.EQ. 1.0) R=0.999
     IF(R.EQ. -1.0) R=-0.999
      R2=1-R*R
      R2=DSQRT(R2)
     B=(BAR-UN)/(SIG*R2)
     IF(B.LT. 0.0) PFP=1.0
     SL=VN/(R2*SIGV)-R*B
```

```
      B2=B*B*R2*R2/2.DO
      IF(B2.GT.112.5) GOTO 11
      B2=DEXP(B2)
      T1=R2*SIGV
      SL2=SL*SL/2.DO
      T2=-T1*SL*2.50628
      B2=1.DO/(6.283185DO*SIG*B2)
      IF(SL.GT.0.0) GOTO 13
      IF(SL2.GT. 115) GOTO 15
      SL=-SL
       MUA=B2*(T1/DEXP(SL2)+T2*(.5+DERF(SL)))
      GOTO 12
   15 CONTINUE
      MUA=B2*T2
      GOTO 12
   13 IF(SL2.GT.112.5) GOTO 11
      MUA=B2*(T1/DEXP(SL2)+T2*(.5-DERF(SL)))
      GOTO 12
   11 MUA=0.DO
   12 MUA=MUA*DELT
      PFPT=1.DO-DEXP(-MUA)
      PFP=PFP+(1.DO-PFP)*PFPT
      IF(N2.NE.N1) GOTO 40
        WRITE(6,150) T,UN,VN,SIG,PFP
         N2=0
   40   N2=N2+1
        T=T+DELT
   20 CONTINUE
  150 FORMAT(F10.6,4X,F10.6,4X,F10.6,4X,F10.6,4X,F10.8)
      IF(IN.NE.0) GOTO 10
       STOP
       END
```

```
SUBROUTINE LOAD(P,T,VP,DELT)
DOUBLE PRECISION P,T,VP,DELT,T1
P=100
 VP=0.0
RETURN
END
```

APPENDIX B

PROGRAM RMDOF

```
C*****************************************************
C*                                                   *
C*   PROGRAM RHDOF                                   *
C*                                                   *
C*   THIS PROGRAM USES A PLANE FRAME FINITE ELEMENT  *
C*   CODE TO EVALUATE THE FIRST AND SECOND MOMENTS OF *
C*   THE RESPONSE OF A MULTI-DEGREE-OF-FREEDOM SYSTEM *
C*                                                   *
C*   THE LOADING MAY BE DETERMINISTIC OR RANDOM      *
C*****************************************************
C
C  INITIALIZE AND MAKE DECLARATIONS ETC
C
      DOUBLE PRECISION RNODE(7,2),K(21,21),VK(21,21),M(21),MI(21)
      DOUBLE PRECISION A1(21,21),A2(21,21),UM1(21),UN(21),UN1(21)
      DOUBLE PRECISION UN2(21),VM1(21),VN(21),EKDN(21),EKDN1(21)
      DOUBLE PRECISION EDN(21,21),EDN1(21,21),EDN2(21,21)
      DOUBLE PRECISION EKDN2(21),ZN(21),ZM1(21)
      DOUBLE PRECISION W1(21,21),W2(21,21)
      DOUBLE PRECISION W3(21,21),W4(21,21),W5(21,21),P(21)
      DOUBLE PRECISION EQN(21,21)
      DOUBLE PRECISION EDNN1(21,21),DIAG(21),AE,COVA,EI,COVE,RHO,Z1
      DOUBLE PRECISION OMEG1,Z2,OMEG2,BETA,ALPHA,DELT,S1,S2,T1,T2
      DOUBLE PRECISION SM
      COMMON/RND/RNODE
      COMMON/GLB/K,M,VK,N
      DO 1 I=1,21
        UN1(I)=0.D0
        UN2(I)=0.D0
        VN(I)=0.D0
        ZN(I)=0.D0
        EKDN1(I)=0.D0
        EKDN2(I)=0.D0
        P(I)=0.D0
```

```
      DO 1 J=1,21
        A1(I,J)=0.D0
        A2(I,J)=0.D0
        K(21,21)=0.D0
        EDNN1(I,J)=0.0D0
        EDN(I,J)=0.D0
        EDN1(I,J)=0.D0
        EQN(I,J)=0.D0
        W1(I,J)=0.D0
        W2(I,J)=0.D0
        W3(I,J)=0.D0
        W4(I,J)=0.D0
        W5(I,J)=0.D0
        VK(I,J)=0.D0
  1   CONTINUE
C
C
C
C
C  READ NUMBER OF NODES NN,NUMBER OF ELEMENTS NEL,NUMBER OF
C  RESTRAINTS NRST.
C
     READ(4,120)  NN,NEL,NRST
     N=3*NN
 120 FORMAT(4I4)
C
C  READ AND STORE NODAL INFORMATION
C
     WRITE(7,140) NN
 140 FORMAT(1H ,'TOTAL NUMBER OF NODES=',I2,/,1H ,'NODE',
    1 5X,'X',12X,'Y')
     DO 10 I=1,NN
       READ(4,130)(RNODE(I,J),J=1,2)
       WRITE(7,150) I,(RNODE(I,J),J=1,2)
  10 CONTINUE
 150 FORMAT(I5,5X,F10.3,5X,F10.3)
```

```
  130 FORMAT(2F10.3)
C
C
C  READ IN ELEMENT DATA AND COMPUTE STIFFNESS AND MASS MATRICES
C
C
      DO 20 LELM=1,NEL
        READ(4,180) N1,N2,AE,COVA,EI,COVE,RHO
        CALL ELEMEN(N1,N2,AE,COVA,EI,COVE,RHO,LELM)
   20 CONTINUE
  180 FORMAT(I2,I2,F10.3,F5.3,F10.3,F5.3,F10.3)
C
C  READ DAMPING PARAMETERS
C
      READ(4,190) OMEG1,Z1,OMEG2,Z2
  190 FORMAT(4F10.3)
      ALPHA=2.D0*(OMEG2*Z2-OMEG1*Z1)/(OMEG2**2.D0-OMEG1**2.D0)
      BETA=2.D0*OMEG1*Z1-ALPHA*OMEG1**2.D0
      WRITE(7,191) ALPHA,BETA
  191 FORMAT(1X,'ALPHA=',E10.4,5X,'BETA=',E10.4)
C
C  READ TIMESTEP AND PROBLEM DURATION
C
      READ(4,200) DELT,TD
  200 FORMAT(2F10.4)
      NT=TD/DELT
      S1=1.D0-ALPHA/DELT
      S2=ALPHA/DELT
      WRITE(7,192) S1,S2
  192 FORMAT(1X,'S1=',E10.4,5X,'S2=',E10.4)
C
      DO 55 I=1,N
        M(I)=M(I)/(DELT**2.D0)
        MI(I)=1.D0/M(I)
   55 CONTINUE
```

```
C
C  COMPUTE THE MATRICES USED IN THE FINITE DIFFERENCE SCHEME
C
      DO 70 I=1,N
        DO 80 J=1,N
          A1(I,J)=-S2*K(I,J)
          A2(I,J)=-S1*K(I,J)
  80    CONTINUE
        A1(I,I)=A1(I,I)+(2.DO-DELT*BETA)*M(I)
        A2(I,I)=A2(I,I)-(1.DO-BETA*DELT)*M(I)
  70 CONTINUE
C  IMPOSE CCNSTRAINT CONDITIONS
C
      DO 50 IL=1,NRST
        READ(4,210) NODE,IDOF
 210    FORMAT(2I4)
        IDOF=3*NODE-3+IDOF
        DO 60 J=1,N
         A1(J,IDOF)=0.DO
         A1(IDOF,J)=0.DO
          VK(J,IDOF)=0.DO
          VK(IDOF,J)=0.DO
          A2(J,IDOF)=0.DO
          A2(IDOF,J)=0.DO
  60    CONTINUE
        MI(IDOF)=0.DO
  50 CONTINUE
C
C  START TIME STEP EVALUATION
C
      NP=20
      TIME=0.DO
```

```
      DO 90 ITL=1,NT
C
C  UPDATE ARRAYS
C
      CALL LOAD(TIME,P,EQN,N)
        DO 201 I=1,N
          UM1(I)=UN(I)
          UN(I)=UN1(I)
          UN1(I)=UN2(I)
          UN2(I)=0.D0
          VM1(I)=VN(I)
          ZM1(I)=ZN(I)
          VN(I)=(UN1(I)-UN(I))/DELT
          ZN(I)=UN(I)+ALPHA*VN(I)
          EKDN(I)=EKDN1(I)
          EKDN1(I)=EKDN2(I)
          DO 250 J=1,N
            EDN(I,J)=EDN1(I,J)
            EDN1(I,J)=EDN2(I,J)
          W1(I,J)=0.D0
          W2(I,J)=0.D0
          W3(I,J)=0.D0
          W4(I,J)=0.D0
          W5(I,J)=0.D0
 250      CONTINUE
 201    CONTINUE
        DO 260 I=1,N
            EKDN2(I)=MI(I)*(-VK(I,I)*ZN(I)+A1(I,I)*EKDN1(I)
     1                   +A2(I,I)*EKDN(I))
 260    CONTINUE
        CALL MAMA(A1,EDN,W1,N)
        CALL MAMA(EDNN1,A2,W2,N)
        DO 280 I=1,N
          DO 279 J=1,N
            EDNN1(I,J)=MI(I)*(W1(I,J)+W2(I,J))
```

```
 279  CONTINUE
            EDNN1(I,I)=MI(I)*(-EKDN(I)*ZM1(I))+EDNN1(I,I)
 280   CONTINUE
C
C
C  COMPUTE MEAN RESPONSE UN2
C
       DO 290 I=1,N
         T1=0.DO
         T2=0.DO
         DO 300 J=1,N
           T1=T1+A1(I,J)*UN1(J)
           T2=T2+A2(I,J)*UN(J)
 300     CONTINUE
         UN2(I)=MI(I)*(P(I)-S1*EKDN(I)-S2*EKDN1(I)+T1+T2)
 290   CONTINUE
C
C  COMPUTE DIAG VECTOR
C
       DO 310 I=1,N
         DIAG(I)=0.DO
         DO 310 J=1,N
           DIAG(I)=DIAG(I)+VK(I,J)*(ZN(J)*ZN(J))
 310   CONTINUE
 101 FORMAT(1X,6(E8.2,2X))
C
C  COMPUTE THE EXPECTED VALUE OF K*U*D
C
       DO 320 I=1,N
         DO 319 J=1,N
            W1(I,J)=0.DO
           W2(I,J)=0.DO
 319       CONTINUE
         W1(I,I)=EKDN1(I)*ZN(I)
         W2(I,I)=EKDN(I)*ZN(I)
```

```
      320   CONTINUE
              CALL MAMA(W1,A1,W1,N)
              CALL MAMA(W2,A2,W2,N)
              CALL MAMA(A1,EDN1,W3,N)
              CALL MAMA(A2,EDN,W4,N)
              CALL MAMA(W3,A1,W3,N)
              CALL MAMA(W4,A2,W4,N)
              CALL MAMA(A1,EDNN1,W5,N)
              CALL MAMA(W5,A2,W5,N)
              DO 330 I=1,N
                DO 340 J=1,N
                  EDN2(I,J)=0.D0
                  EDN2(I,J)=MI(I)*MI(J)*(EQN(I,J)-W1(I,J)-W2(J,I)
         1         -W1(J,I)-W2(I,J)+W3(I,J)+W4(I,J)+W5(I,J)+W5(J,I))
      340         CONTINUE
                  EDN2(I,I)=MI(I)*MI(I)*DIAG(I)+EDN2(I,I)
      330   CONTINUE
             DO 335 I=1,N
            DO 335 J=1,I
            SM=(EDN2(I,J)+EDN2(J,I))/2.D0
            EDN2(I,J)=SM
            EDN2(J,I)=SM
      335 CONTINUE
C
C   WRITE OUT DESIRED RESULTS
            IF(NP.NE. 20) GOTO 92
            NP=0
            WRITE(7,91) TIME,UN(4),UN(7),EDN(4,4),EDN(4,7),EDN(7,7)
       91       FORMAT(6(E9.3,2X))
       92 CONTINUE
            NP=NP+1
C
              TIME=TIME+DELT
       90 CONTINUE
            END
```

```
C
C
C
      SUBROUTINE ELEMEN(N1,N2,AE,COVA,EI,COVE,RHO,LELH)
C
C  THIS SUBROUTINE COMPUTES THE ELEMENT STIFFNESS AND MASS
C  MATRICES,THE VARIANCE OF THE STIFFNESS MATRIX AND SUMS
C  THE GLOBAL MATRICES.
C
C  COMMON BLOCKS AND TYPE DECLARATIONS
C
      DOUBLE PRECISION KE(6,6),ME(6,6),K(21,21),M(21),VKE(6,6)
      DOUBLE PRECISION VK(21,21),W1(6,6),AE,COVE,EI,COVI
      DOUBLE PRECISION L,L2,L3,RHO,T(6,6),TT(6,6),RNODE(7,2)
      DOUBLE PRECISION X1,X2,Y1,Y2,XDIF,YDIF,SNB,CSB,C1,C2
      DOUBLE PRECISION W2(6,6),W3(6,6)
      DOUBLE PRECISION W4(6,6),W5(6,6),W6(6,6)
      DOUBLE PRECISION P,S,G,VP,VS,VG
      INTEGER CON(6)
      COMMON/GLB/K,M,VK,N
      COMMON/RND/RNODE
      DO 1 I=1,6
        DO 1 J=1,6
          W1(I,J)=0.D0
      W2(I,J)=0.D0
      W3(I,J)=0.D0
          T(I,J)=0.D0
          VKE(I,J)=0.D0
          ME(I,J)=0.D0
          KE(I,J)=0.D0
          W4(I,J)=0.D0
           W5(I,J)=0.D0
          W6(I,J)=0.D0
    1 CONTINUE
```

```
        IROT=0
C
C   FORM CONNECTIVITY VECTOR
C
        CON(1)=3*N1-2
        CON(2)=3*N1-1
        CON(3)=3*N1
        CON(4)=3*N2-2
        CON(5)=3*N2-1
        CON(6)=3*N2
        X1=RNODE(N1,1)
        Y1=RNODE(N1,2)
        X2=RNODE(N2,1)
        Y2=RNODE(N2,2)
        XDIF=X2-X1
        YDIF=Y2-Y1
        L2=XDIF*XDIF+YDIF*YDIF
        L=USQRT(L2)
        L3=L*L2
C
C   CHECK TO SEE IF ROTATION IS REQUIRED AND SET FLAG
C
        IF(YDIF.NE. 0.0) IRUT=1
C
C   COMPUTE AXIAL TERMS
C
        C1=AE/L
        VC1=C1*C1*COVA*COVA
        KE(1,1)=C1
        KE(1,4)=-C1
        KE(4,4)=C1
        VKE(1,1)=VC1
        VKE(1,4)=VC1
        VKE(4,4)=VC1
```

```
C
C  AXIAL MASS
C
       C2=RHO*L/2.DO
       ME(1,1)=C2
       ME(4,4)=C2
C
C  ASSEMBLE BENDING PORTIONS
C
       P=12.DO*EI/L3
       S=6.DO*EI/L2
       G=4.DO*EI/L
       VP=P*P*COVE*COVE
       VS=S*S*COVE*COVE
       VG=G*G*COVE*COVE
       KE(2,2)=P
       KE(2,3)=S
       KE(2,5)=-P
       KE(2,6)=S
       KE(3,3)=G
       KE(3,5)=-S
       KE(3,6)=G/2.DO
       KE(5,5)=P
       KE(5,6)=-S
       KE(6,6)=G
C
C  VARIANCE TERMS
C
       VKE(2,2)=VP
       VKE(2,3)=VS
       VKE(2,5)=VP
       VKE(2,6)=VS
       VKE(3,3)=VG
       VKE(3,5)=VS
```

```
      VKE(3,6)=VG/4.DO
      VKE(5,5)=VP
      VKE(5,6)=VS
      VKE(6,6)=VG
C
C  MASS TERMS
C
      ME(2,2)=C2
      ME(3,3)=C2*L2/12.DO
      ME(5,5)=C2
      ME(6,6)=C2*L2/12.DO
C
C  FILL IN LOWER PORTION OF MATRICES
C
      DO 10 I=1,6
        DO 10 J=1,I
          KE(I,J)=KE(J,I)
          VKE(I,J)=VKE(J,I)
 10   CONTINUE
C
C  PERFORM ROTATIONS IF NECCESARY
C
      IF(IROT.NE.1) GOTO 100
        CSB=XDIF/L
        SNB=YDIF/L
        T(1,1)=CSB
        T(1,2)=-SNB
        T(2,1)=SNB
        T(2,2)=CSB
        T(3,3)=1.DO
        T(4,4)=CSB
        T(4,5)=-SNB
```

```
      T(5,4)=SNB
      T(5,5)=CSB
      T(6,6)=1.D0
    DO 50 I=1,6
     DO 50 J=1,6
      DO 50 N3=1,6
       W1(I,J)=W1(I,J)+T(N3,I)*KE(N3,J)
       W2(I,J)=W2(I,J)+T(N3,I)*ME(N3,J)
       W3(I,J)=W3(I,J)+T(N3,I)*T(N3,I)*VKE(N3,J)
 50 CONTINUE
    DO 60 I=1,6
     DO 60 J=1,6
       DO 60 N3=1,6
       W4(I,J)=W4(I,J)+W1(I,N3)*T(N3,J)
       W5(I,J)=W5(I,J)+W2(I,N3)*T(N3,J)
       W6(I,J)=W6(I,J)+W3(I,N3)*T(N3,J)*T(N3,J)
 60 CONTINUE
    DO 70 I=1,6
      DO 70 J=1,6
       KE(I,J)=W4(I,J)
       ME(I,J)=W5(I,J)
       VKE(I,J)=W6(I,J)
 70 CONTINUE
100 CONTINUE
C
C  ASSEMBLE GLOBAL MATRICES
    DO 110 I=1,6
      II=CON(I)
      M(II)=M(II)+ME(I,I)
      DO 110 J=1,6
        JI=CON(J)
        K(II,JI)=K(II,JI)+KE(I,J)
        VK(II,JI)=VK(II,JI)+VKE(I,J)
110 CONTINUE
```

```
      WRITE(7,15) LELM
   15 FORMAT(1X,'STIFFNESS MATRIX ELEMENT',I2)
      DO 5 I=1,6
        WRITE(7,17)(KE(I,J),J=1,6)
    5 CONTINUE
      WRITE(7,17)(ME(I,I),I=1,6)
   17 FORMAT(1X,6(E9.3,2X))
   18 FORMAT(1X,'MASS MATRIX',/,6(E9.3,2X))
      RETURN
      END
C  THIS SUBROUTINE PERFORMS THE OPERATION
C  A*B=C. WHERE A,B,AND C ARE N*N MATRICES
      SUBROUTINE MAMA(A,B,C,N)
      DOUBLE PRECISION A(21,21),B(21,21),C(21,21),D(21,21)
      DO 20 I=1,N
        DO 20 J=1,N
          D(I,J)=0.D0
   20 CONTINUE
      DO 30 I=1,N
      DO 30 J=1,N
      DO 30 K=1,N
        D(I,J)=A(I,K)*B(K,J)+D(I,J)
   30 CONTINUE
      DO 40 I=1,N
      DO 40 J=1,N
        C(I,J)=D(I,J)
   40 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE LOAD(T,P,EQN,N)
     DOUBLE PRECISION P(21),EQN(21,21),T
C
C  DEFINE FURTHER VARIABLES HERE
C
C
C  USER DEFINED LOADING FUNCTIONS
C
     RETURN
     END
OFF
13:31:35 03/31/85 60237
CONNECTED 00:11:41
CPU TIME 0.0
```

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