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CONFIDENCE INTERVALS FOR CEP WHEN THE ERRORS ARE ELLIPTICAL NORMAL

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Approximate confidence intervals are developed for CEP when the delivery errors are elliptical normal. Their accuracies are determined through Monte Carlo sampling. The procedures for confidence interval computation are illustrated via numerical examples.

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FOREWORD

The work documented in this technical report was performed at the Naval Surface Weapons Center (NSWC) by the Mathematical Statistics Staff (K106), Space and Surface Systems Division, Strategic Systems Department. The date of completion was October 1983.

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CHAPTER 1

INTRODUCTION

A common parameter for describing the accuracy of a weapon is the circular probable error, generally referred to as CEP. CEP is simply the bivariate analog of the univariate probable error and measures the radius of a mean-centered circle which includes 50% of the bivariate probability. In the case of circular normal errors where the error variances are the same in both directions, CEP can be expressed as a function of the common miss distance standard deviation. Also, CEP estimators based on observed miss distances are easily formulated and can be used to construct confidence intervals for CEP. In the case of elliptical normal errors, CEP cannot be expressed explicitly as a function of the miss distance standard deviations. Here, one must obtain CEP by numerical methods or by referring to tabular values. This has led to the development of a number of approximations by which CEP can be expressed as a function of the miss distance standard deviations. While CEP estimators based on observed miss distances are easily formulated from these approximations, their probability distributions are too complicated to be useful for CEP confidence intervals. In this report, these probability distributions are approximated with distributions which are more practical for the formulation and application of CEP confidence intervals. Approximate CEP confidence intervals are then formulated and their accuracy determined through Monte Carlo sampling.

The first part of this report is tutorial in the development of CEP and discusses the commonly used approximations for the elliptical case. The development of approximate confidence intervals begins with Chapter 4.

CHAPTER 2

REVIEW OF CIRCULAR CASE

In general, it will be assumed that the errors in the X and Y directions are independent with mean zero and variances σ_x^2 and σ_y^2, respectively. Under the circular normal assumption, $\sigma_x^2 = \sigma_y^2 = \sigma^2$ and the bivariate distribution of errors is given by

$$f_c(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2 + y^2)/2\sigma^2} \quad , -\infty < x,\ y < \infty \tag{2.1}$$

where the subscript c denotes circular. The distribution of the radial miss distance is derived by first obtaining the distribution of the polar variables $(R,\ \Theta)$ where

$$X = R \cos \Theta$$

$$Y = R \sin \Theta.$$

This is found to be

$$g_c(r,\theta) = \frac{r}{2\pi\sigma^2} e^{-\ r^2/2\sigma^2} \quad , 0 < r < \infty,\ 0 < \theta < 2\pi. \tag{2.2}$$

The distribution of $R = (X^2 + Y^2)^{\frac{1}{2}}$ is now obtained from (2.2) by using the marginal rule; i.e.,

$$g_c(r) = \int_0^{2\pi} g_c(r,\theta)d\theta = \frac{r}{\sigma^2} e^{-\ r^2/2\sigma^2} \quad , r > 0. \tag{2.3}$$

This is the well-known Rayleigh distribution (see Lindgren (1968)) with cumulative distribution function

$$P(R < r) = G_c(r) = 1 - e^{-r^2/2\sigma^2}. \tag{2.4}$$

By definition, $G_c(\text{CEP}) = .5$ and the solution of (2.4) yield the frequently used expression

$$\text{CEP} = [-2\ \ell n(.50)]^{\frac{1}{2}}\ \sigma = 1.1774\ \sigma. \tag{2.5}$$

Consider now that σ (and hence, CEP) is unknown and is estimated from n observed miss distances. These miss distances will be designated as (x_i, y_i), i = 1, ..., n in the X and Y directions, respectively. Moranda (1959) has shown that the maximum likelihood estimator for σ is

$$\hat{\sigma} = \left\{ \sum_{i=1}^{n} (x_i^2 + y_i^2)/2n \right\}^{\frac{1}{2}}, \tag{2.6}$$

and the corresponding estimator for CEP is simply $\hat{\mathrm{CEP}} = 1.1774\ \hat{\sigma}$. This estimator is biased for CEP, i.e., the expectation of $\hat{\mathrm{CEP}}$ is not equal to CEP. However, the bias is small and the unbiasing factor is cumbersome. Therefore, it will be retained in its slightly biased form.

To place confidence limits on CEP, it will be necessary to examine the probability distribution of $\hat{\mathrm{CEP}}$. It is well-known under normal theory (see Mood and Graybill (1963)) that if $\{W_i\}$, i = 1, ..., n is a random sample from a normal population with mean μ and variance σ^2, then $\Sigma(W_i - \mu)^2/\sigma^2$ has a chi-square distribution with n degrees of freedom. One can consider $\{x_i\}$, i = 1, ..., n and $\{y_i\}$, i = 1, ..., n to be a random sample of size 2n from a normal population with mean zero and variance σ^2. Therefore,

$$\sum_{i=1}^{n} \frac{\left(x_i^2 + y_i^2\right)}{\sigma^2} = \frac{2n\hat{\sigma}^2}{\sigma^2} = \frac{2n\hat{\mathrm{CEP}}^2}{\mathrm{CEP}^2} \sim \chi^2_{2n} \tag{2.7}$$

where "~" designates "is distributed as" and χ^2_{2n} designates a chi-square probability distribution with 2n degrees of freedom. The 100 $(1 - \alpha)$% confidence limits are now easily constructed using the probability statement

$$\mathrm{Prob}\left\{ \chi^2_{2n,\alpha/2} < \frac{2n\hat{\mathrm{CEP}}^2}{\mathrm{CEP}^2} < \chi^2_{2n,1-\alpha/2} \right\} = 1 - \alpha \quad . \tag{2.8}$$

In this expression, $\chi^2_{\upsilon,\alpha}$ designates the 100α percentage point for a chi-square with υ degrees of freedom. Tabular values for integral υ can be found in the back of most statistics texts. A more complete table is found in Hald (1952). Manipulating the inequality in (2.8) leads to the following 100 $(1 - \alpha)$% confidence limits for CEP:

$$\left[\frac{\hat{\mathrm{CEP}}}{\left(\chi^2_{2n,1-\alpha/2}/2n\right)^{\frac{1}{2}}}, \frac{\hat{\mathrm{CEP}}}{\left(\chi^2_{2n,\alpha/2}/2n\right)^{\frac{1}{2}}} \right] \quad . \tag{2.9}$$

The interpretation here is that one is 100 $(1 - \alpha)$% confident that the interval in (2.9) contains the population CEP. This formula is valid only for the case where the errors are known to be circular, i.e., the case where $\sigma_x^2 = \sigma_y^2 = \sigma^2$.

Before leaving this review of the circular case, it will be instructive to work through an example. Suppose confidence limits on CEP are desired from the ten round sample shown in Table 2-1. We first need to compute $\hat{\sigma}$ in (2.6). One notes that the sum under the radical in (2.6) can be expressed as

$$\left(\frac{\Sigma x_i^2}{n} + \frac{\Sigma y_i^2}{n}\right)/2.$$

TABLE 2-1. 10 HYPOTHETICAL MISS DISTANCES (FEET)

x	y
42	-123
-13	-12
-50	14
-70	169
-191	-58
117	-79
158	99
16	-18
101	170
27	65

The two components are independent estimates of the common variance σ^2. If they differ significantly, it would cast doubt on the circular normal assumption. These components will be referred to as s_x^2 and s_y^2 so that $\hat{\sigma}$ in (2.6) becomes

$$\hat{\sigma} = \left[\left(s_x^2 + s_y^2\right)/2\right]^{\frac{1}{2}}.$$

For this example, one finds

$$s_x^2 = \Sigma x_i^2/n = 9565.3$$

$$s_y^2 = \Sigma y_i^2/n = 9688.5$$

$$\hat{\sigma} = [(9565.3 + 9688.5)/2]^{\frac{1}{2}} = 98.12$$

$$\widehat{CEP} = 1.1774\ \hat{\sigma} = 115.53.$$

To form confidence limits, the computations in (2.9) are required. Let us consider 95% limits so $\alpha = .05$ and the tabular values required are

$$\chi^2_{20,.025} = 9.59$$

$$\chi^2_{20,.975} = 34.20.$$

These would both be divided by 2n = 20 to form the terms under the radical in (2.9). One could also use a table of chi-square percentage points divided by the degrees of freedom here to avoid the latter step. Such a table is in Hald (1952) and provides

$$\chi^2_{20,.025}/20 = .4796$$

$$\chi^2_{20,.975}/20 = 1.7085 \quad .$$

The 95% confidence limits on CEP can now be completed and are found to be

$$\left(\frac{115.53}{(1.7085)^{\frac{1}{2}}} \; , \; \frac{115.53}{(.4796)^{\frac{1}{2}}} \right) = (88.39, \; 166.82)$$

The units are feet, the same as the miss distance units in Table 2-1. The interpretation is that one is 95% confident that the true (or population) CEP lies in the interval (88.39, 166.82). The result is valid only if the probability distribution of miss distances follows a circular normal distribution. Application of (2.9) when the probability distribution is elliptical can lead to serious errors. A discussion of the elliptical case begins with Chapter 3.

CHAPTER 3

CEP DERIVATION AND APPROXIMATIONS FOR ELLIPTICAL ERRORS

In the elliptical case, the error variances are unequal and the bivariate distribution of errors is given by

$$f_E(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left[(x/\sigma_x)^2 + (y/\sigma_y)^2\right]}, \quad -\infty < x,y < \infty$$

where the subscript E denotes elliptical. The distribution of the radial error R for this case was derived by Chew and Boyce (1961). They proceeded, as in the circular case, by first obtaining the distribution of the polar variables (R,θ). This was found to be

$$g_E(r,\theta) = \frac{r}{2\pi\sigma_x\sigma_y} e^{-ar^2} e^{(-br^2\cos 2\theta)}, \quad \begin{array}{l} 0 < r < \infty \\ 0 < \theta < 2\pi \end{array} \tag{3.1}$$

where

$$a = \frac{\sigma_y^2 + \sigma_x^2}{(2\sigma_x\sigma_y)^2}, \quad b = \frac{\sigma_y^2 - \sigma_x^2}{(2\sigma_x\sigma_y)^2}.$$

Using the marginal rule, the distribution of R was obtained by integrating $g_E(r,\theta)$ in (3.1) with respect to θ between 0 and 2π. This integration cannot be expressed in tractable form, so they expressed their result in terms of a modified Bessel function as

$$g_E(r) = \frac{r}{\sigma_x\sigma_y} e^{-ar^2} I_0(br^2), \quad 0 < r < \infty. \tag{3.2}$$

In this expression, the subscript E denotes elliptical and I_0 is a modified Bessel function of the first kind and zero order, i.e.,

$$I_0(x) = \frac{1}{\pi}\int_0^{\pi} e^{-x\cos\theta}\, d\theta.$$

The cumulative distribution function for R is denoted by

$$P(R < r) = G_E(r) = \int_0^r g_E(t)\, dt. \tag{3.3}$$

However, $G_E(r)$ cannot be expressed in tractable form because $g_E(t)$ cannot be so expressed. This means that the radius of the 50% circle for the elliptical case cannot be expressed by a simple formula as it was in the circular case. One has to solve $G_E(CEP) = .5$ by numerical methods or by referring to tables prepared by Harter (1960), DiDonato and Jarnagin (1962), and others. To avoid using these tables or numerical procedures for CEP evaluation, a number of approximations have been developed over the years. Five of the most common are shown below; they are designated as CEP_1 through CEP_5:

$$CEP_1 = 1.1774 \left(\frac{\sigma_x^2 + \sigma_y^2}{2} \right)^{\frac{1}{2}}$$

$$CEP_2 = 1.1774 \left(\frac{\sigma_x + \sigma_y}{2} \right)$$

$$CEP_3 = \left(2\, \chi^2_{\upsilon,.50}/\upsilon \right)^{\frac{1}{2}} \left(\frac{\sigma_x^2 + \sigma_y^2}{2} \right)^{\frac{1}{2}}$$

$$\upsilon = \frac{\left(\sigma_x^2 + \sigma_y^2\right)^2}{\sigma_x^4 + \sigma_y^4}$$

$$CEP_4 = .565\, \sigma_{max} + .612\, \sigma_{min}, \quad \sigma_{min}/\sigma_{max} \geq .25$$

$$= .667\, \sigma_{max} + .206\, \sigma_{min}, \quad \sigma_{min}/\sigma_{max} < .25$$

$$CEP_5 = \left[2^{\frac{1}{3}} \left(1 - \frac{2}{9\upsilon} \right) \right]^{\frac{3}{2}} \left(\frac{\sigma_x^2 + \sigma_y^2}{2} \right)^{\frac{1}{2}} .$$

CEP_1 and CEP_2 were taken from Groves (1961); CEP_3 was formulated using the chi-square approximation for calculating hit probabilities provided by Grubbs (1964). It was also derived independent of the Grubbs approximation by Thomas and Taub (1978). CEP_4 is a piece-wise linear combination of standard deviations which is commonly used in the missile community; CEP_5 was formulated by Terzian (1974) using the Wilson-Hilferty approximation for calculating hit probabilities provided by Grubbs (1964). Plots of each approximation versus the true CEP as a function of $\sigma_{min}/\sigma_{max}$ are shown in Figures 3-1 through 3-5. These give a fairly good indication of how well each performs. It is seen that CEP_1 deteriorates rapidly as we depart from the circular case (for which CEP_1 degenerates to $1.1774\ \sigma$), CEP_2 is reasonably good if the ratio $\sigma_{min}/\sigma_{max}$ is not less than about .2; CEP_3 appears good for all ratios, and CEP_4 and CEP_5 appear good to a lesser extent for all ratios.

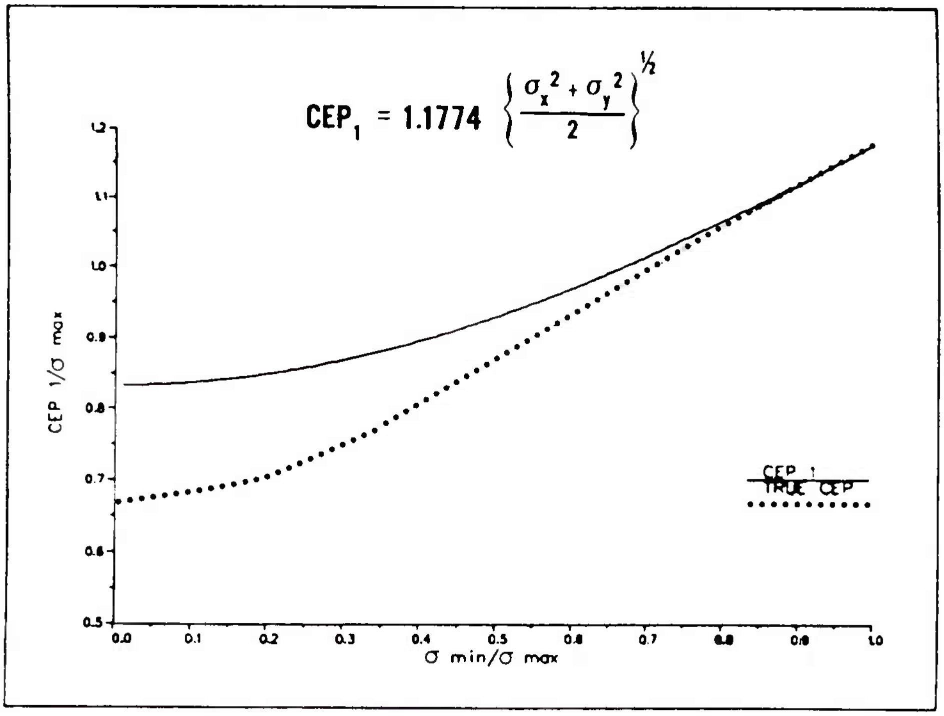

FIGURE 3-1. CEP_1 APPROXIMATION VERSUS TRUE CEP

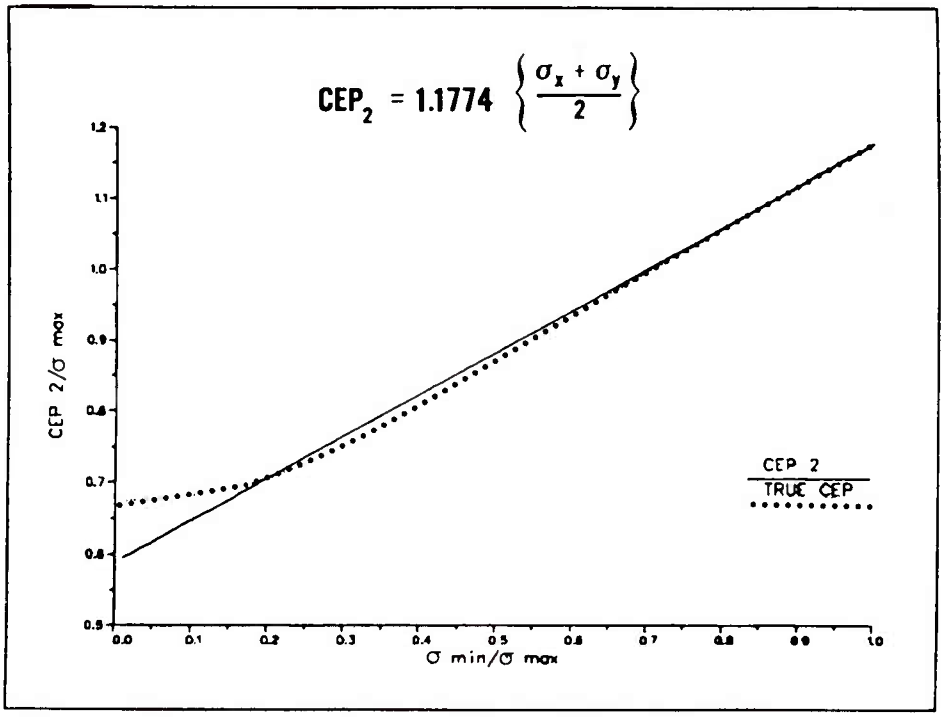

FIGURE 3-2. CEP_2 APPROXIMATION VERSUS TRUE CEP

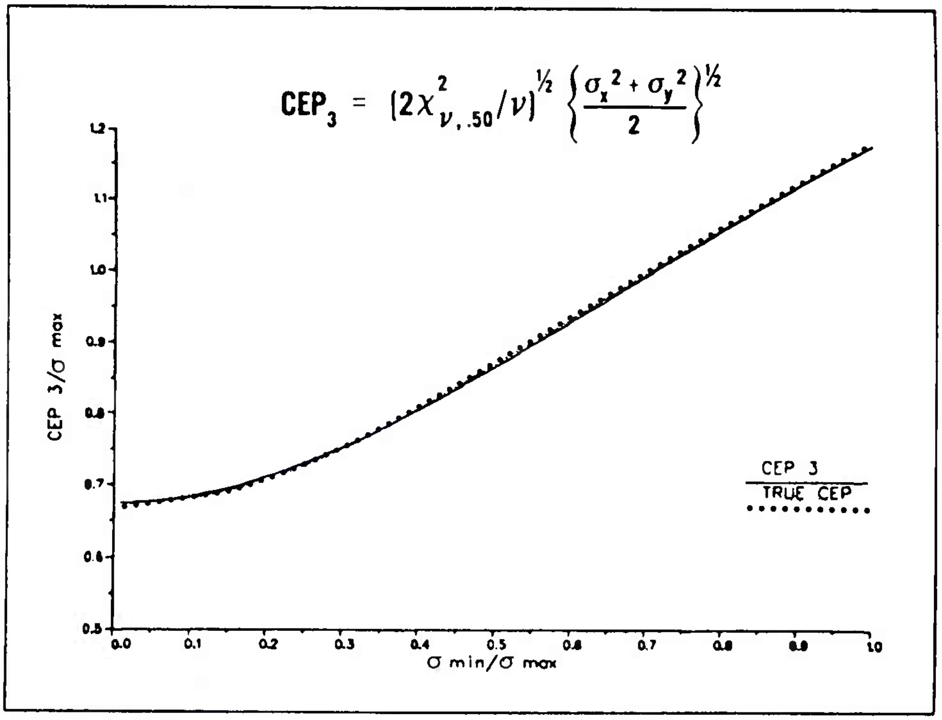

FIGURE 3-3. CEP_3 APPROXIMATION VERSUS TRUE CEP

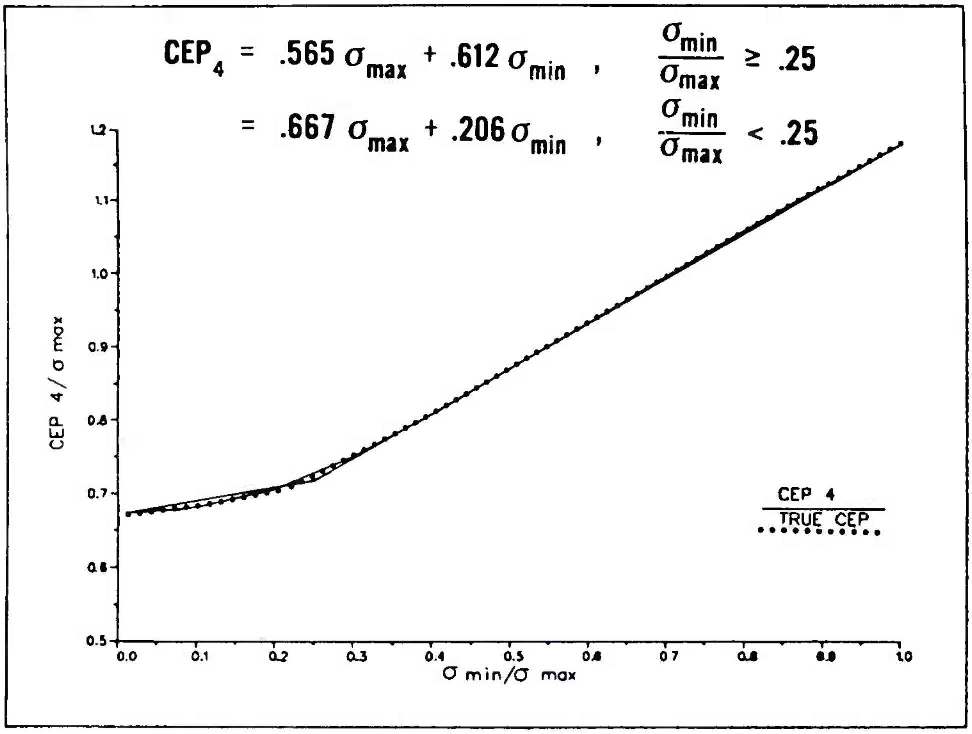

FIGURE 3-4. CEP_4 APPROXIMATION VERSUS TRUE CEP

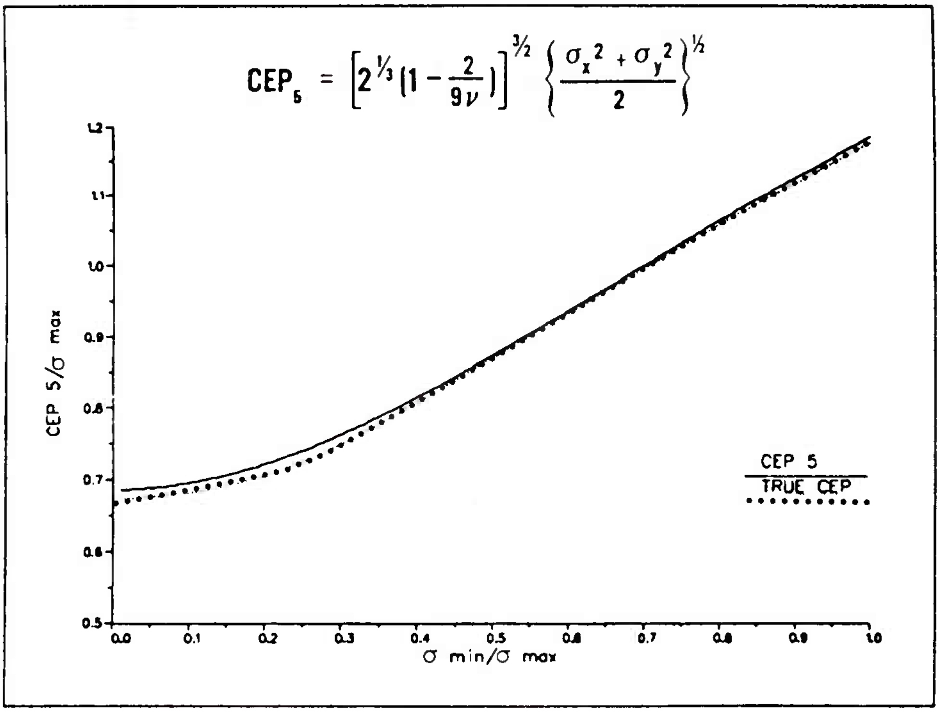

FIGURE 3-5. CEP_5 APPROXIMATION VERSUS TRUE CEP

To recap the elliptical case thus far, for known values of σ_x and σ_y, one can compute the exact CEP by solving $G_E(CEP) = .50$ or compute an approximate CEP by using an approximation such as CEP_1 to CEP_5. Consider now that σ_x and σ_y (and hence CEP) are unknown and are to be estimated from n observed miss distances. These miss distances will be designated as before, as (x_i, y_i), $i = 1, \ldots, n$ in the X and Y directions, respectively. The maximum likelihood estimators for σ_x and σ_y pose no problem. They are given by

$$s_x = \hat{\sigma}_x = \left\{ \sum_{i=1}^{n} x_i^2/n \right\}^{\frac{1}{2}} , \quad s_y = \hat{\sigma}_y = \left\{ \sum_{i=1}^{n} y_i^2/n \right\}^{\frac{1}{2}} \qquad (3.4)$$

as shown in Lindgren (1968). These estimators are slightly biased and as before, they will not be corrected due to the cumbersome nature of the correction factor. They can now be substituted for the unknown σ's in (3.3) to obtain a numerical estimate of CEP by solving $G_E(CEP) = .50$ or they can be substituted into CEP_1 to CEP_5 to obtain estimates of the approximate CEP. These latter estimators will be referred to as $\widehat{CEP}_1$ to $\widehat{CEP}_5$ and have the appeal of being explicitly expressible. Hence, estimates of CEP for the elliptical case can be rather easily obtained.

The problem of obtaining confidence limits for CEP in this case is more complex. The complexity is based on the fact that to find confidence limits for a parameter, one needs information regarding the probability distribution of the estimator for the parameter. As previously noted, $\hat{CEP}$ can be obtained by solving $G_E(\hat{CEP}) = .50$. Symbolically, we can write

$$\hat{CEP} = G_E^{-1}(.50) \tag{3.5}$$

but to obtain it requires recursive numerical integration or the use of previously noted tables. Hence, the formulation of confidence limits based on this estimator holds little promise for a practical solution. Therefore, we shall consider the formulation of confidence limits based on the estimators of the approximate CEP. The distributions of these estimators are extremely complicated since they involve radicals and linear combinations of sample variances and standard deviations. Hence, these distributions were approximated and confidence limits formulated on the basis of the approximate distributions. This development is provided in Chapter 4.

CHAPTER 4

APPROXIMATE DISTRIBUTIONS OF CEP ESTIMATORS

The five estimators for CEP fall into two classes. One class involves the square root of linear combinations of sample variances and the other involves linear combinations of sample standard deviations. $\widehat{CEP}_1$, $\widehat{CEP}_3$, and $\widehat{CEP}_5$ fall into the first class and can be written in the form

$$\widehat{CEP}_i = K_i \left(\frac{s_x^2 + s_y^2}{2}\right)^{\frac{1}{2}} , \quad i = 1, 3, 5 \tag{4.1}$$

where

$$K_1 = 1.1774$$

$$K_3 = \left(2\,\chi^2_{\upsilon,.50}/\upsilon\right)^{\frac{1}{2}}$$

$$K_5 = \left[2^{1/3}\,(1 - 2/9\upsilon)\right]^{3/2} .$$

$\widehat{CEP}_2$ and $\widehat{CEP}_4$ fall into the second class and can be written in the form

$$\widehat{CEP}_i = a_1\, s_{max} + a_2\, s_{min} , \quad i = 2, 4 \tag{4.2}$$

where for $i = 2$, $\quad a_1 = a_2 = 1.1774/2$

and for $i = 4$, $\quad a_1 = .565$ and $a_2 = .612$ when $s_{min}/s_{max} \geq .25$

$\quad a_1 = .667$ and $a_2 = .206$ when $s_{min}/s_{max} < .25$.

The distribution of the square of each estimator can be approximated by a chi-square distribution with appropriate degrees of freedom. The following is the rationale for these approximations. The squares of $\widehat{CEP}_1$, $\widehat{CEP}_3$, and $\widehat{CEP}_5$ are linear combinations of sample variances. Satterthwaite (1946) has shown that one can approximate the distribution of such linear combinations with a chi-square distribution with degrees of freedom chosen so the approximate distribution has a variance equal to that of the exact distribution. Here, one is approximating the distribution of a linear combination of sample variances with a chi-square. A natural extension is to approximate the distribution of a linear combination of sample standard deviations with a chi-distribution (see Appendix C). Hence, the distributions of $\widehat{CEP}_2$ and $\widehat{CEP}_4$ were approximated by a chi-distribution with degrees of freedom chosen so the approximate distribution has a variance equal to that of the exact distribution. Since the square of a chi-variable has a chi-square distribution, the squares of $\widehat{CEP}_2$ and $\widehat{CEP}_4$ also have approximate chi-square distributions. The major task is to find the appropriate degrees of freedom for each class of estimator.

First, one sees that $\widehat{CEP}_1$ has the same form as the maximum likelihood estimator for CEP in the circular case. In that case, the degrees of freedom associated with $\widehat{CEP}$ were 2n. These same degrees of freedom will be retained here for $\widehat{CEP}_1$. This will eventually show how poorly $\widehat{CEP}_1$ performs as an estimator for CEP when the error distribution is elliptical vice circular.

Next, consider $\widehat{CEP}_3$ and $\widehat{CEP}_5$ of the first class. Each has form

$$CEP_i = K_i \left(\frac{s_x^2 + s_y^2}{2} \right)^{\frac{1}{2}} .$$

To obtain υ', the degrees of freedom for our chi-square, we need to equate the variance of $\upsilon' \widehat{CEP}_i^2 / CEP_i^2$ with $2\upsilon'$ (the variance of a chi-square with υ' degrees of freedom) and solve for υ'. Now

$$\frac{\upsilon' \widehat{CEP}_i^2}{CEP_i^2} = \frac{\upsilon' \ K_i^2 \left(s_x^2 + s_y^2 \right)}{2 \ CEP_i^2}$$

and the variance of this expression is

$$\frac{(\upsilon')^2 \ K_i^4}{4 \ CEP_i^4} \left(\frac{2\sigma_x^4}{n} + \frac{2\sigma_y^4}{n} \right) .$$

Upon substituting $K_i \left(\frac{\sigma_x^2 + \sigma_y^2}{2} \right)^{\frac{1}{2}}$ for CEP_i, this becomes

$$\frac{(\upsilon')^2 \left(\sigma_x^4 + \sigma_y^4 \right)}{n \left(\sigma_x^2 + \sigma_y^2 \right)^2} .$$

Equating this expression to $2\upsilon'$ and solving for υ' yields

$$\upsilon' = \frac{n \left(\sigma_x^2 + \sigma_y^2 \right)^2}{\left(\sigma_x^4 + \sigma_y^4 \right)} = n \ \upsilon \qquad (4.3)$$

where υ was previously defined to be

$$\upsilon = \frac{\left(\sigma_x^2 + \sigma_y^2\right)^2}{\sigma_x^4 + \sigma_y^4} . \tag{4.4}$$

Consider next, $\hat{CEP}_2$ and $\hat{CEP}_4$ of the second class. Each of these estimators has the form

$$\hat{CEP}_i = a_1 \, s_{max} + a_2 \, s_{min} \qquad i = 2, 4 .$$

To obtain υ^*, the degrees of freedom for the chi-square in this class, we equate the variance of $\frac{(\upsilon^*)^{\frac{1}{2}} \hat{CEP}_i}{CEP_i}$ with $\upsilon^* - 2\left[\frac{\Gamma\left(\frac{\upsilon^* + 1}{2}\right)}{\Gamma\left(\frac{\upsilon^*}{2}\right)}\right]^2$ (the variance of a chi-random variable with υ^* degrees of freedom) and solve for υ^*. Now,

$$\frac{(\upsilon^*)^{\frac{1}{2}} \hat{CEP}_i}{CEP_i} = \frac{(\upsilon^*)^{\frac{1}{2}}\left(a_1 \, s_{max} + a_2 \, s_{min}\right)}{\left(a_1 \, \sigma_{max} + a_2 \, \sigma_{min}\right)}$$

and the variance of this expression is

$$\frac{\upsilon^* \left(a_1^2 \, V\left(s_{max}\right) + a_2^2 \, V\left(s_{min}\right)\right)}{\left(a_1 \, \sigma_{max} + a_2 \, \sigma_{min}\right)^2} . \tag{4.5}$$

If we denote the function H(x) as

$$H(x) = \sqrt{\frac{2}{x}} \; \frac{\Gamma\left(\frac{x + 1}{2}\right)}{\Gamma\left(\frac{x}{2}\right)} , \tag{4.6}$$

then the variance of a sample standard deviation based on n observations, i.e., the variance of s_x or s_y shown in (3.4) can be expressed as

$$V(s_x) = V(s_y) = \sigma^2 \, [1 - H^2(n)].$$

Using this notation in (4.5) and equating the latter to the variance of a chi-random variable yields

$$\frac{\left(a_1^2 \, \sigma_x^2 + a_2^2 \, \sigma_y^2\right)\left(1 - H^2(n)\right)}{\left(a_1 \, \sigma_x + a_2 \, \sigma_y\right)^2} = 1 - H^2(\upsilon^*)$$

or

$$H(\upsilon^*) = \left[1 - \left\{1 - H^2(n)\right\} \frac{\left(a_1^2 \, \sigma_x^2 + a_2^2 \, \sigma_y^2\right)}{\left(a_1 \, \sigma_x + a_2 \, \sigma_y\right)^2}\right]^{\frac{1}{2}} . \tag{4.7}$$

The value of υ^* cannot be computed explicitly but is easily obtained using the following procedure.

Evaluate the right-hand side of (4.7) using estimates of σ_x and σ_y obtained from n sample data points and values of a_1 and a_2 determined by (4.2). Call this value w. Refer to Appendix B which contains tabled values of x and H(x). Enter the table and find the value of x for which H(x) = w. This value of x is υ^*. An example which incorporates this procedure begins in Chapter 5.

Clearly, υ' and υ^* may take on fractional (non-integral) values. However, this poses no problem. Although the question of fractional degrees of freedom is rarely addressed in standard statistics texts, an extensive table of chi-square percentage points with fractional degrees of freedom has been generated by DiDonato and Hageman (1977). Also, one can obtain such percentage points using the MDCHI subroutine available in IMSL (1982).

For each of the five CEP estimators, it has been shown that the distribution of

$$\frac{\upsilon_i \, \widehat{CEP}_i^2}{CEP_i^2} \qquad i = 1, 2, \ldots 5 \tag{4.8}$$

can be approximated by a chi-square distribution with υ_i degrees of freedom where υ_i is either 2n, υ', or υ^* defined previously. Since the form of a confidence interval for a chi-square random variable is well-known, construction of confidence intervals for CEP, using (4.8), is straightforward.

CHAPTER 5

APPROXIMATE CEP CONFIDENCE INTERVALS

An approximate 100 (1 - α)% confidence interval for the true (population) CEP can be constructed using the probability statement

$$\text{Prob}\left\{ \chi^2_{\upsilon_i,\ \alpha/2} < \frac{\upsilon_i \ \widehat{CEP}_i^2}{CEP_i^2} < \chi^2_{\upsilon_i,\ 1-\alpha/2} \right\} = 1 - \alpha \ . \tag{5.1}$$

The subscript i is used to indicate the approximation on which the estimate $\widehat{CEP}_i$ and the degrees of freedom, υ_i, are based. Rewriting (5.1) in terms of CEP_i yields

$$\left(\frac{\widehat{CEP}_i}{\left(\chi^2_{\upsilon_i, 1-\alpha/2}/\upsilon_i\right)^{\frac{1}{2}}} < CEP_i < \frac{\widehat{CEP}_i}{\left(\chi^2_{\upsilon_i, \alpha/2}/\upsilon_i\right)^{\frac{1}{2}}} \right) . \tag{5.2}$$

However, CEP_i represents an approximation to the true CEP for any i. Therefore, (5.2) may be considered an approximate confidence interval for CEP and expressed as

$$\left(\frac{\widehat{CEP}_i}{\left(\chi^2_{\upsilon_i, 1-\alpha/2}/\upsilon_i\right)^{\frac{1}{2}}} < CEP < \frac{\widehat{CEP}_i}{\left(\chi^2_{\upsilon_i, \alpha/2}/\upsilon_i\right)^{\frac{1}{2}}} \right) . \tag{5.3}$$

In the following example, approximate confidence intervals will be computed for CEP using two CEP estimators, $\widehat{CEP}_3$ and $\widehat{CEP}_4$. Using (3.4), estimates of σ_x and σ_y can be computed for the 12 sample miss distances given in Table 5-1.

TABLE 5-1. 12 HYPOTHETICAL MISS DISTANCES (FEET)

x	y
-163	-363
104	-56
-47	224
-13	-61
-84	-267
53	-85
93	383

TABLE 5-1. (Cont.)

x	y
197	-147
-266	61
135	626
107	187
-112	-11

For these data, one finds

$$s_x^2 = \sum_{i=1}^{n} x_i^2/n = 17{,}505.00$$

$$s_y^2 = \sum_{i=1}^{n} y_i^2/n = 72{,}191.75 \quad .$$

As previously defined,

$$\hat{CEP}_3 = \left(2\chi^2_{\upsilon,.50}/\upsilon\right)^{\frac{1}{2}} \left(\frac{s_x^2 + s_y^2}{2}\right)^{\frac{1}{2}} \quad .$$

Since υ assumes values between 1 and 2 inclusively, a table of the $(2\chi_{\upsilon,.50}/\upsilon)^{\frac{1}{2}}$ factor is readily constructed using chi-square percentage points taken from DiDonato and Hageman (1977). Table 5-2 is a short table that has been prepared to facilitate computation.

TABLE 5-2. MULTIPLYING FACTORS OF $\left(\frac{s_x^2 + s_y^2}{2}\right)^{\frac{1}{2}}$ FOR CEP_3 APPROXIMATION

υ	$\left(2\ \chi^2_{\upsilon,.50}/\upsilon\right)^{\frac{1}{2}}$
1.0	.9538
1.1	.9928
1.2	1.0258
1.3	1.0542
1.4	1.0789
1.5	1.1005
1.6	1.1195
1.7	1.1365
1.8	1.1516
1.9	1.1652
2.0	1.1774

$$\upsilon = \frac{\left(\sigma_x^2 + \sigma_y^2\right)^2}{\sigma_x^4 + \sigma_y^4}$$

The 12 sample miss distances in Table 5-1 are used to obtain estimates of υ and υ' given by $\hat{\upsilon}$ and $\hat{\upsilon}'$ below:

$$\hat{\upsilon} = \frac{\left(s_x^2 + s_y^2\right)^2}{s_x^4 + s_y^4} = 1.46 \qquad \hat{\upsilon}' = n\hat{\upsilon} = 17.52 \quad .$$

Interpolation in Table 5-2 gives $\left(2\ \chi^2_{\hat{\upsilon},.50}/\hat{\upsilon}\right)^{\frac{1}{2}} = 1.0919$ so that

$$\widehat{CEP}_3 = (1.0919)\ (211.77) = 231.23 \quad .$$

To form confidence limits, the computations in (5.3) are needed. If 95% limits are considered, the chi-square tabular values for $\alpha = .05$ are obtained via interpolation in DiDonato and Hageman (1977) and are

$$\chi^2_{\hat{\upsilon}',.025} = 7.91$$

$$\chi^2_{\hat{\upsilon}',.975} = 30.89 \quad .$$

The approximate 95% confidence interval for CEP based on approximation 3 is, therefore

$$\left(\frac{231.23}{(30.89/17.52)^{\frac{1}{2}}} < CEP < \frac{231.23}{(7.91/17.52)^{\frac{1}{2}}}\right)$$

or

$$(174.14 < CEP < 344.13) \quad .$$

The interpretation here is that one is approximately 95% confident that the true CEP lies in the computed interval.

Let us next consider the approximate 95% confidence interval obtained by using $\widehat{CEP}_4$. Before using estimator 4, one must compute the ratio $c = s_{min}/s_{max}$ to determine which half of the piece-wise approximation should be used. In this case, $c = .49$ so that

$$\widehat{CEP}_4 = .565\ s_{max} + .612\ s_{min} = 232.78.$$

This is reasonably close to the 231.23 obtained for $\widehat{CEP}_3$.

To determine $\hat{\upsilon}^*$, an estimate of υ^*, evaluate

$$H(\hat{\upsilon}^*) = \left[1 - (1 - H^2(n))\ \frac{\left(\overline{.565}^2\ s^2_{max} + \overline{.612}^2\ s^2_{min}\right)}{\left(.565\ s_{max} + .612\ s_{min}\right)^2}\right]^{\frac{1}{2}} \quad .$$

$H^2(n)$ may be determined using tabled values of the gamma function provided in the National Bureau of Standards Applied Mathematics Series document by Salzer (1951). An abbreviated version is given in Appendix A. $H(n)$ may also be read directly from the table provided in Appendix B.

Now

$$H^2(n) = (2/12)\left(\frac{\Gamma(6.5)}{\Gamma(6)}\right)^2 = .9592$$

and

$$H(\hat{\upsilon}^*) = .9888 \; .$$

Entering Appendix B with .9888 and interpolating between .9887 and .9890 yields $\hat{\upsilon}^* = 22.17$. From DiDonato and Hageman (1977), obtain

$$\chi^2_{\hat{\upsilon}^*,.025} = 11.10$$

$$\chi^2_{\hat{\upsilon}^*,.975} = 37.00$$

via interpolation. An approximate 95% confidence interval for CEP using the fourth estimator is, therefore, given by

$$\left(\frac{232.78}{(37.00/22.17)^{\frac{1}{2}}} < \text{CEP} < \frac{232.78}{(11.10/22.17)^{\frac{1}{2}}}\right)$$

or

$$(180.19 < \text{CEP} < 328.98).$$

One notes that these two intervals are different. Had the other three estimators been used to construct confidence intervals, they too would have been different. We now have the problem of deciding which estimator to use for constructing confidence intervals. This will be discussed in the next chapter.

Before leaving this chapter, something needs to be said regarding the assumption of zero means. Throughout the development in the report, it has been assumed that the errors have zero mean in both directions. An error has been assumed to be a miss distance from a target, and zero mean implies there is no bias, i.e., the target coincides with the distribution mean. There are applications where the errors are not miss distances, per se, but deviations from the mean impact point. This occurs when there is bias in either or both directions or when there is no target, i.e., the firings are conducted to estimate dispersion without regard to a target. In either case, the impact distribution is no longer centered on the target but on an unknown point (μ_x, μ_y), and CEP is the radius of the 50% circle which is centered on this point vice the target. To apply the methodology in this report to these cases, the squares of s_x and s_y in (3.4) must be modified to read

$$s_x^2 = \hat{\sigma}_x^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n - 1} = \frac{\sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2/n}{n - 1}$$

$$s_y^2 = \hat{\sigma}_y^2 = \frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{n - 1} = \frac{\sum_{i=1}^{n} y_i^2 - (\sum_{i=1}^{n} y_i)^2/n}{n - 1} \quad . \tag{5.4}$$

In (5.4), $\bar{x}$ and $\bar{y}$ are the averages of the impact point locations in the x and y directions, respectively. In addition, the degrees of freedom associated with each estimator must be reduced by replacing n with n - 1 in each as follows:

ESTIMATOR	MODIFIED d.f. FOR NON-ZERO MEANS
CEP_1	$2\ (n - 1)$
CEP_3, CEP_5	$(n - 1)\upsilon$
CEP_2, CEP_4	$H(\upsilon^*) = \left[1 - \left\{1 - H^2\ (n - 1)\right\} \frac{(a_1^2\sigma_x^2 + a_2^2\sigma_y^2)}{(a_1\sigma_x + a_2\sigma_y)^2}\right]^{\frac{1}{2}}$

This reduction in degrees of freedom means a slight reduction in precision. In effect, it requires n + 1 deviations from the mean to provide the same precision as n miss distances.

CHAPTER 6

MONTE CARLO VERIFICATION

To ascertain if the formulated approximations produced confidence intervals with confidence close to $1 - \alpha$, a Monte Carlo simulation was written for the CDC 6700. This would also provide a means of comparing the estimators with respect to their confidence level and their expected confidence interval length. One replicate of the simulation entailed generating a sample of n miss distances from a bivariate normal distribution with zero mean and variances σ_x^2 and σ_y^2. These n values were then used to compute a CEP confidence interval using the five estimators $\widehat{CEP}_1$ to $\widehat{CEP}_5$. The length of each interval was computed, and for each interval, it was determined whether the interval contained the true (population) CEP. This process was replicated N = 10,000 times. The proportion of replicates in which the confidence interval contained the true CEP provided an estimate of the confidence associated with each estimator, and the average interval length for each provided an estimate of the expected length.

The values of the parameters used in the simulation are:

n = number of miss distances = sample size = 5, 10, 20

N = number of replicates = 10,000

$1 - \alpha$ = nominal confidence level = .95

$c = \sigma_{min}/\sigma_{max}$ = 1.0, .75, .50, .35, .20, .05.

It was necessary to run a large number of replicates to ensure results with reasonable precision. The 10,000 replicates used provided estimates of confidence with an error of less than .01 with probability .95. Because of the large N, the sample sizes were restricted to small values to keep the computer time within bounds. A nominal value of $\alpha = .05$ ($\Longrightarrow$ a nominal confidence level of .95) was used in the construction of all confidence intervals within the simulation. It follows that if the confidence intervals were exact (vice approximate), the Monte Carlo confidence estimates should be within .01 of .95 or .95 ± .01 with probability .95. Hence, any confidence estimate which departs seriously from .95 ± .01 will reflect poorly on the estimator which produced it. One additional comment is required before discussing the results. The simulated confidence levels depend only on the ratio of the sigmas so that it was not necessary to vary both standard deviations. The larger was designated σ_y and set equal to unity while the smaller was designated σ_x and set equal to the ratio c. While the average confidence interval lengths are dependent on both sigmas, they were only constructed for $\sigma_x = c$, $\sigma_y = 1$. This is all that is required for comparison purposes. If such lengths are needed for values of σ_x and σ_y other than c and 1, they can be obtained by multiplying the tabled entry for appropriate c by $\sigma_y = \sigma_{max}$.

The results of the simulations are set out in Tables 6-1 and 6-2. Let us first discuss the simulated confidence levels shown in Table 6-1. One first notes that all five estimators provide confidence within (or nearly within) the sampling variations (±.01) of .95 when c = 1. This is the circular case and all the estimators except $\widehat{CEP}_5$ degenerate properly to the maximum likelihood estimator for CEP. Hence, the result that all do well for c = 1 is not unexpected. Next, one notes that as c departs from unity (that is, as the impact distribution departs from circularity), the confidence associated with $\widehat{CEP}_1$ departs seriously from .95. For example, with n = 10 and c = .20 (5 to 1 ratio of the sigmas) the confidence associated with $\widehat{CEP}_1$ is only .689. This means that if one were to use circular theory to construct a 95% confidence interval for CEP when the distribution was elliptical with a 5 to 1 ratio of the sigmas, his interval would have confidence of less than .7! This rules out $\widehat{CEP}_1$ unless one is nearly certain that the impact distribution is circular normal. This result is also not unexpected, but it does quantify how poorly the circular estimator performs in the elliptical case.

In general, the others do reasonably well unless c is small. One notes this especially for $\widehat{CEP}_2$ when c = .05; the confidence falls from .930 for n = 5 to .912 for n = 20. It would continue to decrease as n increases due to the error in CEP approximation for small c (see Figure 3-2). The distribution of $\widehat{CEP}_2$ becomes more concentrated about the approximate CEP as n increases. If the approximation is in serious error (which it is for small c), then the distribution is concentrated about the wrong value. The simulation was run for n = 100 at c = .05 with a resulting confidence estimate of only .714. We see the same behavior at larger c values but to a lesser extent. For example, at c = .35, the confidence estimate is .942 for n = 20 but dips to .922 for n = 100. The upshot here is that $\widehat{CEP}_2$ would be a problem for small c or even moderate c if the sample is large enough. With regard to $\widehat{CEP}_3$, one sees that small values of c pose no problem. In fact, the confidence for $\widehat{CEP}_3$ is asymptotic to .95 at c = 0. Also, for values of c around .5, the confidence estimates are slightly higher than .95. It tends to peak out at about .97. Selected runs for n = 100 show that this result changes very little with n. There is a slight price to pay for this extra confidence, and this will be addressed when Table 6-2 is discussed. $\widehat{CEP}_4$ provides confidence close to .95 for all values of c except those where CEP_4 departs from CEP (see Figure 3-4). At those values, there is a reduction in confidence which increases with n but not as severely as for $\widehat{CEP}_2$. The performance of $\widehat{CEP}_5$ is not poor with respect to confidence. However,

TABLE 6-1. SIMULATED CONFIDENCE LEVELS

n = 5

C	$\widehat{CEP}_1$	$\widehat{CEP}_2$	$\widehat{CEP}_3$	$\widehat{CEP}_4$	$\widehat{CEP}_5$
1.0	.950	.947	.963	.946	.963
.75	.941	.947	.965	.945	.965
.50	.894	.941	.968	.944	.967
.35	.830	.937	.963	.939	.961
.20	.753	.932	.952	.923	.950
.05	.714	.930	.950	.936	.948

n = 10

C	$\widehat{CEP}_1$	$\widehat{CEP}_2$	$\widehat{CEP}_3$	$\widehat{CEP}_4$	$\widehat{CEP}_5$
1.0	.947	.945	.955	.944	.955
.75	.935	.944	.958	.943	.959
.50	.876	.941	.967	.943	.966
.35	.789	.939	.967	.941	.965
.20	.689	.939	.959	.931	.955
.05	.640	.931	.951	.943	.948

n = 20

C	$\widehat{CEP}_1$	$\widehat{CEP}_2$	$\widehat{CEP}_3$	$\widehat{CEP}_4$	$\widehat{CEP}_5$
1.0	.952	.952	.956	.951	.956
.75	.938	.951	.960	.951	.960
.50	.858	.945	.970	.948	.969
.35	.724	.942	.970	.947	.968
.20	.567	.946	.961	.937	.955
.05	.506	.912	.950	.946	.946

TABLE 6-2. AVERAGE CONFIDENCE INTERVAL LENGTHS

n = 5

C	$\hat{CEP}_1$	$\hat{CEP}_2$	$\hat{CEP}_3$	$\hat{CEP}_4$	$\hat{CEP}_5$
1.0	1.213	1.160	1.305	1.145	1.317
.75	1.071	1.028	1.175	1.014	1.186
.50	.950	.923	1.118	.918	1.131
.35	.897	.889	1.129	.917	1.144
.20	.856	.876	1.149	.985	1.168
.05	.841	.912	1.176	1.054	1.196

n = 10

C	$\hat{CEP}_1$	$\hat{CEP}_2$	$\hat{CEP}_3$	$\hat{CEP}_4$	$\hat{CEP}_5$
1.0	.792	.776	.817	.768	.823
.75	.698	.685	.735	.676	.741
.50	.622	.614	.697	.601	.705
.35	.587	.586	.693	.586	.702
.20	.564	.573	.694	.636	.706
.05	.553	.579	.695	.662	.707

n = 20

C	$\hat{CEP}_1$	$\hat{CEP}_2$	$\hat{CEP}_3$	$\hat{CEP}_4$	$\hat{CEP}_5$
1.0	.537	.533	.545	.531	.549
.75	.475	.472	.492	.467	.496
.50	.423	.421	.465	.411	.470
.35	.400	.401	.460	.393	.466
.20	.384	.388	.455	.429	.462
.05	.378	.387	.453	.441	.461

there is little rationale for its use in constructing confidence intervals. The Wilson-Hilferty approximation avoids the use of chi-square percentage points for fractional degrees of freedom in forming $\widehat{CEP}_4$. However, they are needed in the computation of the interval, so little effort is saved.

Let us now discuss the average confidence interval lengths in Table 6-2. As previously noted, these lengths depend on both σ_x and σ_y but were computed only for values of the ratio of σ_x to σ_y for comparison purposes. However, we need not compare all five since some were eliminated as viable candidates in our discussions of Table 6-1. $\widehat{CEP}_1$ was eliminated because its confidence eroded seriously as c departed from unity. $\widehat{CEP}_2$ had a less serious but similar problem, and $\widehat{CEP}_5$ was eliminated because it offered no improvement over $\widehat{CEP}_3$ and only a slight reduction in computation. This leaves only $\widehat{CEP}_3$ and $\widehat{CEP}_4$ to discuss here. One notes that the average lengths are uniformly less for $\widehat{CEP}_4$ than for $\widehat{CEP}_3$. At mid values of c, this is due in part to the inflated confidence inherent in the approximation of the distribution of $\widehat{CEP}_3$, i.e., the higher the confidence, the longer the confidence length. However, not all of the difference in length can be attributed to higher confidence. A study by Taub and Thomas (1982) shows that the variance of $\widehat{CEP}_4$ is less than the variance of $\widehat{CEP}_3$, and this is the primary reason for the difference in length. Even so, $\widehat{CEP}_4$ suffers from the bias caused by the error in approximation shown in Figure 3-4. This has an effect on the confidence level associated with $\widehat{CEP}_4$ for some values of c, but it would not be appreciable for small n.

In summary, we can state that the logical choice lies between $\widehat{CEP}_3$ and $\widehat{CEP}_4$. The third holds for all values of c, regardless of n, and is easy to implement. The fourth offers somewhat shorter confidence lengths but it is cumbersome to implement and would provide a reduced confidence level for some values of n and c.

It would be highly desirable to have a CEP estimator with a variance as small or smaller than $\widehat{CEP}_4$ which would have negligible bias for all $0 < c \leq 1$, and which would avoid the cumbersomeness of a piece-wise linear function. The authors have several ideas along this line and hope to explore their merits in the near future.

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APPENDIX A

TABLE OF $\Gamma(n)$ AND $\Gamma(n + \frac{1}{2})$ FOR $n = 1, \ldots 100$

Excerpted from National Bureau of Standards

Applied Mathematics Series - 16

by

Herbert E. Salzer

n	$n!$	$\Gamma(n+\frac{1}{2})$	n	$n!$	$\Gamma(n+\frac{1}{2})$
0	1. 00000 00000 00000 (0)	1. 7724 539 (0)	50	3. 04140 93201 71338 (64)	4. 2904 629 (63)
1	1. 00000 00000 00000 (0)	8. 8622 693 (−1)	51	1. 55111 87532 87382 (66)	2. 1666 838 (65)
2	2. 00000 00000 00000 (0)	1. 3293 404 (0)	52	8. 06581 75170 94388 (67)	1. 1158 421 (67)
3	6. 00000 00000 00000 (0)	3. 3233 510 (0)	53	4. 27488 32840 60026 (69)	5. 8581 712 (68)
4	2. 40000 00000 00000 (1)	1. 1631 728 (1)	54	2. 30843 69733 92414 (71)	3. 1341 216 (70)
5	1. 20000 00000 00000 (2)	5. 2342 778 (1)	55	1. 26964 03353 65828 (73)	1. 7080 963 (72)
6	7. 20000 00000 00000 (2)	2. 8788 528 (2)	56	7. 10998 58780 48635 (74)	9. 4799 344 (73)
7	5. 04000 00000 00000 (3)	1. 8712 543 (3)	57	4. 05269 19504 87722 (76)	5. 3561 629 (75)
8	4. 03200 00000 00000 (4)	1. 4034 407 (4)	58	2. 35056 13312 82879 (78)	3. 0797 937 (77)
9	3. 62880 00000 00000 (5)	1. 1929 246 (5)	59	1. 38683 11854 56898 (80)	1. 8016 793 (79)
10	3. 62880 00000 00000 (6)	1. 1332 784 (6)	60	8. 32098 71127 41390 (81)	1. 0719 992 (81)
11	3. 99168 00000 00000 (7)	1. 1899 423 (7)	61	5. 07580 21387 72248 (83)	6. 4855 951 (82)
12	4. 79001 60000 00000 (8)	1. 3684 337 (8)	62	3. 14699 73260 38794 (85)	3. 9886 410 (84)
13	6. 22702 08000 00000 (9)	1. 7105 421 (9)	63	1. 98260 83154 04440 (87)	2. 4929 006 (86)
14	8. 71782 91200 00000 (10)	2. 3092 318 (10)	64	1. 26886 93218 58842 (89)	1. 5829 919 (88)
15	1. 30767 43680 00000 (12)	3. 3483 861 (11)	65	8. 24765 05920 82471 (90)	1. 0210 298 (90)
16	2. 09227 89888 00000 (13)	5. 1899 985 (12)	66	5. 44344 93907 74431 (92)	6. 6877 450 (91)
17	3. 55687 42809 60000 (14)	8. 5634 974 (13)	67	3. 64711 10918 18869 (94)	4. 4473 504 (93)
18	6. 40237 37057 28000 (15)	1. 4986 121 (15)	68	2. 48003 55424 36831 (96)	3. 0019 615 (95)
19	1. 21645 10040 88320 (17)	2. 7724 323 (16)	69	1. 71122 45242 81413 (98)	2. 0563 436 (97)
20	2. 43290 20081 76640 (18)	5. 4062 430 (17)	70	1. 19785 71669 96989 (100)	1. 4291 588 (99)
21	5. 10909 42171 70944 (19)	1. 1082 798 (19)	71	8. 50478 58856 78623 (101)	1. 0075 570 (101)
22	1. 12400 07277 77608 (21)	2. 3828 016 (20)	72	6. 12344 58376 88609 (103)	7. 2040 324 (102)
23	2. 58520 16738 88498 (22)	5. 3613 036 (21)	73	4. 47011 54615 12684 (105)	5. 2229 235 (104)
24	6. 20448 40173 32394 (23)	1. 2599 063 (23)	74	3. 30788 54415 19386 (107)	3. 8388 487 (106)
25	1. 55112 10043 33099 (25)	3. 0867 705 (24)	75	2. 48091 40811 39540 (109)	2. 8599 423 (108)
26	4. 03291 46112 66056 (26)	7. 8712 649 (25)	76	1. 88549 47016 66050 (111)	2. 1592 564 (110)
27	1. 08888 69450 41835 (28)	2. 0858 852 (27)	77	1. 45183 09202 82859 (113)	1. 6518 312 (112)
28	3. 04888 34461 17139 (29)	5. 7361 843 (28)	78	1. 13242 81178 20630 (115)	1. 2801 692 (114)
29	8. 84176 19937 39702 (30)	1. 6348 125 (30)	79	8. 94618 21307 82975 (116)	1. 0049 328 (116)
30	2. 65252 85981 21911 (32)	4. 8226 969 (31)	80	7. 15694 57046 26380 (118)	7. 9892 157 (117)
31	8. 22283 86541 77923 (33)	1. 4709 226 (33)	81	5. 79712 60207 47368 (120)	6. 4313 187 (119)
32	2. 63130 83693 36935 (35)	4. 6334 061 (34)	82	4. 75364 33370 12842 (122)	5. 2415 247 (121)
33	8. 68331 76188 11886 (36)	1. 5058 570 (36)	83	3. 94552 39697 20659 (124)	4. 3242 579 (123)
34	2. 95232 79903 96041 (38)	5. 0446 209 (37)	84	3. 31424 01345 65353 (126)	3. 6107 553 (125)
35	1. 03331 47966 38614 (40)	1. 7403 942 (39)	85	2. 81710 41143 80550 (128)	3. 0510 883 (127)
36	3. 71993 32678 99012 (41)	6. 1783 994 (40)	86	2. 42270 95383 67273 (130)	2. 6086 805 (129)
37	1. 37637 53091 22635 (43)	2. 2551 158 (42)	87	2. 10775 72983 79528 (132)	2. 2565 086 (131)
38	5. 23022 61746 66011 (44)	8. 4566 842 (43)	88	1. 85482 64225 73984 (134)	1. 9744 450 (133)
39	2. 03978 82081 19744 (46)	3. 2558 234 (45)	89	1. 65079 55160 90846 (136)	1. 7473 838 (135)
40	8. 15915 28324 78977 (47)	1. 2860 502 (47)	90	1. 48571 59644 81761 (138)	1. 5639 085 (137)
41	3. 34525 26613 16381 (49)	5. 2085 035 (48)	91	1. 35200 15276 78403 (140)	1. 4153 372 (139)
42	1. 40500 61177 52880 (51)	2. 1615 290 (50)	92	1. 24384 14054 64131 (142)	1. 2950 336 (141)
43	6. 04152 63063 37384 (52)	9. 1864 981 (51)	93	1. 15677 25070 81642 (144)	1. 1979 060 (143)
44	2. 65827 15747 88449 (54)	3. 9961 267 (53)	94	1. 08736 61566 56743 (146)	1. 1200 422 (145)
45	1. 19622 22086 54802 (56)	1. 7782 764 (55)	95	1. 03299 78488 23906 (148)	1. 0584 398 (147)
46	5. 50262 21598 12089 (57)	8. 0911 574 (56)	96	9. 91677 93487 09497 (149)	1. 0108 100 (149)
47	2. 58623 24151 11682 (59)	3. 7623 882 (58)	97	9. 61927 59682 48212 (151)	9. 7543 169 (150)
48	1. 24139 15592 53607 (61)	1. 7871 344 (60)	98	9. 42689 04488 83248 (153)	9. 5104 590 (152)
49	6. 08281 86403 42676 (62)	8. 6676 018 (61)	99	9. 33262 15443 94415 (155)	9. 3678 021 (154)
50	3. 04140 93201 71338 (64)	4. 2904 629 (63)	100	9. 33262 15443 94415 (157)	9. 3209 631 (156)

APPENDIX B

TABLES OF x AND H(x) FOR x = .1 TO 400

TABLE 1

X	H(X)	X	H(X)	X	H(X)	X	H(X)
.10	.3712	1.95	.8837	3.80	.9370	5.65	.9569
.15	.4411	2.00	.8862	3.85	.9378	5.70	.9573
.20	.4950	2.05	.8887	3.90	.9385	5.75	.9577
.25	.5386	2.10	.8911	3.95	.9393	5.80	.9580
.30	.5748	2.15	.8933	4.00	.9400	5.85	.9584
.35	.6056	2.20	.8955	4.05	.9407	5.90	.9587
.40	.6322	2.25	.8976	4.10	.9414	5.95	.9590
.45	.6555	2.30	.8996	4.15	.9421	6.00	.9594
.50	.6760	2.35	.9015	4.20	.9427	6.05	.9597
.55	.6942	2.40	.9034	4.25	.9434	6.10	.9600
.60	.7105	2.45	.9052	4.30	.9440	6.15	.9603
.65	.7252	2.50	.9069	4.35	.9446	6.20	.9606
.70	.7385	2.55	.9086	4.40	.9452	6.25	.9609
.75	.7507	2.60	.9102	4.45	.9458	6.30	.9613
.80	.7617	2.65	.9118	4.50	.9464	6.35	.9615
.85	.7719	2.70	.9133	4.55	.9469	6.40	.9618
.90	.7812	2.75	.9147	4.60	.9475	6.45	.9621
.95	.7899	2.80	.9161	4.65	.9480	6.50	.9624
1.00	.7979	2.85	.9175	4.70	.9486	6.55	.9627
1.05	.8053	2.90	.9188	4.75	.9491	6.60	.9630
1.10	.8122	2.95	.9201	4.80	.9496	6.65	.9632
1.15	.8187	3.00	.9213	4.85	.9501	6.70	.9635
1.20	.8247	3.05	.9225	4.90	.9506	6.75	.9638
1.25	.8304	3.10	.9237	4.95	.9511	6.80	.9640
1.30	.8357	3.15	.9248	5.00	.9515	6.85	.9643
1.35	.8407	3.20	.9259	5.05	.9520	6.90	.9645
1.40	.8454	3.25	.9270	5.10	.9524	6.95	.9648
1.45	.8499	3.30	.9280	5.15	.9529	7.00	.9650
1.50	.8541	3.35	.9290	5.20	.9533	7.05	.9653
1.55	.8581	3.40	.9300	5.25	.9538	7.10	.9655
1.60	.8619	3.45	.9310	5.30	.9542	7.15	.9657
1.65	.8654	3.50	.9319	5.35	.9546	7.20	.9660
1.70	.8689	3.55	.9328	5.40	.9550	7.25	.9662
1.75	.8721	3.60	.9337	5.45	.9554	7.30	.9664
1.80	.8752	3.65	.9345	5.50	.9558	7.35	.9667
1.85	.8781	3.70	.9354	5.55	.9562	7.40	.9669
1.90	.8810	3.75	.9362	5.60	.9566	7.45	.9671

TABLE 1

X	H(X)	X	H(X)	X	H(X)	X	H(X)
7.50	.9673	9.35	.9737	16.00	.9845	31.00	.9920
7.55	.9675	9.40	.9738	16.25	.9847	32.00	.9922
7.60	.9677	9.45	.9739	16.50	.9850	33.00	.9925
7.65	.9679	9.50	.9741	16.75	.9852	34.00	.9927
7.70	.9681	9.55	.9742	17.00	.9854	35.00	.9929
7.75	.9683	9.60	.9743	17.25	.9856	36.00	.9931
7.80	.9685	9.65	.9745	17.50	.9858	37.00	.9933
7.85	.9687	9.70	.9746	17.75	.9860	38.00	.9934
7.90	.9689	9.75	.9747	18.00	.9862	39.00	.9936
7.95	.9691	9.80	.9749	18.25	.9864	40.00	.9938
8.00	.9693	9.85	.9750	18.50	.9866	41.00	.9939
8.05	.9695	9.90	.9751	18.75	.9868	42.00	.9941
8.10	.9697	9.95	.9752	19.00	.9869	43.00	.9942
8.15	.9699	10.00	.9754	19.25	.9871	44.00	.9943
8.20	.9700	10.25	.9759	19.50	.9873	45.00	.9945
8.25	.9702	10.50	.9765	19.75	.9874	46.00	.9946
8.30	.9704	10.75	.9770	20.00	.9876	47.00	.9947
8.35	.9706	11.00	.9776	20.50	.9879	48.00	.9948
8.40	.9707	11.25	.9781	21.00	.9882	49.00	.9949
8.45	.9709	11.50	.9785	21.50	.9884	50.00	.9950
8.50	.9711	11.75	.9790	22.00	.9887	60.00	.9958
8.55	.9712	12.00	.9794	22.50	.9890	70.00	.9964
8.60	.9714	12.25	.9798	23.00	.9892	80.00	.9969
8.65	.9716	12.50	.9802	23.50	.9894	90.00	.9972
8.70	.9717	12.75	.9806	24.00	.9896	100.00	.9975
8.75	.9719	13.00	.9810	24.50	.9899	110.00	.9977
8.80	.9720	13.25	.9813	25.00	.9901	120.00	.9979
8.85	.9722	13.50	.9817	25.50	.9902	130.00	.9981
8.90	.9724	13.75	.9820	26.00	.9904	140.00	.9982
8.95	.9725	14.00	.9823	26.50	.9906	150.00	.9983
9.00	.9727	14.25	.9826	27.00	.9908	175.00	.9986
9.05	.9728	14.50	.9829	27.50	.9910	200.00	.9988
9.10	.9730	14.75	.9832	28.00	.9911	225.00	.9989
9.15	.9731	15.00	.9835	28.50	.9913	250.00	.9990
9.20	.9732	15.25	.9838	29.00	.9914	300.00	.9992
9.25	.9734	15.50	.9840	29.50	.9916	350.00	.9993
9.30	.9735	15.75	.9843	30.00	.9917	400.00	.9994

APPENDIX C

THE CHI DISTRIBUTION

A discussion of the chi-square probability distribution can be found in most probability and statistics textbooks. It is a special case of the more general gamma distribution and its density function is given by

$$f_X(x) = \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\,n/2}} x^{n/2 - 1} e^{-x/2}, \quad 0 < x < \infty \quad .$$

Rarely, however, is the CHI distribution discussed in detail. Because this distribution is critical to the development of the approximate distributions for $\widehat{CEP}_2$ and $\widehat{CEP}_4$, the following is a brief introduction to the CHI distribution.

If X is defined as a chi-square random variable, then $Y = \sqrt{X}$ is distributed as a CHI random variable whose density function is written as

$$f_Y(y) = \frac{2}{\Gamma\left(\frac{n}{2}\right)2^{\frac{n}{2} - 1}} x^{n - 1} e^{\frac{-x^2}{2}} \quad , \quad 0 < x < \infty \quad .$$

The mean of Y is given by

$$E(Y) = \frac{\Gamma\left(\frac{n + 1}{2}\right)2^{\frac{1}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

and the variance of Y is given by

$$V(Y) = n - 2\left[\frac{\Gamma\left(\frac{n + 1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]^2 .$$

In general, the r^{th} moment of Y is given by

$$E(Y^r) = \frac{\Gamma\left(\frac{n + r}{2}\right)2^{r/2}}{\Gamma\left(\frac{n}{2}\right)} \quad .$$

If s is the sample standard deviation with υ degrees of freedom, then the previous results can be used to establish that

$$E(s) = \sqrt{\frac{2}{\upsilon}}\ \frac{\Gamma\left(\frac{\upsilon + 1}{2}\right)}{\Gamma\left(\frac{\upsilon}{2}\right)}$$

and

$$V(s) = \sigma^2 - \frac{2\sigma^2}{\upsilon}\left\{\frac{\Gamma\left(\frac{\upsilon + 1}{2}\right)}{\Gamma\left(\frac{\upsilon}{2}\right)}\right\}^2 .$$

These results are needed to determine the degrees of freedom associated with $\widehat{CEP}_2$ and $\widehat{CEP}_4$. Additional information on the CHI distribution may be found in Krutchkoff (1970).

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