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SHEAR DEFLECTION IN A THREE-POINT BEND BEAM OF A SOLID CIRCULAR CROSS-SECTION

BOAZ AVITZUR

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US ARMY ARMAMENT RESEARCH AND DEVELOPMENT CENTER
LARGE CALIBER WEAPON SYSTEMS LABORATORY
BENÉT WEAPONS LABORATORY
WATERVLIET N.Y. 12189

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An equation, correlating the elastic deflection of a simply supported beam of a circular cross-section, with the applied load and beam's material properties and dimensions is being offered here. The contributions due to the bending moment and due to shear stresses are computed and compared.

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INTRODUCTION

It is customary in calculating the deflection of a simply supported beam under load, to compute that part of the deflection resulting from the accumulated curvature caused by the bending moment. These calculations are based on the assumption that planes normal to the beam's neutral plane remain planar and normal to the (curved) neutral surface. In these calculations, one replaces the equation describing the radius of curvature,

$$\frac{1}{\rho} = \frac{\frac{d^2y}{dt^2}}{[1 + (\frac{dy}{dt})^2]^{3/2}} \quad (1)$$

where $\rho \equiv$ radius of curvature, by

$$\frac{1}{\rho} = \frac{d^2y}{dt^2} \quad (2)$$

In the treatment of the bending of beams within the elastic limit, the smallest radius of curvature is

$$\frac{1}{\rho} = \frac{\sigma_o}{c \cdot E} \quad (3)$$

where $\sigma_o \equiv$ the material's yield strength, $E \equiv$ the material's modulus of elasticity, and $c \equiv$ the outermost fiber (the one farthest from the netural plane). This suggests that the smallest radius of curvature is proportional to the ratio E/σ_o, of the material's properties, which for most structural materials, is in the order of 10^2. Therefore, the assumption that dy/dt is one of small values and thus $(dy/dt)^2 \approx 0$ is a valid one.

Analyzing the equation for the combined deflection (ref 1) due to bending moment and due to shear, one can see that for most structural members with large length over height ratio, the contribution of shear to the deflection can be neglected. Haggag and Underwood (ref 1) offer the following for the deflection at midpoint in a symmetrically loaded beam of a rectangular cross-section. They suggest that

$$\delta_{shear} = \frac{3}{5}(1+\nu)\frac{P\ell}{bhE} \tag{4}$$

and

$$\delta_{bend} = \frac{P\ell^3}{4bh^3E} \tag{5}$$

where $\delta \equiv$ deflection of beam's midpoint

$P \equiv$ load at beam's midpoint

$\nu \equiv$ beam material's Poisson's ratio

$E \equiv$ beam material's Modulus of Elasticity

$b \equiv$ beam's cross-sectional width

$h \equiv$ beam's cross-sectional height

$\ell \equiv$ distance between beam's supports.

Thus the ratio between the two portions of the deflection is

$$\frac{\delta_{shear}}{\delta_{bend}} = 2.4(1+\nu)\left(\frac{h}{\ell}\right)^2 \tag{6}$$

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[1] F. M. Haggag and J. H. Underwood, "Compliance of a Three-Point Bend Specimen at Load Line," ARDC Memorandum Report No. ARLCB-MR-84035, Benet Weapons Laboratory, Watervliet, NY, October 1984.

For most structural members where $(\ell/h) > 30$, the deflection due to shear is less than one-third of a percent. This portion of the deflection becomes a significant part of the total deflection in short, laboratory size, three-point bending test specimens. Such three-point bending tests are performed in studying the material's fracture toughness and/or in calculating its modulus of elasticity. This report presents similar equations for the evaluation of deflection due to shear, as compared to that due to bending moment, for a symmetrically loaded beam having a uniform solid circular cross-section.

DISCUSSION AND DERIVATIONS

Castigliano's theorem (ref 2), equating the internal deformation energy with that of the outside work done on the structural member, is being invoked here. However, since there is no method for directly measuring either the stress distribution or the strain distribution in the interior of the member, one has to resort to some assumptions concerning these distributions. The assumptions made here (for the circular beam) are as follows:

1. We are considering here linear elastic response of the material only, and as such invoke Hooke's stress-strain relation.

2. It is being assumed that planes normal to the beam's neutral plane remain planar and normal to that neutral surface at all times.

3. As a consequence of the above two assumptions, it is being assumed that the only normal-stress component is in the axial direction and it varies

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[2]A. C. Ugural and S. K. Fenster, Advanced Strength and Applied Elasticity, American Elsevier, NY, 1981, pp. 146-148.

linearly from zero at the neutral surface to its maximum at the outermost fiber (and from zero at the cross-sections over the supports to its maximum where the moment is maximum, which is under the concentrated load).

4. The only components of shear stress are $\tau_y = \tau_x$ which act on planes parallel to the neutral surface (with an equal component or planes normal to the neutral surface).

5. It is being assumed here that the σ_{yx} component of shear τ_y is uniform across any line on the cross-section parallel to the neutral surface and at any given distance, $y < R_o$ from it, where $R_o \equiv$ the beam's cross-sectional radius.

The shear force in the x direction acting on any shear surface b·xx at distance y from the neutral surface is being balanced by the axial forces as assumed in assumption 3 above, acting on the volume above or below that surface. (The volume is A·Δx where A is being defined in Figure 1 (ref 3). Thus

$$\sigma_{xy} = \frac{2\int_y^{R_o}\int_o^{\sqrt{R_o^2-y^2}} \frac{\Delta\sigma_{xx}}{\Delta x}\,dzdy}{b} = \frac{\frac{2}{I}\int_y^{R_o}\int_o^{\sqrt{R_o^2-y^2}} \frac{\Delta M}{\Delta x}\,ydzdy}{2\sqrt{R_o^2-y^2}} = \frac{\frac{P}{I}\int_y^{R_o}\int_o^{\sqrt{R_o^2-y^2}} ydzdy}{2\sqrt{R_o^2-y^2}}$$

$$= \frac{2(R_o^2-y^2)}{3\pi R_o^4}P \tag{7}$$

From Figure 1,

$$\frac{\tau_x}{\sigma_{xy}} = \frac{\ell_z}{Y} \tag{8}$$

[3]F. Panlilio, Elementary Theory of Structural Strength, John Wiley & Sons, Inc., NY, 1963, p. 165.

where

$$\ell_z = \sqrt{Y^2+z^2} = \sqrt{\frac{(R_o^2-y^2)^2}{y^2} + z^2} \tag{9}$$

Thus

$$\frac{\tau_x}{\sigma_{xy}} = \sqrt{\frac{\frac{(R_o^2-y^2)^2}{y^2} + z^2}{\frac{R_o^2-y^2}{y}}} = \sqrt{\frac{(R_o^2-y^2)^2+y^2z^2)}{R_o^2-y^2}} = \sqrt{\frac{y^2z^2}{(R_o^2-y^2)^2} + 1} \tag{10}$$

or

$$\tau_x = \sqrt{\frac{y^2z^2}{(R_o^2-y^2)^2} + 1} \cdot \sigma_{xy} = \frac{2(R_o^2-y^2)}{3\pi R_o^4} \cdot \sqrt{\frac{y^2z^2}{(R^2-y^2)^2} + 1} \cdot P$$

$$= \frac{2}{3\pi R_o^4} \sqrt{(R_o^2-y^2)^2+y^2z^2}\, P \tag{11}$$

The local strain energy, due to shear, is

$$u_{shear} = \tau_x\gamma_x = 2\,\frac{1+\nu}{E} \cdot \tau_x^2 = \frac{8(1+\nu)}{9\cdot\pi^2\cdot E\cdot R_o^8} \cdot [(R_o^2-y^2)^2 + y^2z^2] \cdot P^2 \tag{12}$$

or the total shear energy over the whole volume becomes

$$U_{shear} = \frac{8(1+\nu)P^2}{9\pi^2 ER_o^8} \cdot 8 \cdot \int_0^{\ell/2}\int_0^{R_o}\int_0^{\sqrt{R_o^2-y^2}} [(R_o^2-y^2)^2+y^2z^2]\cdot dz\cdot dy\cdot dx \tag{13}$$

for which

$$\int_0^{\sqrt{R_o^2-y^2}} [(R_o^2-y^2)^2 + y^2z^2]dz = [(R_o^2-y^2)^2z + \frac{y^2z^3}{3}]_0^{\sqrt{R_o^2-y^2}} =$$

$$(R_o^2-y^2)^{5/2} + \frac{y^2}{3}(R_o^2-y^2)^{3/2} = (R_o^2-y^2)^{3/2}[R_o^2 - \frac{2}{3}y^2] \tag{14}$$

and

$$\int_0^{R_o}(R_o^2-y^2)^{3/2}[R_o^2 - \frac{2}{3}y^2]dy = R_o^2\int_0^{R_o}(R_o^2-y^2)^{3/2}\,dy - \frac{2}{3}\int_0^{R_o} y^2(Ro^2-y^2)dy \tag{15}$$

for which

$$R_o^2\int_o^{R_o}(R_o^2-y^2)^{3/2}dy = \frac{1}{4}\left[y\sqrt{(R_o^2-y^2)^3} + \frac{3R_o^2y}{2}\sqrt{R_o^2-y^2} + \frac{3R_o^4}{2}\sin^{-1}\frac{y}{R_o}\right]_o^{R_o} = \frac{3\pi}{16}R_o^6 \tag{16}$$

and

$$-\frac{2}{3}\int_o^{R_o}y^2(R_o^2-y^2)^{3/2}dy = \left[\frac{1}{9}y\sqrt{(R_o^2-y^2)^5} - \frac{R_o^2y}{36}\sqrt{(R_o^2-y^2)^3} - \frac{R_o^4y}{24}\sqrt{R_o^2-y^2} - \frac{R_o^6}{24}\sin^{-1}\frac{y}{R_o}\right]_o^{R_o} = -\frac{\pi R_o^6}{48} \tag{17}$$

Thus

$$\int_o^{R_o}(R_o^2-y^2)^{3/2}[R_o^2 - \frac{2}{3}y^2]dy = (\frac{3}{16} - \frac{1}{48})\pi R_o^6 = \frac{\pi}{6}R_o^6 \tag{18}$$

and

$$\frac{\pi}{6}R_o^6\int_o^{\ell/2}dx = \frac{\pi}{6}R_o^6\,x\Big|_o^{\ell/2} = \frac{\pi}{12}R_o^6\ell$$

or

$$U_{shear} = \frac{64(1+\nu)P^2}{9\pi^2ER_o^8}\int_o^{\ell/2}\int_o^{R_o}\int_o^{\sqrt{R_o^2-y^2}}[(R_o^2-y^2)^2 + y^2z^2]dz\cdot dy\cdot dx = \frac{16(1+\nu)P^2}{27\pi ER_o^2}\ell \tag{19}$$

The local strain energy due to normal stresses or due to the bending moment is

$$u_{bend} = \sigma_{xx}\varepsilon_{xx} = \frac{\sigma_{xx}^2}{E} \tag{20}$$

where

$$\sigma_{xx} = \frac{M}{I}y = \frac{P}{2I}xy \tag{21}$$

and where

$$I = \frac{\pi}{4}R_o^4 \tag{22}$$

Thus

$$u_{bend} = \frac{4P^2}{\pi^2 R_o{}^8 E} x^2 y^2 \tag{23}$$

Thus, the total bending energy, U_{bend}, is derived as follows:

$$U_{bend} = \frac{4P^2}{\pi^2 R_o{}^8 E} 8 \int_o^{\ell/2}\int_o^{R_o}\int_o^{\sqrt{R_o{}^2-y^2}} x^2y^2 dzdydx = \frac{32P^2}{\pi^2 R_o{}^8 E}\int_o^{\ell/2}\int_o^{R_o} x^2y^2\sqrt{R_o{}^2-y^2}\, dydx \tag{24}$$

where

$$\int_o^{R_o} x^2y^2\sqrt{R_o{}^2-y^2}\, dy = x^2\left[\frac{y}{4}\sqrt{(R_o{}^2-y^2)^3} + \frac{R_o{}^2}{8}\left(y\sqrt{R_o{}^2-y^2} + R_o{}^2 \sin^{-1}\frac{y}{R_o}\right)\right]_o^{R_o} =$$

$$\frac{\pi}{2} x^2 \frac{R_o{}^4}{8} = \frac{\pi}{16} x^2 R_o{}^4 \tag{25}$$

and

$$\frac{\pi R_o{}^4}{16}\int_o^{\ell/2} x^2 dx = \frac{\pi R_o{}^4}{48} x^3\Big|_o^{\ell/2} = \frac{\pi}{384} R_o{}^4 \ell^3 \tag{26}$$

Thus

$$U_{bend} = \frac{P^2\ell^3}{12\pi E R_o{}^4} \tag{27}$$

and

$$U_{total} = \frac{P^2\ell}{108\pi E R_o{}^2}\left[9\left(\frac{\ell}{R_o}\right)^2 + 64(1+\nu)\right] \tag{28}$$

Invoking Castigliano's theorem,

$$\frac{P^2\ell}{108\pi E R_o{}^2}\left[9\left(\frac{\ell}{R_o}\right)^2 + 64(1+\nu)\right] = P\delta \tag{29}$$

from which

$$\delta = \frac{P\ell}{108\pi E R_o{}^2}\left[9\left(\frac{\ell}{R_o}\right)^2 + 64(1+\nu)\right] \tag{30}$$

or

$$\frac{\delta_{shear}}{\delta_{bend}} = \frac{64(1+\nu)}{9}\left(\frac{R_o}{\ell}\right)^2 = \frac{16(1+\nu)}{9}\left(\frac{D_o}{\ell}\right)^2 \tag{31}$$

where $D_o \equiv$ beam's cross-sectional direction $= 2 \cdot R_o$.

CONCLUSION

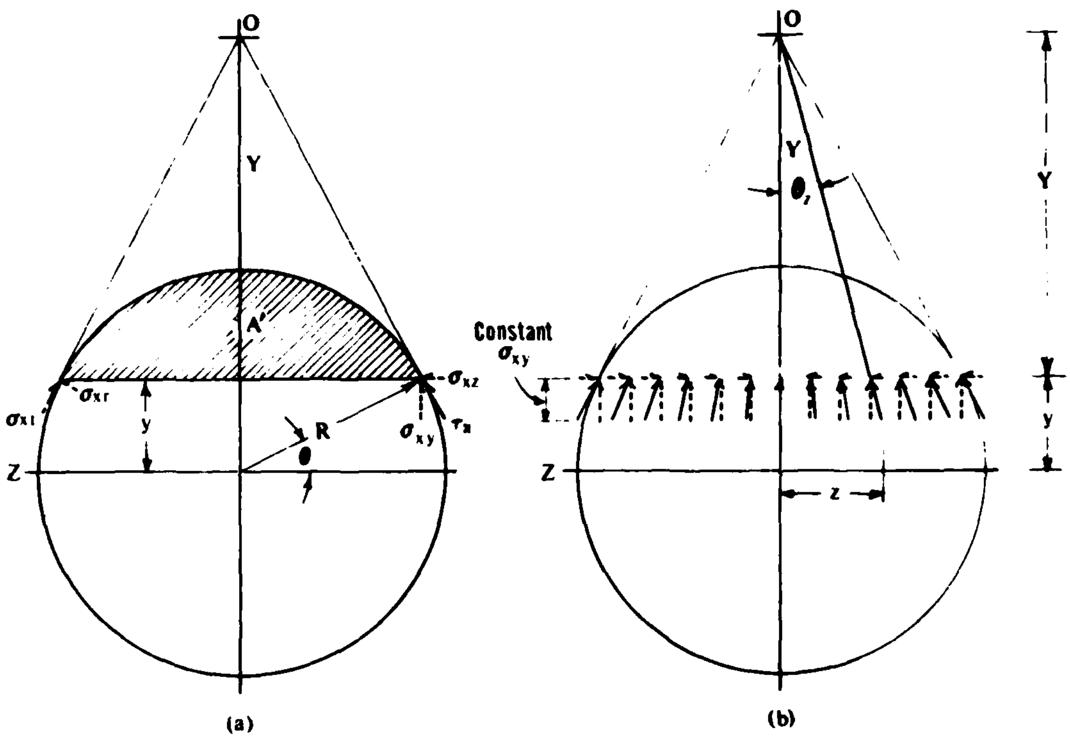
As in the case of a rectangular beam, the ratio between δ_{shear} and δ_{bend} diminishes as the square of the ratio of height over the distance between the supports diminishes. However, this ratio is being multiplied by a different coefficient, $16/9(1+\nu)$ for a solid circular cross-section compared with $2.4(1+\nu)$ for a solid rectangular cross-section.

REFERENCES

1. F. M. Haggag and J. H. Underwood, "Compliance of a Three-Point Bend Specimen at Load Line," ARDC Memorandum Report No. ARLCB-MR-84035, Benet Weapons Laboratory, Watervliet, NY, October 1984.

2. A. C. Ugural and S. K. Fenster, Advanced Strength and Applied Elasticity, American Elsevier, NY, 1981, pp. 146-148.

3. F. Panlilio, Elementary Theory of Structural Strength, John Wiley & Sons, Inc., NY, 1963, p. 165.



(Figure from Reference 3)

Figure 1. Shear stresses in a circular beam under bending forces.

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