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Variability of Measures of Weapons Effectiveness

Volume I: Methodology and Application to Fragment Sensitive Targets in the Absence of Delivery Error

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<p>The problem of computing the uncertainty associated with the probability of kill P_{kf} when using the two-parameter Carleton damage function is considered. It is assumed that the input parameters are not known exactly, and that the measure of uncertainty in these parameters is given. Two types of estimation procedures are used: the Taylor's series estimation procedure and a subjective estimation procedure. Two inferential estimation procedures are also discussed.</p>			
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PREFACE

This report describes work done in the summer of 1983 by Dr B. D. Sivazlian, Department of Industrial and Systems Engineering, the University of Florida, Gainesville, Florida 32611 under Contract No. F08635-83-C-0202 with the Air Force Armament Laboratory (AFATL), Armament Division, Eglin Air Force Base, Florida 32542. The program manager was Mr. Daniel A. McInnis (DLYW).

The work was initiated under a 1982 USAF-SCEEE Summer Faculty Research Program sponsored by the Air Force Office of Scientific Research conducted by the Southeastern Center for Electrical Engineering Education under Contract No. F49620-82-C-0035.

This work addresses itself to the problem of computing the uncertainty associated with the probability of kill when using the two-parameter Carleton damage function as specified in the Joint Munitions Effectiveness Manual/Air-to-Surface (JMEM/AS) open end methods as described in 61 JTCG/ME-3-7 (Revised 15 May 1980).

The two-parameter Carleton damage function approximates the probability of kill due to fragmentation of an exploding weapon in the absence of blast effect and delivery error. Further, it assumes that at the center of the exploding weapon, the probability of kill is unity. It thus excludes weapon/target situations in which such probability of kill results in a number less than unity.

The author has benefited from helpful discussions with several people. Particular thanks are due to Mr Jerry Bass, Mr Daniel McInnis, Mr Charles Reynolds, and Ms Katherine H. Douglas, all from DLYW who have read the report and have contributed to it through helpful comments.

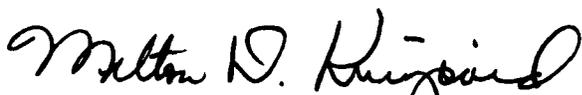
The report is the first of a series dealing with the uncertainty associated with various weapon effectiveness indices, and details

methodologies and techniques used in computing such uncertainties in the presence of error in the input parameters.

The Public Affairs Office has reviewed this report, and it is releasable to the National Technical Information Service (NTIS), where it will be available to the general public, including foreign nationals.

This technical report has been reviewed and is approved for publication.

FOR THE COMMANDER



MILTON D. KINGCAID, Colonel, USAF
Chief, Analysis and Strategic Defense Division

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SECTION 1
INTRODUCTION

The present report discusses methodologies for estimating the probability of kill, P_k , taken as a measure of weapons effectiveness. In addition, the report addresses itself to the important question of estimating the variability in P_k in the presence of uncertainty.

Section II elaborates on the techniques available to determine the mean and the variance of P_k , denoted, respectively, by $E[P_k]$ and $\text{Var}[P_k]$, when P_k is expressed as a mathematical function of given input parameters, say X and Y . It is assumed here that none of X or Y are known precisely but are subject to estimation error. Such error may arise if, for example, X and Y are measured subjectively, or X and Y are obtained through some type of inferential estimation procedure such as the use of multiple regression scheme.

Section III applies the methodology to the particular situation of fragment sensitive targets in the absence of blast and aiming error. The probability of kill in such an instance can be approximated by the Carleton damage function

$$P_{kf} = \exp\left[-\left(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2}\right)\right].$$

Here it is assumed that the weapon explodes at $(0,0)$ and the target is located at (x,y) . R_x and R_y are two parameters identifying the weapon's radii. The general approach assumes that estimates of R_x and R_y , as well as the errors in such estimates, are available, and the problem addresses itself to estimating P_{kf} and its variance.

Section IV goes one step beyond and looks at the problem of actually estimating R_x and R_y as well as their covariance matrix. It is assumed that experimental results obtained from fragmentation field data of an exploding weapon are used to estimate the values of R_x and R_y in the Carleton damage function. Two methods are explored, both based on the theory of linear multiple regression.

Some concluding statements are made in Section V.

At this point two remarks are in order. The first remark concerns the Carleton damage function. In the cases considered in this report, it is implicitly assumed that, at the point of weapon explosion, the probability of kill is unity. Further, one assumes that the equiprobability contour lines in the (x,y) plane are ellipses with axes coinciding with the x and y axes. These conditions are fairly reasonably satisfied when the impact angle of the weapon is close to 90 degrees and the x and y axes are taken to be the directions of the weapon range and deflection, respectively. Thus, one may use the two-parameter Carleton damage function as a model for this situation. When the impact angle is smaller than 90 degrees, the equiprobability contour lines do not, in general, follow elliptic patterns and are not in general symmetrical about the deflection axis. However, because of its simplicity, the Carleton damage function is still used in practice.

When the probability of kill at the point of impact is less than unity, the three-parameter Carleton damage function given by

$$P_{kf} = D_0 \exp\left[-D_0\left(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2}\right)\right]$$

is a suggested model (see e.g., [6]). Notice that in this case, since $0 < P_{kf} \leq 1$, one must necessarily have $0 < D_0 \leq 1$. In what follows, only the two-parameter Carleton damage function is considered ($D_0=1$).

The second remark concerns the fragmentation field data for computing P_{kf} . It will be assumed that such data are available and that they have been processed to exhibit in matrix form the value of P_{kf} at a target point (x,y) , when the weapon impacts at $(0,0)$ with a given angle and at a given velocity.

SECTION II

METHODOLOGY

1. Background

In many practical situations, a measure of the effectiveness of an exploding weapon upon a target is usually expressed by the probability of kill P_k . The computation of this P_k value is most often undertaken by using a mathematical expression or model which is a function of the target position relative to the center of explosion, as well as a function of one or more parameters. These parameters are, in general, established either subjectively or experimentally or both, and they account for a number of factors which may be present at the time of the explosion, such as weapon characteristics, impact angle, impact velocity, nature of target, etc. An example of a P_k function is the Carleton damage function which was previously introduced. Thus, given a set of conditions, the formula allows one to compute a P_k value for any location of the target relative to the weapon. However, one must realize that this formula, like any other, is only an approximation whose usefulness is dictated by how accurately it represents reality. This is due to the fact that (a) the mathematical model is not an exact replica of the actual situation and (b) the parameters in the model are estimates (random variables) rather than exact quantities.

A natural question that may be raised in this context is how good is the P_k value computed from the mathematical model; or, precisely what is the error of the P_k value? To answer this question, one should note that P_k being a function of the estimates is itself a random variable. Theoretically speaking, given the joint distribution of the parameters, the distribution of P_k could be obtained, thus providing a means for calculating, for any confidence level, interval estimates for P_k . However, in practice this is not

possible due to several reasons. First, from an economic point of view, inadequate data are available to describe fully the joint distribution of the parameters. Second, P_k is, in general, a complex function of the parameters, and thus, characterizing its distribution becomes very difficult, if not impossible. Finally, the intent is not to set up accurate confidence intervals on P_k , but rather to provide the decision maker with adequate information on how P_k behaves statistically in the presence of estimation error in the input parameters. Thus, very often the parameters will often be characterized by their first two moments. Similarly, the establishment of the first two moments of P_k will be adequate for our purpose.

For argument sake, suppose that the probability of kill, P_k , is a function of the two input parameters X and Y so that

$$P_k = P_k (X, Y) . \quad (1)$$

Suppose that each of the parameters X and Y are estimates subject to error. It is required to determine $E[P_k]$ and $\text{Var}[P_k]$.

If the joint distribution of X and Y is known and is, say, $f_{X,Y} (x,y)$, then

$$E[P_k] = \int_{x,y} P_k (x,y) f_{X,Y} (x,y) dx dy$$

and
$$E[P_k^2] = \int_{x,y} P_k^2 (x,y) f_{X,Y} (x,y) dx dy .$$

From these two expressions one can obtain

$$\text{Var}[P_k] = E[P_k^2] - E^2[P_k].$$

It should be noted that although it is impossible to completely characterize the function $f_{X,Y} (x,y)$, nevertheless, there exists methodologies to arrive at approximate estimates of $E[P_k]$ and $\text{Var}[P_k]$. We shall digress on two such methods which are:

- a. the subjective estimation procedure
- b. the Taylor's series estimation procedure.

Each of these methods assumes that certain statistical information is available on all input parameters which enter in the computation of P_k .

2. The Subjective Estimation Procedure

In the subjective estimation procedure, it is assumed that the uncertainty level of each input parameter is provided as subjective information. A lower and upper bound value for each parameter is obtained. The value of a particular parameter is assumed to take equally likely values between its two extreme points. This is equivalent to assuming that each parameter is a random variable uniformly distributed over its range of values. Further, the parameters are assumed to be mutually independent random variables. With this statistical information, the evaluation of $E[P_k]$ and $E[P_k^2]$ are reduced to the computation of a set of definite integrals. Thus, referring to (1), if X is assumed to be uniformly distributed in the interval $[x_1, x_2]$, while Y is assumed to be uniformly distributed in the interval $[y_1, y_2]$, then, if X and Y are mutually independent, it follows that

$$E[P_k] = \frac{1}{(x_2-x_1)(y_2-y_1)} \int_{y_1}^{y_2} \int_{x_1}^{x_2} P_k(x,y) dx dy$$

$$E[P_k^2] = \frac{1}{(x_2-x_1)(y_2-y_1)} \int_{y_1}^{y_2} \int_{x_1}^{x_2} P_k^2(x,y) dx dy .$$

Note here that under these assumptions, the mean and variance of X and Y are given, respectively, by

$$E[X] = \frac{x_1+x_2}{2} \quad ; \quad \text{Var}[X] = \frac{(x_2-x_1)^2}{12}$$

$$E[Y] = \frac{y_1+y_2}{2} \quad ; \quad \text{Var}[Y] = \frac{(y_2-y_1)^2}{12} .$$

Although the method often provides expressions for $E[P_k]$ and $E[P_k^2]$ which can be analytically manipulated to arrive at closed form expressions, nevertheless, it has certain inherent disadvantages which should be stated at this stage. First, there is no guarantee that closed form expressions can be obtained for $E[P_k]$ and $E[P_k^2]$. If such expressions are derivable, they are usually fairly complex in form. Second, the method does not make any allowance for incorporating a dependency factor between X and Y , if such dependency is known to exist. A third disadvantage of the method lies in the fact that it is not possible to segregate the contribution of the variance of each parameter component to the variance of the probability of kill P_k . Finally, the numerical computation of $\text{Var}[P_k]$ requires that the computed values of $E[P_k]$ and $E[P_k^2]$ be carried to several significant digits.

3. The Taylor's Series Estimation Procedure

In the Taylor's series estimation procedure, the assumption is made that the statistical moments of the input parameters are known and that $P_k(X,Y)$ is a differentiable function of X and Y . The expression for the probability of kill P_k is expanded as a Taylor's series about the expected value of the input parameters. Only first order terms are assumed significant in the derivation that follows. However, in general, one could use a procedure parallel to that outlined here if higher order terms beyond the first are to be included.

Let $\bar{X} = E[X]$ and $\bar{Y} = E[Y]$, then

$$P_k(X,Y) = P_k(\bar{X},\bar{Y}) + (X-\bar{X}) \left. \frac{\partial P_k}{\partial X} \right|_{\bar{X},\bar{Y}} + (Y-\bar{Y}) \left. \frac{\partial P_k}{\partial Y} \right|_{\bar{X},\bar{Y}} . \quad (2)$$

Taking expectations on both sides of (2) yields

$$E[P_k(X, Y)] = P_k(\bar{X}, \bar{Y}) .$$

Thus, in the Taylor's series estimation procedure, the probability of kill computed at the mean values of the parameters is taken as an estimate of the mean of the probability of kill. This estimate is an approximation which is adequate for most practical problems. Further accuracy may be obtained by the inclusion of higher order terms in the Taylor's series expansion. For example, the addition of the second order term to (2) yields

$$\frac{1}{2!} \left\{ (X-\bar{X})^2 \frac{\partial^2 P_k}{\partial X^2} \Big|_{\bar{X}, \bar{Y}} + 2(X-\bar{X})(Y-\bar{Y}) \frac{\partial^2 P_k}{\partial X \partial Y} \Big|_{\bar{X}, \bar{Y}} + (Y-\bar{Y})^2 \frac{\partial^2 P_k}{\partial Y^2} \Big|_{\bar{X}, \bar{Y}} \right\}$$

which upon taking expectations gives

$$\frac{1}{2} \text{Var}[X] \frac{\partial^2 P_k}{\partial X^2} \Big|_{\bar{X}, \bar{Y}} + \text{Cov}[X, Y] \frac{\partial^2 P_k}{\partial X \partial Y} \Big|_{\bar{X}, \bar{Y}} + \frac{1}{2} \text{Var}[Y] \frac{\partial^2 P_k}{\partial Y^2} \Big|_{\bar{X}, \bar{Y}} .$$

This last expression would be added to $P_k(\bar{X}, \bar{Y})$ to improve the accuracy of the value of $E[P_k(X, Y)]$.

To obtain an estimate of the variance of P_k , write (2) as

$$P_k(X, Y) - P_k(\bar{X}, \bar{Y}) = (X-\bar{X}) \frac{\partial P_k}{\partial X} \Big|_{\bar{X}, \bar{Y}} + (Y-\bar{Y}) \frac{\partial P_k}{\partial Y} \Big|_{\bar{X}, \bar{Y}} .$$

Squaring both sides yields

$$\begin{aligned} [P_k(X, Y) - P_k(\bar{X}, \bar{Y})]^2 &= (X-\bar{X})^2 \left(\frac{\partial P_k}{\partial X} \right)^2 \Big|_{\bar{X}, \bar{Y}} + (Y-\bar{Y})^2 \left(\frac{\partial P_k}{\partial Y} \right)^2 \Big|_{\bar{X}, \bar{Y}} \\ &+ 2(X-\bar{X})(Y-\bar{Y}) \left(\frac{\partial P_k}{\partial X} \right) \left(\frac{\partial P_k}{\partial Y} \right) \Big|_{\bar{X}, \bar{Y}} . \end{aligned} \quad (3)$$

Taking expectations on both sides of (3) and remembering that $P_k(\bar{X}, \bar{Y}) = E[P_k(X, Y)]$ yields

$$\begin{aligned} \text{Var}[P_k(X, Y)] = & \text{Var}[X] \left(\frac{\partial P_k}{\partial X} \right)^2 \Big|_{\bar{X}, \bar{Y}} + \text{Var}[Y] \left(\frac{\partial P_k}{\partial Y} \right)^2 \Big|_{\bar{X}, \bar{Y}} \\ & + 2 \text{Cov}[X, Y] \left(\frac{\partial P_k}{\partial X} \right) \left(\frac{\partial P_k}{\partial Y} \right) \Big|_{\bar{X}, \bar{Y}} . \end{aligned} \quad (4)$$

Formula (4) provides a mean for computing the uncertainty in the value of $P_k(X, Y)$ as a function of the uncertainty in the values of the input parameters X and Y , namely $\text{Var}[X]$, $\text{Var}[Y]$ and $\text{Cov}[X, Y]$. Of course, in case when X and Y are independently distributed, the covariance term vanishes and expression (4) involves only the variance components.

The advantages of this method are three fold. First, it is possible to obtain fairly simple expressions for $E[P_k]$ and $\text{Var}[P_k]$. Second, in case when the input parameters are correlated, the uncertainty in P_k can reflect the extent of this correlation through a covariance term. Finally, the contribution to the P_k variance of each variance component can be identified and segregated. With an objective towards reducing the P_k variance, the methodology allows one to break down the P_k variance into its components and to identify those input parameters with the largest variance contribution. It should be noted that the method fails in cases where P_k is not a differentiable function of the input parameters.

As was mentioned earlier, when using the Taylor's series estimation procedure, a decision has to be made on how many terms are to be retained in the expansion. As a first approximation, only first order terms are usually retained. Improved accuracy in both $E[P_k]$ and $\text{Var}[P_k]$ could be obtained by the inclusion of second and higher order terms. This however, would generate

cumbersome mathematical expressions and would require a knowledge of the higher moments of X and Y. In what follows, only first order terms are considered in order to maintain the simplicity of the expressions derived. An extensive discussion of the Taylor's series estimation procedure is included in [2].

4. Remarks

a. Experience has shown that in cases where X and Y are independently distributed, the subjective estimation procedure and the Taylor's series estimation procedure provide results for $E[P_k]$ and $\text{Var}[P_k]$ which are in close numerical agreement. This will be verified later on in this report when the methods are applied to fragment sensitive targets in the presence of no aiming error.

b. In establishing confidence intervals for P_k , a two standard deviation ($\alpha=2$) two-sided confidence interval is selected. In such a case, Chebyshev's inequality guarantees at least a 75 percent confidence interval since for $\alpha=2$ we have

$$P\{E[P_k] - \alpha\sqrt{\text{Var}[P_k]} \leq P_k \leq E[P_k] + \alpha\sqrt{\text{Var}[P_k]}\} \geq 1 - \frac{1}{\alpha^2} = .75 .$$

SECTION III

APPLICATION OF THE ESTIMATION PROCEDURES

1. The Carleton Damage Function

For fragment sensitive targets, in the absence of aiming error, the Carleton damage function may be used as a model to measure the value of the probability of kill due to fragmentation, P_{kf} , at an arbitrary point (x,y) in the plane given that the weapon bursts at the origin $(0,0)$. This damage function has the form

$$P_{kf} = \exp \left[-\left(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2} \right) \right] \quad (5)$$

where R_x and R_y are two parameters known as weapon radii. R_x is the weapon radius in the direction of range which is the direction of the main weapon axis (the x-axis). R_y is the weapon radius in the direction of deflection (y-axis) perpendicular to the x-axis. In general, $R_y > R_x$.

It is assumed now that the parameters R_x and R_y are not known exactly and that they are subject to estimation errors. Under these conditions, it is required to estimate the error in P_{kf} . To do this, one needs to evaluate both $E[P_{kf}]$ and $\text{Var}[P_{kf}]$ in order to specify an interval estimation for P_{kf} for any (x,y) . The estimates of R_x and R_y are assumed to be given either in the form of a minimum value and a maximum value through a subjective estimation procedure, or in the form of a vector mean and a covariance matrix through an inferential estimation procedure. Given the estimates of R_x and R_y , the problem is to determine $E[P_{kf}]$ and $\text{Var}[P_{kf}]$. We shall consider next the application of each of the two methods discussed in Section II to this particular problem.

2. The Use of the Subjective Estimation Procedure

Here it is assumed that R_x and R_y are uniformly and independently distributed over the respective ranges $R_{x_1} < R_x < R_{x_2}$ and $R_{y_1} < R_y < R_{y_2}$. The quantities R_{x_1} , R_{x_2} , R_{y_1} , and R_{y_2} are supposed to be known. They may be determined, for example, through a subjective procedure in which individuals are requested to provide a lower and upper bound on the values of R_x and R_y based on their judgement and their experience. The main objective is to determine $E[P_{kf}]$ and $\text{Var}[P_{kf}]$.

a. Estimation of $E[P_{kf}]$

Using (5), the expectation of P_{kf} is given by:

$$\begin{aligned} E[P_{kf}] &= \frac{1}{(R_{y_2} - R_{y_1})(R_{x_2} - R_{x_1})} \int_{R_{y_1}}^{R_{y_2}} \int_{R_{x_1}}^{R_{x_2}} \exp\left[-\left(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2}\right)\right] dR_x dR_y \\ &= \frac{1}{(R_{y_2} - R_{y_1})(R_{x_2} - R_{x_1})} \int_{R_{y_1}}^{R_{y_2}} \exp\left(-\frac{y^2}{R_y^2}\right) dR_y \int_{R_{x_1}}^{R_{x_2}} \exp\left(-\frac{x^2}{R_x^2}\right) dR_x. \quad (6) \end{aligned}$$

To proceed further, one needs to evaluate integrals of the form

$$J(A,B) = \int_A^B \exp\left(-\frac{k^2}{u^2}\right) du.$$

Now making the change in variable $v = \frac{1}{u}$ in this integral expression yields

$$J(A,B) = \int_{\frac{1}{B}}^{\frac{1}{A}} e^{-k^2 v^2} \cdot \frac{1}{v^2} dv.$$

Integrating by parts yields

$$\begin{aligned}
 J(A,B) &= -\frac{1}{v} e^{-k^2 v^2} \left[\frac{1}{A} - \int \left(-\frac{1}{v}\right) (-2k^2 v) e^{-k^2 v^2} dv \right. \\
 &\quad \left. \frac{1}{B} \right] \\
 &= B \exp\left(-\frac{k^2}{B^2}\right) - A \exp\left(-\frac{k^2}{A^2}\right) - 2k^2 \int \frac{1}{v} e^{-k^2 v^2} dv .
 \end{aligned}$$

In the integral expression let $kv = \frac{w}{\sqrt{2}}$; then $dv = \frac{1}{k\sqrt{2}} dw$ and

$$\begin{aligned}
 J(A,B) &= B \exp\left(-\frac{k^2}{B^2}\right) - A \exp\left(-\frac{k^2}{A^2}\right) - k\sqrt{2} \int \frac{\frac{k\sqrt{2}}{A} e^{-\frac{w^2}{2}}}{\frac{k\sqrt{2}}{B}} dw \\
 &= B \exp\left(-\frac{k^2}{B^2}\right) - A \exp\left(-\frac{k^2}{A^2}\right) - k\sqrt{2} \left[\int_0^{\frac{k\sqrt{2}}{A}} e^{-\frac{w^2}{2}} dw - \int_0^{\frac{k\sqrt{2}}{B}} e^{-\frac{w^2}{2}} dw \right] \\
 &= B \exp\left(-\frac{k^2}{B^2}\right) - A \exp\left(-\frac{k^2}{A^2}\right) - 2k\sqrt{\pi} \left[\frac{1}{\sqrt{2\pi}} \int_0^{\frac{k\sqrt{2}}{A}} e^{-\frac{w^2}{2}} dw \right. \\
 &\quad \left. - \frac{1}{\sqrt{2\pi}} \int_0^{\frac{k\sqrt{2}}{B}} e^{-\frac{w^2}{2}} dw \right] .
 \end{aligned}$$

Let
$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{w^2}{2}} dw .$$

Thus, $\phi(z)$ is the area under the standardized normal curve to the right of $z=0$. These values are readily available, and can be obtained from existing tables (see e.g., [1]). The expression for $J(A,B)$ can be written as:

$$J(A,B) = B \exp\left(-\frac{k^2}{B^2}\right) - A \exp\left(-\frac{k^2}{A^2}\right) - 2k\sqrt{\pi} \left[\phi\left(\frac{k\sqrt{2}}{A}\right) - \phi\left(\frac{k\sqrt{2}}{B}\right) \right] . \quad (7)$$

The expression for $E[P_{kf}]$ given in (6) can be written using (7) as

$$\begin{aligned}
 E[P_{kf}] = & \frac{1}{(R_{y_2} - R_{y_1})(R_{x_2} - R_{x_1})} \cdot \{R_{x_2} \exp(-\frac{x^2}{R_{x_2}^2}) - R_{x_1} \exp(-\frac{x^2}{R_{x_1}^2}) \\
 & - 2x \sqrt{\pi} [\phi(\frac{x\sqrt{2}}{R_{x_1}}) - \phi(\frac{x\sqrt{2}}{R_{x_2}})] \{R_{y_2} \exp(-\frac{y^2}{R_{y_2}^2}) \\
 & - R_{y_1} \exp(-\frac{y^2}{R_{y_1}^2}) - 2y\sqrt{\pi} [\phi(\frac{y\sqrt{2}}{R_{y_1}}) - \phi(\frac{y\sqrt{2}}{R_{y_2}})]\} . \quad (8)
 \end{aligned}$$

b. Estimation of $\text{Var}[P_{kf}]$

The following expression results after squaring both sides of (5)

$$P_{kf}^2 = \exp[-(\frac{2x^2}{R_x^2} + \frac{2y^2}{R_y^2})] . \quad (9)$$

It is thus evident that $E[P_{kf}^2]$ can be obtained directly from $E[P_{kf}]$ given in (8) by replacing x and y , respectively, by $x\sqrt{2}$ and $y\sqrt{2}$. This substitution yields

$$\begin{aligned}
 E[P_{kf}^2] = & \frac{1}{(R_{y_2} - R_{y_1})(R_{x_2} - R_{x_1})} \cdot \{R_{x_2} \exp(-\frac{2x^2}{R_{x_2}^2}) - R_{x_1} \exp(-\frac{2x^2}{R_{x_1}^2}) \\
 & - 2x\sqrt{\pi} [\phi(\frac{2x}{R_{x_1}}) - \phi(\frac{2x}{R_{x_2}})] \{R_{y_2} \exp(-\frac{2y^2}{R_{y_2}^2}) \\
 & - R_{y_1} \exp(-\frac{2y^2}{R_{y_1}^2}) - 2y\sqrt{2\pi} [\phi(\frac{2y}{R_{y_1}}) - \phi(\frac{2y}{R_{y_2}})]\} . \quad (10)
 \end{aligned}$$

The expression for $\text{Var}[P_{kf}]$ can be obtained by using (8) and (10) in the formula

$$\text{Var}[P_{kf}] = E[P_{kf}^2] - \{E[P_{kf}]\}^2 . \quad (11)$$

c. Example

It is required to determine an estimate of the probability of kill P_{kf} for a fragment sensitive target located at $x=50$ ft and $y=100$ ft when a weapon detonates at $(0,0)$. The blast effect is neglected, and it is assumed that the Carleton damage function given by expression (5) can reasonably be used to compute P_{kf} . A subjective evaluation of the weapon radii identifies the value of R_x to be at least 80 ft but no more than 90 ft. Similarly, the value of R_y is judged to be at least 160 ft but no more than 180 ft.

To proceed with the calculations, it is assumed that the parameters R_x and R_y are independently and uniformly distributed over the respective ranges $80 < R_x < 90$ and $160 < R_y < 180$. Further, the requirement is to calculate not only $E[P_{kf}]$, but also $\text{Var}[P_{kf}]$ in order to place confidence bounds on P_{kf} . Note that $R_{x_1} = 80$ ft, $R_{x_2} = 90$ ft, $R_{y_1} = 160$ ft, and $R_{y_2} = 180$ ft.

The expression for $E[P_{kf}]$ is given by (8). Substituting for the numerical values of $x, y, R_{x_1}, R_{x_2}, R_{y_1}$ and R_{y_2} yields

$$\begin{aligned}
 E[P_{kf}] &= \frac{1}{(180-160)(90-80)} \cdot \{90 \exp[-\left(\frac{50}{90}\right)^2] - 80 \exp[-\left(\frac{50}{80}\right)^2]\} \\
 &\quad - (2)(50)\sqrt{\pi} \left[\Phi\left(\frac{50\sqrt{2}}{80}\right) - \Phi\left(\frac{50\sqrt{2}}{90}\right) \right] \{180 \exp[-\left(\frac{100}{180}\right)^2] \\
 &\quad - 160 \exp[-\left(\frac{100}{160}\right)^2] - (2)(100)\sqrt{\pi} \left[\Phi\left(\frac{100\sqrt{2}}{160}\right) - \Phi\left(\frac{100\sqrt{2}}{180}\right) \right]\} \\
 &= \frac{1}{(20)(10)} \{66.099,930,47 - 54.130,707,69 \\
 &\quad - 100\sqrt{\pi} [\Phi(.883,883,5) - \Phi(.785,674,2)]\} \\
 &\quad \cdot \{132.199,860,9 - 108.261,415,4 - 200\sqrt{\pi} [\Phi(.883,833,5) \\
 &\quad - \Phi(.785,674,2)]\} . \tag{12}
 \end{aligned}$$

The values of the function $\phi(\cdot)$ are obtained from the tabulated values in [1] using linear interpolation. As an example, to compute $\phi(.883,883,5)$, the following two values are read from the table:

$$\phi(.90) = .315,939,88$$

$$\phi(.88) = .310,570,35 .$$

It thus follows that

$$\begin{aligned} \phi(.883,883,5) &= \phi(.88) + \frac{[\phi(.90) - \phi(.88)][.883,883,5 - .88]}{(.90 - .88)} \\ &= .310,570,35 + \frac{(.005,369,53)(.003,883,5)}{.02} \\ &= .311,612,98 . \end{aligned}$$

One obtains similarly

$$\begin{aligned} \phi(.785,674,2) &= \phi(.78) + \frac{[\phi(.80) - \phi(.78)][.785,674,2 - .78]}{(.80 - .78)} \\ &= .282,304,56 \\ &\quad + \frac{(.288,144,60 - .282,304,56)(.005,674,2)}{.02} \\ &= .283,961,44. \end{aligned}$$

Substituting for these numerical values in (12) yields

$$E[P_{k_f}] = .499,582,48.$$

In a similar fashion, using (10) one obtains for $E[P_{k_f}^2]$

$$\begin{aligned}
E[P_{kf}^2] &= \frac{1}{(180-160)(90-80)} \{ 90 \exp[-2 \left(\frac{50}{90}\right)^2] - 80 \exp[-2 \left(\frac{50}{80}\right)^2] \\
&\quad - (2)(50) \sqrt{2\pi} \left[\phi\left(\frac{(2)(50)}{80}\right) - \phi\left(\frac{(2)(50)}{90}\right) \right] \} \\
&\quad \{ 180 \exp[-2 \left(\frac{100}{180}\right)^2] - 160 \exp[-2 \left(\frac{100}{160}\right)^2] \\
&\quad - (2)(100) \sqrt{2\pi} \left[\phi\left(\frac{(2)(100)}{160}\right) - \phi\left(\frac{(2)(100)}{180}\right) \right] \} \\
&= \frac{1}{(20)(10)} \{ 48.546,675,65 - 36.626,668,94 \\
&\quad - 100 \sqrt{2\pi} \left[\phi(1.25) - \phi(1.111,111,1) \right] \} \\
&\quad \{ 97.093,351,30 - 73.253,337,88 \} \\
&\quad - 200 \sqrt{2\pi} \left[\phi(1.25) - \phi(1.111,111,1) \right] \} .
\end{aligned}$$

From the tables one obtains after using linear interpolation

$$\phi(1.25) = .394,338,81$$

and $\phi(1.111,111,1) = .366,727,93$.

Substituting these numerical values in the expression for $E[P_{kf}^2]$ yields

$$E[P_{kf}^2] = .249,898,556,3 .$$

Using (11), the following numerical value for $\text{Var}[P_{kf}]$ is obtained

$$\begin{aligned}
\text{Var}[P_{kf}] &= E[P_{kf}^2] - \{E[P_{kf}]\}^2 \\
&= .249,898,556,3 - (.499,582,48)^2 \\
&= .000,315,902 .
\end{aligned}$$

The standard deviation of P_{kf} is:

$$\begin{aligned}
\sigma_{P_{kf}} &= \sqrt{\text{Var}[P_{kf}]} \\
&= \sqrt{.000,315,902} = .0178 .
\end{aligned}$$

Thus, using Chebyshev's inequality, with at least a 75 percent confidence interval, we have

$$\begin{aligned}
P_{kf} &= E[P_{kf}] \pm 2\sigma_{P_{kf}} \\
&= .4996 \pm .0356 .
\end{aligned}$$

3. The Use of the Taylor's Series Estimation Procedure

Referring once more to the Carleton damage function given by expression (5), the parameters R_x and R_y are assumed to be subject to estimation error and thus cannot be determined accurately. These parameters are to be specified at least in terms of their first two moments, that is, in terms of their statistical means

$$\bar{R}_x = E[R_x] \quad \text{and} \quad \bar{R}_y = E[R_y] \tag{13}$$

and in terms of a covariance matrix defined by

$$\begin{pmatrix} \text{Var}[R_x] & \text{Cov}[R_x, R_y] \\ \text{Cov}[R_x, R_y] & \text{Var}[R_y] \end{pmatrix} \tag{14}$$

In case R_x and R_y are subjectively estimated in terms of their ranges, say $R_{x_1} < R_x < R_{x_2}$ and $R_{y_1} < R_y < R_{y_2}$, and assuming that R_x and R_y are independently and uniformly distributed over such ranges, the statistical means and the covariance matrix are given, respectively, by:

$$\bar{R}_x = \frac{R_{x_1} + R_{x_2}}{2} \quad \text{and} \quad \bar{R}_y = \frac{R_{y_1} + R_{y_2}}{2} \quad (15)$$

$$\begin{pmatrix} \frac{(R_{x_2} - R_{x_1})^2}{12} & 0 \\ 0 & \frac{(R_{y_2} - R_{y_1})^2}{12} \end{pmatrix} \quad (16)$$

On the other hand, it is possible that the parameters R_x and R_y are estimated using a specific inferential procedure, such as a linear multiple regression analysis. In such a case, one may, for example, use experimental data to fit the Carleton damage function, and as a result of such an analysis, derive estimates for the first two moments of R_x and R_y (see Section IV).

We next show how one can use the estimates of R_x and R_y as expressed in (13) and (14) to derive expressions for $E[P_{kf}]$ and $\text{Var}[P_{kf}]$.

a. Estimation of $E[P_{kf}]$

As a first approximation one can write (see Section II).

$$E[P_{kf}] = P_{kf}(\bar{R}_x, \bar{R}_y) .$$

Using expression (5) for the Carleton damage function one obtains

$$E[P_{kf}] = \exp\left[-\left(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2}\right)\right] . \quad (17)$$

b. Estimation of Var[P_{kf}]

The procedure outlined in Section II is used to arrive at an expression for Var[P_{kf}]. Expanding the expression for P_{kf} = P_{kf}(R_x, R_y) as given in (5) as a Taylor's series about the point (\bar{R}_x, \bar{R}_y) and retaining only first order terms results in:

$$P_{kf}(R_x, R_y) = P_{kf}(\bar{R}_x, \bar{R}_y) + (R_x - \bar{R}_x) \left(\frac{\partial P_{kf}}{\partial R_x} \right)_{\bar{R}_x, \bar{R}_y} \\ + (R_y - \bar{R}_y) \left(\frac{\partial P_{kf}}{\partial R_y} \right)_{\bar{R}_x, \bar{R}_y} .$$

Transposing and squaring both sides results in

$$[P_{kf}(R_x, R_y) - P_{kf}(\bar{R}_x, \bar{R}_y)]^2 = (R_x - \bar{R}_x)^2 \left(\frac{\partial P_{kf}}{\partial R_x} \right)_{\bar{R}_x, \bar{R}_y}^2 \\ + (R_y - \bar{R}_y)^2 \left(\frac{\partial P_{kf}}{\partial R_y} \right)_{\bar{R}_x, \bar{R}_y}^2 + 2(R_x - \bar{R}_x)(R_y - \bar{R}_y) \left[\left(\frac{\partial P_{kf}}{\partial R_x} \right) \left(\frac{\partial P_{kf}}{\partial R_y} \right) \right]_{\bar{R}_x, \bar{R}_y} . \quad (18)$$

Taking expectations on both sides of (18), one obtains as a first approximation

$$\text{Var}[P_{kf}] = \text{Var}[R_x] \left(\frac{\partial P_{kf}}{\partial R_x} \right)_{\bar{R}_x, \bar{R}_y}^2 + \text{Var}[R_y] \left(\frac{\partial P_{kf}}{\partial R_y} \right)_{\bar{R}_x, \bar{R}_y}^2 \\ + 2 \text{Cov}[R_x, R_y] \left[\left(\frac{\partial P_{kf}}{\partial R_x} \right) \left(\frac{\partial P_{kf}}{\partial R_y} \right) \right]_{\bar{R}_x, \bar{R}_y} . \quad (19)$$

Now from expression (5)

$$\frac{\partial P_{kf}}{\partial R_x} = \frac{2x^2}{R_x^3} \exp\left[-\left(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2}\right)\right] \quad (20)$$

$$\text{and } \frac{\partial P_{kf}}{\partial R_y} = \frac{2y^2}{R_y^3} \exp\left[-\left(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2}\right)\right] . \quad (21)$$

Substituting (20) and (21) in (19) yields

$$\begin{aligned} \text{Var}[P_{kf}] &= \frac{4x^4}{R_x^6} \left\{ \exp\left[-2\left(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2}\right)\right] \right\} \text{Var}[R_x] \\ &+ \frac{4y^4}{R_y^6} \left\{ \exp\left[-2\left(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2}\right)\right] \right\} \text{Var}[R_y] \\ &+ \frac{8x^2y^2}{R_x^3 R_y^3} \left\{ \exp\left[-2\left(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2}\right)\right] \right\} \text{Cov}[R_x, R_y] . \end{aligned} \quad (22)$$

C. Example

Consider the data given in the example of Section III.2.c. where $80 < R_x < 90$, $160 < R_y < 180$, $x = 50$, $y = 100$. Using the expression (15) and (16) yields

$$\bar{R}_x = \frac{R_{x1} + R_{x2}}{2} = \frac{80+90}{2} = 85 \text{ ft}$$

$$\bar{R}_y = \frac{R_{y1} + R_{y2}}{2} = \frac{160+180}{2} = 170 \text{ ft}$$

$$\text{Var}[R_x] = \frac{(R_{x2} - R_{x1})^2}{12} = \frac{(90-80)^2}{12} = \frac{100}{12} \text{ ft}^2$$

$$\text{Var}[R_y] = \frac{(R_{y2} - R_{y1})^2}{12} = \frac{(180-160)^2}{12} = \frac{400}{12} \text{ ft}^2 .$$

By assumption, $\text{Cov}[R_x, R_y] = 0$.

Using these numerical values in (7) and in (22), we obtain

$$\begin{aligned} E[P_{kf}] &= \exp\left[-\left(\frac{50}{85}\right)^2 + \left(\frac{100}{170}\right)^2\right] \\ &\approx .500 \end{aligned}$$

$$\begin{aligned}
\text{Var}[P_{kf}] &= \frac{(4)(50)^4}{(85)^6} \left\{ \exp\left[-2\left(\frac{(50)^2}{(85)^2} + \frac{(100)^2}{(170)^2}\right)\right] \right\} \left(\frac{100}{12}\right) \\
&+ \frac{(4)(100)^4}{(170)^6} \left\{ \exp\left[-2\left(\frac{(50)^2}{(85)^2} + \frac{(100)^2}{(170)^2}\right)\right] \right\} \left(\frac{400}{12}\right) \\
&= .000,138,4 + .000,138,4 \\
&= .000,276,8 .
\end{aligned}$$

The standard deviation of P_{kf} is

$$\sigma_{P_{kf}} = \sqrt{.000,276,8} = .0166 .$$

Using Chebyshev's inequality with at least a 75 percent confidence interval we have

$$\begin{aligned}
P_{kf} &= E[P_{kf}] \pm 2\sigma_{P_{kf}} \\
&= .500 \pm .033.
\end{aligned}$$

This last result is fairly close to the numerical result in the example of Section III.2.c. The computational simplicity of this last method as compared to the subjective estimation procedure should be pointed out. In addition, one notes that the variance contribution of the input parameters R_x and R_y , respectively, to the total P_{kf} variance are equal. Of course, this is coincidental and is due to the particular numerical data used. In general, the variance contributions will be different. The methodology provides a means for segregating the variance component of each of the input parameters entering in the computation of P_{kf} , and from that point of view is definitely more advantageous than the subjective estimation procedure.

SECTION IV
INFERENCE ESTIMATION PROCEDURES

1. Background

The next logical question that arises consists in the way the statistical characteristics of the input parameters R_x and R_y are established. It was seen that one of the methods is the so-called subjective method. The other method is the inferential one in which experimental data are used to estimate the values of the parameters R_x and R_y when P_{kf} is expressed as a mathematical function of these parameters. Two regression schemes will be discussed through which one could compute such values as $E[R_x]$, $E[R_y]$, $\text{Var}[R_x]$, $\text{Var}[R_y]$ and $\text{Cov}[R_x, R_y]$, all of which are used in the Taylor's series estimation procedure to arrive at values of $E[P_{kf}]$ and $\text{Var}[P_{kf}]$.

2. A Linear Multiple Regression Scheme: Method 1

The starting point is the Carleton damage function which was defined by expression (5) and which is repeated here:

$$P_{kf} = \exp\left[-\left(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2}\right)\right]. \quad (23)$$

It is assumed here that, through the analysis of the fragmentation data, a value of P_{kf} is specified at a given location (x,y) when the weapon bursts at $(0,0)$. The main objective is to estimate the values of R_x and R_y .

Suppose that there are n data points. For the i th data point ($i=1,2,\dots,n$) P_{kfi} represents the probability of kill due to fragmentation at the point (x_i, y_i) . In what follows it shall always be assumed that $0 < P_{kfi} \leq 1$ for all i . In case $P_{kfi} = 0$ such data point is discarded.

Taking logarithms on both sides of (23) yields

$$-\ln p_{kf} = \ln\left(\frac{1}{p_{kf}}\right) = \frac{x^2}{R_x^2} + \frac{y^2}{R_y^2}. \quad (24)$$

In (24) let

$$z = \ln\left(\frac{1}{p_{kf}}\right), \quad u = x^2, \quad v = y^2 \quad (25)$$

$$a = \frac{1}{R_x^2} \quad \text{and} \quad b = \frac{1}{R_y^2}. \quad (26)$$

Substituting these values in (23) one obtains

$$z = au + bv. \quad (27)$$

For the i th data point, we shall assume that

$$\epsilon_i = z_i - au_i - bv_i \quad (i=1,2,\dots,n) \quad (28)$$

are independently distributed, and further that

$$\begin{aligned} E[\epsilon_i] &= 0 \\ \text{Var}[\epsilon_i] &= \sigma^2. \end{aligned} \quad (29)$$

The parameters a , b , and σ^2 are to be estimated.

At this stage one may be tempted to assume that $z = \ln\left(\frac{1}{p_{kf}}\right)$ is normally distributed with mean $(au+bv)$ and variance σ^2 , and attempt to use the maximum likelihood technique to arrive at estimates of a , b , and σ^2 . However, since $0 < p_{kf} \leq 1$, it follows that $0 \leq z < \infty$, and hence the assumption of normality of z may not be justified. However, the least square technique may still be used for the estimation of a , b , and σ^2 . Using (26) and the Taylor's series estimation procedure, it will be shown how one can obtain estimates of R_x and R_y from the estimates of a and b .

a. A Least Square Technique

Using (28) the classical least square technique may be used to arrive at estimates of a , b , and σ^2 . Note here that through the transformation (25) and (26) a non-linear multiple regression scheme was changed into a linear multiple regression scheme. One is thus led to determine a and b so as to minimize the function

$$\sum_{i=1}^n \epsilon_i^2 = \sum_{i=1}^n (z_i - au_i - bv_i)^2. \quad (30)$$

The specific steps of the technique are well known (see e.g., [5]) and will not be presented here. The two normal equations are

$$\sum_{i=1}^n u_i (z_i - au_i - bv_i) = 0$$

$$\sum_{i=1}^n v_i (z_i - au_i - bv_i) = 0$$

which reduce to

$$\left. \begin{aligned} a \sum_{i=1}^n u_i^2 + b \sum_{i=1}^n u_i v_i &= \sum_{i=1}^n u_i z_i \\ a \sum_{i=1}^n u_i v_i + b \sum_{i=1}^n v_i^2 &= \sum_{i=1}^n v_i z_i \end{aligned} \right\} \quad (31)$$

The following notations are now introduced; let

$$a_{11} = \sum_{i=1}^n u_i^2, \quad a_{12} = a_{21} = \sum_{i=1}^n u_i v_i, \quad a_{22} = \sum_{i=1}^n v_i^2 \quad (32)$$

and $c_1 = \sum_{i=1}^n u_i z_i, \quad c_2 = \sum_{i=1}^n v_i z_i. \quad (33)$

All these quantities can be numerically computed. The normal equations (31) can be written as

$$\left. \begin{aligned} a a_{11} + b a_{12} &= c_1 \\ a a_{21} + b a_{22} &= c_2 \end{aligned} \right\} \quad (34)$$

Consider the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \quad (35)$$

Its inverse is

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}^2} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}. \quad (36)$$

Clearly

$$\begin{pmatrix} a \\ b \end{pmatrix} = A^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

and the estimates \hat{a} and \hat{b} of a and b are immediately given by the equations

$$\hat{a} = \frac{a_{22}c_1 - a_{12}c_2}{a_{11}a_{22} - a_{12}^2} \quad (37)$$

$$\hat{b} = \frac{-a_{12}c_1 + a_{11}c_2}{a_{11}a_{22} - a_{12}^2}. \quad (38)$$

It can be verified that \hat{a} and \hat{b} are unbiased estimators of a and b so that $E[\hat{a}] = a$ and $E[\hat{b}] = b$. Knowing \hat{a} and \hat{b} , one can obtain an unbiased estimate of σ^2 (see e.g., [4]) which is given by

$$\hat{\sigma}^2 = \frac{1}{n-2} \sum_{i=1}^n (z_i - \hat{a}u_i - \hat{b}v_i)^2. \quad (39)$$

Further, one obtains [5]:

$$\text{Var}[\hat{a}] = \frac{a_{22}}{a_{11}a_{22}-a_{12}^2} \hat{\sigma}^2 \quad (40)$$

$$\text{Var}[\hat{b}] = \frac{a_{11}}{a_{11}a_{22}-a_{12}^2} \hat{\sigma}^2 \quad (41)$$

$$\text{Cov}[\hat{a}, \hat{b}] = -\frac{a_{12}}{a_{11}a_{22}-a_{12}^2} \hat{\sigma}^2 \quad (42)$$

or more succinctly, the covariance matrix of (\hat{a}, \hat{b}) is given by

$$\begin{pmatrix} \text{Var}[\hat{a}] & \text{Cov}[\hat{a}, \hat{b}] \\ \text{Cov}[\hat{a}, \hat{b}] & \text{Var}[\hat{b}] \end{pmatrix} = \frac{\hat{\sigma}^2}{a_{11}a_{22}-a_{12}^2} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{12} & a_{11} \end{pmatrix} \quad (43)$$

b. Estimates of R_x and R_y

It is clear that in the Taylor's series estimation procedure, the determination of $E[P_{kf}]$ and $\text{Var}[P_{kf}]$ requires as input the values of

$$\bar{R}_x = E[R_x], \quad \bar{R}_y = E[R_y] \quad (44)$$

and the covariance matrix

$$\begin{pmatrix} \text{Var}[R_x] & \text{Cov}[R_x, R_y] \\ \text{Cov}[R_x, R_y] & \text{Var}[R_y] \end{pmatrix}. \quad (45)$$

The intent now is to use the results of the previous least square analysis which yielded \hat{a} and \hat{b} , the estimates of a and b , respectively, as well as

$$\begin{pmatrix} \text{Var}[\hat{a}] & \text{Cov}[\hat{a}, \hat{b}] \\ \text{Cov}[\hat{a}, \hat{b}] & \text{Var}[\hat{b}] \end{pmatrix}$$

to obtain (44) and (45). This can be accomplished through the functional relationship given by (26).

Before one proceeds any further, it should be noted that the notation \hat{a} is used to denote both the estimator of a as a random variable and the particular numerical estimate of a obtained from the least square analysis and given by (37). The same holds true for \hat{b} . In the sequel, the meaning of the notation \hat{a} or \hat{b} should be clear from the context.

Since a and b are not known exactly but are, in general, random variables, it follows from (26) that R_x and R_y are themselves random variables given, respectively, by

$$R_x = \frac{1}{\sqrt{a}} \quad \text{and} \quad R_y = \frac{1}{\sqrt{b}} . \quad (46)$$

Expand now $R_x = a^{-1/2}$ as a Taylor's series about the point a . One obtains

$$\begin{aligned} R_x &= R_x \Big|_{\hat{a}=a} + (\hat{a}-a) \frac{dR_x}{d\hat{a}} \Big|_{\hat{a}=a} + \frac{1}{2!} (\hat{a}-a)^2 \frac{d^2R_x}{d\hat{a}^2} \Big|_{\hat{a}=a} \\ &= \frac{1}{\sqrt{a}} - \frac{1}{2a^{3/2}} (\hat{a}-a) + \frac{3}{8a^{5/2}} (\hat{a}-a)^2 . \end{aligned} \quad (47)$$

Recall that \hat{a} is an unbiased estimator of a , i.e., $E[\hat{a}] = a$. Taking expectations on both sides of (47) yields:

$$\begin{aligned} \bar{R}_x = E[R_x] &= \frac{1}{\sqrt{a}} - \frac{1}{2a^{3/2}} E[(\hat{a}-a)] + \frac{3}{8a^{5/2}} E[(\hat{a}-a)^2] \\ &= \frac{1}{\sqrt{a}} + \frac{1}{8a^{5/2}} \text{Var}[\hat{a}] . \end{aligned} \quad (48)$$

Since a is an unknown quantity, its numerical estimate \hat{a} given by (37) is used instead. The value of $\text{Var}[\hat{a}]$ is given by (40).

Subtracting (48) from (47) one obtains the following up to and including first order terms:

$$R_x - \bar{R}_x = \frac{1}{2a^{3/2}} (\hat{a} - a) . \quad (49)$$

Squaring both sides of (49) and taking expectations yields

$$\text{Var}[R_x] = \frac{1}{4a^3} \text{Var}[\hat{a}] . \quad (50)$$

Similarly one obtains

$$\bar{R}_y = \frac{1}{\sqrt{b}} + \frac{3}{8b^{5/2}} \text{Var}[\hat{b}] \quad (51)$$

$$R_y - \bar{R}_y = -\frac{1}{2b^{3/2}} (\hat{b} - b) \quad (52)$$

$$\text{Var}[R_y] = \frac{1}{4b^3} \text{Var}[\hat{b}] . \quad (53)$$

Multiplying (49) by (52) and taking expectation yields

$$\text{Cov}[R_x, R_y] = \frac{1}{4(ab)^{3/2}} \text{Cov}[\hat{a}, \hat{b}] . \quad (54)$$

Expressions (44) and (45) are thus completely specified. The following is a summary of the results obtained when a and b are replaced, respectively, by their estimate \hat{a} and \hat{b}

$$\bar{R}_x = \frac{1}{\sqrt{a}} + \frac{3}{8a^{5/2}} \text{Var}[\hat{a}], \quad \bar{R}_y = \frac{1}{\sqrt{b}} + \frac{3}{8b^{5/2}} \text{Var}[\hat{b}] \quad (55)$$

$$\begin{pmatrix} \text{Var}[R_x] & \text{Cov}[R_x, R_y] \\ \text{Cov}[R_x, R_y] & \text{Var}[R_y] \end{pmatrix} = \begin{pmatrix} \frac{\text{Var}[\hat{a}]}{4a^3} & \frac{\text{Cov}[\hat{a}, \hat{b}]}{4(\hat{a}, \hat{b})^{3/2}} \\ \frac{\text{Cov}[\hat{a}, \hat{b}]}{4(\hat{a}, \hat{b})^{3/2}} & \frac{\text{Var}[\hat{b}]}{4b^3} \end{pmatrix} \quad (56)$$

3. Another Linear Multiple Regression Scheme: Method 2

This section digresses on a second linear multiple regression scheme to determine estimates of the parameters R_x and R_y of the Carleton damage function

$$P_{kf} = \exp \left[- \left(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2} \right) \right].$$

In this instance, it is assumed that one knows, a priori, values R_{x_0} and R_{y_0} close to the estimates of R_x and R_y , respectively. Specific procedures to obtain such initial values of R_{x_0} and R_{y_0} will be discussed later.

The expression for P_{kf} is expanded as a Taylor's series about the point (R_{x_0}, R_{y_0}) to yield up to its first order terms

$$\begin{aligned} P_{kf} &= P_{kf} \Big|_{R_{x_0}, R_{y_0}} + (R_x - R_{x_0}) \frac{\partial P_{kf}}{\partial R_x} \Big|_{R_{x_0}, R_{y_0}} + (R_y - R_{y_0}) \frac{\partial P_{kf}}{\partial R_y} \Big|_{R_{x_0}, R_{y_0}} \\ &= \exp \left[- \left(\frac{x^2}{R_{x_0}^2} + \frac{y^2}{R_{y_0}^2} \right) \right] + (R_x - R_{x_0}) \frac{2x^2}{R_{x_0}^3} \exp \left[- \left(\frac{x^2}{R_{x_0}^2} + \frac{y^2}{R_{y_0}^2} \right) \right] \\ &\quad + (R_y - R_{y_0}) \frac{2y^2}{R_{y_0}^3} \exp \left[- \left(\frac{x^2}{R_{x_0}^2} + \frac{y^2}{R_{y_0}^2} \right) \right]. \end{aligned} \quad (57)$$

Multiplying both sides of (57) by $\exp \left(\frac{x^2}{R_{x_0}^2} + \frac{y^2}{R_{y_0}^2} \right)$ and transposing one obtains

$$P_{kf} \cdot \exp \left(\frac{x^2}{R_{x_0}^2} + \frac{y^2}{R_{y_0}^2} \right) - 1 = (R_x - R_{x_0}) \frac{2x^2}{R_{x_0}^3} + (R_y - R_{y_0}) \frac{2y^2}{R_{y_0}^3}. \quad (58)$$

Let now

$$z = P_{kf} \cdot \exp \left(\frac{x^2}{R_{x_0}^2} + \frac{y^2}{R_{y_0}^2} \right) - 1 \quad (59)$$

$$a = R_x - R_{x_0} \quad , \quad b = R_y - R_{y_0} \quad (60)$$

$$u = \frac{2x^2}{R_{x_0}^3} \quad , \quad v = \frac{2y^2}{R_{y_0}^3} \quad (61)$$

Equation (58) becomes:

$$z = au + bv \quad (62)$$

a. The Least Square Technique

Clearly z is a linear function of u and v . The method just described approximates what would have resulted in a nonlinear regression scheme with a linear regression scheme using the Taylor's series expansion and retaining only first order terms. The variables u and v are obtained from the known coordinates (x,y) at which P_{kf} is measured experimentally. The numerical values of z at a particular (u,v) are deduced from the known values of x,y and P_{kf} . Under these conditions, one is again dealing with a linear multiple regression scheme, and the values of a and b , and hence R_x and R_y , can be estimated using least square techniques as described in the previous section.

The output results of the least square approach will consist of the estimates \hat{a} and \hat{b} of a and b , respectively, as well as the covariance matrix

$$\begin{pmatrix} \text{Var}[\hat{a}] & \text{Cov}[\hat{a},\hat{b}] \\ \text{Cov}[\hat{a},\hat{b}] & \text{Var}[\hat{b}] \end{pmatrix} \quad (63)$$

From (60) it follows that

$$\hat{R}_x = R_{x_0} + \hat{a} \quad , \quad \hat{R}_y = R_{y_0} + \hat{b} \quad (64)$$

and, of course,

$$\begin{pmatrix} \text{Var}[\hat{R}_x] & \text{Cov}[\hat{R}_x, \hat{R}_y] \\ \text{Cov}[\hat{R}_x, \hat{R}_y] & \text{Var}[\hat{R}_y] \end{pmatrix} = \begin{pmatrix} \text{Var}[\hat{a}] & \text{Cov}[\hat{a}, \hat{b}] \\ \text{Cov}[\hat{a}, \hat{b}] & \text{Var}[\hat{b}] \end{pmatrix} \quad (65)$$

It is possible to obtain better approximations for the estimates in (64) and (65) by retaining higher order terms in the Taylor's series expansion. This, however, would considerably add to the complexity of the problem. A more natural approach consists in using the initial values of R_{x_0} and R_{y_0} to obtain estimates R_{x_1} and R_{y_1} using the linear regression scheme. This could be followed by a second iteration in which R_{x_1} and R_{y_1} would be used as initial values to obtain improved estimates R_{x_2} and R_{y_2} using the same linear regression scheme and so forth, assuming that the procedure will converge to some value of R_x and R_y .

It should be noted that one of the advantages of the present multiple regression scheme over Method 1 discussed in Section IV-2 is that values of $P_{kf} = 0$ need not be discarded from the data in the least square analysis, provided that they be counted as additional observation points.

b. Initialization of the values of R_x and R_y

The previous method on the linearization of a non-linear multiple regression scheme using Taylor's series expansion procedure is predicated on the assumption that initial values R_{x_0} and R_{y_0} for the two weapon radii, which are close to the estimated values, can be obtained. Next, three possible avenues are discussed that could be used to obtain such initial values using experimental data. Such data is usually in the form of a matrix giving values of P_{kf} for various values of (x,y) when the weapon detonates at $(0,0)$.

In the first procedure, the values of R_{x_0} and R_{y_0} are obtained from a knowledge of the mean area of effectiveness (MAE) of the weapon as well as the impact angle I . The following relation is used:

$$MAE = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_{kf}(x,y) dx dy = \pi R_{x_0} R_{y_0} \quad (66)$$

The value of the double integral is obtained from field data giving $P_{kf}(x,y)$ at specific values of x and y . Assume now that R_{x_0} is the weapon radius in the direction of range and R_{y_0} is the weapon radius in the direction of deflection. Let I be the weapon impact angle. Then the ratio R_{x_0}/R_{y_0} satisfies the following experimental relation (see e.g., [3]):

$$\frac{R_{x_0}}{R_{y_0}} = 1 - 0.8 \cos I . \quad (67)$$

Using (66) and (67) values of R_{x_0} and R_{y_0} can be obtained.

In the second procedure one uses the following relations to obtain R_{x_0} and R_{y_0} , respectively, (see e.g., [6])

$$R_{x_0}^2 = \int_0^{\infty} 2x P_{kf}(x,0) dx$$

$$R_{y_0}^2 = \int_0^{\infty} 2y P_{kf}(0,y) dy .$$

Thus $R_{x_0}^2$ is obtained by multiplying each value of the damage function on the x -axis by $2x$ and integrating the result over all possible values of x .

Similarly, $R_{y_0}^2$ is obtained by multiplying each value of the damage function along the y -axis by $2y$ and integrating it over all possible values of y .

To use the third procedure, one notes that in the expression for the Carleton damage function, the equiprobability contour line corresponding to a probability of kill of $P_{kf} = e^{-1} = .3678794$, has for equation

$$\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2} = 1 .$$

Thus the two weapon radii R_x and R_y correspond to the semi-axes of an ellipse, and these values can be determined from the P_{kf} matrix obtained from the fragmentation field data.

SECTION V
CONCLUSIONS

It has been customary to evaluate weapon effectiveness by the use of a single index number such as the probability of kill P_k . In the present report procedures have been developed for estimating the error in the P_k value knowing the error in the input parameters used to compute P_k .

For fragment sensitive targets, the Carleton damage function appears to be a reasonable model to compute P_k in the absence of blast and aiming error. This function usually contains the two parameters R_x and R_y which define the weapon radii. The report has addressed itself to the specific problem of utilizing the inherent errors in R_x and R_y to compute the error in P_k .

Methodologies have been developed to identify the causes of error in the parameters when field data are used to estimate R_x and R_y . These methodologies are based on well established statistical techniques such as linear multiple regression, and their use has provided estimates for R_x and R_y as well as estimates for $\text{Var}[R_x]$, $\text{Var}[R_y]$, and $\text{Cov}[R_x, R_y]$. In turn, these estimates can be used to compute $E[P_k]$ and $\text{Var}[P_k]$. Knowing the first two moments of P_k , Chebyshev's inequality provides one with a means for setting up confidence intervals for P_k values.

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