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Variability of Measures of Weapons Effectiveness

Volume VIII: Effectiveness Indices in Multiple and Guided Weapons

B D Sivazlian

UNIVERSITY OF FLORIDA
DEPARTMENT OF INDUSTRIAL AND SYSTEMS ENGINEERING
GAINESVILLE, FLORIDA 32611

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Mathematical models are developed for obtaining expressions for the probability of kill, P_k, for multiple weapons under stick delivery, for laser guided bombs, and for guided weapons. It is assumed that the damage function can be approximated by the Carleton damage function and that delivery errors are present having predefined distributions. The variability in P_k is computed.

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PREFACE

This report describes work done during the summer of 1984 by Dr B. D. Sivazlian, principal investigator, of the Department of Industrial and Systems Engineering, the University of Florida, Gainesville, Florida 32611, under Contract No. F08635-83-C-0202 with the Air Force Armament Laboratory (AFATL), Armament Division, Eglin Air Force Base, Florida 32542. The program manager was Mr Daniel A. McInnis (DLYW).

This work addresses itself to the problem of determining the probability of kill, P_k, of multiple weapons under stick delivery (single pass and multiple passes). Explicit expressions for P_k are not, however, derived. For laser guided bombs and guided weapons, mathematical models have been developed for obtaining explicit expressions for P_k as well as its variance. It is assumed that for each weapon, the damage function can be approximated by the three-parameter Carleton damage function. It will be seen that the assumptions related to the distribution of the delivery errors constitute the main elements which bring forth the difference between the various models.

The author has benefited from helpful discussions with Mr Jerry Bass, Mr Daniel McInnis and Mr Charles Reynolds who have contributed to the report through their comments.

The Public Affairs Office has reviewed this report, and it is releasable to the National Technical Information Service (NTIS), where it will be available to the general public, including foreign nationals.

This technical report has been reviewed and is approved for publication.

FOR THE COMMANDER

Milton D. Kingcaid

MILTON D. KINGCAID, Colonel, USAF
Chief, Analysis and Strategic Defense Division

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SECTION I

INTRODUCTION

1. Objective of the Study

In this report, analytical models are formulated for obtaining the probability of kill, P_k, of a point target for a variety of situations. These situations can be roughly divided into two categories, namely, multiple weapons delivery (Sections II and III) and single guided weapon delivery (Section IV and V).

In Section II, the general problem of the stick delivery of weapons is formulated. An exact expression for P_k is obtained. However, this expression is in the form of a multiple integral which is very dificult to evaluate. As a consequence, an approximation is suggested and the reader is referred to [3] for further details.

In Section III, the problem of multiple weapons delivery with independent passes is described. In general, at each pass, weapons are delivered in stick, and thus, a formal approach to the problem would have to rely on the results of Section II. No specific methodology is provided to tackle the general problem. The only situation that is worked out in detail is the case of singly delivered weapon per pass.

Sections IV and V consider the problem of guided weapon delivery. The delivery of guided weapons differ from the delivery of general purpose bombs in that the delivery error distributions have different characteristics.

In its simplest form, for guided weapons, the delivery error in the direction of range has the same normal distribution as for the delivery error in the direction of deflection. The two-standard deviations are equal. Given that the two delivery errors are unbiased and that they are independently distributed, it immediately follows that the distribution of errors in the radial direction follows the so-called circular or Rayleigh distribution.

Experimental evidence, however, has shown otherwise and two attempts are made in the JTCG/ME (Air-to-Surface) manuals to correct for this situation. In the early edition (1980) [1], a modified radial distribution is proposed for laser-guided bombs. This situation is studied in detail in Section IV. In a more recent edition (1983) [2], a normal delivery error with bias in the range direction is proposed for general guided weapons. This situation is studied in Section V. Section VI consists of concluding remarks.

2. Assumptions

The following assumptions are made:

a. Each of the target and weapon is idealized as a point. Depending on the particular situation considered, the weapon may or may not be aimed directly at the target.

b. The direction of the weapon delivery range and deflection are respectively parallel to the (x-y) coordinate system on the ground plane. Since the coordinate system can be arbitrarily selected, there is no loss in generality in making this specific assumption. The position of the target has coordinates (u,v).

c. For stick delivery, the weapons are subject to ballistic and aiming errors in each of the x and y directions. For single weapon release, a delivery error in each of the x and y directions is assumed. In each case, the probability density function of these errors must be specified.

d. The probability of kill due to fragmentation at a point (u,v) given that the weapon impacts at (x,y) is given by the three-parameter Carleton damage function

$$D(u-x,\ v-y) = D_0 \exp[-D_0[(\frac{u-x}{R_x})^2 + (\frac{v-y}{R_y})^2]$$

The parameters R_x and R_y are, respectively, the weapon radii in the x and y directions. The parameters D_0 is the maximum probability of kill which occurs at the point of weapon impact

e. Fragmentation does not contribute to the delivery error.

SECTION II

MULTIPLE WEAPONS: STICK DELIVERY

1. Introduction

We consider the stick delivery of n weapons (n=1,2,...) for which the following assumptions are made:

a. Each weapon is subject to ballistic errors which are assumed to be normally distributed, independent of each other and independent in each of the x (range) and y (deflection) directions.

b. The entire stick pattern is subject to aiming error which is assumed to be normally and independently distributed in each of the x and y directions, and independently distributed from the ballistic errors. Equivalently, it can be stated that each of the n weapons is subject to the same aiming error.

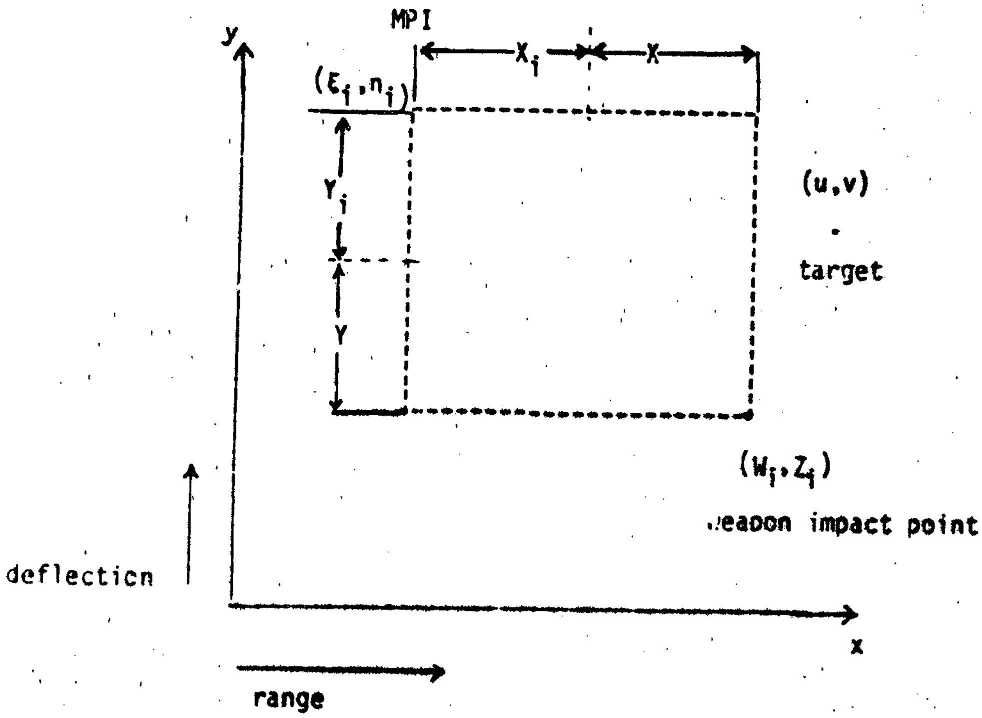

Figure 1. Geometry of Target and Weapon

Let (ξ_i, η_i) be the mean point of impact (MPI) of the ith weapon $(i=1,2,\ldots,n)$. The ballistic errors in the directions of range and deflection are, respectively, X_i and Y_i. The random variables X_i and Y_i are assumed to be independent each with zero mean and having the respective normal probability density functions

$$f_{X_i}(x_i) = \frac{1}{\sigma_1\sqrt{2\pi}} \exp[-\frac{x_i^2}{2\sigma_1^2}] \tag{1}$$

and

$$f_{Y_i}(y_i) = \frac{1}{\sigma_2\sqrt{2\pi}} \exp[-\frac{y_i^2}{2\sigma_2^2}] \tag{2}$$

For each of the i weapons, the aiming errors in the directions of range and deflection are, respectively, X and Y. The random variables X and Y are assumed to be independent each with zero mean and having the respective normal probability density functions

$$g_X(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp[-\frac{x^2}{2\sigma_x^2}] \tag{3}$$

and

$$g_Y(y) = \frac{1}{\sigma_y\sqrt{2\pi}} \exp[-\frac{y^2}{2\sigma_y^2}] \tag{4}$$

The random variables $\{X_i\}$, $\{Y_i\}$, X, and Y are mutually independent.

Let (W_i, Z_i) be the actual impact point of the ith weapon. This is the result of the combined effect of the ballistic error and aiming error. This combined effect is the sum of the ballistic error and aiming error in each of the x and y directions (see Figure 1). Clearly, for $i=1,2,\ldots,n$

$$W_i = \xi_i + X_i + X \tag{5}$$

$$Z_i = \eta_i + Y_i + Y \tag{6}$$

The sequence of random variables $\{W_i\}$ and $\{Z_i\}$ are mutually independent and are normally distributed with respective means

$$E[W_i] = \xi_i, \qquad E[Z_i] = \eta_i \tag{7}$$

and respective variances

$$Var[W_i] = \sigma_1^2 + \sigma_x^2; \quad Var[Z_i] = \sigma_2^2 + \sigma_y^2 \tag{8}$$

The random variables in the sequence $\{W_i\}$, $i=1,\ldots,n$, are not mutually independent since all of them depend on the common random variable X. Clearly for $i \neq j$

$$\begin{aligned} Cov[W_i, W_j] &= E[(W_i - \xi_i)(W_j - \xi_j)] \\ &= E[(X_i + X)(X_j + X)] \\ &= E[X^2] = \sigma_x^2 \end{aligned} \tag{9}$$

Thus, the sequence of random variables $\{W_i\}$ are jointly normally distributed with mean vector

$$\underline{\xi} = (\xi_1, \xi_2, \ldots, \xi_n) \tag{10}$$

and variance-covariance matrix

$$\underline{\Omega} = \begin{bmatrix} \sigma_1^2 + \sigma_x^2 & \sigma_x^2 & \cdots & \sigma_x^2 \\ \sigma_x^2 & \sigma_1^2 + \sigma_x^2 & \cdots & \sigma_x^2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \sigma_x^2 & \sigma_x^2 & \cdots & \sigma_1^2 + \sigma_x^2 \end{bmatrix} \tag{11}$$

or

$$\underline{\Omega} = \sigma_1^2 \, \underline{I} + \sigma_x^2 \, 1'\underline{1} \tag{12}$$

where $\underline{I}$ is the (nxn) identity matrix and where the n-dimensional row vector $\underline{1}$ is defined as

$$\underline{1} = (1, 1, \ldots, 1) \tag{13}$$

If we define $\underline{W} = (W_1, W_2, \ldots, W_n)$, then the joint probability density function of the W_i's is $N(\underline{W}; \underline{\xi}, \underline{\Omega})$

Similarly, for $i \neq j$

$$\begin{aligned} Cov[Z_i, Z_j] &= E[(Z_i - \eta_i)(Z_j - \eta_j)] \\ &= E[(Y_i + Y)](Y_j + Y)] \\ &= E[Y^2] = \sigma_y^2 \end{aligned} \tag{14}$$

Thus, the sequence of random variables $\{Z_i\}$, i=1,2,...,n are jointly normally distributed with mean vector

$$\underline{\eta} = (\eta_1, \eta_2, \ldots, \eta_n) \tag{15}$$

and variance-covariance matrix

$$\Lambda = \begin{bmatrix} \sigma_2^2 + \sigma_y^2 & \sigma_y^2 & \cdots & \sigma_y^2 \\ \sigma_y^2 & \sigma_2^2 + \sigma_y^2 & \cdots & \sigma_y^2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \sigma_y^2 & \sigma_y^2 & \cdots & \sigma_2^2 + \sigma_y^2 \end{bmatrix} \tag{16}$$

$$= \sigma_2^2 \underline{I} + \sigma_y^2 \underline{1}'\underline{1} \tag{17}$$

If we define $\underline{Z} = (Z_1, Z_2, \ldots, Z_n)$, then the joint probability density function of the Z_i's is $N(\underline{Z}; \underline{\eta}, \underline{\Lambda})$.

2. The Model

For a target located at (u,v), it is required to determine the probability of kill of the target assuming that if a weapon impacts at (w,z), the damage function is of the form

$$D(u-w,\ v-z) = D_0 \exp[-D_0[(\frac{u-w}{R_x})^2 + (\frac{v-z}{R_y})^2]] \tag{18}$$

The conditional probability of kill given that the first weapon impacts at (w_1, z_1), the second weapon impacts at (w_2, z_2),..., and the nth weapon impacts at (w_n, z_n) is

$$\{1 - \prod_{i=1}^{n} [1 - D(u-w_i,\ v-z_i)]\} \tag{19}$$

The unconditional probability of kill at (u,v) is

$$P_k = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{2n} \text{[Probability of kill at } u,v) \mid i\text{th weapon impacts at } (w_i, z_i)$$

$$\text{for } i=1,2,\ldots,n]\text{[Probability that the } i\text{th weapon impacts between } (w_i, z_i) \text{ and } (w_i+dw_i,\ z_i+dz_i) \text{ for } i=1,2,\ldots,n]$$

$$= \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{2n} \{1 - \prod_{i=1}^{n} [1 - D(u-w_i,\ v-z_i)]\}\ N(\underline{w};\ \underline{\xi},\ \underline{\Omega})$$

$$N(\underline{z};\ \underline{\eta},\ \underline{\Lambda}) \prod_{i=1}^{n} dw_i\ dz_i \tag{20}$$

Note that P_k, in general, is going to be a function of the location of the target, namely (u,v), and also a function of the cordinates of the MPI of all n weapons. It is customary to assume that the location of the taget coincides with the origin, so that $u=0=v$, and hence, P_k is only a function of $\underline{\xi}$ and $\underline{\eta}$ and one can write:

$$P_k(\underline{\xi}, \underline{\eta}) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{2n} \{1 - \prod_{i=1}^{n} [1 - D(w_i, z_i)]\}\ N(\underline{w},\ \underline{\xi},\ \underline{\Omega})$$

$$N(\underline{z},\ \underline{\eta},\ \underline{\Lambda}) \prod_{i=1}^{n} dw_i\ dz_i \tag{21}$$

3. Remarks

There are two problems to be addressed here. The first problem is that of obtaining the optimum stick pattern that will maximize the probability of kill $P_k(\underline{\xi},\underline{\eta})$. By an optimum stick pattern is meant the determination of the optimum locations of the MPI's of each of the n weapons delivered in such a way that the probability of kill is maximized. As one of the possible courses of action one could select the MPI's to coincide with the location of the point target, that is to set

$$\xi_i = 0 = \eta_i, \quad i = 1,2,\ldots,n$$

But this course of action does not necessarily maximize the probability of kill. Note, however, that the weapons are dropped from specific locations on the plane. There are usually three such locations: the two wings and the airplane centerline. As many as 100 of these weapons may be released in a single stick delivery. Thus, from a practical point of view, the variables ξ_i and η_i are selected according to a constrained path.

The second problem consists in obtaining a computable expression for $P_k(\underline{\xi}, \underline{\eta})$. The 2n- tuple integral could theoretically be evaluated (since the function $D(w_i, z_i)$ is of an exponential form). However, it is extremely difficult to proceed with the integrations particularly for large number of weapons. The complexity of the expression for $P_k(\underline{\xi},\underline{\eta})$ has led to the development of an alternate method for rederiving $P_k(\underline{\xi},\underline{\eta})$. The other method relies on a decomposition principle which results in a substantially more simplified expression for $P_k(\underline{\xi},\underline{\eta})$ ultimately reduced to a double integral (for details see [3]).

SECTION III

MULTIPLE WEAPONS: INDEPENDENT PASSES

1. Introduction

It is conceivable that when delivering weapons to a given target, an aircraft makes several independent passes over the target and releases one or more weapons at each pass. Multiple weapons are assumed to be released according to a stick delivery pattern. In general, the number of passes is a random variable and the number of weapons delivered at each pass is a random variable. The determination of the probability of kill of the target will then depend on the probability of kill of the target at each delivery as well as the statistical characteristics of the number of weapons released at each pass and the number of passes.

To solve this problem, it is necessary to know the following:

i. The relationship between the probability of kill of the target and the number of weapons released at a given pass. This type of functional relationship would be the result of an analysis of the stick delivery of weapons.

ii. The probability distribution function of the number of weapons released.

iii. The probability distribution function of the number of passes.

In general, let

M = random variable denoting the number of passes;

N_i = random variable denoting the number of weapons released at the ith pass $(i=1,2,\ldots,M)$;

$P_k(N_i)$ = probability of kill for N_i stick delivered weapons per pass;

The net target probability of kill is

$$P_{kf}^{(M)} = 1 - \prod_{i=1}^{M} [1 - P_k(N_i)] \tag{22}$$

Even when M and N_i are deterministic, explicit expressions for $P_{kf}^{(M)}$ cannot, in general, be obtained unless the function $P_k(\cdot)$ is known. However, for single weapon release (N_i=1 for all i) and M fixed, a closed form expression for P_{kf} can be obtained. This particular case is investigated in detail in the paragraphs that follows.

2. The Model

The development of the model for single weapon delivery per pass is rather straightforward and consists essentially in determining the probability of kill of the target per pass and then applying formula (22).

It wil be assumed that the delivery errors in each of the x(range) direction and y (deflection) direction are independently and normally distributed with respective standard deviations σ_x and σ_y. Let

(u,v) = cordinates of the point target;

(x,y) = cordinates of the point at which weapon impacts.

It is assumed that the weapon is aimed at the target. Then, the probability density function of the delivery error in the x direction is

$$f_1(x-u) = \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left[-\frac{(x-u)^2}{2\sigma_x^2}\right] \tag{23}$$

and the probability density function of the delivery error in the y direction is

$$f_2(y-v) = \frac{1}{\sqrt{2\pi}\,\sigma_y} \exp\left[-\frac{(y-v)^2}{2\sigma_y^2}\right] \tag{24}$$

If we assume that the damage function is represented by the three parameter Carleton damage function, then the probability of kill at (u,v) given that the weapon impacts at (x,y) is

$$D(u-x,\ v-y) = D_0 \exp\{- D_0[\frac{(u-x)^2}{R_x^2} + \frac{(v-y)^2}{R_y^2}]\} \tag{25}$$

To determine the probability of kill of the target for a single weapon in the presence of delivery errors, the laws of conditional probabilities are used and one has

P_{kf} = Probability of kill at (u,v)

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\text{Probability of kill at } (u,v) | \text{weapon impacts at } (x,y)]$$
$$[\text{Probability weapon impacts between } (x,y) \text{ and } (x+dx.\ y+dy)]$$

Using (23), (24), and (25), one obtains

$$P_{kf} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(u-x,\ v-y)\ f_1(x-u)\ f_2(y-v)\ dx\ dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D_0 \exp\{- D_0[\frac{(u-x)^2}{R_x^2} + \frac{(v-y)^2}{R_y^2}]\}$$

$$\cdot \frac{1}{\sqrt{2\pi}\ \sigma_x} \exp[- \frac{(x-u)^2}{2\sigma_x^2}] \cdot \frac{1}{\sqrt{2\pi}\ \sigma_y} \exp[- \frac{(y-v)^2}{2\sigma_y^2}]\ dx\ dy \tag{26}$$

This double integral can be easily evaluated, and it is found that

$$P_{kf} = D_0 \frac{R_x}{\sqrt{R_x^2 + 2D_0\ \sigma_x^2}} \cdot \frac{R_y}{\sqrt{R_y^2 + 2D_0\ \sigma_y^2}} \tag{27}$$

Since all passes are assumed to be independent, then from (22) the net target probability of kill for M passes is

$$P_{kf}^{(M)} = 1 - (1 - P_{kf})^M \tag{28}$$

Using (27) in (28) results in

$$P_{kf}^{(M)} = 1 - \left[1 - D_0 \frac{R_x}{\sqrt{R_x^2 + 2D_0 \sigma_x^2}} \cdot \frac{R_y}{\sqrt{R_y^2 + 2D_0 \sigma_y^2}}\right]^M \tag{29}$$

3. Estimation of $E[P_{kf}^{(M)}]$ and $Var[P_{kf}^{(M)}]$

The Taylor's series estimation procedure will be used to obtain approximate expressions for $E[P_{kf}^{(M)}]$ and $Var[P_{kf}^{(M)}]$ in terms of the means and variances of the five input parameters D_0, R_x, R_y, σ_x, and σ_y.

Let $\bar{D}_0$, $\bar{R}_x$, $\bar{R}_y$, $\bar{\sigma}_x$, and $\bar{\sigma}_y$ refer, respectively, to the mean of D_0, R_x, R_y, σ_x and σ_y. Expanding $P_{kf}^{(M)}$ about the point $(\bar{D}_0, \bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y)$ in Taylor's series and retaining only first order terms one obtains

$$\begin{aligned} P_{kf}^{(M)}(D_0, R_x, R_y, \sigma_x, \sigma_y) &= P_{kf}^{(M)}(\bar{D}_0, \bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y) \\ &+ (D_0 - \bar{D}_0)\frac{\partial P_{kf}^{(M)}}{\partial D_0} + (R_x - \bar{R}_x)\frac{\partial P_{kf}^{(M)}}{\partial R_x} + (R_y - \bar{R}_y)\frac{\partial P_{kf}^{(M)}}{\partial R_y} \\ &+ (\sigma_x - \bar{\sigma}_x)\frac{\partial P_{kf}^{(M)}}{\partial \sigma_x} + (\sigma_y - \bar{\sigma}_y)\frac{\partial P_{kf}^{(M)}}{\partial \sigma_y} \end{aligned} \tag{30}$$

All the partial derivatives of $P_{kf}^{(M)}$ with respect to the five variables D_0, R_x, R_y, σ_x and σ_y are to be evaluated at the mean values of the variables. Approximate expressions for $E[P_{kf}^{(M)}]$ and $Var[P_{kf}^{(M)}]$ are now provided.

a. Estimation of $E[P_{kf}^{(M)}]$

Taking expectation on both sided of (30) yields as a first approximation

$$E[P_{kf}^{(M)}(D_0, R_x, R_y, \sigma_x, \sigma_y)] = P_{kf}^{(M)}(\bar{D}_0, \bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y)$$

From (28)

$$E[P_{kf}^{(M)}] = [1 - \bar{D}_0 \frac{\bar{R}_x}{\sqrt{\bar{R}_x^2 + 2\bar{D}_0 \bar{\sigma}_x^2}} \cdot \frac{\bar{R}_y}{\sqrt{\bar{R}_y^2 + 2\bar{D}_0 \bar{\sigma}_y^2}}]^M \qquad (31)$$

For simplicity the bar (-) notation will be omitted in the sequel. Let

$$Q_1^2 = R_x^2 + 2D_0 \sigma_x^2 \qquad (32)$$

$$Q_2^2 = R_y^2 + 2D_0 \sigma_y^2 \qquad (33)$$

Then, from (31)

$$E[P_{kf}^{(M)}] = 1 - (1 - D_0 R_x R_y Q_1^{-1} Q_2^{-1})^M \qquad (34)$$

b. Estimation of $Var[P_{kf}^{(M)}]$

First, expression (30) is written as

$$P_{kf}^{(M)}(D_0, R_x, R_y, \sigma_x, \sigma_y) - P_{kf}^{(M)}(\bar{D}_0, \bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y)$$

$$= (D_0 - \bar{D}_0)\frac{\partial P_{kf}^{(M)}}{\partial D_0} + (R_x - \bar{R}_x)\frac{\partial P_{kf}^{(M)}}{\partial R_x} + (R_y - \bar{R}_y)\frac{\partial P_{kf}^{(M)}}{\partial R_y}$$

$$+ (\sigma_x - \bar{\sigma}_x) \frac{\partial P_{kf}^{(M)}}{\partial \sigma_x} + (\sigma_y - \bar{\sigma}_y) \frac{\partial P_{kf}^{(M)}}{\partial \sigma_y} \quad (35)$$

Squaring and taking expectations on both sides of (35) yields

$$\begin{aligned} \mathrm{Var}[P_{kf}^{(M)}] &= \mathrm{Var}[D_0] \left(\frac{\partial P_{kf}^{(M)}}{\partial D_0}\right)^2 + \mathrm{Var}[R_x] \left(\frac{\partial P_{kf}^{(M)}}{\partial R_x}\right)^2 \\ &+ \mathrm{Var}[R_y] \left(\frac{\partial P_{kf}^{(M)}}{\partial R_y}\right)^2 + \mathrm{Var}[\sigma_x] \left(\frac{\partial P_{kf}^{(M)}}{\partial \sigma_x}\right)^2 \\ &+ \mathrm{Var}[\sigma_y] \left(\frac{\partial P_{kf}^{(M)}}{\partial \sigma_y}\right)^2 + \text{covariance terms} \end{aligned} \quad (36)$$

Again, it should be noted that all partial derivatives are evaluated at the point $(\bar{D}_0, \bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y)$.

Now, in general, from (28)

$$\frac{\partial P_{kf}^{(M)}}{\partial (\cdot)} = M(1 - P_{kf})^{M-1} \frac{\partial P_{kf}}{\partial (\cdot)} \quad (37)$$

P_{kf} here is the probability of kill of a single weapon

Expression for $\partial P_{kf}/\partial(\cdot)$ can be obtained from the results in Section V in which biased delivery error for single weapon is considered. For the present Section, results for $\partial P_{kf}/\partial(\cdot)$ are obtained by setting in all the partial derivatives of Section V $B_x=0=B_y$. This immediately yields from (115) and (116) $E_1=1=E_2$.

From (117)

$$\frac{\partial P_{kf}}{\partial D_0} = R_x R_y Q_1^{-1} Q_2^{-1} [1 - D_0 \sigma_x^2 Q_1^{-2} - D_0 \sigma_y^2 Q_2^{-2}] \quad (38)$$

Q_1 and Q_2 are defined in (32) and (33) and they are the same as (113) and (114)

From (118)

$$\frac{\partial P_{kf}}{\partial R_x} = D_0 R_y Q_1^{-1} Q_2^{-1} [1 - Q_1^{-2} R_x^2] \tag{39}$$

From (119)

$$\frac{\partial P_{kf}}{\partial R_y} = D_0 R_x Q_1^{-1} Q_2^{-1} [1 - Q_2^{-2} R_y^2] \tag{40}$$

From (120)

$$\frac{\partial P_{kf}}{\partial \sigma_x} = -2 D_0^2 R_x R_y Q_1^{-3} Q_2^{-1} \sigma_x \tag{41}$$

From (121)

$$\frac{\partial P_{kf}}{\partial \sigma_y} = -2 D_0^2 R_x R_y Q_1^{-1} Q_2^{-3} \sigma_y \tag{42}$$

The explicit expression for $Var[P_{kf}^{(M)}]$ can be obtained from the following relation obtained from (36) and (37):

$$\begin{aligned} Var[P_{kf}^{(M)}] = M(1 - P_{kf})^{M-1} \{ & Var[D_0] \left(\frac{\partial P_{kf}}{\partial D_0}\right)^2 + Var[R_x] \left(\frac{\partial P_{kf}}{\partial R_x}\right)^2 \\ & + Var[R_y] \left(\frac{\partial P_{kf}}{\partial R_y}\right)^2 + Var[\sigma_x] \left(\frac{\partial P_{kf}}{\partial \sigma_x}\right)^2 \\ & + Var[\sigma_y] \left(\frac{\partial P_{kf}}{\partial \sigma_y}\right)^2 \} + \text{covariance terms} \end{aligned} \tag{43}$$

c. Numerical Example

An aircraft makes M=2 passes over a given target and at each pass drops a general purpose bomb. The weapon is aimed at the target. The parameters of the Carleton damage function are

$$D_0 = 1, \qquad R_x = 120 \text{ ft} \quad \text{and} \quad R_y = 200 \text{ ft}$$

These parameters are subject to estimation errors and their variances are given by

$$Var[D_0] = .002, \quad Var[R_x] = 50 \text{ ft}^2. \text{ and } Var[R_y] = 70 \text{ ft}^2.$$

The aiming errors in the x (range) and y (deflection) directions are independently and normally distributed with

$$\sigma_x = 30 \text{ ft} \quad ; \qquad \sigma_y = 20 \text{ ft.}$$

$$Var[\sigma_x] = 4 \text{ ft}^2; \quad Var[\sigma_y] = 4 \text{ ft}^2$$

we have

$$Q_1^2 = (120)^2 + (2)(1)(30)^2 = 16{,}200$$

$$Q_1 = 127.279{,}22$$

$$Q_2^2 = (200)^2 + (2)(1)(20)^2 = 40{,}800$$

$$Q_2 = 201.990{,}10$$

$$P_{kf} = D_0 R_x R_y Q_1^{-1} Q_2^{-1}$$

$$P_{kf} = \frac{(1)(120)(200)}{(127.279,22)(201.990,10)} = .93352$$

From (34)

$$E[P_{kf}^{(2)}] = 1 - (1 - P_{kf})^2$$

$$= 1 - (1 - .93352)^2 = .9956$$

From (38)

$$\frac{\partial P_{kf}}{\partial D_0} = \frac{(120)(200)}{(127.279,22)(201.990,10)} [1 - \frac{(1)(30)^2}{16,000} - \frac{(1)(20)^2}{40,800}]$$

$$= .933,52 [1 - .055,555,5 - .009,803,8]$$

$$= .872,505,7$$

From (39)

$$\frac{\partial P_{kf}}{\partial R_x} = \frac{(1)(200)}{(127.279,222)(201.990,10)} [1 - \frac{(120)^2}{16,200}]$$

$$= .007,779,3 (1 - .888,888,8)$$

$$= .000,864,3$$

From (40)

$$\frac{\partial P_{kf}}{\partial R_y} = \frac{(1)(120)}{(127.279,22)(201.990,10)} [1 - \frac{(200)^2}{40,800}]$$

$$= .004,667,6 (1 - .980,392,1)$$

$$= .000,091,5$$

From (41)

$$\frac{\partial P_{kf}}{\partial \sigma_x} = - \frac{(2)(1)^2(120)(200)(30)}{(127.279,22)^3(201.990,10)}$$

$$= - .003,457,4$$

From (42)

$$\frac{\partial P_{kf}}{\partial \sigma_y} = - \frac{(2)(1)^2(120)(200)(20)}{(127.279,22)(201.990,10)^3}$$

$$= - .000,915,2$$

Assuming all the covariance terms to be zero, then using (43) we obtain

$$\mathrm{Var}[P_{kf}^{(2)}] = 2\ (1 - .93352)\ \{(.002)(.872,505,7)^2 + (50)(.000,864,3)^2$$

$$+ (70)(.000,091,5)^2 + (4)(-.003,457,4)^2$$

$$+ (4)(-.000,915,2)^2\} = .000,214,2$$

$$\sigma_{P_{kf}^{(2)}} = .014,6$$

SECTION IV

LASER GUIDED BOMBS

1. Introduction

For laser-guided bombs, the distances between the mean points of impact (MPI) and the actual impact points cannot be accurately represented by the circular or Rayleigh distribution function. It becomes necessary to devise some other distribution function to represent the delivery errors distribution in the range and deflection directions. Let:

σ_x = standard deviation of delivery errors in the x or range direction;

σ_y = standard deviation of delivery errors in the y or deflection direction;

$F_R(r)$ = the distribution function of the delivery errors R in the radial direction;

α = a positive constant such that $0 < \alpha < 1$.

Then, it has been found experimentally [1] that $F_R(r)$ is more accurately represented by a function of the form

$$F_R(r) = \alpha \left[1 - \exp\left(-\frac{r^2}{2\sigma_x^2}\right)\right] + (1-\alpha)\left[1 - \exp\left(-\frac{r^2}{2\sigma_y^2}\right)\right], \quad 0 < r < \infty \tag{44}$$

$F_R(r)$ is a proper distribution function since $F(0) = 0$ and $F(\infty) = 1$. The corresponding density function is

$$f_R(r) = \frac{\alpha r}{\sigma_x^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) + \frac{(1-\alpha) r}{\sigma_y^2} \exp\left(-\frac{r^2}{2\sigma_y^2}\right), \quad 0 < r < \infty \tag{45}$$

In order to find the joint density function of the delivery errors X and Y in the range and deflection directions, say $f_{X,Y}(x,y)$, it is necessary to make certain assumptions about the random variables R and Θ which represent respectively the delivery errors in the radial direction R and the delivery errors in the argument Θ. The following shall be assumed:

a. The delivery error is equally likely to be at any point in the interval $0 < \theta < 2\pi$, so that the marginal density function of θ is

$$f_{\Theta}(\theta) = \frac{1}{2\pi}, \quad 0 < \theta < 2\pi \tag{46}$$

b. The random variables R and Θ are independently distributed.

The joint probability function of R and θ is:

$$f_R(r)\; f_{\Theta}(\theta) = \frac{1}{2\pi}\left[\frac{\alpha r}{\sigma_x^2}\ \exp\left(-\frac{r^2}{2\sigma_x^2}\right) + \frac{(1-\alpha)r}{\sigma_y^2}\ \exp\left(-\frac{r^2}{2\sigma_y^2}\right)\right],$$

$$0 < r < \infty, \quad 0 < \theta < 2\pi \tag{47}$$

The probability that the weapon will impact in the interval (r, r+dr) and (θ, θ+dθ) is:

$$f_R(r)\; f_{\Theta}(\theta)\ dr\ d\theta = \frac{1}{2\pi}\left[\frac{\alpha}{\sigma_x^2}\ \exp\left(-\frac{r^2}{2\sigma_x^2}\right)\right.$$

$$\left. + \frac{(1-\alpha)}{\sigma_y^2}\ \exp\left(-\frac{r^2}{2\sigma_y^2}\right)\right]\ r\ dr\ d\theta,\ 0 < r < \infty,\ 0 < \theta < 2\pi \tag{48}$$

Changing to Cartesian coordinates, one obtains the probability that the weapon will impact in the interval (x, x+dx), (y, y+dy) or

$$f_{X,Y}(x,y)\ dx\ dy = \frac{1}{2\pi}\left\{\frac{\alpha}{\sigma_x^2}\ \exp\left[-\frac{(x^2+y^2)}{2\sigma_x^2}\right]\right.$$

$$\left. + \frac{(1-\alpha)}{\sigma_y^2}\ \exp\left[-\frac{(x^2+y^2)}{2\sigma_y^2}\right]\right\}\ dx\ dy,\ -\infty < x < \infty,\ -\infty < y < \infty \tag{49}$$

Thus, the joint probability density function of X and Y is

$$f_{X,Y}(x,y) = \alpha \{\frac{1}{\sqrt{2\pi}\,\sigma_x} \exp[-(\frac{x^2}{2\sigma_x^2})] \cdot \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp[-(\frac{y^2}{2\sigma_x^2})]\}$$

$$+ (1-\alpha) \{\frac{1}{\sqrt{2\pi}\,\sigma_y} \exp[-(\frac{x^2}{2\sigma_y^2})] \cdot \frac{1}{\sqrt{2\pi}\,\sigma_y} \exp[-(\frac{y^2}{2\sigma_y^2})]\}$$

$$0 < \alpha < 1, -\infty < x < \infty, -\infty < y < \infty \tag{50}$$

Clearly X and Y are not in general independent. X and Y are independent only in the special cases when $\alpha=0$ and $\alpha=1$. However, $f_{X,Y}(x,y)$ is a proper density function. It may be verified that $f_{X,Y}(x,y)$ is a unimodal function, achieving a maximum at the point (0,0). It may be noted that $f_{X,Y}(x,y)$ is given as a convex combination of density functions of two pairs of independently distributed random variables.

Let the MPI of the weapon be the coordinates (u,v) of the target, in other words the weapon is aimed at the target. Let (dx, dy) be the infinitesimal rectangle close to the point (x,y) at which the weapon impacts. Define the random variables X and Y which measure respectively the distances between the target point and the weapon impact point along the abscissa and the ordinate. Then, from (50) the probability that the weapon will impact in the rectangle dx dy is given by:

$$f_{X,Y}(x-u,y-v)dxdy = \alpha \{\frac{1}{\sqrt{2\pi}\,\sigma_x^2} \exp[-\frac{(x-u)^2 - (y-v)^2}{2\sigma_x^2}]\}$$

$$+ (1-\alpha) \{\frac{1}{2\pi\,\sigma_y^2} \exp[-\frac{(x-u)^2 + (y-v)^2}{2\sigma_y^2}]\} \, dx\, dy \tag{51}$$

2. The Model

To determine the probability of kill of the target for a single weapon in the presence of delivery error, th laws of conditional probabilities are used and we have the probability of kill at (u,v) as

$$P_{kf} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\text{Probability of kill at } (u,v) \mid \text{weapon impacts at } (x,y)]$$
$$[\text{Probability that the weapon impacts between } (x,y) \text{ and } (x+dx,\ y+dy)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D\,(u-x,\ v-y)\ f_{X,Y}(x-u,\ v-y)\ dx\ dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D_0\ \exp[-\ D_0[(\frac{u-x}{R_x})^2 + (\frac{v-y}{R_y})^2]$$
$$(\alpha\ \{\frac{1}{2\Pi\ \sigma_x^2}\ \exp[-\ \frac{(x-u)^2 + (y-v)^2}{2\sigma_x^2}]\}$$
$$+ (1-\alpha)\ \{\frac{1}{2\Pi\ \sigma_y^2}\ \exp[-\ \frac{(x-u)^2 + (y-v)^2}{2\sigma_y^2}]\})\ dx\ dy \qquad (52)$$

Making the change in variables $w = x-u$ and $z = y-v$ yields

$$P_{kf} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D_0\ \exp[-\ D_0(\frac{w^2}{R_x^2} + \frac{z^2}{R_y^2})]\ (\alpha\ \{\frac{1}{2\pi\ \sigma_x^2}\ \exp[-\ \frac{(w^2 + z^2)}{2\sigma_x^2}]\}$$
$$+ (1-\alpha)\ \{\frac{1}{2\pi\ \sigma_y^2}\ \exp[-\ \frac{(w^2 + z^2)}{2\sigma_y^2}]\})\ dx\ dy \qquad (53)$$

This integral may be expressed as a sum involving products of two single integrals:

$$P_{kf} = \frac{\alpha\ D_0}{2\pi\ \sigma_x^2} \int_{-\infty}^{\infty} \exp[-(\frac{D_0\ w^2}{R_x^2} + \frac{w^2}{2\sigma_x^2})]\ dw.\ \int_{-\infty}^{\infty} \exp[-\ (\frac{D_0 z^2}{R_y^2} + \frac{z^2}{2\sigma_x^2})]\,dz$$

$$+ \frac{(1-\alpha)\, D_0}{2\pi\sigma_y^2} \int_{-\infty}^{\infty} \exp[-(\frac{D_0\, w^2}{R_x^2} + \frac{w^2}{2\sigma_y^2})]\, dw \cdot \int_{-\infty}^{\infty} \exp[-(\frac{D_0\, z^2}{R_y^2} + \frac{z^2}{2\sigma_y^2})]\, dz \tag{54}$$

Consider any one of the four integrals. For example, let

$$I_{xx} = \int_{-\infty}^{\infty} \exp[-(\frac{D_0}{R_x^2} + \frac{1}{2\sigma_x^2})\, w^2]\, dw \tag{55}$$

Let $$\frac{\theta}{\sqrt{2}} = (\frac{D_0}{R_x^2} + \frac{1}{2\sigma_x^2})^{\frac{1}{2}}\, w \tag{56}$$

Then

$$\begin{aligned} I_{xx} &= \frac{1}{\sqrt{2}} (\frac{D_0}{R_x^2} + \frac{1}{2\sigma_x^2})^{-\frac{1}{2}} \int_0^{\infty} \exp(-\frac{\theta^2}{2})\, d\theta \\ &= \frac{1}{\sqrt{2}} (\frac{D_0}{R_x^2} + \frac{1}{2\sigma_x^2})^{-\frac{1}{2}} \sqrt{2\pi} \\ &= \sqrt{2\pi}\, R_x\, \sigma_x\, (R_x^2 + 2D_0\, \sigma_x^2)^{-\frac{1}{2}} \end{aligned} \tag{57}$$

The other three integrals may be obtained in a similar fashion. One then immediately obtains for P_{kf}

$$\begin{aligned} P_{kf} &= \alpha\, D_0 \cdot \frac{R_x}{\sqrt{R_x^2 + 2D_0\, \sigma_x^2}} \cdot \frac{R_y}{\sqrt{R_y^2 + 2D_0\, \sigma_x^2}} \\ &+ (1-\alpha)\, D_0\, \frac{R_x}{\sqrt{R_x^2 + 2D_0\, \sigma_y^2}} \cdot \frac{R_y}{\sqrt{R_y^2 + 2D_0\, \sigma_y^2}} \end{aligned} \tag{58}$$

3. Estimation of $E[P_{kf}]$ and $Var[P_{kf}]$

Recall that P_{kf} is a function of the six parameters α, D_0, R_x, R_y, σ_x and σ_y. Let $\bar{\alpha}$, $\bar{D}_0$, $\bar{R}_x$, $\bar{R}_y$, $\bar{\sigma}_x$ and $\bar{\sigma}_y$ refer respectively to the mean of α, D_0, R_x,

R_y, σ_x and σ_y. Expanding P_{kf} about the point $(\bar{\alpha}, \bar{D}_0, \bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y)$ one obtains up to the first order terms:

$$P_{kf}(\alpha, D_0, R_x, R_y, \sigma_x, \sigma_y) = P_{kf}(\bar{\alpha}, \bar{D}_0, \bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y)$$

$$+ (\alpha - \bar{\alpha}) \frac{\partial P_{kf}}{\partial \alpha} + (D_0 - \bar{D}_0) \frac{\partial P_{kf}}{\partial D_0} + (R_x - \bar{R}_x) \frac{\partial P_{kf}}{\partial R_x}$$

$$+ (R_y - \bar{R}_y) \frac{\partial P_{kf}}{\partial R_y} + (\sigma_x - \bar{\sigma}_x) \frac{\partial P_{kf}}{\partial \sigma_x} + (\sigma_y - \bar{\sigma}_y) \frac{\partial P_{kf}}{\partial \sigma_y} \qquad (59)$$

Note that the partial derivatives of P_{kf} with respect to the six variables α, D_0, R_x, R_y, σ_x , and σ_y are to be evaluated at the mean values of the variables. We now provide approximate expressions for $E[P_{kf}]$ and $Var[P_{kf}]$.

a. Estimation of $E[P_{kf}]$

Taking expectations on both sides of (59) yields as a first approximation

$$E[P_{kf}(\alpha, D_0, R_x, R_y, \sigma_x \sigma_y)] = P_{kf}(\bar{\alpha}, \bar{D}_0, \bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y)$$

$$= \bar{\alpha}\, \bar{D}_0 \cdot \frac{\bar{R}_x}{\sqrt{\bar{R}_x^2 + 2\bar{D}_0\, \bar{\sigma}_x^2}} \cdot \frac{\bar{R}_y}{\sqrt{\bar{R}_y^2 + 2\bar{D}_0\, \bar{\sigma}_x^2}}$$

$$+ (1 - \bar{\alpha})\, \bar{D}_0 \cdot \frac{\bar{R}_x}{\sqrt{\bar{R}_x^2 + 2\bar{D}_0\, \bar{\sigma}_y^2}} \cdot \frac{\bar{R}_y}{\sqrt{\bar{R}_y^2 + 2\bar{D}_0\, \bar{\sigma}_y^2}} \qquad (60)$$

b. Estimation of $Var[P_{kf}]$

First, expression (59) is written as

$$P_{kf}(\alpha, D_0, R_x, R_y, \sigma_x, \sigma_y) - P_{kf}(\bar{\alpha}, \bar{D}_0, \bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y)$$

$$= (\alpha - \bar{\alpha}) \frac{\partial P_{kf}}{\partial \alpha} + (D_0 - \bar{D}_0) \frac{\partial P_{kf}}{\partial D_0} + (R_x - \bar{R}_x) \frac{\partial P_{kf}}{\partial R_x}$$

$$+ (R_y - \bar{R}_y) \frac{\partial P_{kf}}{\partial R_y} + (\sigma_x - \bar{\sigma}_x) \frac{\partial P_{kf}}{\partial \sigma_x} + (\sigma_y - \bar{\sigma}_y) \frac{\partial P_{kf}}{\partial \sigma_y} \tag{61}$$

Squaring and taking expectations on both sides of (61) yields

$$\mathrm{Var}[P_{kf}] = \mathrm{Var}[\alpha] \left(\frac{\partial P_{kf}}{\partial \alpha}\right)^2 + \mathrm{Var}[D_0] \left(\frac{\partial P_{kf}}{\partial D_0}\right)^2 + \mathrm{Var}[R_x] \left(\frac{\partial P_{kf}}{\partial R_x}\right)^2$$

$$+ \mathrm{Var}[R_y] \left(\frac{\partial P_{kf}}{\partial R_y}\right)^2 + \mathrm{Var}[\sigma_x] \left(\frac{\partial P_{kf}}{\partial \sigma_x}\right)^2 + \mathrm{Var}[\sigma_y] \left(\frac{\partial P_{kf}}{\partial \sigma_y}\right)^2$$

$$+ \text{covariance terms} \tag{62}$$

Again, it should be noted that all the partial derivatives are evaluated at the point $(\bar{\alpha}, \bar{D}_0, \bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y)$. We now obtain the expressions for the partial derivatives

$$\frac{\partial P_{kf}}{\partial \alpha} = D_0 R_x R_y (R_x^2 + 2D_0 \sigma_x^2)^{-\frac{1}{2}} (R_y^2 + 2D_0 \sigma_x^2)^{-\frac{1}{2}}$$

$$- D_0 R_x R_y (R_x^2 + 2 D_0 \sigma_y^2)^{-\frac{1}{2}} (R_y^2 + 2D_0 \sigma_y^2)^{-\frac{1}{2}} \tag{63}$$

$$\frac{\partial P_{kf}}{\partial D_0} = \alpha R_x R_y [(R_x^2 + 2D_0 \sigma_x^2)^{-\frac{1}{2}} (R_y^2 + 2D_0 \sigma_x^2)^{-\frac{1}{2}}$$

$$- D_0 \sigma_x^2 (R_x^2 + 2D_0 \sigma_x^2)^{-\frac{3}{2}} (R_y^2 + 2 D_0 \sigma_x^2)^{-\frac{1}{2}}$$

$$- D_0 \sigma_x^2 (R_x^2 + 2D_0 \sigma_x^2)^{-\frac{1}{2}} (R_y^2 + 2D_0 \sigma_x^2)^{-\frac{3}{2}}]$$

$$+ (1-\alpha)\, R_x\, R_y \,[(R_x^2 + 2D_0\, \sigma_y^2)^{-\frac{1}{2}} (R_y^2 + 2\, D_0\, \sigma_y^2)^{-\frac{1}{2}}$$

$$- D_0\, \sigma_y^2\, (R_x^2 + 2D_0\, \sigma_y^2)^{-\frac{3}{2}}\, (R_y^2 + 2\, D_0\, \sigma_y^2)^{-\frac{1}{2}}$$

$$- D_0\, \sigma_y^2\, (R_x^2 + 2D_0\, \sigma_y^2)^{-\frac{1}{2}}\, (R_y^2 + 2D_0\, \sigma_y^2)^{-\frac{3}{2}}] \tag{64}$$

$$\frac{\partial P_{kf}}{\partial R_x} = \alpha\, D_0\, R_y\, (R_y^2 + 2D_0\, \sigma_x^2)^{-\frac{1}{2}} [(R_x^2 + 2D_0\, \sigma_x^2)^{-\frac{1}{2}}$$

$$- R_x^2\, (R_x^2 + 2D_0\, \sigma_x^2)^{-\frac{3}{2}}] + (1-\alpha)\, D_0\, R_y\, (R_y^2 + 2D_0\, \sigma_y^2)^{-\frac{1}{2}}$$

$$[(R_x^2 + 2D_0\, \sigma_y^2)^{-\frac{1}{2}} - R_x^2\, (R_x^2 + 2D_0\, \sigma_y^2)^{-\frac{3}{2}}] \tag{65}$$

$$\frac{\partial P_{kf}}{\partial R_y} = \alpha\, D_0\, R_x\, (R_x^2 + 2D_0\, \sigma_x^2)^{-\frac{1}{2}} [(R_y^2 + 2D_0\, \sigma_x^2)^{-\frac{1}{2}}$$

$$- R_y^2\, (R_y^2 + 2D_0\, \sigma_x^2)^{-\frac{3}{2}}] + (1-\alpha)\, D_0\, R_x\, (R_x^2 + 2D_0\, \sigma_y^2)^{-\frac{1}{2}}$$

$$[(R_y^2 + 2D_0\, \sigma_y^2)^{-\frac{1}{2}} - R_y^2\, (R_y^2 + 2D_0\, \sigma_y^2)^{-\frac{3}{2}}] \tag{66}$$

$$\frac{\partial P_{kf}}{\partial \sigma_x} = \alpha\, D_0\, R_x\, R_y\, [- 2D_0\, \sigma_x\, (R_y^2 + 2D_0\, \sigma_x^2)^{-\frac{1}{2}} (R_x^2 + 2D_0\, \sigma_x^2)^{-\frac{3}{2}}$$

$$- 2D_0\, \sigma_x\, (R_x^2 + 2D_0\, \sigma_x^2)^{-\frac{1}{2}} (R_y^2 + 2D_0\, \sigma_x^2)^{-\frac{3}{2}}] \tag{67}$$

$$\frac{\partial P_{kf}}{\partial \sigma_y} = (1-\alpha)\, D_0\, R_x\, R_y\, [- 2D_0\, \sigma_y\, (R_y^2 + 2D_0\, \sigma_y^2)^{-\frac{1}{2}} (R_x^2 + 2D_0\, \sigma_y^2)^{-\frac{3}{2}}$$

$$- 2D_0\, \sigma_y\, (R_x^2 + 2D_0\, \sigma_y^2)^{-\frac{1}{2}} (R_y^2 + 2D_0\, \sigma_y^2)^{-\frac{3}{2}}] \tag{68}$$

c. Numerical Example

Assume that for a laser guided weapon, the parameters of the Carleton damage function are

$$D_0 = 1, \quad R_x = 120 \text{ ft} \quad \text{and} \quad R_y = 200 \text{ ft}$$

These parameters are subject to estimation errors and their variances are given by

$$\mathrm{Var}[D_0] = .002, \quad \mathrm{Var}[R_x] = 50 \text{ ft}^2 \text{ and } \mathrm{Var}[R_y] = 70 \text{ ft}^2$$

The guided weapon is aimed at a point target. The aiming in the x(range) and y (deflection) directions have a joint probability density function given by (51) where

$$\alpha = .75; \quad \sigma_x = 30 \text{ ft}; \quad \sigma_y = 20 \text{ ft}$$

$$\mathrm{Var}[\alpha] = .001; \quad \mathrm{Var}[\sigma_x] = 4 \text{ ft}^2; \quad \mathrm{Var}[\sigma_y] = 4 \text{ ft}^2$$

To obtain the numerical values of the expressions for $E[P_{kf}]$ and the partial derivatives, the following quantities are introduced

$$Q_1^2 = R_x^2 + 2D_0\,\sigma_x^2 \tag{69}$$

$$Q_2^2 = R_y^2 + 2D_0\,\sigma_y^2 \tag{70}$$

$$Q_3^2 = R_x^2 + 2D_0\,\sigma_y^2 \tag{71}$$

$$Q_4^2 = R_y^2 + 2D_0\,\sigma_x^2 \tag{72}$$

Substituting for the values of the parameters yields

$$Q_1^2 = (120)^2 + (2)(1)(30)^2 = 16,200$$

$$Q_1 = 127.279,22$$

$$Q_2^2 = (200)^2 + (2)(1)(20)^2 = 40,800$$

$$Q_2 = 201.990,10$$

$$Q_3^2 = (120)^2 + (2)(1)(20)^2 = 15,200$$

$$Q_3 = 123.288,28$$

$$Q_4^2 = (200)^2 + (2)(1)(30)^2 = 41,800$$

$$Q_4 = 204.450,48$$

Omitting for simplicity the symbol for averages one obtains

$$E[P_{kf}] = \alpha\ D_0\ R_x\ R_y\ Q_1^{-1}\ Q_4^{-1} + (1-\alpha)\ D_0\ R_x\ R_y\ Q_3^{-1}\ Q_2^{-1}$$

$$= D_0\ R_x\ R_y\ [\alpha\ Q_1^{-1}\ Q_4^{-1} + (1-\alpha)\ Q_3^{-1}\ Q_2^{-1}] \qquad (73)$$

$$= (1)(120)(200)\ [\frac{.75}{(127.279,22)(204.450,48)}$$

$$+ \frac{.25}{(123.288,28)(201.990,10)}]$$

$$= .932,649,1$$

Now

$$\frac{\partial P_{kf}}{\partial \alpha} = D_0\ R_x\ R_y\ [Q_1^{-1}\ Q_4^{-1} - Q_3^{-1}\ Q_2^{-1}] \qquad (74)$$

$$= (1)(120)(200)\ [\frac{1}{(127.279,22)(204,450,48)}$$

$$- \frac{1}{(123.288,28)(201,990,10)}] = -.041,452,8$$

$$\frac{\partial P_{kf}}{\partial D_0} = \alpha\ R_x\ R_y\ [Q_1^{-1}\ Q_4^{-1} - D_0\sigma_x^2\ Q_1^{-3}\ Q_4^{-1} - D_0\sigma_x^2\ Q_1^{-1}\ Q_4^{-3}]$$

$$+ (1-\alpha)\ R_x\ R_y\ [Q_3^{-1}\ Q_2^{-1} - D_0\ \sigma_y^2\ Q_3^{-3}\ Q_2^{-1} - D_0\ \sigma_y^2\ Q_3^{-1}\ Q_2^{-3}]$$

$$= \alpha\ R_x\ R_y\ Q_1^{-1}\ Q_4^{-1}\ [1 - D_0\ \sigma_x^2\ Q_1^{-2} - D_0\ \sigma_x^2\ Q_4^{-2}]$$

$$+ (1-\alpha)\ R_x\ R_y\ Q_2^{-1}\ Q_3^{-1}\ [1 - D_0\ \sigma_y^2\ Q_3^{-2} - D_0\ \sigma_y^2\ Q_2^{-2}] \qquad (75)$$

$$= \frac{(.75)(120)(200}{(127.\ 279,22)(204.450,48)}\ [1 - \frac{(1)(30)^2}{16,200} - \frac{(1)(30)^2}{41,800}]$$

$$+ \frac{(.25)(120)(200)}{(201.990,10)(123.288,28)}\ [1 - \frac{(1)(20)^2}{15,200} - \frac{(1)(20)^2}{40,800}]$$

$$= (.691,714,4)(.922,913,3) + (.240,934,7)(.963,8802)$$

$$= .870,6246$$

$$\frac{\partial P_{kf}}{\partial R_x} = \alpha\ D_0\ R_y\ Q_4^{-1}\ [Q_1^{-1} - R_x^2\ Q_1^{-3}] + (1-\alpha)\ D_0\ R_y\ Q_2^{-1}\ [Q_3^{-1} - R_x^2\ Q_3^{-3}]$$

$$= \alpha\ D_0\ R_y\ Q_1^{-1}\ Q_4^{-1}\ [1 - R_x^2\ Q_1^{-2}] + (1-\alpha)\ D_0\ R_y\ Q_2^{-1}\ Q_3^{-1}[1 - R_x^2\ Q_3^{-2}] \qquad (76)$$

$$= \frac{(.75)(1)(200)}{(127.279,22)(204.450,48)} [1 - \frac{(120)^2}{16,200}]$$

$$+ \frac{(.25)(1)(200)}{(201.990,10)(123.288,28)} [1 - \frac{(120)^2}{15,200}]$$

$$= (.005,764,2)(.111,111,1) + (.002,007,7)(.052,631,5)$$

$$= .000,746,1$$

$$\frac{\partial P_{kf}}{\partial R_y} = \alpha D_0 R_x Q_1^{-1} [Q_4^{-1} - R_y^2 Q_4^{-3}] + (1-\alpha) D_0 R_x Q_3^{-1} [Q^{-1}_2 - R_y^2 Q_2^{-3}]$$

$$= \alpha D_0 R_x Q_1^{-1} Q_4^{-1} [1 - R_y^2 Q_4^{-2}] + (1-\alpha) D_0 R_x Q_2^{-1} Q_3^{-1} [1 - R_y^2 Q_2^{-2}] \quad (77)$$

$$= \frac{(.75)(1)(120)}{(127.279,22)(204.450,48)} [1 - \frac{(200)^2}{41,800}]$$

$$+ \frac{(.25)(1)(120)}{(201.990,10)(123.288,28)} [1 - \frac{(200)^2}{40,800}]$$

$$= (.0003,458,5)(.043,062,2) + (.001,204,6)(.019,607,8)$$

$$= .000,172,5$$

$$\frac{\partial P_{kf}}{\partial \sigma_x} = - 2\alpha D_0^2 \sigma_x R_x R_y [Q_4^{-1} Q_1^{-3} + Q_1^{-1} Q_4^{-3}]$$

$$= - 2\alpha D_0^2 \sigma_x R_x R_y Q_1^{-1} Q_4^{-1} [Q_1^{-2} + Q_4^{-2}] \quad (78)$$

$$= - \frac{(2)(.75)(1)^2(30)(120)(200)}{(127.279,22)(204.450.48)} [\frac{1}{16,200} + \frac{1}{41,800}]$$

$$= - .003,554,7$$

$$\frac{\partial P_{kf}}{\partial \sigma_y} = -2(1-\alpha)\, D_0^2\, \sigma_y\, R_x\, R_y\, [Q_2^{-1}\, Q_3^{-3} + Q_3^{-1}\, Q_2^{-3}]$$

$$= -2(1-\alpha)\, D_0^2\, \sigma_x\, R_x\, R_y\, Q_2^{-1}\, Q_3^{-1}\, [Q_3^{-2} + Q_2^{-2}] \qquad (79)$$

$$= \frac{(2)(.25)(1)^2(20)(120)(200)}{(201.990,10)(123.288,28)} \left[\frac{1}{15,200} + \frac{1}{40,800}\right]$$

$$= -.000,870,2$$

To determine the variance of P_{kf}, we assume α, D_0, R_x, R_y, σ_x and σ_y to be independent so that

$$\mathrm{Var}[P_{kf}] = (\mathrm{Var}[\alpha]) \left(\frac{\partial P_{kf}}{\partial \alpha}\right)^2 + (\mathrm{Var}[D_0]) \left(\frac{\partial P_{kf}}{\partial D_0}\right)^2 + (\mathrm{Var}[R_x]) \left(\frac{\partial P_{kf}}{\partial R_x}\right)^2$$

$$+ (\mathrm{Var}[R_y]) \left(\frac{\partial P_{kf}}{\partial R_y}\right)^2 + (\mathrm{Var}[\sigma_x]) \left(\frac{\partial P_{kf}}{\partial \sigma_x}\right)^2 + (\mathrm{Var}[\sigma_y]) \left(\frac{\partial P_{kf}}{\partial \sigma_y}\right)^2$$

$$= (.001)(-.041,452,8)^2 + (.002)(.870,624,6)^2$$

$$+ (50)(.000,746,1)^2 + (70)(.000,172,5)^2$$

$$+ (4)(-.003,554,7)^2 + (4)(-.000,870,2)^2$$

$$= .001,601,1$$

$$\sigma_{P_{kf}} = .0400$$

A two-standard deviation confidence interval on P_{kf} is

$$P_{kf} = E[P_{kf}] \pm 2\,\sigma_{P_{kf}}$$

$$= .9326 \pm .0800$$

We note, in particular, that the contribution to the total variance of D_0 is 94.7%.

SECTION V

GUIDED WEAPONS

1. Introduction

In the most recent JTCG/ME manual on the "Derivation of JMEM/AS Open-End Methods" (1983)[2], the methodology for computing range and deflection delivery errors for guided weapons is outlined. It is assumed that in the range direction, the aimpoint is offset by a negative value, that is the center of the distribution of impact points is short of the aimpoint. No such offset occurs in the deflection direction. The amount of offset is a correction factor which can be introduced as a bias in the distribution of delivery errors in the range direction. This bias is assumed to be a known input to the problem. The distribution of delivery errors in each of the x(range) and y (deflection) directions are assumed to be normally and independently distributed with the same standard deviation ($\sigma_x = \sigma_y = \sigma$).

The problem treated in this section is quite general and considers the presence of biases in both the range and deflection directions with $\sigma_x \neq \sigma_y$. By setting the bias in the deflection direction equals to zero and $\sigma_x = \sigma_y = \sigma$, the distribution of delivery errors for guided weapons is obtained. By setting the biases in both direction equal to zero, the distribution of delivery errors for general purpose bombs is obtained.

2. Weapon Delivery with Offset or Bias

In studying the delivery of a guided weapon (idealized as a point) upon a point target, several factors have to be accounted for in developing an expression for the probability of kill of the target. Among such factors, the coordinates of the various interplaying variables should be considered, particularly if the weapon is not aimed directly at the target or the presence of bias in the delivery of weapon cannot be avoided.

Let then;

(u,v) = location of the point target on the ground plane;

(ξ,η) = coordinates of the desired mean point of impact (MPI) of the weapon, or the coordinates of the point at which the weapon is aimed or aimpoint;

(x,y) = coordinates of the actual weapon impact point;

(ℓ,m) = coordinates of the center of impact points (CIP).

A little elaboration is needed on the last set of coordinates defined. If there is no bias in the delivery of the weapon, then, the CIP coincides with the MPI. If on the other hand bias exists, then the CIP will be different from the MPI. Thus, in general, $\ell \neq \xi$ and $m \neq \eta$.

Assume now that the x direction is the direction of the range and the y direction is the direction of deflection. Suppose now that there is a delivery error when releasing the weapon. This error is measured in each of the x and y directions. The delivery error in the x direction is the distance along the abscissa between the weapon impact point and the center of impact points or $(x-\ell)$. Similarly, the delivery error in the y direction is the distance along the ordinate between the weapon impact point and the center of impact points or $(y-m)$. Assume $(x-\ell)$ and $(y-m)$ to be independently and normally distributed with respective standard deviations σ_x and σ_y. Thus,

$$f_1\,(x-\ell) = \frac{1}{\sqrt{2\pi}\,\sigma_x}\,\exp\left[-\frac{(x-\ell)^2}{2\sigma_x^2}\right] \tag{80}$$

$$f_2\ (y-m) = \frac{1}{\sqrt{2\pi}\ \sigma_y}\ \exp[-\frac{(y-m)^2}{2\sigma_y^2}] \tag{81}$$

Assume that the bias in the x direction and the bias in the y direction are known. Thus, let

$b_x = \xi - \ell$ = bias in the x direction or range offset,

$b_y = \eta - m$ = bias in the y direction or deflection offset.

Thus,

$$\begin{aligned} \ell &= \xi - b_x \\ m &= \eta - b_y \end{aligned} \tag{82}$$

and it becomes possible to express the probability density functions of the delivery errors in terms of the MPI and the offset. Thus

$$f_1\ (x-\xi+b_x) = \frac{1}{\sqrt{2\pi}\ \sigma_x}\ \exp[-\frac{(x-\xi+b_x)^2}{2\sigma_x^2}] \tag{83}$$

$$f_2\ (y-\eta+b_y) = \frac{1}{\sqrt{2\pi}\ \sigma_y}\ \exp[-\frac{(y-\eta+b_y)^2}{2\sigma_y^2}] \tag{84}$$

3. The Model

If we assume that the damage function is represented by the Carleton damage function, then the probability of kill at (u,v) given that the weapon impacts at (x,y) is

$$D(u-x,\ v-y) = D_0\ \exp\{-D_0\ [(\frac{u-x}{R_x})^2 + (\frac{v-y}{R_y})^2]\} \tag{85}$$

To determine the probability of kill of the target in the presence of delivery error, the laws of conditional probabilities are used and we have:

P_{kf} = Probability of kill at (u,v)

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{[Probability of kill at (u,v)|weapon impacts at (x,y)]}.$$

$$\text{[Probability weapon impacts between (x,y) and (x+dx, y+dy)]}.$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(u-x,\ v-y)\ f_1\ (x-\xi+b_x)\ f_2(y-\eta + b_y)\ dx\ dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D_0 \exp\{-D_0 [\frac{(u-x)^2}{R_x^2} + \frac{(v-y)^2}{R_y^2}]\}$$

$$\cdot \frac{1}{\sqrt{2\pi}\ \sigma_x} \exp[-\frac{(x-\xi + b_x)^2}{2\sigma_x^2}] \cdot \frac{1}{2\pi\sigma_y} \exp[-\frac{(y-\eta + b_y)^2}{2\sigma_y^2}]\ dx\ dy \qquad (86)$$

In general, thus, P_{kf} is a function of

a. the location of the target (u,v);

b. the coordinates of the MPI (ξ,η);

c. the biases b_x and b_y in the x and y directions.

To evaluate the double integral in the expression for P_{kf}, write P_{kf} as the product of two single integrals.

$$P_{kf} = D_0 \cdot \frac{1}{\sqrt{2\pi}\ \sigma_x} \cdot \frac{1}{\sqrt{2\pi}\ \sigma_y}$$

$$\cdot \int_{-\infty}^{\infty} \exp\{-[\frac{D_0(u-x)^2}{R_x^2} + \frac{(x-\xi + b_x)}{2\sigma_x^2}]\}\ dx$$

$$\cdot \int_{-\infty}^{\infty} \exp\{-[\frac{D_0(v-y)^2}{R_y^2} + \frac{(y-\eta + b_y)^2}{2\sigma_y^2}]\}\ dy \qquad (87)$$

Let

$$w = x-\xi + b_x \tag{88}$$

$$z = y-\eta + b_y \tag{89}$$

Then

$$P_{kf} = \frac{D_0}{2\pi\sigma_x\sigma_y} \int_{-\infty}^{\infty} \exp\{-[D_0(\frac{(u-\xi + b_x)-w)^2}{R_x}) + \frac{w^2}{2\sigma_x^2}]\} \, dx$$

$$\cdot \int_{-\infty}^{\infty} \exp\{-[D_0 (\frac{(v-\eta + b_y)-z}{R_y})^2 + \frac{z^2}{2\sigma_y^2}]\} \, dy$$

$$= \frac{R_x R_y}{2q_1 q_2} \exp\{-\tfrac{1}{2} [(\frac{u-\xi + b_x}{q_1})^2 + (\frac{v-\eta + b_y}{q_2})^2]\}$$

$$= P_{kf}(u-\xi + b_x, v-\eta + b_y) \tag{90}$$

where

$$q_1^2 = \frac{R_x^2}{2D_0} + \sigma_x^2 \tag{91}$$

$$q_2^2 = \frac{R_y^2}{2D_0} + \sigma_y^2 \tag{92}$$

4. Special Cases

Several special cases are now considered

a. Case of no bias

Here $b_x=0$ and $b_y=0$ and one obtains

$$P_{kf} (u-\xi, u-\eta) = \frac{R_x R_y}{2q_1 q_2} \exp\{-\tfrac{1}{2} [(\frac{u-\xi}{q_1})^2 + (\frac{v-\eta}{q_2})^2]\} \tag{93}$$

b. Case of no bias in the y direction (guided weapons)

This is the case corresponding to guided weapons and the expression for P_{kf} becomes

$$P_{kf}\,(u-\xi+b_x,\ v-\eta) = \frac{R_x R_y}{2q_1 q_2}\exp\{-\frac{1}{2}[(\frac{u-\xi+b_x}{q_1})^2 + (\frac{v-\eta}{q_2})^2]\} \tag{94}$$

c. Case of no bias in the y direction and weapon is aimed at target.

Here, the desired MPI coincides with the point target ($u=\xi$, $v=\eta$):

$$P_{kf}\,(b_x,\ b_y) = \frac{R_x R_y}{2q_1 q_2}\exp\{-\frac{1}{2}\ [(\frac{b_x}{q_1})^2 + (\frac{b_y}{q_2})^2]\} \tag{95}$$

5. Estimation of $E[P_{kf}]$ and $Var[P_{kf}]$

Let

$$B_x = u-\xi + b_x \tag{96}$$

$$B_y = v-\eta + b_y \tag{97}$$

The expression for P_{kf} becomes, in general,

$$P_{kf}\,(B_x,\ B_y) = \frac{R_x R_y}{2q_1 q_2}\exp\{-\frac{1}{2}[(\frac{B_x}{q_1})^2 + (\frac{B_y}{q_2})^2]\} \tag{98}$$

where

$$q_1^2 = \frac{R_x^2}{2\ D_0} + \sigma_x^2 \tag{99}$$

$$q_2^2 = \frac{R_y^2}{2\ D_0} + \sigma_y^2 \tag{100}$$

Note here that P_{kf} is a function of seven parameters namely D_0, R_x, R_y, σ_x, σ_y, B_x and B_y. We shall assume that B_x and B_y are precisely known and are not subject to estimation error, but that the five other parameters D_0, R_x, R_y, σ_x and σ_y may be subject to estimation error. It shall be assumed that both the means and the variances of the parameters are known. First we obtain an explicit expression for P_{kf}. Substituting for q_1 and q_2 in $P_{kf}(B_x, B_y)$ yields

$$P_{kf}(B_x, B_y) = \frac{R_x R_y}{2\sqrt{\frac{R_x^2}{2D_0} + \sigma_x^2}\sqrt{\frac{R_y^2}{2D_0} + \sigma_y^2}}$$

$$\cdot \exp\left\{-\frac{1}{2}\left[\frac{B_x^2}{\frac{R_x^2}{2D_0} + \sigma_x^2} + \frac{B_y^2}{\frac{R_y^2}{2D_0} + \sigma_y^2}\right]\right\}$$

$$= D_0 \frac{R_x}{\sqrt{R_x^2 + 2D_0\sigma_x^2}} \exp\left[-\frac{1}{2}\left(\frac{2D_0B_x^2}{R_x^2 + 2D_0\sigma_x^2}\right)\right]$$

$$\cdot \frac{R_y}{\sqrt{R_y^2 + 2D_0\sigma_y^2}} \exp\left[-\frac{1}{2}\left(\frac{2D_0B_y^2}{R_y^2 + 2D_0\sigma_y^2}\right)\right] \tag{101}$$

The Taylor's series estimation procedure will be used to obtain approximate expressions for $E[P_{kf}]$ and $Var[P_{kf}]$ in terms of the means and variances of the five input parameters D_0, R_x, R_y, σ_x and σ_y.

Let $\bar{D}_0$, $\bar{R}_x$, $\bar{R}_y$, $\bar{\sigma}_x$ and $\bar{\sigma}_y$ refer respectively to the mean of D_0, R_x, R_y, σ_x and σ_y. Expanding P_{kf} about the point $(\bar{D}_0, \bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y)$ one obtains up to the first order terms:

$$P_{kf}(D_0, R_x, R_y, \sigma_x, \sigma_y) = P_{kf}(\bar{D}_0, \bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y)$$

$$+ (D_0 - \bar{D}_0) \frac{\partial P_{kf}}{\partial D_0} + (R_x - \bar{R}_x) \frac{\partial P_{kf}}{\partial R_x} + (R_y - \bar{R}_y) \frac{\partial P_{kf}}{\partial R_y}$$

$$+ (\sigma_x - \bar{\sigma}_x) \frac{\partial P_{kf}}{\partial \sigma_x} + (\sigma_y - \bar{\sigma}_y) \frac{\partial P_{kf}}{\partial \sigma_y} \qquad (102)$$

Note that the partial derivative of P_{kf} with respect to the five variables D_0, R_x, R_y, σ_x and σ_y are to be evaluated at the mean values of the variables. Approximate expressions for $E[P_{kf}]$ and $Var[P_{kf}]$ are now provided.

a. Estimation of $E[P_{kf}]$

Taking expectations on both sides of (102) yields as a first approximation

$$E[P_{kf}(D_0, R_x, R_y, \sigma_x, \sigma_y)] = P_{kf} (\bar{D}_0, \bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y)$$

$$= \frac{\bar{R}_x \bar{R}_y}{2 \bar{q}_1 \bar{q}_2} \exp\{- \frac{1}{2} [(\frac{B_x}{\bar{q}_1})^2 + (\frac{B_y}{\bar{q}_2})^2]\} \qquad (103)$$

where $$\bar{q}_1^2 = \frac{\bar{R}_x^2}{2 \; D_0} + \bar{\sigma}_x^2 \qquad (104)$$

$$\bar{q}_2^2 = \frac{\bar{R}_y^2}{2 \; \bar{D}_0} + \bar{\sigma}_y^2 \qquad (105)$$

b. Estimation of $Var[P_{kf}]$

First expression (102) is written as

$$P_{kf} (D_0, R_x, R_y, \sigma_x, \sigma_y) - P_{kf} (\bar{D}_0, \bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y)$$

$$= (D_0 - \bar{D}_0) \frac{\partial P_{kf}}{\partial D_0} + (R_x - \bar{R}_x) \frac{\partial P_{kf}}{\partial R_x} + (R_y - \bar{R}_y) \frac{\partial P_{kf}}{\partial R_y}$$

$$+ (\sigma_x - \bar{\sigma}_x) \frac{\partial P_{kf}}{\partial \sigma_x} + (\sigma_y - \bar{\sigma}_y) \frac{\partial P_{kf}}{\partial \sigma_y} \tag{106}$$

Squaring and taking expectations on both sides of (106) yields

$$\begin{aligned} \mathrm{Var}[P_{kf}] = {} & \mathrm{Var}[D_0] \left(\frac{\partial P_{kf}}{\partial D_0}\right)^2 + \mathrm{Var}[R_x] \left(\frac{\partial P_{kf}}{\partial R_x}\right)^2 \\ & + \mathrm{Var}[R_y] \left(\frac{\partial P_{kf}}{\partial R_y}\right)^2 + \mathrm{Var}[\sigma_x] \left(\frac{\partial P_{kf}}{\partial \sigma_x}\right)^2 \\ & + \mathrm{Var}[\sigma_y] \left(\frac{\partial P_{kf}}{\partial \sigma_y}\right)^2 + \text{covariance terms} \end{aligned} \tag{107}$$

Again, it should be noted that all partial derivatives are evaluated at the point $(\bar{D}_0, \bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y)$. We now otbain the expressions for the partial derivatives

a. $\underline{\partial P_{kf}/\partial D_0}$

$$\begin{aligned} \frac{\partial P_{kf}}{\partial D_0} = {} & \{R_x (R_x^2 + 2 D_0 \sigma_x^2)^{-\frac{1}{2}} \exp[- D_0 B_x^2 (R_x^2 + 2 D_0 \sigma_x^2)^{-1}] \\ & \cdot R_y (R_y^2 + 2 D_0 \sigma_y^2)^{-\frac{1}{2}} \exp[- D_0 B_y^2 (R_y^2 + 2 D_0 \sigma_y^2)^{-1}]\} \\ & + \{- D_0 R_x \sigma_x^2 (R_x^2 + 2 D_0 \sigma_x^2)^{-\frac{3}{2}} \exp[- D_0 B_x^2 (R_x^2 + 2 D_0 \sigma_x^2)^{-1}] \\ & \cdot R_y (R_y^2 + 2 D_0 \sigma_y^2)^{-\frac{1}{2}} \exp[- D_0 B_y^2 (R_y^2 + 2 D_0 \sigma_y^2)^{-1}]\} \\ & + \{D_0 R_x (R_x^2 + 2 D_0 \sigma_x^2)^{-\frac{1}{2}} [- B_x^2 (R_x^2 + 2 D_0 \sigma_x^2)^{-1} + 2 D_0 B_x^2 \sigma_x^2 (R_x^2 + 2 D_0 \sigma_2^2)^{-2}] \\ & \cdot \exp[- D_0 B_x^2 (R_x^2 + 2 D_0 \sigma_x^2)^{-1}] \cdot R_y (R_y^2 + 2 D_0 \sigma_y^2)^{-\frac{1}{2}} \end{aligned}$$

$$\cdot \exp[- D_0 B_y^2 (R_y^2 + 2 D_o \sigma_y^2)^{-1}]\}$$

$$+ \{D_0 R_x (R_x^2 + 2 D_0 \sigma_x^2)^{-\frac{1}{2}} \exp[- D_0 B_x^2 (R_x^2 + 2 D_0 \sigma_x^2)^{-1}]$$

$$\cdot \{- R_y \sigma_y^2 (R_y^2 + 2 D_0 \sigma_y^2)^{-\frac{3}{2}}\} \exp[- D_0 B_y^2 (R_y^2 + 2 D_0 \sigma_y^2)^{-1}]\}$$

$$+ \{D_0 R_x (R_x^2 + 2 D_0 \sigma_x^2)^{-\frac{1}{2}} \exp[- D_0 B_x^2 (R_x^2 + 2 D_0 \sigma_x^2)^{-1}]$$

$$\cdot R_y (R_y^2 + 2 D_0 \sigma_y^2)^{-\frac{1}{2}} [- B_y^2 (R_y^2 + 2 D_0 \sigma_y^2)^{-1}$$

$$+ 2 D_0 B_y^2 \sigma_y^2 (R_y^2 + 2 D_0 \sigma_y^2)^{-2}] \exp[- D_0 B_y^2 (R_y^2 + 2 D_0 \sigma_y^2)^{-1}]\} \qquad (108)$$

b. <u>$\partial P_{kf}/\partial R_x$</u>

$$\frac{\partial P_{kf}}{\partial R_x} = D_0 R_y (R_y^2 + 2 D_0 \sigma_y^2)^{-\frac{1}{2}} \exp[- D_0 B_y^2 (R_y^2 + 2 D_0 \sigma_y^2)^{-1}]$$

$$\{(R_x^2 + 2 D_0 \sigma_x^2)^{-\frac{1}{2}} \exp[- D_0 B_x^2 (R_x^2 + 2 D_0 \sigma_x^2)^{-1}]$$

$$- R_x^2 (R_x^2 + 2 D_0 \sigma_x^2)^{-\frac{3}{2}} \exp[- D_0 B_x^2 (R_x^2 + 2 D_0 \sigma_x^2)^{-1}]$$

$$+ 2 D_0 B_x^2 R_x^2 (R_x^2 + 2 D_0 \sigma_x^2)^{-\frac{1}{2}} (R_x^2 + 2 D_0 \sigma_x^2)^{-2}$$

$$\cdot \exp[- D_0 B_x^2 (R_x^2 + 2 D_0 \sigma_x^2)^{-1}]\} \qquad (109)$$

c. <u>$\partial P_{kf}/\partial R_y$</u>

$$\frac{\partial P_{kf}}{\partial R_y} = D_0 R_x (R_x^2 + 2 D_0 \sigma_x^2)^{-\frac{1}{2}} \exp[- D_0 B_x^2 (R_x^2 + 2 D_0 \sigma_x^2)^{-1}]$$

$$\cdot \{(R_y^2 + 2\,D_0\,\sigma_y^2)^{-\frac{1}{2}}\ \exp[-\,D_0\,B_y^2\,(R_y^2 + 2\,D_0\,\sigma_y^2)^{-1}]$$

$$-\,R_y^2\,(R_y^2 + 2\,D_0\,\sigma_y^2)^{-\frac{3}{2}}\ \exp[-\,D_0\,B_y^2\,(R_y^2 + 2\,D_0\,\sigma_y^2)^{-1}]$$

$$+\,2\,D_0\,B_y^2\,R_y^2\,(R_y^2 + 2\,D_0\,\sigma_y^2)^{-\frac{1}{2}}\,(R_y^2 + 2\,D_0\,\sigma_y^2)^{-2}$$

$$\cdot \exp[-\,D_0\,B_y^2\,(R_y^2 + 2\,D_0\,\sigma_y^2)^{-1}]\} \qquad (110)$$

d. $\partial P_{kf}/\partial\sigma_x$

$$\frac{\partial P_{kf}}{\partial\sigma_x} = D_0\,R_x\,R_y\,(R_y^2 + 2\,D_0\,\sigma_y^2)^{-\frac{1}{2}}\ \exp[-\,D_0\,B_y^2\,(R_y^2 + 2\,D_0\,\sigma_y^2)^{-1}]$$

$$\cdot \{\,-2\,D_0\,\sigma_x\,(R_x^2 + 2\,D_0\,\sigma_x^2)^{-\frac{3}{2}}\ \exp[-\,D_0\,B_x^2\,(R_x^2 + 2\,D_0\,\sigma_x^2)^{-1}]$$

$$+\,4\,D_0^2\,B_x^2\,\sigma_x\,(R_x^2 + 2\,D_0\,\sigma_x^2)^{-\frac{1}{2}}\,(R_x^2 + 2\,D_0\,\sigma_x^2)^{-2}$$

$$\cdot \exp[-\,D_0\,B_x^2\,(R_x^2 + 2\,D_0\,\sigma_x^2)^{-1}]\} \qquad (111)$$

e. $\partial P_{kf}/\partial\sigma_y$

$$\frac{\partial P_{kf}}{\partial\sigma_y} = D_0\,R_x\,R_y\,(R_x^2 + 2\,D_0\,\sigma_x^2)^{-\frac{1}{2}}\ \exp[-\,D_0\,B_x^2\,(R_x^2 + 2\,D_0\,\sigma_x^2)^{-1}]$$

$$\cdot \{-2\,D_0\,\sigma_y\,(R_y^2 + D_0\,\sigma_y^2)^{-\frac{3}{2}}\ \exp[-\,D_0\,B_y^2\,(R_y^2 + 2\,D_0\,\sigma_y^2)^{-1}]$$

$$+\,4\,D_0^2\,B_y^2\,\sigma_y\,(R_y^2 + 2\,D_0\,\sigma_y^2)^{-\frac{1}{2}}\,(R_y^2 + 2\,D_0\,\sigma_y^2)^{-2}$$

$$\cdot \exp[-\,D_0\,B_y^2\,(R_y^2 + 2\,D_0\,\sigma_y^2)^{-1}]\} \qquad (112)$$

c. Numerical Example

Assume that for a guided weapon the parameters of the Carleton damage function are

$$D_0 = 1, \quad R_x = 120 \text{ ft and } R_y = 200 \text{ ft}$$

These parameters are subject to estimation errors and their variances are given by

$$\mathrm{Var}[D_0] = .002, \quad \mathrm{Var}[R_x] = 50 \text{ ft}^2 \text{ and } \mathrm{Var}[R_y] = 70 \text{ ft}^2$$

The guided weapon is aimed at a point target. Thus,

$$u = \xi \text{ and } v = \eta$$

The aiming errors in the x (range) and y (deflection) directions are normally and independently distributed with the same standard deviation

$$\sigma_x = \sigma_y = \sigma = 20 \text{ ft}$$

and $$\mathrm{Var}[\sigma] = 4 \text{ ft}^2$$

The bias in the range direction is $b_x = 20$ ft. There is no bias in the deflection direction: $b_y = 0$ ft. It, thus, follows that

$$B_x = u-\xi+b_x = 20 \text{ ft}$$

$$B_y = v-\eta+b_y = 0 \text{ ft.}$$

Assume that all the covariance terms are zero. For simplicity, the bar (—) notation for expectation will be omitted.

To obtain the numerical values for $\bar{E}[P_{kf}]$ and the partial derivatives, the following quantities are introduced:

$$Q_1^2 = R_x^2 + 2\, D_0\, \sigma_x^2 \tag{113}$$

$$Q_2^2 = R_y^2 + 2\, D_0\, \sigma_y^2 \tag{114}$$

$$E_1 = \exp\{-\, D_0\, B_x^2\, [R_x^2 + 2\, D_0\, \sigma_x^2]^{-1}\}$$

$$= \exp[-\, D_0\, B_x^2\, Q_1^{-2}] \tag{115}$$

$$E_2 = \exp\{-\, D_0\, B_y^2\, [R_y^2 + 2\, D_0\, \sigma_y^2]^{-1}\}$$

$$= \exp[-\, D_0\, B_y^2\, Q_2^{-2}] \tag{116}$$

Substituting for the values of the parameters yields

$$Q_1^2 = (120)^2 + (2)(1)(20)^2$$

$$= 15,200$$

Thus, $\quad Q_1 = 123,288,28$

$$Q_2^2 = (200)^2 + (2)(1)(20)^2$$

$$= 40,800$$

Thus, $Q_2 = 201.990,10$

$$E_1 = \exp[-\frac{(1)(20)^2}{15,200}] = .974,027,4$$

$$E_2 = \exp[-\frac{(1)(0)^2}{40,800}] = 1$$

Using (113), (114), (115) and (16) in (103), the following expression for $E[P_{kf}]$ is obtained

$$E[P_{kf}] = D_0 R_x Q_1^{-1} E_1 R_y Q_2^{-1} E_2$$

$$= \frac{(1)(120)(.974,027,4)(200)(1)}{(123.288,28)(201.990,10)}$$

$$= .938,708$$

The partial derivatives are now computed. They are first rewritten using (113), (114), (115) and (116).

From (108)

$$\frac{\partial P_{kf}}{\partial D_0} = \{R_x Q_1^{-1} E_1 R_y Q_2^{-1} E_2\} + \{- D_0 R_x \sigma_x^2 Q_1^{-3} E_1 R_y Q_2^{-1} E_2\}$$

$$+ \{[- D_0 R_x B_x^2 Q_1^{-3} + 2 D_0^2 B_x^2 R_x \sigma_x^2 Q_1^{-5}] E_1 R_y Q_2^{-1} E_2$$

$$+ \{- D_0 R_x Q_1^{-1} E_1 R_y \sigma_y^2 Q_2^{-3} E_2\}$$

$$+ \{D_0 R_x Q_1^{-1} E_1 R_y Q_2^{-1}[-B_y^2 Q_2^{-2} + 2 D_0 B_y^2 \sigma_y^2 Q_2^{-4}] E_2\}$$

$$= R_x R_y Q_1^{-1} Q_2^{-1} E_1 E_2 \{1 - D_0 \sigma_x^2 Q_1^{-2} - D_0 B_x^2 Q_1^{-2} + 2 D_0^2 B_x^2 \sigma_x^2 Q_1^{-4}$$

$$- D_0 \sigma_y^2 Q_2^{-2} + D_0 B_y^2 Q_2^{-2} [-1 + 2 D_0 \sigma_y^2 Q_2^{-2}]\} \qquad (117)$$

$$\frac{\partial P_{kf}}{\partial D_0} = \frac{(120)(200)(.974,027,4)(1)}{(123.288,28)(201.990,10)} \{1 - \frac{(1)(20)^2}{15,200} - \frac{(1)(20)^2}{15,200}$$

$$+ \frac{(2)(1)^2(20)^2(20)^2}{(15,200)^2} - \frac{(1)(20)^2}{40,800}\}$$

$$= .938,708 [1 - .026,315,7 - .026,315,7 + .001,385 - .009,803,9]$$

$$= .881,399,5$$

From (109)

$$\frac{\partial P_{kf}}{\partial R_x} = D_0 R_y Q_2^{-1} E_2 [Q_1^{-1} E_1 - R_x^2 Q_1^{-3} E_1 + 2 D_0 B_x^2 R_x^2 Q_1^{-5} E_1]$$

$$= D_0 R_y Q_1^{-1} Q_2^{-1} E_1 E_2 [1 - Q_1^{-2} R_x^2 (1 - 2 D_0 B_x^2 Q_1^{-2})] \qquad (118)$$

$$= \frac{(1)(200)(.974,027,4)(1)}{(123.288,28)(201.990,10)} [1 - \frac{(120)^2}{15,200} (1 - \frac{(2)(1)(20)^2}{15,200})]$$

$$= .007,822,5 [1- .947,368,4 (1 - .052,631,5)]$$

$$= .000,801,7$$

From (110)

$$\frac{\partial P_{kf}}{\partial R_y} = D_0 R_x Q_1^{-1} E_1 [Q_2^{-1} E_2 - R_y^2 Q_2^{-3} E_2 + 2 D_0 B_y^2 R_y^2 Q_2^{-5} E_2]$$

$$= D_0 R_x Q_1^{-1} Q_2^{-1} E_1 E_2 [1 - Q_2^{-2} R_y^2 (1 - 2 D_0 B_y^2 Q_2^{-2})] \qquad (119)$$

$$= \frac{(1)(120)(.974,027,4)(1)}{(123.288,28)(201.990.10)} \left[1 - \frac{(200)^2}{40,800}(1-0)\right]$$

$$= .004,693,5\ (1 - .980,392,1)$$

$$= .000,092$$

From (111)

$$\frac{\partial P_{kf}}{\partial \sigma_x} = D_0\ R_x\ R_y\ Q_2^{-1}\ E_2\ (-2\ D_0\ \sigma_x\ Q_1^{-3}\ E_1 + 4\ D_0^2\ B_x^2\ \sigma_x\ Q_1^{-5}\ E_1)$$

$$= 2\ D_0^2\ R_x\ R_y\ Q_1^{-3}\ Q_2^{-1}\ \sigma_x\ E_1\ E_2\ (-1 + 2\ D_0\ B_x^2\ Q_1^{-2}) \qquad (120)$$

$$= \frac{(2)(1)^2(120)(200)(20)(.974,027,4)(1)}{(123.288,28)^3\ (201.990,10)} \left[-1 + \frac{(2)(1)(20)^2}{15,200}\right]$$

$$= .002,470,2\ (-1 + .052,631,5)$$

$$= -.002,340,1$$

From (112)

$$\frac{\partial P_{kf}}{\partial \sigma_y} = D_0\ R_x\ R_y\ Q_1^{-1}\ E_1\ (-2\ D_0\ \sigma_y\ Q_2^{-3}\ E_2 + 4\ D_0^2\ B_y^2\ \sigma_y\ Q_2^{-5}\ E_2)$$

$$= 2\ D_0^2\ R_x\ R_y\ Q_1^{-1}\ Q_2^{-3}\ \sigma_y\ E_1\ E_2\ (-1 + 2\ D_0\ B_y^2\ Q_2^{-2}) \qquad (121)$$

$$= \frac{(2)(1)^2(120)(200)(20)(.974,027,4)(1)}{(123.288,28)(201.990,10)^3}(-1)$$

$$= -.000,920,3$$

To determine the variance of P_{kf}, expression (107) is used

$$\mathrm{Var}[P_{kf}] = \mathrm{Var}[D_0]\left(\frac{\partial P_{kf}}{\partial D_0}\right)^2 + \mathrm{Var}[R_x]\left(\frac{\partial P_{kf}}{\partial R_x}\right)^2 + \mathrm{Var}[R_y]\left(\frac{\partial P_{kf}}{\partial R_y}\right)^2$$

$$+ \mathrm{Var}[\sigma_x]\left(\frac{\partial P_{kf}}{\partial \sigma_x}\right)^2 + \mathrm{Var}[\sigma_y]\left(\frac{\partial P_{kf}}{\partial \sigma_y}\right)^2 \qquad (122)$$

$$= (.002)(.881,399,5)^2 + (50)(.000,801,7)^2$$

$$+ (70)(.000.092)^2 + (4)(-.002,3401)^2$$

$$+ (4)(-.000,9203)^2$$

$$= .001,611,7$$

Thus, $\sigma_{P_{kf}} = .0402$

A two-standard deviation confidence interval on $E[P_{kf}]$ is

$$P_{kf} = E[P_{kf}] \pm 2\,\sigma_{P_{kf}}$$

$$= .9387 \pm .0804$$

It is noted, in particular, that the contribution of $\mathrm{Var}[D_0]$ to the total variance is about 96.4%.

SECTION VI

CONCLUSIONS AND RECOMMENDATIONS

In this report, the modeling of multiple weapons delivery has been attempted. Details related to the stick delivery of weapons are described in a separate report [3]. In addition, the modeling of the delivery of guided weapons has been undertaken under two differnt sets of assumptions for the distribution of delivery error.

Although the underlying methodology for tackling these different situations is the same, nevertheless, the statistical characteristics of the distribution of delivery errors constitute the main element which brings forth the differences between the various models.

The solutions to the problems related to multiple weapons are far from being complete. For example, the approximate solution to the general stick delivery problem as partially reported in [3] needs some verification using the exact approach presented in Section II in this report. In addition, the multiple weapon delivery under independent passes remains to be solved. Questions such as the effect of the number of weapons delivered per pass or the number of passes on the variability in P_k have not been answered. Finally, it is conceivable that the general problem of weapon delivery be approached as a single mathematical model in which the distribution of the delivery error takes a general form. The different situations studied so far as separate problems may then be recovered as special cases to this general problem.

REFERENCES

[1] JTCG/ME, "Derivation of JMEM/AS Open-End Methods" (Air-to-Surface), 61 JTCG/ME-3-7, May 1980.

[2] JTCG/ME, "Derivation of JMEM/AS Open-End Methods" (Air-to-Surface), Revised, 61 JTCG/ME-3-7, November 1983.

[3] Sivazlian, B. D. "Variability of Measures of Weapons Effectiveness" Volume VII: Effectiveness Indices in Stick Delivery of Weapons, Air Force Armament Laboratory, AFATL/DLYW, Eglin Air Force Base, FL 32542, November 1984.

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