AD-A153 790 MOMENT INEQUALITIES FOR REAL AND VECTOR P-STABLE STOCHASTIC INTEGRALS(U) NORTH CAROLINA UNIV AT CHAPEL HILL DEPT OF STATISTICS W A WOYCZYNSKI DEC 84 TR-87 1/1

UNCLASSIFIED AFOSR-TR-85-0403 F49620-82-C-0009 F/G 12/1 NL

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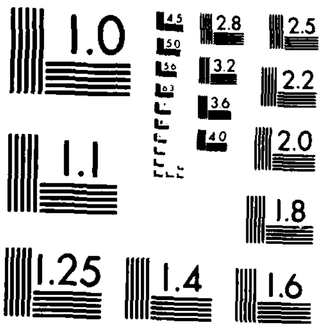

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University of North Carolina
Chapel Hill, North Carolina

Moment Inequalities for Real and Vector p-stable Stochastic Integrals

by

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and

W.A. Woyczynski

Technical Report No. 87

December 1984

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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): TR No. 87	5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 85-0403
6a. NAME OF PERFORMING ORGANIZATION: University of North Carolina; 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION: Air Force Office of Scientific Research
6c. ADDRESS (City, State and ZIP Code): Center for Stochastic Processes, Department of Statistics, Phillips Hall 039-A, Chapel Hill NC 27514	7b. ADDRESS (City, State and ZIP Code): Directorate of Mathematical & Information Sciences, Bolling AFB DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR; 8b. OFFICE SYMBOL (If applicable): NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: F49620-82-C-0009
8c. ADDRESS (City, State and ZIP Code): Bolling AFB DC 20332-6448	10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
61102F	2304	A5	

11. TITLE (Include Security Classification): MOMENT INEQUALITIES FOR REAL AND VECTOR p-STABLE STOCHASTIC INTEGRALS

12. PERSONAL AUTHOR(S): J. Rosinski and W.A. Woyczynski

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
Technical	FROM ______ TO ______	DEC 84	18

16. SUPPLEMENTARY NOTATION: FIFTH INTERNATIONAL CONFERENCE ON PROBABILITY IN BANACH SPACES, MEDFORD, MASS. 1984

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR.	

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

In the present paper the authors obtain moment inequalities for single and double stochastic integrals with respect to p-stable motion (Section 3). The proofs are based on the authors' own work on the structure of single and multiple p-stable integrals (cf. [12], [11] and [13]) which is summarized in some detail in Section 2, and on the work of R.F. Bass and M. Cranston [1] on inequalities for moments of exit times of a p-stable motion. The results, as stated in [1], do not apply directly to the situation in which the authors want to use them, in particular, because one dimensional processes are explicitly excluded there. So, the authors offer the needed variation of their result in Section 3 and, for the sake of completeness, provide its full proof in the Appendix.

In Section 4 the authors propose an extension of the theory of stochastic integration with respect to a p-stable motion, to the case when the latter takes values in a Banach space.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL: MAJ Brian W. Woodruff	22b. TELEPHONE NUMBER (Include Area Code): (202) 767- 5027; 22c. OFFICE SYMBOL: NM

DD FORM 1473, 83 APR — EDITION OF 1 JAN 73 IS OBSOLETE.

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

FIFTH INTERNATIONAL CONFERENCE
ON PROBABILITY IN BANACH SPACES
MEDFORD, MASS. 1984
SPRINGER'S LECTURE NOTES IN MATH

MOMENT INEQUALITIES FOR REAL AND VECTOR p-STABLE STOCHASTIC INTEGRALS[1]

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and
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1. Introduction.

~~In~~ the ~~present~~ paper ~~we obtain~~ moment inequalities for single and double stochastic integrals with respect to p-stable motion ~~(Section 3)~~. The proofs are based on ~~our own~~ work on the structure of single and multiple p-stable integrals (cf. [12], [11] and [13]) which is summarized in some detail in Section 2, and on the work of R.F. Bass and M. Cranston [1] ~~on~~ inequalities for moments of exit times of a p-stable motion. ~~Their~~ results, as stated in [1], do not apply directly to the situation in which ~~we~~ want to use them, in particular, because one dimensional processes are explicitly excluded there. So, ~~we~~ offer the needed variation of their result in Section 3 and, for the sake of completeness, provide its full proof in the Appendix.

In Section 4 ~~we~~ propose an extension of the theory of stochastic integration with respect to a p-stable motion, to the case when the latter takes values in a Banach space.

2. Single and double p-stable integrals.

Let (Ω, F, P) be a probability space and let $(F_t)_{t>0}$ be a right continuous, increasing family of P-complete sub-σ-fields of F. Let $0 < p < 2$. We will denote by $(M(t))_{t>0}$ an (F_t)-p-stable motion i.e. an (F_t)-adapted process with $M(0)=0$, sample paths a.s. in $D[0,\infty)$ and

$$E\{\exp [i\lambda(M(t) - M(s))] | F_s\} = \exp [-(t-s)|\lambda|^p]$$

for every $0 \le s \le t$, and $\lambda \in \mathbb{R}$. For a simple (F_t)-adapted process F such that

$$F(t,\omega) = \begin{cases} \phi_0(\omega) & \text{for } t=0 \\ \phi_i(\omega) & \text{for } t_i < t \le t_{i+1},\ i=0,1,\dots, \end{cases}$$

[1] Research supported by AFOSR Grant No. F49620 82 C 0009
[2] On leave from Wroclaw University

the stochastic integral is defined as usual:

$$\int_0^t F(s,\omega)dM(s,\omega) = \sum_{i=0}^{n-1} \phi_i(\omega)(M(t_{i+1},\omega)-M(t_i,\omega))+\phi_n(\omega)(M(t,\omega)-M(t_n,\omega)),$$

if $t_n \leq t \leq t_{n+1}$, $n=0,1,2,\ldots$. Clearly, the above integral is a process with sample paths a.s. in $D[0,\infty)$.

DEFINITION 2.1. An (F_t)-adapted measurable process $F=(F(t,\omega))_{t\geq 0}$ is said to be M-integrable if there exists a sequence (F_n) of simple (F_t)-adapted processes such that for each $T > 0$:

(i) $F_n \to F$ in measure $dPdt$ on $\Omega \times [0,T]$ as $n \to \infty$,

(ii) $\int_0^t F_n dM$ converge a.s. uniformly in $t \in [0,T]$ as $n \to \infty$,

and the limiting process in (ii) does not depend on the choice of a sequence (F_n) satisfying conditions (i) and (ii). This limit process (with sample paths a.s. in $D[0,\infty)$) will be denoted by $\int_0^t F(s)dM(s)$, $t \geq 0$.

THEOREM 2.1. ([12]) *The process* F *is* M-*integrable if and only if* $F \in L^p_{a.s.}$ *i.e. if*

$$P\ \{ \int_0^T |F(t,\omega)|^p dt < \infty \} = 1$$

for each $T > 0$.

The sufficiency in the above theorem (which also follows from a general result of O. Kallenberg [5]) is obtained by means of a pathwise construction which parallels a known Brownian integral construction and which depends on the following inequality for simple processes F: there exists a constant $c = c(p) > 0$ such that for each $T > 0$

$$c^{-1}||F||^p_{p,T} \leq \sup_{\lambda>0} \lambda^p P\ \{\sup_{t\leq T} |\int_0^t FdM| > \lambda\} \leq c||F||^p_{p,T},$$

where

$$||F||^p_{p,T} = E\int_0^T |F(s,\omega)|^p ds.$$

The inequality implies that the mapping $F \to \int FdM$ extends to an isomorphic embedding of $L^p(L^p)$ into a Lorentz space $\Lambda^p(L^\infty)$. The upper estimate was obtained by E. Giné and M.B. Marcus in [4].

The proof of necessity uses device of the inner clock for p-stable stochastic integrals the usefulness thereof is established by the following:

THEOREM 2.2 ([12]). *Let* $F \in L^p_{a.s.}$ *be such that*

$$\tau(u) \stackrel{df}{=} \int_0^u |F|^p dt \to \infty \quad \text{a.s.}$$

as $u \to \infty$. *Then, if*

$$\tau^{-1}(t) = \inf\{\, u: \tau(u) > t\} \text{ and } A_t = F_{\tau^{-1}(t)} ,$$

then the time-changed stochastic integral

$$\tilde{M}(t) = \int_0^{\tau^{-1}(t)} F(s)dM(s)$$

is an (A_t)*-p-stable motion.*

The above theorem can also be used to establish properties of integrals which are "pathwise inherited" from the properties of p-stable motion itself. For example, the above result immediately yields the following corollary to the classical Khinchine's result on the local behavior of processes with stationary and independent increments:

THEOREM 2.3 ([12]). *Let* F *be as in Theorem 2.2 and suppose that* $\phi: (0,\infty) \to \mathbb{R}^+$ *is such that* $t^{1/p}\phi(t)$ *is increasing and* $\lim_{t\to 0}\phi(t)=\infty$. *Then*

$$\int_0^t F(s)dM(s) = o(\tau^{1/p}(t)\phi(\tau(t))) \quad \text{a.s.}$$

as $t \to 0$, *if and only if*

$$\int_0^1 t^{-1}\phi^{-p}(t)\, dt < \infty .$$

Theorem 2.1 implies that the necessary and sufficient condition for existence of the double integral

(2.1) $$\int_0^T\left(\int_0^t f(s,t)dM(s)\right)dM(t)$$

is that

$$(2.2) \qquad P\{\int_0^T |F(t)|^p dt < \infty\} = 1,$$

where

$$F(t) = \int_0^t f(s,t)dM(s), \qquad t \in [0,T] .$$

The condition (2.2) is equivalent to the property that the integral operator

$$L^{p'}[0,T] \ni \phi \to \int_0^T f(s,t)\chi_\Delta(s,t)\phi(t)\, dt \in L^p[0,T],$$

where $1/p'+1/p=1$, and $\Delta = \{(s,t): 0 \le s < t \le T\}$, is Θ_p-radonifying (or, by Kwapien-Maurey Theory, completely summing) (cf. [7], [3]). The above equivalence follows, in particular, from the following result which gives a natural necessary condition for f to satisfy (2.2). Although this result may have been known in the folklore, we were unable to locate a published proof of it and decided to provide our own proof below. Proof of Thm 2.4 relies on Prop. 2.1.

PROPOSITION 2.1. *Let T be a measurable space, μ be a σ-finite measure on T, and let*

$$X(t) = \int_0^1 f(t,s)dM(s) , \qquad t \in T,$$

be a p-stable process, where $f: T \times [0,1] \to \mathbb{R}$ is a jointly measurable function. Then if

$$\int_T |X(t)|^p \mu(dt) < \infty \qquad a.s.$$

then

$$\int_0^1 \int_T |f(t,s)|^p \mu(dt)\, ds < \infty .$$

Proof. Observe that $\forall q < p \; \exists \; C \; \forall \; t_1,\dots,t_n \in T$

$$(2.3) \qquad \left(\int_0^1 \frac{1}{n} \sum_{i=1}^n |f(t_i,s)|^p ds\right)^{1/p} \le C\left(E\left(\frac{1}{n} \sum_{i=1}^n |X(t_i)|^p\right)^{q/p}\right)^{1/q} .$$

Note, that (2.3) is just a special case of the "stable-cotype-p" inequality (valid in an arbitrary Banach space E, cf. e.g. [7], Cor. 7.3.5):

$$\left(\int_0^1 \|\vec{f}(s)\|_E^p \, ds\right)^{1/p} \le C(E\| \int_0^1 \vec{f}(s)dM(s)\|_E^q)^{1/q},$$

where $\vec{f}$ is taken to be as follows:

$$(2.4) \qquad \vec{f}: [0,1] \ni s \to \sum_{i=1}^{n} I_{[\frac{i-1}{n},\frac{i}{n}]}(u) f(t_i,s) \in L_p[(0,1),du] \equiv E.$$

Assume initially that $\mu(T) = 1$, and define (on a new probability space (Ω_U,P_U)) a sequence of i.i.d. random variables $U_n: \Omega_U \to T$, $n=1,2,\ldots$, such that $\mathcal{L}(U_n) = \mu$. Then, by (2.3) for each $\omega_U \in \Omega_U$

$$C(E_X [\frac{1}{n}\sum_{i=1}^{n} |X(U_i)|^p]^{q/p})^{1/q}$$

$$(2.5) \qquad \ge \left(\int_0^1 \frac{1}{n}\sum_{i=1}^{n} |f(U_i,s)|^p ds\right)^{1/p}$$

$$= \left(\frac{1}{n}\sum_{i=1}^{n} \|f(U_i,.)\|^p_{L^p((0,1),ds)}\right)^{1/p}.$$

Now, by the Kolmogorov's Law of Large Numbers, for each ω such that $\int |X(t,\omega)|^p d\mu(t) < \infty$ we get that

$$\frac{1}{n}\sum_{i=1}^{n} |X(U_i,\omega)|^p \underset{P_U\text{-a.s.}}{\to} E_U\, X(U_1,\omega)|^p = \int_T |X(t,\omega)|^p d\mu(t) < \infty$$

as $n \to \infty$. By Fubini's Theorem, for P_U-almost all ω_U's

$$\frac{1}{n}\sum_{i=1}^{n} |X(U_i(\omega_U),.)|^p \to \int |X(t)|^p d\mu(t) \quad P\text{-a.s.}$$

Since for any p-stable random vectors $Y,Y_1,Y_2,\ldots$ with values in a Banach space E, and any $q < p$, $\|Y_n\|_E \to \|Y\|_E$ in P if and only if $E\|Y_n\|_E^q \to E\|Y\|_E^q$, (cf. [7], Prop. 7.3.11) we can use the same idea as in (2.4) to obtain that P_U-a.s.

$$E_X(\frac{1}{n}\sum_{1}^{n} |X(U_i)|^p)^{q/p} \to E_X(\int_T |X(t)|^p d\mu(t))^{q/p}$$

as $n \to \infty$. By (2.5)

$$\sup_n \frac{1}{n}\sum_{1}^{n} \|f(U_i,.)\|^p_{L^p(ds)} < \infty, \qquad P_U\text{-a.s.},$$

and by Kolmogorov's Law of Large Numbers

$$\int_T \int_0^1 |f(t,s)|^p ds\mu(dt) = E_U ||f(U_i,.)||^p_{L^p(ds)} < \infty ,$$

which completes the proof in the case $\mu(T) = 1$.

Notice now, that letting $n \to \infty$ in (2.5) one immediately obtains the inequality

$$(\int_T \int_0^1 |f(t,s)|^p ds\mu(dt))^{1/p} \le C(E(\int_T |X(t)|^p d\mu(t))^{q/p})^{1/q}$$
$$= C(E||X(.)||^q_{L^p(T)})^{1/q} < \infty$$

from which the extension to σ-finite μ's follows. Q.E.D.

The necessary condition for a.s. p-integrability of sample paths of p-stable processes established by Proposition 2.1 is, however, not sufficient and the following result gives full analytic description of kernels f which have the property (2.2):

THEOREM 2.4. [12]. *Let $1 \le p < 2$ and let the parameter set T be a separable metric space equipped with a σ-finite Borel measure μ. For a measurable symmetric* p*-stable process*

$$X(t) = \int_0^1 f(s,t)dM(s) \quad , \qquad t \in T,$$

we have that

$$P\{ \int_T |X(t)|^p \mu(dt) < \infty\} = 1,$$

if and only if

$$A_p(f) \stackrel{df}{=}$$
$$\int_T \int_0^1 |f(s,t)|^p [1+\log_+ \frac{|f(s,t)|^p \int_T \int_0^1 |f(s,t)|^p \mu(dt)ds}{\int_0^1 |f(s,t)|^p ds \int_T |f(s,t)|^p \mu(dt)}] ds\mu(dt) < \infty .$$

The proof depends on the following two facts:

(i) $X(t)$ has sample paths a.s. in $L^p(T,\mu)$ if and only if the series $\Sigma j^{-1/p}\ r_j V_j(t)$ converges a.s. in $L^p(T,\mu)$, where (r_j) are Rademacher r.v.'s and V_j are independent copies of a process $V(t)$, $t \in T$, which has sample paths a.s. in a sphere of $L^p(T,\mu)$, and which has finite dimensional distributions completely determined by f (cf [12] but the idea really goes back to [8], Remark 3.15);

(ii) for i.i.d. symmetric r.v.'s $X, X_1, X_2, \ldots$ ·

$$c^{-1}E|X|^p\left(1+\log_+\frac{|X|}{E|X|^p}\right)\le E\left|\sum_{j=1}^{\infty} j^{-1/p}X_j\right|^p\le cE|X|^p\left(1+\log_+\frac{|X|^p}{E|X|^p}\right),$$

where $0 < p < q$, and $c = c(p)$ is a numerical constant (cf [3]).

COROLLARY 2.1 ([12]). *Let $1 \le p < 2$. The double integral*

$$\iint_{\Delta} f(s,t)dM(s)dM(t) \quad ,$$

$\Delta = \{(s,t)\colon 0 \le s < t \le T\}$, *exists if and only if* $A_p(fI_{\Delta}) < \infty$.

For f constant on rectangles the double integral (2.1) becomes a random quadratic form

$$Q^2(a) = \sum_{i<j} a(i,j)\, M_i M_j \quad ,$$

where $M_1, M_2, \ldots$ are i.i.d. p-stable random variables, and one immediately obtains from Corollary 2.1 the following

COROLLARY 2.2 ([3]). *Let $1 \le p < 2$. $Q^2(a)$ converges a.s. if and only if*

$$\sum_{i<j}|a(i,j)|^p\left[1+\log_+\frac{|a(i,j)|^p}{\sum_{l=1}^{i-1}|a(l,j)|^p\sum_{l=j+1}^{\infty}|a(i,l)|^p}\right] < \infty .$$

Although we don't have at this point a good theory of n-tuple p-stable integrals for $n \ge 3$, the theorem below, concerning general multilinear random forms may be considered as a step towards such a theory.

THEOREM 2.5 ([13]). *Let $0 < p < 2$. Let $X, X_1, X_2, \ldots$ be i.i.d. with symmetric distributions such that*

$$\lim_{x\to\infty} x^p P\{|X| > x\} = c > 0.$$

If

$$N_p^{(k)}(a) \stackrel{df}{=} \sum_{i_1<i_2<\dots<i_k} |a(i_1,\dots,i_k)|^p(1+\log_+^{k-1}|a(i_1,\dots,i_k)|^{-1}) < \infty$$

then the sequence

$$O_n^{(k)}(a) = \sum_{i_1<i_2<\dots<i_k\le n} a(i_1,\dots,i_k)X_{i_1}\dots X_{i_k}, \quad n=1,2,\dots,$$

converges unconditionally (i.e. $O_n^{(k)}(\varepsilon a)$ *converges for all* $\varepsilon: \mathbb{N}^k \to \{-1,1\}$) *in* L^q *for every* $q<p$ *to a* $O^{(k)}(a)$ *which, for all* $x>0$ *satisfies the following inequality*

$$P\{|O^{(k)}(a)|>x\} \le D_{k,p}x^{-p}(1 + \log_+^{k-1} x)N_p^{(k)}(a),$$

where $D_{k,p}$ *is a constant.*

The proof relies on the tail estimation for $O_n^{(k)}(a)$ which uses the fact that

$$\lim_{x\to\infty} x^p(p \log x)^{1-k}P\{|X_1\cdot X_2\cdot\ldots\cdot X_k|>x\} = c^k/(k-1)! \;.$$

For further results in this direction see a recent paper by W. Krakowiak and J. Szulga [6].

3. Moment inequalities for exit times of stable processes and for p-stable stochastic integrals.

In this section we present a version of R.F. Bass and M. Cranston's [1] inequalities for moments of exit times of a stable process X in the case when X takes values in a separable Banach space E. As corollaries we also obtain moment inequalities for single and double stochastic integrals with respect to p-stable motion.

Recall that a non-zero E-valued stochastic process $X(t)$, $t \ge 0$, is said to be a *symmetric* p-*stable Lévy process*, $0 < p < 2$, if

(i) X has independent and stationary increments,
(ii) $X(t) - X(s) \stackrel{\mathcal{D}}{=} |t - s|^{1/p}X(1)$ for every $t,s \ge 0$,
(iii) $X(.,\omega) \in D_E[0,\infty)$ and $X(0) = 0$ a.s.

The characteristic functional of $X(t)$ can be written in the form

$$(3.1) \qquad E \exp[ix^*X(t)] = \exp[-t \int_U |x^*x|^p m(dx)],$$

where m is a unique, finite, symmetric (i.e. $m(-B) = m(B)$ for every Borel set $B \subset U$) positive measure on the unit sphere U of E. Such an m is called the spectral measure of X. The distribution of X(t) is infinitely divisible without the Gaussian component and with Lévy measure represented in polar coordinates as $tc_p m(dx)dr/r^{1+p}$, $(r,x) \in (0,\infty) \times U$, where $c_p > 0$ depends only on p.

Let X(t), $t \geq 0$, by a symmetric p-stable Lévy process in E and let $\{A_t\}_{t\geq 0}$ be a right continuous filtration such that X(t) is A_t-measurable and $\sigma(X(u)-X(t))$ is independent of A_t for every $u>t\geq 0$.

We will say that a continuous function $\Phi:[0,\infty) \to [0,\infty)$ *grows more slowly than* λ^p, $p > 0$, if there exist constants c, α_0 and $q < p$ such that

$$\Phi(\alpha\lambda) \leq c\alpha^q \Phi(\lambda)$$

for all $\lambda > 0$ and all $\alpha \geq \alpha_0$.

The proof of the following version of a theorem of R.F. Bass and M. Cranston [1] is supplied in the Appendix.

Theorem 3.1. *If Φ grows more slowly than λ^p, $0 < p < 2$, then there exist positive constants c_1 and c_2 depending only on p, c, α_0 and q such that for every finite $\{A_t\}$-stopping time T*

$$c_1 E\Phi(\tau^{1/p}) \leq E\Phi(X^*(\tau-)) \leq E\,(X^*(\tau)) \leq c_2 E\Phi(\tau^{1/p}),$$

where $X^*(\tau-) = \sup_{t<\tau} ||X(t)||$ *and* $X^*(\tau) = \sup_{t\leq\tau} ||X(t)||$.

Define now, a p-stable process starting at $x \in E$ by means of the formula

$$Y_x(t) = x + X(t), \qquad t \geq 0,\ x \in E.$$

Then, with the help of Theorem 3.1, one can obtain the following

COROLLARY 3.1. *If Φ grows more slowly than λ^p, $0 < p < 2$, then there exist positive constants d_1 and d_2 depending only on p, c, α_0 and q such that for every finite $\{A_t\}$-stopping time τ and $x \in E$*

$$d_1 E\Phi((||x||^p+\tau)^{1/p}) \le E\Phi(Y_x^*(\tau-)) \le E\Phi(Y_x^*(\tau)) \le d_2 E\Phi((||x||^p+\tau)^{1/p}).$$

Proof. An elementary application of the triangle inequality yields that

$$\frac{1}{3}(X^*(\tau) + ||x||) \le Y_x^*(\tau) \le X^*(\tau) + ||x||.$$

Similar inequality holds for $X^*(\tau-)$ and $Y_x^*(\tau-)$. Hence, with $L \sim R$ standing for the inequalities $C^{-1}L \le R \le CL$, where C is a positive constant depending perhaps on p, c, α_0, and q, we have

$$\Phi(Y_x^*(\tau)) \sim \Phi(X^*(\tau) + ||x||) = \Phi_{||x||}(X^*(\tau))$$

where $\Phi_u(\lambda) = \Phi(\lambda + u)$, and an analogous result obtains for $X^*(\tau-)$ and $Y_x^*(\tau-)$. Since $\Phi_u(\alpha\lambda) \le c\alpha^q\Phi_u(\lambda)$ for all $\lambda > 0$, $u \ge 0$ and $\alpha \ge \alpha_0 \vee 1$, Theorem 3.1 gives that

$$E\Phi_{||x||}(\tau^{1/p}) \sim E\Phi_{||x||}(X^*(\tau-)) \sim E\Phi_{||x||}(X^*(\tau))$$

which concludes the proof. Q.E.D.

We shall apply now the above theorem to obtain moment estimates for stochastic integrals.

THEOREM 3.2. *Let* $M(t)$ *be a real* (F_t)-*p-stable motion and let* $F \in L^p_{a.s.}$. *If* Φ *grows more slowly than* λ^p *then there exist positive constants* c_1 *and* c_2 *depending only on* p, c, α_0 *and* q *such that for each* u

$$c_1 E\Phi((\int_0^u |F|^p dt)^{1/p}) \le E\Phi(\sup_{t<u} |\int_0^t F(s)dM(s)|)$$
$$\le E\Phi(\sup_{t\le u} |\int_0^t F(s)dM(s)|) \le c_2 E\Phi((\int_0^u |F(t)|^p dt)^{1/p})$$

Proof. Since u is fixed here, we can always extend F in such a way that

$$\tau(u) = \int_0^u |F|^p dt \to \infty \qquad \text{a.s.}$$

Therefore, by Theorem 2.2

$$X(t) = \int_0^{\tau^{-1}(t)} F(s)dM(s)$$

is an (A_t)-p-stable motion, where

$$A_t = F_{\tau^{-1}(t)}$$

and

$$\tau^{-1}(t) = \inf\{u: \tau(u) > t\} .$$

Applying Theorem 3.1 to $X(t)$ and $\tau = \tau(u)$ one immediately obtains our result. Q.E.D.

Taking $f(s,t)$ such that $A_p(f) < \infty$ and substituting in the above Theorem

$$F(t) = \int_0^t f(s,t)dM(s)$$

one immediately obtains from Corollary 2.1 the following result:

COROLLARY 3.3. *If* Φ *grows more slowly than* λ^p *then there exist positive constants* c_1 *and* c_2 *depending only on* p, c, α_0 *and* q *such that for each* $u > 0$

$$c_1 E\Phi\left[\left(\int_0^u \left|\int_0^t f(s,t)dM(s)\right|^p dt\right)^{1/p}\right]$$

$$\leq E\Phi\left[\sup_{v \leq u}\left|\int_0^v\int_0^t f(s,t)dM(s)dM(t)\right|\right]$$

$$\leq c_2 E\Phi\left[\left(\int_0^u \left|\int_0^t f(s,t)dM(s)\right|^p dt\right)^{1/p}\right].$$

The following theorem summarizes recent results concerning moment inequalities for double p-stable stochastic integrals:

THEOREM 3.3. *Let* $1 < q < p \leq 2$. *Then there exist positive constants* c_1, c_2 *and* c_3 *depending only on* p, q, *such that*

$$c_1(A_p(f))^{q/p}$$

$$\le c_2 E\left[\int_0^u \left|\int_0^t f(s,t)dM(s)\right|^p dt\right]^{q/p}$$

$$\le E\left|\int_0^u \int_0^t f(s,t)dM(s)dM(t)\right|^q$$

$$\le E \sup_{v\le u}\left|\int_0^v \int_0^t f(s,t)dM(s)dM(t)\right|^q$$

$$\le c_3(A_p(f))^{q/p}.$$

The proof of the two sided estimate between the first and second quantities has been recently obtained by J. Rosinski [10], between the second and third quantities by T. McConnell and M. Taqqu [9].

In this situation Theorem 3.3 follows directly from Corollary 3.3.

4. Integration with respect to a vector-valued p-stable motion.

Let X be a symmetric p-stable Lévy process, $0 < p < 2$, with values in a separable Banach space E (see Section 3), and let $\{F_t\}_{t\ge 0}$ be a right continuous filtration such that $X(t)$ is F_t-measurable and $\sigma(X(u)-X(t))$ is independent of F_t for every $u>t\ge 0$.

THEOREM 4.1. *For each real process* $F \in L^p_{a.s.}$ *there exists an* E-*valued process* $Y(t)$ *(denoted* $\int_0^t F(s)dX(s)$*) with sample paths in* $D_E[0,\infty)$ *such that for each* $x^* \in E^*$ *and* $t \in \mathbb{R}^+$ *we have that*

$$x^*Y(t) = \int_0^t F(s)d(x^*X(s)) \qquad \text{a.s.}$$

Proof. Without loss of generality we can assume that

$$\tau(t) = \int_0^t |F(s)|^p ds \to \infty$$

a.s. as $t \to \infty$. For each $x^* \in E^*$ the process

$a(x^*)x^*X(t)$, $t \ge 0$, where $a(x^*) = (\int_U |x^*x|^p m(dx))^{-1/p} < \infty$

and $a(x^*)=0$ otherwise, is a real p-stable motion (see (3.1)). By Theorem 2.2, for any fixed $x^* \in E^*$, the real processes

$$Z_{x^*}(t) = \int_0^{\tau^{-1}(t)} F(s)d(x^*X(s)),\ t \geq 0, \text{ and } x^*X(t),\ t \geq 0,$$

have the same finite dimensional distributions. Moreover, $Z_{x^*}(t)$ is A_t-measurable and the increments $Z_{x^*}(t+h) - Z_{x^*}(t)$, $h \geq 0$, are independent of A_t.

Observe now, that for any fixed $t \geq 0$, $Z_{x^*}(t)$, $x^* \in E^*$, is a linear process on E^*, equidistributed with the linear decomposable process $x^*X(t)$, $x^* \in E^*$. Therefore, there exists an E-valued random vector $\tilde{X}(t)$ such that for each $x^* \in E^*$, $x^*\tilde{X}(t) = Z_{x^*}(t)$ a.s. Also, by the above remarks, the process $\tilde{X}(t)$ is (A_t)-adapted and the increments $\tilde{X}(t+h)-\tilde{X}(t)$, $h \geq 0$, are independent of A_t. Therefore $\tilde{X}(t)$, $t \geq 0$, has the same finite dimensional distributions as $X(t)$, $t \geq 0$, and we can select a modification of $\tilde{X}$ (also denoted by $\tilde{X}$) with all sample paths in $D_E[0,\infty)$. Hence $Y(t)$, $t \geq 0$, defined by the formula

$$Y(t) = \tilde{X}(\tau(t)),$$

has sample paths in $D_E[0,\infty)$ and satisfies, for any $x^* \in E^*$ and $t \geq 0$, the formula

$$x^*Y(t) = \int_0^t F(s)d(x^*X(s)) \qquad \text{a.s.}$$

Q.E.D.

Remark. The above construction, with obvious modifications, works for an E-valued Brownian motion as well.

The following result, besides providing moment estimates for the integral $\int F(s)dX(s)$, shows that the latter exists also in the strong sense. It follows immediately from the construction given in the proof of Theorem 4.1 and from Theorem 3.1.

THEOREM 4.2. *If Φ grows more slowly than λ^p, $0 < p < 2$, then there exist positive constants c_1 and c_2 depending only on p, c, u_0 and q such that*

$$c_1 E\Phi\left(\left(\int_0^t |F(s)|^p ds\right)^{\frac{1}{p}}\right) \leq E\Phi\left(\sup_{u \leq t} \left\|\int_0^u F(s)dX(s)\right\|\right) \leq c_2 E\Phi\left(\left(\int_0^t |F(s)|^p ds\right)^{\frac{1}{p}}\right),$$

Appendix

Proof of Theorem 3.1. (cf. [1]). Clearly, it suffices to prove that

(A.1) $$c_1 E(\phi^{1/p}) \leq E\Phi(X^*(\tau-)),$$

and

(A.2) $$E\Phi(X^*(\tau)) \leq c_2 E\Phi(\tau^{1/p}).$$

To obtain (A.1), it is enough to show that for $\beta>1$, $\delta>0$ and $\lambda>0$

$$P[t^{1/p} > \beta\lambda,\ X^*(\tau-) \leq \delta\lambda] \leq c(\beta,\delta)P[\tau^{1/p}>\lambda],$$

where $c(\beta,\lambda) \to 0$ as either $\beta \to \infty$ or $\delta \to 0$ (see Burkholder (1973), Lemma 7.1; the assumption $\Phi(0)=0$ is not necessary in this case).

Setting $a = \lambda^p$ and $b = (\beta\lambda)^p$ one obtains

$$P[\tau^{1/p}>\beta\lambda,\ X^*(\tau-)\leq\delta\lambda] = P[\tau>b,\ X^*(\tau-) \leq \delta\lambda]\leq P[\tau>a, ||X(b)-X(a)||\leq 2\delta\lambda]$$

$$= P[\tau>a]P[||X(b)-X(a)||\leq 2\delta\lambda]$$

$$= P[\tau^{1/p}>\lambda]P[||X(1)||\leq \frac{2\delta}{(\beta^p - 1)^{1/p}}],$$

which proves (A.1).

To obtain (A.2), we define an $\{A_t\}$-stopping time

$$\sigma = \inf\{t>0: ||X(t \wedge \tau)|| > \lambda\}.$$

Then we have that

$$P[X^*(\tau) > \beta\lambda,\ \tau^{1/p} \leq \delta\lambda] =$$

$$= P[X^*(\tau)>\beta\lambda, \tau^{1/p}\leq\delta\lambda, ||X(\sigma)||<\tfrac{\beta\lambda}{2}]+P[||X^*(\tau)||>\beta\lambda, \tau^{1/p}\leq\delta\lambda, ||X(\sigma)||\geq\tfrac{\beta\lambda}{2}]$$

$$= I+J.$$

Put $a = (\delta\lambda)^p$. For $\beta > 2$ we have

$$I = P[X^*(\tau) > \beta\lambda, \tau^{1/p} \leq \sigma\lambda,\ \|X(\sigma)\| < \frac{\beta\lambda}{2},\ \sigma < \tau]$$

$$\leq P[\sup_{t \leq a} \|X(\sigma + t) - X(\sigma)\| > \frac{\beta\lambda}{2},\ \sigma < \tau]$$

$$= P[X^*(a) > \frac{\beta\lambda}{2}]\ P\ [\sigma < \tau]$$

$$\leq 2P\ [\|X(1)\| > \frac{\beta}{2\delta}] \cdot P[X^*(\tau) > \lambda]$$

$$\leq 2\left(\frac{2\delta}{\beta}\right)^p \Lambda_p\ (X)\ P[X^*(\tau) > \lambda],$$

where $$\Lambda_p(X) = \sup_{\lambda > 0} \lambda^p P\ [\|X(1)\| > \lambda] < \infty.$$

Next, we obtain an estimate for J. Note that if n is the Lévy measure for X(1) then

$$n(B_R^c) = \int_{(R,\infty)\times U} c_p m(dx)dr/r^{1+p} = c_p m(U)p^{-1}R^{-p} = CR^{-p},$$

where B_R is the ball in E with radius R and center at 0. Let

$$Y(t) = \sum_{s \leq t} I(\|\Delta X(s)\| > R),$$

where $\Delta X(s)$ is the jump process associated with X(s). Then Y(t) is a Poisson process with parameter $n(B_R^c)$, so that $Y(t) - t\ n\ (B_R^c)$ is a martingale (with respect $\{A_t\}$). By the optional sampling theorem, for every bounded $\{A_t\}$ stopping time τ

$$E\sum_{s \leq \tau} I(\|\Delta X(s)\| > R) = EY(\tau) = n(B_R^c)E\tau$$

Let $\sigma_1 = \sigma \wedge \tau \wedge a$. If $s < \sigma_1 \leq \sigma \wedge \tau$ then $\|\Delta X(s)\| \leq 2\lambda$ by definition of σ. Hence, if $R > 2\lambda$ then

$$P[\|\Delta X(\sigma_1)\| > R] = E\sum_{s \leq \sigma} I(\|\Delta X(s)\| > R) = n(B_R^c)\ E\ \sigma_1.$$

Since $\sigma_1 \leq \tau$ we have $X^*(\tau) \geq \|X(\sigma_1)\|$, and, consequently, for $\beta > \sigma$ we obtain that

$$J = P[X^*(\tau) > \beta\lambda, \tau \leq a,\ \|X(\sigma)\| > \frac{\beta\lambda}{2},\ \sigma \leq \tau]$$

$$\leq P[\|\Delta X(\sigma_1)\| > (\frac{\beta}{2} - 1)\lambda] = n(B^c_{(\frac{\beta}{2}-1)\lambda})E\sigma_1$$

$$= C(\tfrac{\beta}{2} - 1)^{-p}\lambda^{-p}E\sigma_1 = 3^p(\tfrac{\beta}{2} - 1)^{-p}n(B^C_{3\lambda})\ E\ \sigma_1$$

$$= 3^p(\tfrac{\beta}{2} - 1)^{-p}P[||\Delta X(\sigma_1)|| > 3\lambda] \le 3^p(\tfrac{\beta}{2} - 1)^{-p}P[||X(\sigma_1)|| > \lambda]$$

$$\le 3^p(\tfrac{\beta}{2} - 1)^{-p}P[X^*(\tau) > \lambda].$$

Putting together estimates for I and J we get that

$$P[X^*(\tau)>\beta\lambda] \le P[X^*(\tau) > \beta\lambda, \tau^{1/p}\le\delta\lambda] + P[\tau^{1/p} > \delta\lambda]$$

$$\le c(\beta,\delta,p)P[X^*(\tau)>\lambda] + P[\tau^{1/p} > \sigma\lambda],$$

where

$$c(\beta,\delta,p) = \beta^{-p}[2^{p+1}\sigma^p\Lambda_p(X) + 3^p(\tfrac{1}{2} - \tfrac{1}{\beta})^{-p}].$$

Therefore

$$E\Phi(\beta^{-1}X^*(\tau)) = \Phi(0) + \int_0^\infty P[X^*(\tau) > \beta\lambda]d\Phi(\lambda) \le$$

$$\le \Phi(0) + c(\beta,\delta,p)\int_0^\infty P[X^*(\tau) >\lambda]d\Phi(\lambda) + \int_0^\infty P[\tau^{1/p}>\delta\lambda]d\Phi(\lambda)$$

$$\le c(\beta,\delta,p)\ E\Phi(X^*(\tau)) + E\Phi(\delta^{-1}\tau^{1/p}).$$

If $\beta > \alpha_0$ and $\delta < \alpha_0^{-1}$ then

$$E\Phi((X^*(\tau)) = E\Phi(\beta\beta^{-1}X^*(\tau)) \le c\beta^q E\Phi(\beta^{-1}X^*(\tau)),$$

and

$$E\Phi(\delta^{-1}\tau^{1/p}) \le c\delta^{-q}E\Phi(\tau^{1/p}).$$

Finally, we obtain the inequality

$$[c^{-1}\beta^{-q}-c(\beta,\delta,p)]\ E\Phi(X^*(\tau)) \le c\delta^{-q}E\Phi(\tau^{1/p})$$

which proves (A.2) since the constant on the left hand side can be made positive by taking β large enough and δ small enough (remember that $q < p$).

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