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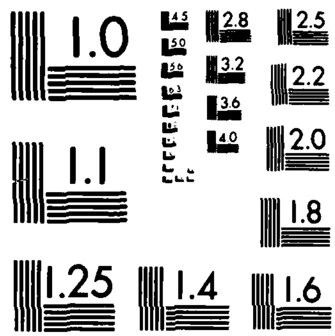

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TRUNCATED DISTRIBUTIONS

by

Ravindra Khattree[1] and Y. Q. Yin[2]

Center for Multivariate Analysis

University of Pittsburgh

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ABSTRACT

An inequality is proved and its interpretation is given. Using the inequality, it is shown, under some mild conditions, that for the univariate truncated distributions, the variance of the truncated distribution increases with the value of the truncation point.

Keywords and Phrases: Truncated distribution, Bound on mean, Monotonicity of Variance, Genetic Selection.

1. INTRODUCTION

The properties of the truncated distributions for the various families of probability densities have been well discussed in the literature. Also, well known are the expressions for mean, variance and higher order moments of truncated distributions, corresponding to certain families. Johnson and Kotz [1] present an excellent account of these properties almost in every chapter of their four-volume reference work on statistical distributions. In this report, ~~we first~~ derive a probability inequality, and then using this inequality, obtain a property of the variance of the subpopulation, obtained by truncating the superpopulation between two points for a certain family of density function bearing some mild conditions.

2. AN INEQUALITY

We start with the notations. Let X be a random variable with the probability density function $f(.) > 0$ and let $F(.)$ be the cumulative distribution function of X. We further assume that X admits the first and second moments m and v respectively.

Let $0 \leq a < b$ be any two points. The probability density of X in the truncated region $a \leq x \leq b$ would be given by

$$g(x) = \frac{f(x)}{F(b) - F(a)} \; ; \; 0 \leq a \leq x \leq b \tag{2.1a}$$

and therefore, the mean and variance are readily seen to be

$$m = \int_a^b x \, g(x)dx \tag{2.1b}$$

$$v = \int_a^b x^2 g(x)dx - m^2 \tag{2.1c}$$

Before we prove the main inequality, we will state and prove the following lemma:

<u>Lemma 1</u>. Let $f(x) > 0$ be a continuous integrable density function. Also, let $f(x)$ be monotinically decreasing function of x for $x \geq 0$.

Then,
$$\int_{-c}^{c} y \, f(y+c+a) \leq 0 \text{ for all } c \geq 0, \; a \geq 0. \tag{2.2}$$

<u>Proof</u>. Consider,

$$\int_{-c}^{c} y \, f(y+c+a)dy$$

$$= \int_{-c}^{0} y \, f(y+c+a)dy + \int_0^c y \, f(y+c+a)dy$$

$$= -\int_0^c y \, f(-y+c+a)dy + \int_0^c y \, f(y+c+a)dy$$

$$= \int_0^c y\{f(y+c+a) - f(-y+c+a)\}dy$$

$$\leq 0, \text{ as } f(y+c+a) \leq f(-y+c+a) \; \forall \; a \geq 0, \text{ and } \forall \; 0 \leq y \leq c.$$

We are now in a position where we can prove our inequality which we state in the following lemma:

Lemma 2. Let $0 \leq a < b$, such that $F(b) - F(a) = \alpha$ is fixed, and let $f(.)$ be as defined as in Lemma 1. Then

$$\frac{a+b}{2} \geq \frac{1}{\alpha} \int_a^b x\, f(x)dx \tag{2.3}$$

Proof. We define $y = x - \frac{a+b}{2}$. Then, the right hand side can be written as

$$\frac{1}{\alpha} \int_{-(\frac{b-a}{2})}^{(\frac{b-a}{2})} (y + \frac{a+b}{2})\, f(y + \frac{a+b}{2})dy$$

$$= \frac{1}{\alpha} \int_{-(\frac{b-a}{2})}^{(\frac{b-a}{2})} y\, f(y + \frac{a+b}{2})dy + \frac{a+b}{2\alpha} \int_{-(\frac{b-a}{2})}^{(\frac{b-a}{2})} f(y + \frac{a+b}{2})\, dy.$$

The first integral in the above expression is nonpositive using Lemma 1 with $c = \frac{b-a}{2}$, while the second integral is easily seen to be equal to α (by writing it again in terms of original variable x.) Hence, (2.3) is established.

Remarks. 1. We will first interpret the inequality (2.3). We notice that the right hand side of (2.3) is mean of the truncated random variable X, $0 \leq a \leq x \leq b$. (See (2.1b)). Hence the inequality states that the mean of the truncated distribution is never more than average of the truncation points, under the assumptions already stated.

2. In case X was originally distributed as standard normal, then (2.3) reduces to another interesting inequality

$$\frac{a+b}{2} \geq \frac{\phi(a) - \phi(b)}{\Phi(b) - \Phi(a)}\ ; \quad 0 \leq a < b \tag{2.4}$$

where $\phi(.)$ and $\Phi(.)$ are respectively ordinate and c.d.f. of standard normal distribution.

(2.4) has another interesting interpretation: if we consider ϕ as a function of Φ, then by mean value theorem, there exists a γ; $\Phi(a) \leq \gamma \leq \Phi(b)$ such that

$$\frac{\phi(a) - \phi(b)}{\Phi(b) - \Phi(a)} = \left.\frac{\partial\phi}{\partial\Phi}\right|_{\Phi(.) = \gamma} = \Phi^{-1}(\gamma) = d, \text{ say.}$$

(2.4) states that such a d, corresponding to γ of mean value theorem, will always be less than or equal to midpoint of a and b.

A different proof of (2.4) has been suggested by Dr. Nitish Mukhopadhyaya of Oklahoma State University in a personal communication.

3. In case f(.) was monotinically increasing, the direction of inequalities in (2.2), (2.3) and (2.4) will be reversed. Similar proof will go through with trivial changes.

3. THE VARIANCE OF THE TRUNCATED DISTRIBUTIONS

Our next result is about the effect of different truncations, but of the same proportion, on the variances of the subpopulation obtained after truncation. The result shows that if a fixed proportion of the original population is truncated by points a and b, $0 \leq a < b$, such that $F(b) - F(a) = \alpha$, a constant, then the truncated subpopulation becomes more and more diverse as we move away from the origin, under some mild conditions. We formally state this result in the following theorem:

Theorem. Let X, f(.) F(.), a, b and α be as in Lemma 2, then v, the variances of X in the truncated population, as a function of a (and hence of b as well) is a monotonically increasing function for $a \geq 0$.

Proof. To prove the theorem, it would be enough to show that the derivative of the variance of the truncated population with respect to a is nonnegative.

Note that

$$\int_a^b f(x)dx = \alpha \tag{3.1}$$

which implies that

$$\frac{\partial b}{\partial a} = \frac{f(a)}{f(b)} \tag{3.2}$$

now using (2.1b) and (2.1c), the variance as a function of a is

$$v_x(a) = \frac{1}{\alpha}\int_a^b x^2\, f(x)dx - \left(\frac{1}{\alpha}\int_a^b x\, f(x)dx\right)^2 \tag{3.3}$$

therefore, using (3.2), we have

$$\frac{\partial v_x(a)}{\partial a} = \frac{1}{\alpha} \left(\int_a^b f(x)dx\right)\left(b^2 f(b) \frac{f(a)}{f(b)} - a^2 f(a)\right)$$

$$- \frac{2}{\alpha^2} \left(\int_a^b x\ f(x)dx\right)\left(bf(b) \frac{f(a)}{f(b)} - af(a)\right)$$

$$= \frac{1}{\alpha} f(a)(b-a)\ \{(a+b) - \frac{2}{\alpha} \int_a^b x\ f(x)dx\}.$$

Note as $b > a$; quantity outside parentheses is positive, while that within parentheses is, using Lemma 2, nonnegative. Hence,

$$\frac{\partial v_x(a)}{\partial a} \geq 0,$$

which proves our theorem.

Remarks. 1. Theorem can easily be stated for monotonically increasing f(.) with trivial changes.

2. As a corollary, it can be seen that for any probability density symmetric about zero; variance of any α-truncation is an increasing function of $|a|$.

4. SOME APPLICATIONS

1. Usually in the problem of genetic selection, selection is made to maximize the average of the unobserved or unobservable criterian variable, but it is made on the basis of observed values of predictors. If we denote the criterian variable by y and the regression of criterian on all the predictors by η, then it is well known that the best strategy is to select all those for which

$$\eta \geq k \tag{4.1}$$

where k is chosen in such a way that proportion of the selected population is α, a predecided value between 0 and 1.

If we assume that all the predictors and criterian are in the original population, distributed jointly as multivariate normal with zero mean, then η will also be normally distributed with zero mean. Writing σ_y^2 and σ_η^2 for variances of y and η respectively in the original population, and W for a truncated region on η axis, we have

$$V(y|\eta \in W) = V(E(y|\eta)|\eta \in W) + E(V(y|\eta)|\eta \in W)$$

or

$$V(y|\eta \in W) = V(\eta|\eta \in W) + \sigma_y^2 - \sigma_\eta^2 \quad . \tag{4.2}$$

(4.2) shows that $V(y|\eta \in W)$ and $V(\eta|\eta \in W)$ differ only by a constant for any region W on η-axis. Now if our policy for selection was as in (4.1), it would lead to a α-proportion subpopulation, even though it maximizes the mean of criterian variable, it is also the most diverse for it. If too much variability is to be avoided and if one seeks a region W, for which $V(\eta|\eta \in W) \leq e$, a prespecified quantity, then the region W, maximizing mean subject to the above constraint, would be:

$$W^*: \quad k_1 \leq \eta \leq k_2 \tag{4.3a}$$

so that

$$V_{\omega^*}(\eta) = e \tag{4.3b}$$

and that

$$P(k_1 \leq \eta \leq k_2) = \alpha. \tag{4.3c}$$

Of course, to control the variability, one has to sacrifice some of the individual units with high values of criterian variable.

2. There may be a situation where, for further experiments, the whole population is to be divided into several groups equal in size on the basis of means of the criterian variable. The theorem says that these groups will differ not only in their mean values but also in the amount of variability and one should possibly take this fact into account while planning for further experiments.

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REFERENCES

[1] JOHNSON, N. L. and KOTZ, S. (1969-1972). Distributions in Statistics (4 Volumes). Wiley, New York.

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