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distribution of the spins can be modelled by 
\[ dQ_n(x) = z_n^{-1} \exp[-H_n(x)] \Pi dP(x), \]
where \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), where \( H_n \) is the Hamiltonian, \( z_n \) is a normalizing constant, and \( P \) is a probability measure on \( \mathbb{R} \). For certain forms of the Hamiltonian \( H_n \), Ellis and Newman (Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 44(1978) 117-139) showed that under appropriate conditions on \( P \), there exists an integer \( r \geq 1 \) such that \( S_n/n^{1-1/2r} \) converges in distribution to a random variable. This limiting random variable is Gaussian if \( r=1 \) and non-Gaussian if \( r \geq 2 \). In this article, utilizing the large deviation local limit theorems for arbitrary sequences of random variables of Chaganty and Sethuraman (Ann. of Probability, 13 (1985)), we obtain similar central limit theorems for a wider class of Hamiltonians \( H_n \) which are functions of moment generating functions of suitable random variables. We also present a number of examples to illustrate our theorems.
Central Limit Theorems in the Area of Large Deviations for Some Dependent Random Variables

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ABSTRACT

A magnetic body can be considered to consist of \( n \) sites, where \( n \) is large. The magnetic spins at these \( n \) sites, whose sum is the total magnetism present in the body, can be modelled by a triangular array of random variables \((X^{(n)}_1, \ldots, X^{(n)}_n)\).

Standard theory of Physics would dictate that the joint distribution of the spins can be modelled by

\[
dQ_n(x) = z_n^{-1} \exp[-H_n(x)] \Pi dP(x_i),
\]

where \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), where \( H_n \) is the Hamiltonian, \( z_n \) is a normalizing constant, and \( P \) is a probability measure on \( \mathbb{R} \). For certain forms of the Hamiltonian \( H_n \), Ellis and Newman (Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 44(1978) 117-139) showed that under appropriate conditions on \( P \), there exists an integer \( r \geq 1 \) such that \( S_n/n^{1-1/2r} \) converges in distribution to a random variable. This limiting random variable is Gaussian if \( r = 1 \) and non-Gaussian if \( r > 2 \). In this article, utilizing the large deviation local limit theorems for arbitrary sequences of random variables of Chaganty and Sethuraman (Ann. of Probability, 13 (1985)), we obtain similar central limit theorems for a wider class of Hamiltonians \( H_n \), which are functions of moment generating functions of suitable random variables. We also present a number of examples to illustrate our theorems.
1. **Introduction.**

In this article we obtain central limit theorems for some dependent random variables which are used to describe the distribution of magnetic spins present in a ferromagnet crystal. A ferromagnet crystal consists of a large number of sites. The amount of magnetic spin present at site \( i \) will be denoted by \( X_i^{(n)} \), \( i = 1, \ldots, n \), where \( n \) is a positive integer. The magnetic spin present at any site interacts with the magnetic spins at its neighboring sites and hence gives rise to some dependency among the random variables \( X_i^{(n)} \)'s. In the Ising model, the joint distribution, at a fixed temperature \( T > 0 \), of the spin random variables \( (X_1^{(n)}, \ldots, X_n^{(n)}) \), is given by

\[
dQ_n(x) = z_n^{-1} \exp \left[ -\frac{H_n(x)}{T} \right] \Pi dP(x_j),
\]

where \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( P \) is a probability measure on \( \mathbb{R} \). The function \( H_n(x) \) is known as the Hamiltonian and it represents the energy of the crystal at the configuration \( x \), and \( z_n \) is a normalizing constant which is also known as the partition function. In many cases, an explicit evaluation of \( z_n \) is very difficult and physicists usually try to evaluate the limiting free spin per state \( \xi(T) \), at the temperature \( T \), defined as follows:

\[
\xi(T) = -\lim_{n \to \infty} \left[ (\log(z_n))/n \right].
\]

For some particular types of Hamiltonians, it has been shown by physicists that there exists a temperature level \( T_c \) such that the function \( \xi(T) \) is infinite or finite according as \( T < T_c \) or \( T \geq T_c \) (see Kac (1968)). A phase transition is said to occur at the critical temperature \( T_c \). As pointed out by Ellis and Newman (1978b), the existence of the critical temperature can be demonstrated in yet another way. We may be able to show that for \( T > T_c \), there is a weak dependence among the random variables \( (X_1^{(n)}, \ldots, X_n^{(n)}) \) and a standard central limit theorem is valid for \( S_n/\sqrt{n} \) and that for \( T = T_c \), there exists a \( \delta \in (1,2) \) such that \( S_n/n^{\delta/2} \) converges to a
non-Gaussian limit and for \( T < T_c \), due to the strong dependence among the \( X_i^{(n)} \)'s, the random variables tend to cluster in several ergodic components. This shows a marked discontinuity in the asymptotic distribution of \( S_n \) as the temperature \( T \) is allowed to vary and represents our approach to demonstrating a phase transition.

In section 2, we consider a special case for the Hamiltonian, \( H_n \), by setting it to be equal to \( -\frac{1}{2n} \sum \sum X_i X_j \). This is known as the Curie-Weiss model. The asymptotic distribution of \( S_n \) for this model when \( P \) (which appears in Theorem 2.1) is symmetric Bernoulli is obtained by Simon and Griffiths (1973). In a two paper series, Ellis and Newman (1977, 1978a, 1978b) extended Theorem 2.1 of Simon and Griffiths to the class of probability measures \( L \), defined in 2.2 (see also Ellis and Rosen (1979)). We state their extension precisely in Theorem 2.6. Recently Jong-Woo Jeon (1979) in his Ph.D. dissertation gave a simpler and statistically motivated proof of Theorem 2.6. The goal of this article is to extend Theorem 2.6 for a larger class of Hamiltonians \( H_n \) and probability measures \( P \). Our main result, Theorem 3.7 is stated in Section 3. The proof of Theorem 3.7 relies on a recent large deviation local limit theorem of Chaganty and Sethuraman (1985), which is restated in Section 3 as Theorem 3.4.

Let \( T_n, n \geq 1 \), be an arbitrary sequence of random variables with analytic moment generating function \( \phi_n(z) \). We assume that \( T_n \) satisfies the conditions of Theorem 3.4. In our generalized model the Hamiltonian \( H_n(x) \) is taken to be equal to \( -\log[\phi_n(s_n/n)] \), where \( s_n = x_1 + \ldots + x_n \). Thus, the joint distribution of the spin random variables \( (X_1^{(n)}, \ldots, X_n^{(n)}) \) is given by

\[
(1.3) \quad dQ_n(x) = z_n^{-1} \phi_n(s_n/n) \prod P(x_j),
\]

where \( P \) is an arbitrary probability measure. Let \( S_n = X_1^{(n)} + \ldots + X_n^{(n)} \). Under appropriate conditions on the probability measure \( P \) we show in Theorems 3.7 and 3.18, there exists an integer \( r \geq 1 \) such that \( S_n/n^{1-1/2r} \) converges in distribution to a random variable \( Y_r^{*} \), which has a non-normal distribution when \( r \geq 2 \) and normal distribution when \( r = 1 \). The technique of our proof is to introduce a new random
variable $W_n$, conditional on which, $X_1^{(n)}, \ldots, X_n^{(n)}$ become i.i.d. It is easy to obtain the limiting distribution of $W_n$ and the conditional asymptotic distribution of $S_n/n^{1-1/2r}$. Using the results of Sethuraman (1961) we deduce the asymptotic distribution of $S_n/n^{1-1/2r}$.

We now briefly give our reasons for calling these theorems on the asymptotic distribution of $S_n$ under $Q_n$, defined in (1.3), as limit theorems in the area of large deviations. A standard technique to obtain the asymptotic distribution of $S_n$ under $Q_n$ is to first obtain the asymptotic distribution of $S_n$ under $P_n$, where

$$dP_n(x) = \Pi dP(x_j)$$

and then to use contiguity arguments, as in LeCam (1960). This technique breaks down completely if $r \geq 2$. For the various models considered in Physics which are described in greater detail in Sections 2 and 3,

$$|L_n(x)| = | \log \frac{dQ_n(x)}{dP_n(x)} | = \left| \frac{H_n(x)}{T} + \log z_n \right|$$

converges to $\infty$ in probability under $P_n$ and thus contiguity arguments are not applicable here. Under $P_n$, $S_n/\sqrt{n}$ has a limiting normal distribution, also, $|L_n(x)|$ is small in the area of ordinary deviations of $S_n$, that is, when $S_n/\sqrt{n}$ is finite, while it is large otherwise. Thus from the point of view of $P_n$, we are looking for the asymptotic distribution of $S_n$, when $P_n$ is modified by $L_n(x)$, which is substantially different from 1 in the area of large deviations of $S_n$. This viewpoint helps in a statistically motivated proof of the asymptotic distribution of $S_n$ under $Q_n$ and describes the background behind the title of this article. One should also note that the normalizing factor on $S_n$ in its asymptotic distribution under $Q_n$ is different from the corresponding factor under $P_n$. 


In a ferromagnetic system with only isotropic pair interactions and with no external magnetic field, the Hamiltonian $H_n$, may be taken to be $\frac{1}{2}\sum a_{ij}x_ix_j$, where $a_{ij} \geq 0$. The Curie-Weiss model assumes that $a_{ij} = \frac{1}{n}$ for all $i$ and $j$, that is to say that each spin interacts equally with every other spin with strength $\frac{1}{n}$ and takes $P$ to be symmetric Bernoulli, i.e., $P(\{-1\}) = P(\{1\}) = \frac{1}{2}$. Replacing $P$ by $P_T(x) = P(x\sqrt{T})$, we get

$$dQ_n(x) = z^{-1}_n \exp\left[\frac{s_n^2}{2n}\right] \Pi dP(x_j),$$

where $s_n = x_1 + \ldots + x_n$. This model has the advantage that the limiting free spin per state can be solved exactly. The existence of the critical temperature and phase transition for this model was demonstrated by Kac (1968). The asymptotic distribution for the total magnetism, $S_n$, for this model was obtained by Simon and Griffiths (1973). This is contained in Theorem 2.1.

**Theorem 2.1** (Simon and Griffiths). Let $X_{j}^{(n)}$, $j = 1, \ldots, n$ be a triangular array of random variables whose joint distribution is given by (2.1) and $P$ be symmetric Bernoulli. Then $S_n/n^{3/4}$ converges in distribution to a random variable whose density function is proportional to $\exp(-y^4/12)$.

Theorem 2.1 was extended to the class of probability measures $L$, which is defined below, by Ellis and Newman (1978b).

**Definition 2.2.** Let $L$ be the class of probability measures $P$ on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \exp(x^2/2) \, dP(x) < \infty.$$  

Fix $P \in L$. It can be shown that condition (2.2) guarantees the existence of the moment generating function (m.g.f.), $m(u)$, of $P$. Let $h(u) = \log m(u)$ be the cumulant generating function (c.g.f.) of $P$. The function $G(u) = u^2/2 - h(u)$ plays an important role in Theorem 2.6 below.
Definition 2.3. A real number $m$ is said to be a global minimum for $G$ if $G(u) \geq G(m)$ for all $u$.

Definition 2.4. A global minimum $m$ for $G$ is said to be of type $r$ if

$$G(u+m) - G(m) = c_{2r} u^{2r}/(2r)! + O(|u|^{2r+1}) \text{ as } u \to 0,$$

where $c_{2r} = G^{(2r)}(m)$ is strictly positive.

Definition 2.5. A probability measure $P$ is said to be pure if $G$ has a unique global minimum.

Let $Y_r$, $r \geq 1$, be a sequence of random variables with density function $p_r(y)$, where

$$p_r(y) = \begin{cases} d_r \exp \left[ -c_{2r} y^{2r}/(2r)! \right] & \text{if } r \geq 2 \\ N(0, (1 - c_2)/c_2) & \text{if } r = 1 \end{cases}$$

and where $d_r$ is the appropriate normalizing constant. With these definitions and notation we are now in a position to state the generalization of Theorem 2.1, due to Ellis and Newman (1978b).

Theorem 2.6. (Ellis and Newman). Let $P \in L$. Let $P$ be pure, that is, let $m$ be the unique global minimum of type $r$ for $G$. Let $X_j^{(n)}$, $j = 1, \ldots, n$, be a triangular array of random variables with joint distribution given by (2.1). Let

$$S_n = X_1^{(n)} + \ldots + X_n^{(n)}.$$

Then

$$G_n = S_n - nm \quad \frac{d}{dY_r},$$

where $Y_r$ is a random variable with density function given by (2.4).

It is easily verified that the symmetric Bernoulli measure is pure and belongs to the class $L$ with the corresponding value of $r$ equal to 2. Thus Theorem 2.6 contains Theorem 2.1.
Note that the moment generating function $M(z)$ of the standard normal is given by $\exp(z^2/2)$. Thus we can write (2.1) as

(2.6) \[ dQ_n(x) = z_n^{-1} [M(s_n/n)]^n \Pi dP(x_j). \]

One might ask the question whether it is possible to obtain limit theorems of the type (2.5) when $[M(z)]^n$ is replaced by the m.g.f. $\phi_n(z)$ of a random variable $T_n$, satisfying some conditions. We answer this question in the affirmative in the next section.

3. Further Extensions of the Curie-Weiss Model.

In this section we propose to extend Theorem 2.6 by enlarging the class of Hamiltonians as well as the class of probability measures $L$. The large deviation local limit theorems for arbitrary sequence $T_n$, $n \geq 1$, of random variables of Chaganty and Sethuraman (1985) (stated below) plays a key role in this extension. The Hamiltonians, $H_n$, in our generalized model (3.13) are taken to be the cumulant generating functions of these random variables $T_n$.

Let $(T_n, n \geq 1)$ be a sequence of non-lattice valued random variables with m.g.f.'s $\phi_n(z)$, $n \geq 1$, which are analytic and non-vanishing for $z$ in $\Omega = \{z : |\text{Re}(z)| < c\}$, with $0 < c < \infty$. Let $I = (-a, a)$ and $\Omega_a = \{z : |\text{Re}(z)| < a\}$, where $0 < a < c$. Let

(3.1) \[ \psi_n(z) = \frac{1}{n} \log \phi_n(z), \text{ for } z \in \Omega \]

(3.2) and \[ \gamma_n(u) = \sup_{|s| < c} [us - \psi_n(s)], \text{ for } u \in I. \]

Let $A_n = \{\psi_n(s); s \in I\}$. For $u \in A_n$, we have $\gamma_n(u) = [us_n - \psi_n(s_n)]$, where $s_n \in I$ satisfies $\psi_n'(s_n) = u$. Let $P$ be a probability measure which satisfies the following condition:

(3.3) \[ \int_{-c}^{c} \exp[\psi_n(x)] \, dP(x) < \infty \text{ for all } n \geq 1. \]

Let $h(u)$ denote the c.g.f. of $P$. It is easy to check that condition (3.3) implies that $h(u)$ is finite for $u \in R_n$, where

(3.4) \[ R_n = \{u : \gamma_n(u) < \infty\}. \]

Let

(3.5) \[ V_n(u) = \begin{cases} \gamma_n(u) - h(u) & \text{for } u \in R_n, \\ \infty & \text{for } u \notin R_n. \end{cases} \]
The function $V_n$ plays the same role as the function $G$ of Section 2.

**Definition 3.1.** Let $L^\ast$ be the class of all probability measures $P$ on $(-c, c)$ satisfying condition (3.3). We assume that there exists $\ell, p_1 > 0$ such that

$$
\int_{R^m} \exp[-\ell V_n(u)] \, du = O(n^{\ell}),
$$

and $V_n$'s have a unique global minimum at some point $m_n$. Furthermore there exists $n_1 > 0$ such that

$$
\inf_{|u| < \delta} [V_n(m_n + u) - V_n(m_n)] = [V_n(m_n + \delta) - V_n(m_n)] \quad \text{for all } 0 < \delta < n_1.
$$

**Remark 3.2.** Condition (3.7) is used mainly in inequality (3.27) of Lemma 3.13. An easily verifiable sufficient condition for (3.7) is

$$
V_n'(u) > 0 \quad \text{for } u > m_n \quad \text{and} \quad V_n'(u) < 0 \quad \text{for } u < m_n.
$$

In all the examples of section 4 we will be verifying (3.8) instead of (3.7).

**Remark 3.3.** Suppose that $R_n = (-\infty, \infty)$. If $\gamma_n(u)/|u|$ converges to $\infty$ as $|u| \to \infty$, then condition (3.3) implies (3.7) as seen below:

$$
\exp[-V_n(u)] = \exp[-\gamma_n(u) + h(u)]
= \exp[-\gamma_n(u)][\int_{|x| < A} \exp[ux] \, dP(x) + \int_{|x| > A} \exp[ux] \, dP(x)]
\leq \exp[-\gamma_n(u) + uA] + \int_{|x| > A} \exp[\psi_n(x)] \, dP(x)
\leq \exp[-u|\gamma_n(u)/|u| - A|] + \int_{|x| > A} \exp[\psi_n(x)] \, dP(x).
$$

The right hand side can be made close to zero first by choosing $A$ and then letting $|u| \to \infty$. This shows that $V_n(u) \to \infty$ as $|u| \to \infty$. Since $m_n$ is the unique global minimum of $V_n$, this also shows that condition (3.7) holds.
Let \( m \in \mathbb{A}_n \). Then there is a \( \tau_n \) in \( I \) such that \( \psi_n'(\tau_n) = m_n \). For \( t \in I \), define

\[
G_n(t) = \psi_n(\tau_n) + itm_n - \psi_n(\tau_n + it).
\]

The following theorem, which provides an asymptotic expansion for the density function \( k_n \) of \( T_n/n \) in terms of the large deviation rate \( \gamma_n \), is due to Chaganty and Sethuraman (1985).

**Theorem 3.4.** Assume the following conditions for \( T_n \):

(A) There exists \( \beta > 0 \) such that \( |\psi_n(z)| < \beta \) for \( z \in \Omega_n \) and \( n \geq 1 \).

(B) There exists \( \alpha > 0 \) such that \( \psi_n''(\tau) \geq \alpha \) for \( \tau \in I \) and \( n \geq 1 \).

(C) There exists \( \eta > 0 \) such that for any \( 0 < \delta < \eta \),

\[
\inf_{|t| \geq \delta} \text{Real}(G_n(t)) = \min[\text{Real}(G_n(\delta)), \text{Real}(G_n(-\delta))], \text{ for } n \geq 1.
\]

(D) There exists \( p > 0 \) such that

\[
\sup_{\tau \in I} \int_{-\infty}^{\infty} \left| \frac{\phi_n(\tau + it)}{\phi_n(\tau)} \right|^{1/n} dt = O(n^p).
\]

Then

\[
k_n(m_n) = \left[ \frac{n}{2\pi \psi_n''(\tau_n)} \right]^{1/2} \exp(-n\gamma_n(m_n))[1 + O(\frac{1}{n})].
\]

**Remark 3.5.** When \( T_n \) is the sum of \( n \) i.i.d. random variables, condition (C) is automatically satisfied and conditions (A), (B), (D) are easy to verify, since they do not depend on \( n \).

**Remark 3.6.** Suppose that \( m \in \cap \mathbb{A}_n \). Then there exists \( \xi_n \in I \) such that \( \psi_n''(\xi_n) = m_n \) for \( n \geq 1 \). In this case one can verify that \( \sqrt[1/n]{\psi_n''(\xi_n)} \sqrt[1/n]{\psi_n''(\tau_n)} = [1 + O(|m_n - m|)] \)

and thus we can rewrite (3.11) as
The cumulant generating function of $P$ is given by

\[(4.6) \quad h(u) = \log(3/2) - 3 \log|u| + |u| + \log[|u|(1 + e^{-2|u|}) - (1 - e^{-2|u|})], \quad u \in \mathbb{R}.\]

Therefore $u \in \mathbb{R}$,

\[(4.7) \quad V(u) = \gamma(u) - h(u)\]

\[= [-1 + \sqrt{1+u^2}] + \log|u| + \log[-1 + \sqrt{1+u^2}] - |u|\]

\[- \log[|u|(1 + e^{-2|u|}) - (1 - e^{-2|u|})] + \log(4/3).\]

**Part (I).** We just need to verify (4.1). Now

\[(4.8) \quad V'(u) = \sqrt{1+(1/u^2)} + (2/u) - u/(ucoth(u) - 1)\]

\[> 1 + (2/u) - u/(ucoth(u) - 1)\]

\[= \left(\frac{u^2 + u}{ucoth(u) - 1}\right)\left(\frac{coth(u) - 1 + ucoth(u) - 2}{(ucoth(u) - 1)}\right) > 0\]

for $u \geq 2$, since $ucoth(u) - 1 > 0$ and $coth(u) > 1$. Again

\[(4.9) \quad V'(u) = \frac{[2 + \sqrt{1+u^2}][ucoth(u) - 1] - u^2}{u[ucoth(u) - 1]} \quad \text{for } u > 0.\]

By differentiation one can verify that

\[(4.10) \quad ucoth(u) - 1 \geq \frac{u^2}{3} - \frac{u^4}{45} \quad \text{for } 0 < u < 7.5\]

and

\[(4.11) \quad \frac{u^2}{3} - \frac{u^4}{45} \geq \frac{u^2}{2 + \sqrt{1+u^2}} \quad \text{for } 0 < u < 2.19.\]

These two inequalities show that $V'(u) > 0$ in the region $0 < u \leq 2$. Since

\[V''(0) = \gamma''(0) - h''(0) = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}, \quad V \text{ has a unique global minimum of type 1.}\]
written down. Part (I) will show that \( V \) has a unique global minimum of type \( r \) at \( m_n = 0 \). Part (II) will verify the four conditions of Theorem 3.4 or Theorem 3.16 depending on whether \( T_n \) is non-lattice or lattice. Part (III) will verify conditions (3.3), (3.6) and (3.8) that insure that \( P \in \mathcal{L}^* \), in the case of continuous \( P \) and conditions (3.3), (3.8) and (3.38) that insure that \( P \in \mathcal{L}_1^* \), in the case of discrete \( P \). These three parts will imply that the random variable \( S_n \), after proper normalization converges in distribution to \( Y^*_r \), for appropriate value of \( r \).

The verifications of parts (II) and (III) are usually very easy. The verification of Part (I) presents more difficulties. However, in all the examples considered, \( V \) is symmetric around the origin and hence Part (I) is verified once we show that

\[
V'(u) > 0 \quad \text{for } u > 0. \tag{4.1}
\]

Note that (4.1) also verifies condition (3.8). The verification of (4.1) is routine in all the examples, but sometimes long, and therefore only the main steps are presented. One should also note that in part (II), condition (C) or (C') is automatically satisfied by Remark 3.5.

Example 4.1. Let \( F \) have p.d.f. \((1/2) \exp(-|x|), -\infty < x < \infty \), and \( P \) have p.d.f. \((3/4)(1-x^2)\) for \(|x| < 1\). Thus the joint distribution \( Q_n \) given in (3.13) becomes

\[
dQ_n(x) = z_n^{-1} (3/4)^n (1-s^2/n^2)^{-n} \prod_{1 \leq j < \infty} (1-x_j^2) \, dx_j. \tag{4.2}
\]

In this case

\[
\phi_n(z) = (1-z^2)^{-n} \tag{4.3}
\]

and

\[
\psi(z) = -\log(1-z^2) \quad \text{for} \quad |\text{Real}(z)| < 1, \tag{4.4}
\]

\[
\gamma(u) = \log 2 + [-1 + \sqrt{1+u^2}] - 2 \log |u| + \log [-1 + \sqrt{1+u^2}], \text{ for } u \in \mathbb{R}. \tag{4.5}
\]
where

\[(3.43) \quad g_n^*(y_n) = \frac{\sqrt{2\pi \psi''(\xi_n)}}{n(1-1/r)/2} k_n(m_n + n^{-1/2}r_{-1/2r}) \exp[n(h(m_n + n^{-1/2}r_{-1/2r}) + \nu_n(m_n))].\]

Imitating the proofs of Lemmas 3.9 thru 3.13, one can show the following:

(i) \(\frac{1}{|h'_n|} g_n^*(y_n) + g(y)\) as \(n \to \infty\), uniformly on bounded intervals of \(y\).

(ii) \(\sum_{\{y_n\leq n\}} 1/4r g_n^*(y_n) + \int g(y) dy\) as \(n \to \infty\).

(iii) \(\sum_{\{y_n\geq n\}} 1/4r g_n^*(y_n) + 0\) as \(n \to \infty\).

The above three steps (i), (ii), (iii) complete the proof of Lemma 3.19.

4. Applications.

In this section we illustrate the main theorems of section 3 with four applications and demonstrate limit theorems in quite complicated situations of dependent variables. The model (3.13) for the joint distribution of \((x_1^{(n)}, \ldots, x_n^{(n)})\) is completely specified if \(T_n\) and \(P\) that arise in it are specified. To simplify matters, in all the examples of this section we let \(T_n\) be the sum of \(n\) i.i.d. random variables with common d.f.\(F\). The four examples below contain all occurrences of lattice and non-lattice \(T_n\), and continuous and discrete \(P\). The limit distribution of the normalized sum \(S_n = x_1^{(n)} + \ldots + x_n^{(n)}\) is normal \((r=1,\) in the notation of Theorems 3.7 and 3.18) in example 4.1 and is non-normal \((r=2)\) in examples 4.2, 4.3 and 4.4. The results of Ellis and Newman (1978b) show that limit distributions with every possible value of \(r > 2\) can also arise in suitable models.

In all the examples below we will specify \(F\) and \(P\) and write down the joint distribution \(Q_n\). The exact expression for \(\phi_n, \psi_n \equiv \phi, \gamma_n \equiv \gamma, h\) and \(\nu_n \equiv \nu\) will be
If $T_n$ satisfies the conditions of Theorem 3.16, then

\[(3.40) \quad (S_n - n\tau_n)/n^{1-1/2r} \overset{d}{\to} \gamma_r^*,\]

where $\gamma_r^*$ and $\tau_n$ are as defined in Theorem 3.7.

The proof of the above theorem parallels the proof of Theorem 3.7. We therefore outline briefly the modifications that need to be done. Note that $dQ_n$ can be written as

\[(3.41) \quad dQ_n(x) = \sum_y \prod_{i=1}^n f_n^*(y), \]

where $f_n^*(y) = \frac{1}{n^{1-1/2r}} k_n(m_n + n^{-1/2r}y) \exp[n h(m_n + n^{-1/2r}y)]$ is a probability mass function of a lattice valued distribution with span $h_n' = h_n/n^{1-1/2r}$. We introduce discrete random variables $W^*_n$ with p.m.f. $f_n^*$. It suffices to show that $W^*_n$ converges weakly to a continuous random variable $W$ with probability density function $f$, defined in (3.35). The rest of the proof is identical to the proof of Theorem 3.7. Note that the span, $h_n'$, of $W^*_n$ converges to zero. By a theorem of Okamoto (1959), the sequence of random variables $W^*_n$ will converge in distribution to $W$, once we prove the following:

**Lemma 3.19.** For $y \in \mathbb{R}$, define $y_n = h_n'[y/h_n']$. Let the probability mass function $f_n^*$ and the probability density function $f$ be as defined above. Then

\[(3.42) \quad \frac{1}{|h_n'|} f_n^*(y_n) \to f(y) \text{ as } n \to \infty,\]

uniformly on bounded intervals of $y$.

**Proof (outline).** Note that $f(y) = g(y)/\int g(y)dy$, where $g(y)$ is as defined in (3.15). We first write

\[f_n^*(y_n) = \frac{g_n^*(y_n)}{\sum g_n^*(y)},\]
There exists $n > 0$ such that for any $0 < \delta < n$,

$$\inf_{\delta \leq |t| \leq \pi/|h_n|} \text{Re}(G_n(t)) = \min\{\text{Re}(G_n(\delta)), \text{Re}(G_n(-\delta))\}$$

for $n \geq 1$, where $G_n(t)$ is defined by (3.10).

There exists $p > 0$ such that $|h_n|^{-1} = O(n^p)$.

Then

$$\frac{\sqrt{n}}{|h_n|} \text{Pr}(T_n = m_n) = \left[ \frac{1}{2\pi} \omega_n^{1/2}(\tau_n) \right]^{1/2} \exp(-n\gamma_n(m_n))[1 + O(\frac{1}{n})].$$

As before for a probability measure $P$ on $R$, define $V_n(u)$ as in (3.5). The class of probability measures that are of interest is defined below.

**Definition 3.17.** Let $L^*_1$ be the class of probability measures $P$ satisfying conditions (3.3), (3.7) and (3.38) (defined below).

$$\sum_{u \in R_n} \exp[-\ell V_n(u)] = O(n^{p_1})$$

for some $\ell, p_1 > 0$.

Note that (3.38) is the appropriate replacement of (3.6) for the lattice valued case.

For Hamiltonians which are functions of the moment generating functions of lattice valued random variables we have the following theorem almost identical to Theorem 3.7.

**Theorem 3.18.** Let $P \in L^*_1$. Let $X_j^{(n)}$, $j = 1, \ldots, n$, be a triangular array of random variables satisfying $|X_j^{(n)}| < \epsilon$ and having a joint distribution given by

$$dQ_n(x) = z_n^{-1} \phi_n(s_n/n) \Pi dP(x_j),$$

where $\phi_n$ is the m.g.f. of the lattice valued random variables $T_n$. Let

$$S_n = X_1^{(n)} + \ldots + X_n^{(n)}.$$ 

Let $V_n$ have a unique global minimum of type $r$ at the point $m_n \in A_n$. Let $m_n$ converge to a point $m$ belonging to the interior of $\partial A_n$. 

The unconditional limiting distribution of \( (S_n - n\tau_n)/n^{1-1/2r} \) is just the mixture of the limiting conditional distribution and \( f(y) \), by Theorem 3.15 of Sethuraman. This completes the proof of Theorem 3.7. ||

Remark 3.14. When \( T_n \) is the sum of independent, normally distributed random variables with mean zero and variance one, \( \phi_n(s_n/n) \) becomes \( \exp[s_n^2/2n] \) and the class of probability measures \( L^* \) reduces to the class \( L \). Thus Theorem 3.7 generalizes Theorem 2.6 to a larger class of Hamiltonians and Probability measures.

We now state the theorem of Sethuraman (1961) which was crucially used to obtain the limiting marginal distribution of \( (S_n - n\tau_n)/n^{1-1/2r} \) in the proof of Theorem 3.7.

**Theorem 3.15** (Sethuraman). Let \( \Lambda_n \) be a sequence of probability measures on \( V \times W \), where \( V \) and \( W \) are topological spaces. Let \( \nu_n \) be the marginal probability measure of \( \Lambda_n \) on \( V \) and \( \nu_n(v,\cdot) \) be the conditional probability measure on \( W \). Suppose that \( \nu_n \) converges to a probability measure \( \nu \) for every measurable set in \( V \) and for almost all \( v \) with respect to \( \nu \), \( \nu_n(v,\cdot) \) converges weakly to \( \nu(v,\cdot) \). Then \( \Lambda_n \) converges weakly to \( \Lambda \), where

\[
\Lambda(A \times B) = \int_A \nu(v,B) \, du(v)
\]

for every measurable rectangular set \( A \times B \).

We now turn our attention to the case where \( T_n, n \geq 1 \), are lattice valued random variables with spans \( h_n, n \geq 1 \). The following theorem, which is analogous to Theorem 3.4, was proved by Chaganty and Sethuraman (1985).

**Theorem 3.16.** Let \( T_n, n \geq 1 \), be a sequence of lattice valued random variables with spans \( h_n, n \geq 1 \). Let \( m_n \) belong to the range of \( T_n/n \). Assume that conditions (A), (B) of Theorem 3.4 hold and replace conditions (C), (D) by the following:
We first note that

\[(3.32) \quad \log E_{M, y} \exp[t(S_n - n\tau_n)/n^{1-1/2r}] = n \left[ -\frac{t\tau_n}{n^{1-1/2r}} + h \left( \frac{t}{n^{1-1/2r}} + m_{n, r}(y) \right) - h(m_{n, r}(y)) \right] = n \left[ -\frac{t\tau_n}{n^{1-1/2r}} + h'(m_{n, r}(y)) \frac{t}{n^{1-1/2r}} + h''(m_{n, r}(y)) \frac{t^2}{2n^{2-1/r}} + o(n^{-1}) \right] = h''(m_{n, r}) ty + \frac{h''(m_{n, r}) t^2}{2n^{1-1/r}} + o(1), \]

since \( \tau_n = h'(m_{n, r}) \). Thus

\[(3.33) \quad \log E_{M, y} \exp[t(S_n - n\tau_n)/n^{1-1/2r}] \rightarrow \begin{cases} h''(m) ty & \text{if } r \geq 2 \\ h''(m) ty + \frac{h''(m) t^2}{2} & \text{if } r = 1. \end{cases} \]

This shows that the limiting distribution of \((S_n - n\tau_n)/n^{1-1/2r}\) given \(W_n = y\) is degenerate at \(h''(m)y\) if \(r > 1\) and \(N(h''(m)y, h''(m))\) if \(r = 1\). Next we note that

\[(3.34) \quad f_n(y) = \frac{1}{n^{1-1/2r}} \int k_{(m_{n, r}(y))} \exp(n h(m_{n, r}(y))) = g_n(y) \int g_n(y) \ dy, \]

where \(g_n(y)\) is as defined in \((3.16)\). By Lemmas 3.9, 3.11 and 3.13 it follows that

\[(3.35) \quad f_n(y) \to f(y) = \frac{g(y)}{\int g(y) dy} \quad \text{as } n \to \infty, \]

where \(g(y) = \exp[-y^{2r}/(2r)!]\). Thus the limiting distribution of \(W_n\) is \(f(y)\).
Hence
\[ \left| \int_{|y| > n^{1/4r}} g_n(y) \, dy \right| = O(n^q) \exp[-(n-\xi)[\frac{c_2 r, n}{(2r)!} \frac{1}{\sqrt{n}} + K n^{-(2r+1)/4r}]] \]
which goes to zero since \(|K_n| \leq K\) for all \(n\). The proof of Lemma 3.13 is now complete.

**Proof of Theorem 3.7.** We first express \(dQ_n\) defined in (3.13) as follows:

\[ dQ_n(x) = z_n^{-1} f_n(s_n/x) \Pi dP(x_j) \]

\[ = z_n^{-1} \int \exp(y s_n) k_n(y) \, dy \Pi dP(x_j). \]

Substituting \(m_n, r(y) = m_n + n^{-1/2r} y\), we get

\[ dQ_n(x) = z_n^{-1} n^{-1/2r} \int \exp(m_n, r(y) s_n) k_n(m_n, r(y)) \, dy \Pi dP(x_j) \]

\[ = z_n^{-1} n^{-1/2r} \int \Pi \exp(x_j m_n, r(y) - h(m_n, r(y)) \, dP(x_j) \]

\[ \cdot k_n(m_n, r(y)) \exp(n h(m_n, r(y))) \, dy \]

\[ = \int \Pi dM_{n, y}(x_j) f_n(y) \, dy, \text{ where} \]

\[ dM_{n, y}(x_j) = \exp(x_j m_n, r(y) - h(m_n, r(y))) \, dP(x_j) \]

And

\[ f_n(y) = z_n^{-1} n^{-1/2r} k_n(m_n, r(y)) \exp(n h(m_n, r(y))). \]

Since \(\int dQ_n(x) = 1\) and \(\int dM_{n, y}(x_j) = 1\) for each \(y\) and \(j\), we have \(\int f_n(y) \, dy = 1\). Thus we can introduce random variables \(W_n\) with probability density function \(f_n(y)\) and the representation (3.29) of \(dQ_n(x)\) shows that given \(W_n = Y, X_j^{(n)}, j = 1, \ldots, n\) are i.i.d. with common distribition \(M_{n, y}(x)\).

We now proceed to obtain the limiting distribution of \((S_n - n r_n)/n^{1-1/2r}\) under \(dM_{n, y}(x)\).
Substituting \( z = n^{-1/2}y \), we get

\[
\left| \int_{|y| > n^{1/4}r} \psi_n(y) \, dy \right| \\
\leq \sqrt{2 \pi} \psi'(\epsilon_n^{1/n}) \, n^{-(1-1/r)/2} \int_{|z| > n^{-1/4}r} \left| \exp[-n(V_n(m_n + z) - V_n(m_n))] \right| \\
\cdot \left| \exp(n \, \gamma_n(m_n + z) \, k_n(m_n + z)) \right| \, dz \\
\leq 0(n^{p + (1+1/r)/2}) \max_{|z| > n^{-1/4}r} \exp[-(n-\ell)(V_n(m_n + z) - V_n(m_n))] \\
\cdot \int \exp(-\ell(V_n(m_n + z) - V_n(m_n))) \, dz.
\]

The last inequality follows from Lemma 3.12. This together with condition (3.6) yields

\[
(3.27) \left| \int_{|y| > n^{1/4}r} \psi_n(y) \, dy \right| \leq 0(n^q) \max_{|z| > n^{-1/4}r} \exp[-(n-\ell)(V_n(m_n + z) - V_n(m_n))]
\]

\[
= 0(n^q) \exp[-(n-\ell)L_n],
\]

where

\[
q = p_1 + p + (1 + 1/r)/2,
\]

and

\[
L_n = \min_{|z| > n^{-1/4}r} [V_n(m_n + z) - V_n(m_n)]. \text{ This minimum is attained at } z = \pm n^{-1/4}r \text{ by condition (3.7). Therefore,}
\]

\[
L_n = \min\{(V_n(m_n + n^{-1/4}r) - V_n(m_n)), (V_n(m_n - n^{-1/4}r) - V_n(m_n))\}
\]

\[
= \frac{c_{2r,n}}{(2r)!} \frac{1}{\sqrt{n}} + K_n \, n^{-(2r+1)/4} r.
\]
\begin{align}
\sup_y |\exp(n\gamma_n(m_n + y)) k_n(m_n + y)| &= O(n^{p+1}) \quad \text{as } n \to \infty. \\
\end{align}

**Proof.** An application of the inversion formula yields (see (2.12) of Chaganty and Sethuraman (1985)),

\begin{align}
|\exp[n((m_n + y)s - \psi_n(s))]| k_n(m_n + y) &= \left| \frac{n}{2\pi} \int_{-\infty}^{\infty} \exp[n(\psi_n(s + it) - \psi_n(s) - it(m_n + y))] \, dt \right| \\
&\leq \frac{n}{2\pi} \int_{-\infty}^{\infty} |\psi_n(s + it)/ \psi_n(s)|^{1/n} \, dt.
\end{align}

Taking supremum with respect to \( s \in I \) and using condition (D) of Theorem 3.4 we get

\[ \sup_y |\exp(n\gamma_n(m_n + y)) k_n(m_n + y)| = O(n^{p+1}). \]

**Lemma 3.13.** Suppose that \( V_n \) has a unique global minimum at the point \( m_n \in A_n \) and let \( g_n \) be as defined in (3.16). Then

\begin{align}
|y| > n^{1/4r} g_n(y) \, dy \to 0 \quad \text{as } n \to \infty.
\end{align}

**Proof.** Let \( m_{n,r}(y) = m_n + n^{-1/2} y. \) By (3.16) we have

\begin{align}
|y| > n^{1/4r} g_n(y) \, dy &= \left[ \frac{2\pi \psi''(\xi_n)}{n} \right]^{1/2} |y| > n^{1/4r} k_n(m_{n,r}(y)) \\
&\cdot \exp[n(h(m_{n,r}(y)) + V_n(m_n))] \, dy \\
&= \left[ \frac{2\pi \psi''(\xi_n)}{n} \right]^{1/2} |y| > n^{1/4r} k_n(m_{n,r}(y)) \\
&\cdot \exp \left[-n(V_n(m_{n,r}(y)) - V_n(m_n)) + n\gamma_n(m_{n,r}(y)) \right] \, dy.
\end{align}
Proof. Note that \( n^{-1/2r(y)} \) converges to zero uniformly in \( y \) for \( |y| < n^{1/4r} \).

Since \( m \) is an interior point of \( nA_n \) there exists \( N_3 \) (independent of \( y \)) such that \( m_n,r(y) = (m_n + n^{-1/2r(y)}) \in A_n \) for \( n \geq N_3 \). Applying Theorem 3.4 for \( n \geq N_3 \), we get

\[
\int_{|y| \leq n^{1/4r}} g_n(y) \, dy = \left[ \frac{2\pi \psi_n''(\xi_n)}{n} \right]^{1/2} \int_{|y| \leq n^{1/4r}} \exp[n(h(m_n,r(y)) + V_n(m_n))] \cdot k_n(m_n,r(y)) \, dy
\]

\[
= \int_{|y| \leq n^{1/4r}} \exp[-n(V_n(m_n,r(y)) - V_n(m_n))] \cdot [1 + O(|m_n,r(y) - m|) + O(n^{-2})] \, dy
\]

\[
= \int_{-\infty}^{\infty} \lambda_n(y) \, dy,
\]

where

\[
\lambda_n(y) = I(|y| \leq n^{1/4r}) \exp[-n(V_n(m_n,r(y)) - V_n(m_n))] \cdot [1 + O(|m_n,r(y) - m|) + O(n^{-2})],
\]

and \( I(\cdot) \) is the indicator function. It follows from Lemma 3.10 that \( |\lambda_n(y)| \) is bounded by an integrable function. We can now conclude from Lemma 3.9 and Lebesgue dominated convergence theorem that

\[
\int_{-\infty}^{\infty} \lambda_n(y) \, dy \rightarrow \int_{-\infty}^{\infty} g(y) \, dy \text{ as } n \rightarrow \infty.
\]

The proof Lemma 3.11 is now complete. ||

The next Lemma 3.12 is needed in the proof of Lemma 3.13.

Lemma 3.12. Let \( T_n, n \geq 1 \) be a sequence of random variables satisfying the conditions of Theorem 3.4. Then
(3.20) \[ n[V_n(m_n + n^{-1/2}r - V_n(m_n))] \geq y^{2r} c_{2r}/2(2r)! \]

for all \( n \geq N \), and \( |y| < n^{1/4r} \).

**Proof.** Let \( 0 < \epsilon < c_{2r}/2 \). Since \( c_{2r,n} \) converges to \( c_{2r} \) we can find \( N_1 \) such that \( c_{2r,n} > c_{2r}/2 + \epsilon \) for all \( n \geq N_1 \). Recall that \( \gamma_n(m_n) = m_n \tau_n - \psi_n(\tau_n) \), where \( \tau_n \) is such that \( \psi_n'(\tau_n) = m_n \). It is easy to verify that \( \gamma_n'(m_n) = \tau_n \) and \( \gamma_n''(m_n) = [\psi_n'(\tau_n)]^{-1} \).

Also, \( \gamma_n(2r+1)(m_n) \) is the ratio of a simple function of \((2r+1)\) derivatives of \( \psi_n \) at \( \tau_n \) to \([\psi_n''(\tau_n)]^{2r} \). Conditions (A) and (B) of Theorem 3.4 imply that all these derivatives are bounded uniformly in \( n \) and that \( \psi_n''(\tau_n) \geq \alpha > 0 \) (see (2.6) of Chaganty and Sethuraman (1985)). Hence \( \gamma_n(2r+1)(m_n) \) is uniformly bounded in \( n \) and consequently \( \gamma_n(2r+1)(m_n) = \gamma_n(2r+1)(m_n) - h(2r+1)(m_n) \) is also uniformly bounded in \( n \). Therefore

\[
V_n(m_n + u) - V_n(m_n) = u^{2r} c_{2r,n}/(2r)! + K_n u^{2r+1}
\]

as \( u \to 0 \), where \( |K_n| \leq K < \infty \) for all \( n \). Thus

\[
n[V_n(m_n + n^{-1/2}r - V_n(m_n)]
\]

\[
= y^{2r} c_{2r,n}/(2r)! + K_n y^{2r+1}/n^{1/2r}
\]

\[
\geq y^{2r} c_{2r}/2(2r)! + y^{2r}[\epsilon/(2r)! - K y/n^{1/2r}]
\]

\[
\geq y^{2r} c_{2r}/2(2r)!
\]

if \( |y| < n^{1/4r} \) and \( n \geq N = \max\{N_1, (K(2r)!/\epsilon)^{4r} \} \). This completes the proof of the Lemma 3.10. ||

**Lemma 3.11.** Let \( g \) and \( g_n \) be as defined in (3.15) and (3.16). Then under the hypothesis of Lemma 3.9 we have

(3.21) \[ \int_{-\infty}^{\infty} g_n(y) \, dy \to \int_{-\infty}^{\infty} g(y) \, dy \text{ as } n \to \infty. \]
For $y \in \mathbb{R}$, let

\begin{equation}
    g(y) = \psi_{\nu}[-y^{2r} c_{2r} / (2r)!] \quad \text{and}
\end{equation}

\begin{equation}
    g_n(y) = \left[ \frac{2 \pi \psi'_n(x_n)}{n} \right]^{1/2} k_n(m_n + n^{-1/2} r y) \exp[n(h(m_n + n^{-1/2} r y) + V_n(m_n))]
\end{equation}

where $x_n$'s are defined as in Remark 3.6. The functions $g_n$'s arise in the proof of Theorem 3.7. Lemma 3.9 shows that $g_n(y)$ converges to $g(y)$ as $n \to \infty$ for each $y$. The next four lemmas, Lemma 3.10 - 3.13, show that

\begin{equation}
    \int_{-\infty}^{\infty} g_n(y) \, dy \to \int_{-\infty}^{\infty} g(y) \, dy \quad \text{as} \quad n \to \infty.
\end{equation}

**Lemma 3.9.** Suppose that $V_n$ has a unique global minimum of type $r$ at the point $m_n \in A_n$. Let $m_n$ converge to $m$, where $m$ is an interior point of $\partial A_n$. Suppose that $V_n^{(2r)}(m_n) = c_{2r,n}^2$ converges to $c_{2r}$ as $n \to \infty$. Then

\begin{equation}
    g_n(y) \to g(y) \quad \text{as} \quad n \to \infty.
\end{equation}

**Proof.** Fix $y \in \mathbb{R}$. Let $m_{n,r}(y) = m_n + n^{-1/2} r y$. Then $m_{n,r}(y)$ converges to $m$ and $m_{n,r}(y) \in A_n$ for sufficiently large $n$. Applying Theorem 3.4 together with Remark 3.6, with $m_n$ replaced by $m_{n,r}(y)$ we get

\begin{equation}
    g_n(y) = \exp[-n V_n(m_{n,r}(y)) + n h(m_{n,r}(y)) + V_n(m_n)]
    \left[ 1 + O(|m_{n,r}(y) - m|) + O(\frac{1}{n}) \right]
    \exp[-n(V_n(m_{n,r}(y)) - V_n(m_n))] [1 + O(|m_{n,r}(y) - m|) + O(\frac{1}{n})]
    \exp[-y^{2r} c_{2r,n}/(2r)! + n o(|y|^{2r}/n)] [1 + O(|m_{n,r}(y) - m|) + O(\frac{1}{n})]
    \to g(y) \quad \text{as} \quad n \to \infty. \quad ||
\end{equation}

**Lemma 3.10.** Suppose that the $V_n$'s have a unique global minimum of type $r$ at the point $m_n \in A_n$. Then there exists $N$ such that
For each integer \( r \geq 1 \), let \( Y_{r}^{*} \) be a random variable with probability density function given by

\[
d_{r} \exp[-c_{2r}^{2}\frac{Y^{2r}}{h''(m)]^{2r} (2r)!} \text{ if } r \geq 2 \]

and

\[
N(0, h''(m)[h''(m) + c_{2r}] / c_{2r}) \text{ if } r = 1, \]

where \( c_{2r} \) is the constant that appears in Theorem 3.7 below and \( d_{r} \) is the normalizing factor. With these assumptions and notation, we are in a position to state the main theorem of this section.

**Theorem 3.7.** Let \( X_{j}^{(n)} \), \( j = 1, \ldots, n \) be a triangular array of random variables satisfying \( |X_{j}^{(n)}| < c \) and having a joint distribution given by

\[
dQ_{n}(x) = z_{n}^{-1} \phi_{n}(s_{n}/n) dP(x_{j}),
\]

where \( \phi_{n} \) is the m.g.f. of \( T_{n} \) and \( P \in L^{*} \). Assume that \( V_{n} \), defined in (3.5), has a unique global minimum of type \( r \) at \( m_{n} \in A_{n} \). Let \( m + m_{n} + V_{n}(n) + c_{2r} \) as \( n \to \infty \), where \( m \) is an interior point of \( A_{n} \). Let \( S_{n} = X_{1}^{(n)} + \ldots + X_{n}^{(n)} \). If \( T_{n} \) satisfies the conditions of Theorem 3.4, then

\[
(S_{n} - nT_{n}) / n^{1-1/2r} \overset{d}{\to} Y_{r}^{*},
\]

where \( \psi'_{n}(r) = m_{n} \) and \( Y_{r}^{*} \) is as defined above.

The proof of the above theorem is postponed until the end of Lemma 3.13.

**Remark 3.8.** The distribution function \( Q_{n}(x) \) is well defined because

\[
z_{n} = \int \exp [n \psi_{n}(s_{n}/n)] dP(x_{j}) \leq c \int \exp [\psi_{n}(x)] dP(x) \right]^{n} < \infty,
\]

wherein we have used condition (3.3) and the fact that \( \psi_{n} \) is a convex function.
Part (II). It is easy to verify that the random variables $T_n, n \geq 1,$ satisfies conditions (A), (B) of Theorem 3.4. Condition (D) also holds because

\begin{equation}
\sup_{|\tau| < a} \int_{-\infty}^{\infty} |\phi(\tau + it)/\phi(\tau)| \, dt \\
\leq \sup_{|\tau| < a} \int_{0}^{\infty} \frac{2(1 - \tau^2)}{[4t^2 \tau^2 + (1 - \tau^2 + \tau^2)]^{\frac{3}{2}}} \, dt \\
\leq 2 + 2 \int_{1}^{\infty} \frac{1}{t^2} \, dt < \infty.
\end{equation}

Part (III). To show $P \in L^*$, we only need to verify conditions (3.3) and (3.6). Condition (3.3) trivially holds since $\exp[\psi(u)]$ is bounded on $(-1, 1)$, the support of $P$. Condition (3.6) on the integrability follows from the fact $V(u) \sim \log |u|$ as $|u| \to \infty$.

We can therefore conclude in this example that

\begin{equation}
S_n/\sqrt{n} \overset{d}{\to} N(0, 1/3).
\end{equation}

Example 4.2. Let $F$ be the distribution function of the sum of two independent and identically distributed uniform random variables on the interval $(-b, b)$ with $b = \sqrt{3}/\sqrt{2}$. Let $P$ be standard normal probability measure. The joint distribution $Q_n$ is given by

\begin{equation}
dQ_n(x) = z_n^{-1}(2\pi)^{-n/2} \left[ \frac{n \sinh(bs_n/n)}{(bs_n)} \right]^{2n} \exp \left[ -\frac{1}{2} \sum_{j=1}^{\infty} \text{Ex}_j^2 \right] \dd x_j.
\end{equation}

In this example

\begin{equation}
\psi_n(z) = [\sinh(bz)/(bz)]^{2n}
\end{equation}

\begin{equation}
\psi(z) = 2[\log(\sinh(bz)) - \log(bz)] \quad \text{for} \quad |\text{Real}(z)| < \infty.
\end{equation}

Also

\begin{equation}
\gamma(u) = \sup_{|s| < \infty} [us - \psi(s)] \quad \text{for} \quad |u| < 2b.
\end{equation}
Thus

\[(4.18) \quad V(u) = \gamma(u) - h(u) = \gamma(u) - (u^2/2) = us_1 - \psi(s_1) - (u^2/2), \quad \text{for } |u| < 2b,\]

where \(\psi'(s_1) = u\).

**Part (I).** Differentiating (4.18) with respect to \(u\), we get

\[(4.19) \quad V'(u) = [s_1 - \psi'(s_1)] = \frac{s_1^2 - 2bs_1 \coth(bs_1) + 2}{s_1}, \quad \text{where } \psi'(s_1) = u.\]

Taking successive derivatives one can show that \(t^2 - 3(t \coth(t) - 1) > 0 \) for \(t > 0\).

Letting \(t = bs_1\), and noting that \(b = \sqrt{3}/\sqrt{2}\), we get

\[(4.20) \quad s_1^2 - 2bs_1 \coth(bs_1) + 2 > 0 \quad \text{for } s_1 > 0.\]

Thus \(V'(u) > 0 \) for \(u > 0\). With our choice of \(b = \sqrt{3}/\sqrt{2}\) one can verify that \(V''(0) = 0\) and \(V^{(4)}(0) = 3/5 > 0\). Therefore \(V\) has a unique global minimum of order 2 at the origin.

**Part (II).** We need to verify that \(T_n\) satisfies conditions (A) thru (D) of Theorem 3.4. Note that \(\psi''(0) = 1\). Hence, there exists \(a > 0\) such that \(\psi''(t) > \frac{1}{2} = a\) for \(|t| < a\). This verifies condition (B). It is easy to check that condition (A) also holds for this choice of \(a\). Consider

\[
\sup_{|\tau| < a} \left| \frac{\phi(\tau + it)}{\phi(\tau)} \right| = \left[ \frac{bt}{\sinh(bt)} \right]^2 \left[ \frac{(\sinh(bt))^2 + (\sin(bt))^2}{b^2(t^2 + t^2)} \right] \leq \frac{c}{t^2}, \quad -\infty < t < \infty,
\]

for some constant \(c\). Therefore
This verifies condition (D).

Part (III). We only need to verify that conditions (3.3) and (3.6). Let 
\( \epsilon > 0 \) be given. There exists a \( \delta > 0 \) such that \( |x| < \delta \) implies \( \left[ \sinh(bx)/(bx) \right]^2 < (1+\epsilon) \).

Therefore

\[
\int_{-\infty}^{\infty} \exp[\psi(x)] \, dP(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \left[ \sinh(bx)/(bx) \right] \exp(-x^2/2) \, dx \\
\leq (1+\epsilon) + (2\pi)^{-1/2} \int_{|x| > \delta} \left[ \sinh(bx)/(bx) \right]^2 \exp(-x^2/2) \, dx \\
\leq (1+\epsilon) + (b\delta)^{-2}(2\pi)^{-1/2} \int_{|x| > \delta} \left[ \sinh(bx) \right]^2 \exp(-x^2/2) \, dx,
\]

which is finite. Hence condition (3.3) is satisfied. The other condition (3.6) also holds since the integral is over a finite range and \( V(u) \geq -h(u) \).

Thus the random variables \( T_n \)'s and \( P \) satisfy the conditions of Theorem 3.7. In this example

\[
(4.23) \quad S_n/n^{3/4} \xrightarrow{d} Y_2^*,
\]

where the p.d.f. of \( Y_2^* \) is given by \( d_2 \exp(-y^4/40), \quad -\infty < y < \infty \).

**Example 4.3.** Let \( F \) be as defined in Example 4.2. Let \( P \) be symmetric Bernoulli, i.e., \( P(\{-1\}) = P(\{1\}) = 1/2 \). The joint distribution \( Q_n \) is given by

\[
(4.24) \quad \text{d}Q_n(x) = z_n^{-1} \left[ \frac{n \sinh(bs_n/n)}{(\sqrt{2} b s_n)} \right]^{2n},
\]

where \( x_j = \pm 1 \) for all \( 1 \leq j \leq n \), and \( b = \sqrt{3}/\sqrt{2} \). Recall that

\[
(4.25) \quad \psi(z) = 2 \log(\sinh(bz)) - 2 \log(bz) \quad \text{for} \quad |\text{Real}(z)| < \infty,
\]

and

\[
(4.26) \quad \gamma(u) = u s_1 - \psi(s_1) \quad \text{if} \quad |u| < 2b,
\]
where \( \psi'(s_1) = u \). The cumulant generating function of \( P \) is given by

\[
(4.27) \quad h(u) = \log \cosh(u), \quad -\infty < u < \infty.
\]

The function \( V(u) = \gamma(u) - h(u) \) is finite if \( u < 2b \). Now for \( u > 0 \),

\[
(4.28) \quad V'(u) = s_1 - h'((\psi'(s_1))
\]

\[
= s_1 - \tanh((\psi'(s_1))
\]

\[
\geq s_1 - \psi'(s_1) > 0,
\]

as shown in Example 4.2. Since the first three derivatives of \( V \) at the origin are equal to zero and \( V^{(4)}(0) = 13/5 \), the point zero is a minimum of order 2 for \( V \). This completes the verification of Part (I). We have already checked Part (II) in Example 4.2. One can easily show that the probability measure \( P \) belongs to the class \( L^* \), completing verification of Part (III). Thus by the conclusion of Theorem 3.7 we get

\[
(4.29) \quad \frac{S_n}{n^{3/4}} \underset{d}{\sim} Y_2^*,
\]

where the p.d.f. of \( Y_2^* \) is given by \( d_2 \exp[-13y^4/120], -\infty < y < \infty \).

**Example 4.4.** Let \( F \) be symmetric Bernoulli distribution and \( P \) be the standard normal probability measure. The joint distribution in this example is given by

\[
(4.30) \quad dQ_n(x) = \frac{1}{(2\pi)^{-n/2}} \cosh(s_n/n)_n \exp[-\frac{1}{2}x_j^2] dx_j.
\]

In this example

\[
(4.31) \quad \phi_n(z) = [\cosh(z)]^n,
\]

and

\[
(4.32) \quad \psi(z) = \log[\cosh(z)], \quad \text{for } |\text{Real}(z)| < \infty.
\]
The large deviation rate \( \gamma(u) \) is finite for \( |u| < 1 \) and is given by

\[
(4.33) \quad \gamma(u) = us_2 - \log[\cosh(s_2)],
\]

where \( \tanh(s_2) = u \). Since the c.g.f. of \( P \) is \( h(u) = u^2/2 \), we get for \( u > 0 \),

\[
(4.34) \quad V'(u) = s_2 - h'(\psi'(s_2))
\]

\[= [s_2 - \tanh(s_2)] > 0\]

since \( s_2 = \tanh^{-1}(u) > 0 \). It is easy to check that the first three derivatives of \( V \) at the origin are zero and \( V^{(4)}(0) = 2 \). Thus zero is the unique global minimum of order 2. This completes the verification of Part (I). It is easy to verify that the lattice random variables \( T_n \)'s satisfy all the conditions of Theorem 3.16.

Part (III). Condition (3.3) is trivially satisfied since \( \cosh(x) \exp(-x^2/2) \) is an integrable function. Also for \( \ell > 0 \),

\[
(4.35) \quad \sum_{u \in \mathbb{R}_n} \exp[-\ell(V(u))] \leq \sum_{u \in \mathbb{R}_n} \exp[\ell h(u)]
\]

\[= \frac{n}{2} \sum_{u \in \mathbb{R}_n} \frac{2}{n} \exp[\ell u^2/2]
\]

\[\sim \frac{n}{2} \int_{-1}^{1} \exp[\ell u^2/2] = O(n) .
\]

Therefore \( P \) belongs to the class \( L_1^* \). Thus by the conclusion of Theorem 3.18 we get

\[
(4.36) \quad S_n / n^{3/4} \overset{d}{\rightarrow} Y^*_2 ,
\]

where \( Y^*_2 \) is distributed as \( d_2 \exp(-y^4/12) \), \( -\infty < y < \infty \).
REFERENCES


5) Jong-Woo Jeon (1979). Central limit theorems in the regions of large deviations with applications to Statistical Mechanics, Ph.D. Dissertation, Department of Statistics, Florida State University, Tallahassee, FL.


