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TECHNICAL REPORT RG-84-17

MISSILE DYNAMICS EQUATIONS FOR GUIDANCE AND CONTROL MODELING AND ANALYSIS

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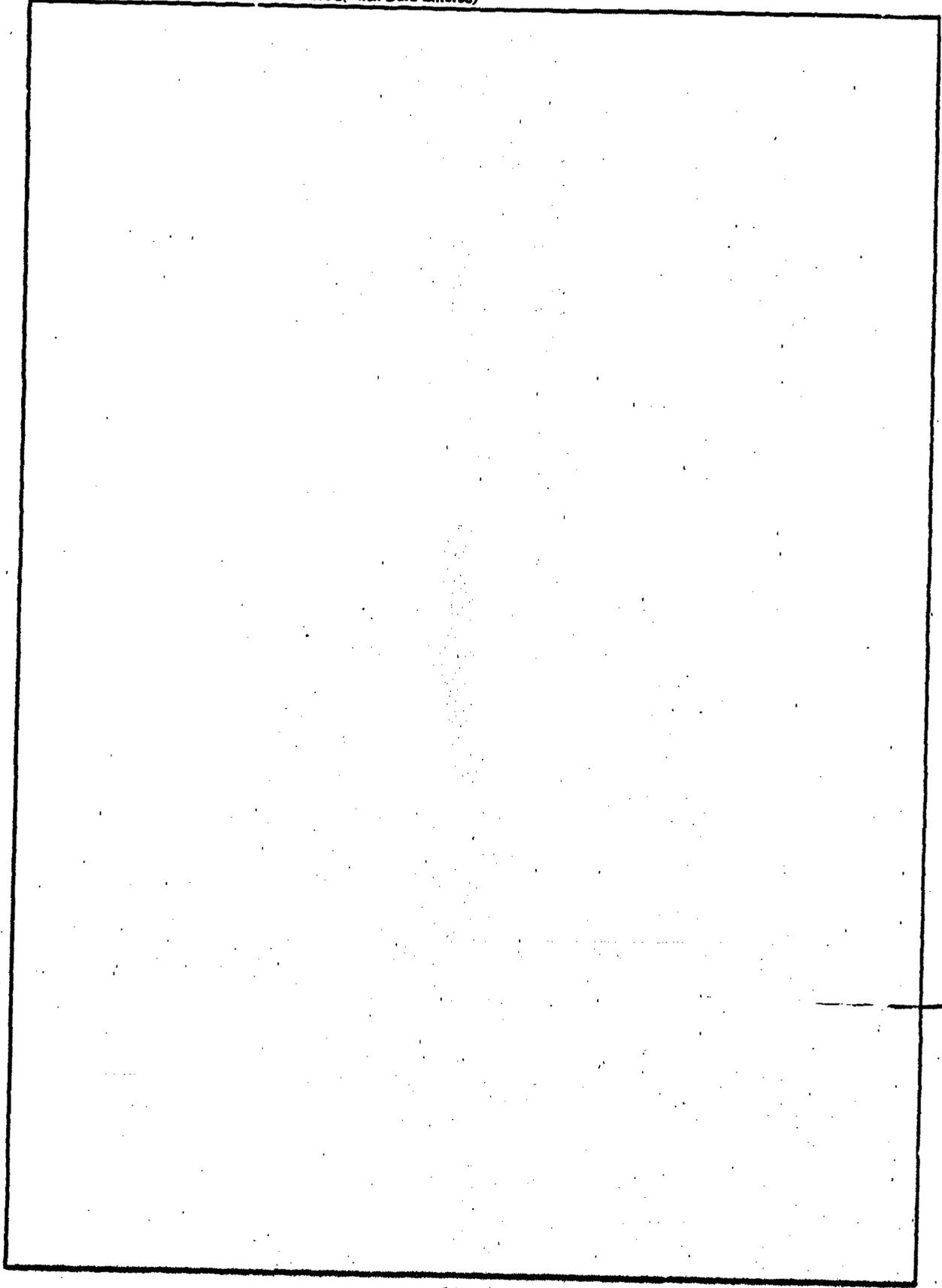
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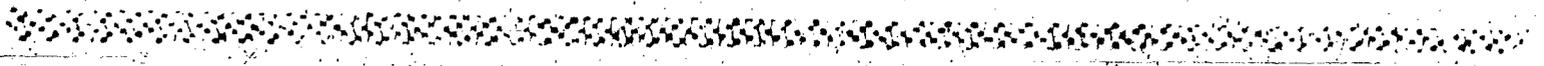


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## I. INTRODUCTION

Simulation is a tool used in the design of new missile systems and in the modification or evaluation of existing systems. A missile simulation allows the engineer to actually try out his design without the expense of building and flying an actual missile. Many simplifications are made of course, but a great deal can be learned through simulation with a substantial savings in time and expense. These simulations are usually put together with emphasis on a particular aspect of the design such as guidance, controlability, component evaluation, etc., and thus vary widely with no single simulation being a general purpose analysis tool that is good for every consideration. A central part of each missile simulation is the missile dynamics equations which govern how the missile will react to a given force or moment. These equations can be implemented in a variety of ways and are often a source of confusion due to the differences and simplifications made. This report deals with this issue by developing several common forms of the missile dynamics equations. Methods of implementation and some of the possible simplifications are also discussed, and a summary is included showing several of the most common forms and their implementation.

## II. DEVELOPMENT OF THE EQUATIONS OF MOTION

### A. Coordinate Systems and Newton's Second Law

The fundamental law governing translational motion is Newton's second law as given below.

$$\bar{F} = d(m\bar{v})/dt \quad (1)$$

This law is only applicable in an inertial coordinate frame and thus care must be taken to define a suitable inertial coordinate frame. At this point it should be noted that an inertial frame can only be defined by the fact that it is a system in which Newton's laws hold. Thus for this article, an inertial frame can be taken to be any frame such that effects due to the movement of the frame are negligible for the purpose of the simulation. Two common types of coordinate frames used in missile simulations are coordinate frames fixed to the earth and coordinate frames that translate with but do not rotate with the earth. An earth fixed coordinate frame is a coordinate frame that rotates and translates with but does not move with respect to the earth, such as the North-East-Down system shown in Figure 1.

One example of the other type is a non-rotating Earth centered coordinate frame. This coordinate frame has its origin at the center of the earth and translates with the earth but does not rotate (see Figure 2). That is the origin of the coordinate system moves with the earth, but the coordinate frame does not rotate with respect to the "fixed" stars. Of these two systems the non-rotating coordinate frame provides the "best" inertial frame for missile

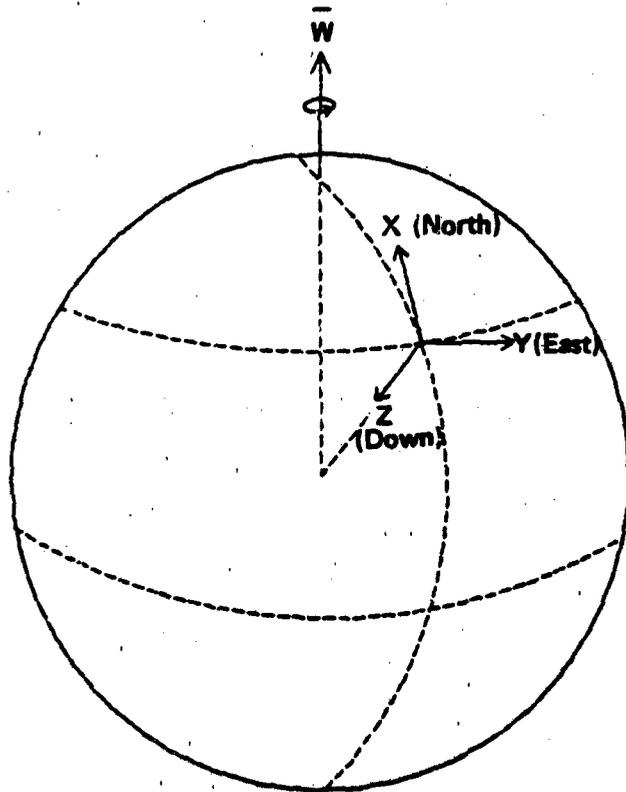


Figure 1. Earth fixed coordinate frame.

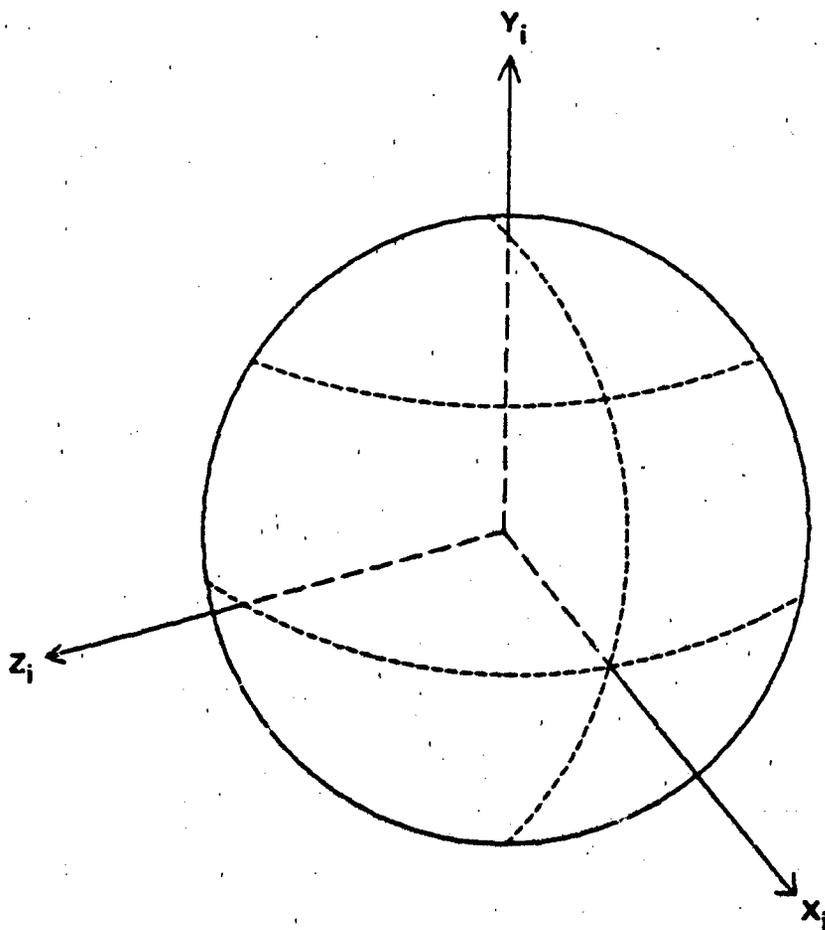


Figure 2. Earth centered coordinate frame.

simulation work. However; the earth fixed coordinate frames can be used in certain cases such as with non-inertially guided missiles when the distances traveled are small compared to the earth's radius and the velocities are small compared to the velocity of escape from the earth. This will be discussed more in Section II, part D.2.

Now assuming that we have decided upon an inertial reference frame, Newton's second law may be used to find the acceleration of the missile with respect to this frame if the forces acting on the missile and the mass of the missile are known. The forces acting on the missile can be separated into three main groups which are: thrust, aerodynamic, and gravitational. Thrust and aerodynamic forces can be evaluated fairly accurately with standard test and design procedures such as static firings to obtain thrust versus time profiles for the engine and wind tunnel measurements to determine the aerodynamic characteristics. Gravitational forces can be calculated from a knowledge of the missile's position relative to the earth. The mass can be estimated from a knowledge of the missile's weight before and after burnout (from test measurements) by using some sort of relationship (often linear) for the decrease in missile mass over the engine's burntime. It is not possible to apply Newton's second law yet, however; as the forces are only known in magnitude and their direction depends on the missile's orientation in the inertial frame. This leads to the use of a missile body frame which is a frame fixed in the missile such as shown in Figure 3. The thrust and aerodynamic forces can be easily evaluated in both magnitude and direction in this frame. All that remains then is to develop the relationship between the inertial reference frame and the missile body frame. To completely specify the location of the body frame in the inertial reference frame, six independent coordinates are needed. These are often referred to as degrees of freedom, with a rigid body, in general, having six degrees of freedom (6-DOF). (Note that a rigid body is an object that does not bend.) The six degrees of freedom are commonly made up of three translational coordinates and three rotational coordinates. The translational coordinates normally chosen are the location of the body frame origin in the inertial frame. Usually the missile center of mass (c.m.) is collocated with the origin of the body frame. This simplifies the calculations involved in evaluating the missile's acceleration. The rotational coordinates must then specify how the axes of the body frame are inclined with respect to the inertial reference frame. This is often done using Euler angles which are three angles that specify the rotational orientation of the body frame with respect to the inertial reference frame. These will be discussed in greater detail in Section II, part C.

#### B. Rotational Equations

It is apparent from the above discussion that to completely describe the motion of a missile in flight both its translational and rotational motion must be known. The translational motion is governed by equation 1 which is the standard form of Newton's second law. It would seem logical then that there would be a similar equation governing rotational motion, which is in fact the case. Thus the rotational equation of motion can be stated as:

$$\bar{M} = \dot{\bar{H}}$$

(2)

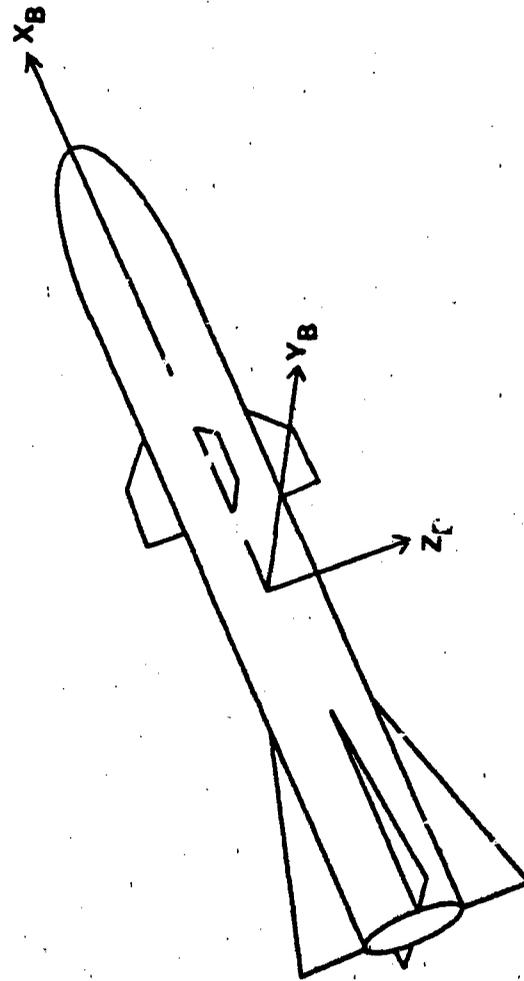


Figure 3. Missile body coordinate frame.

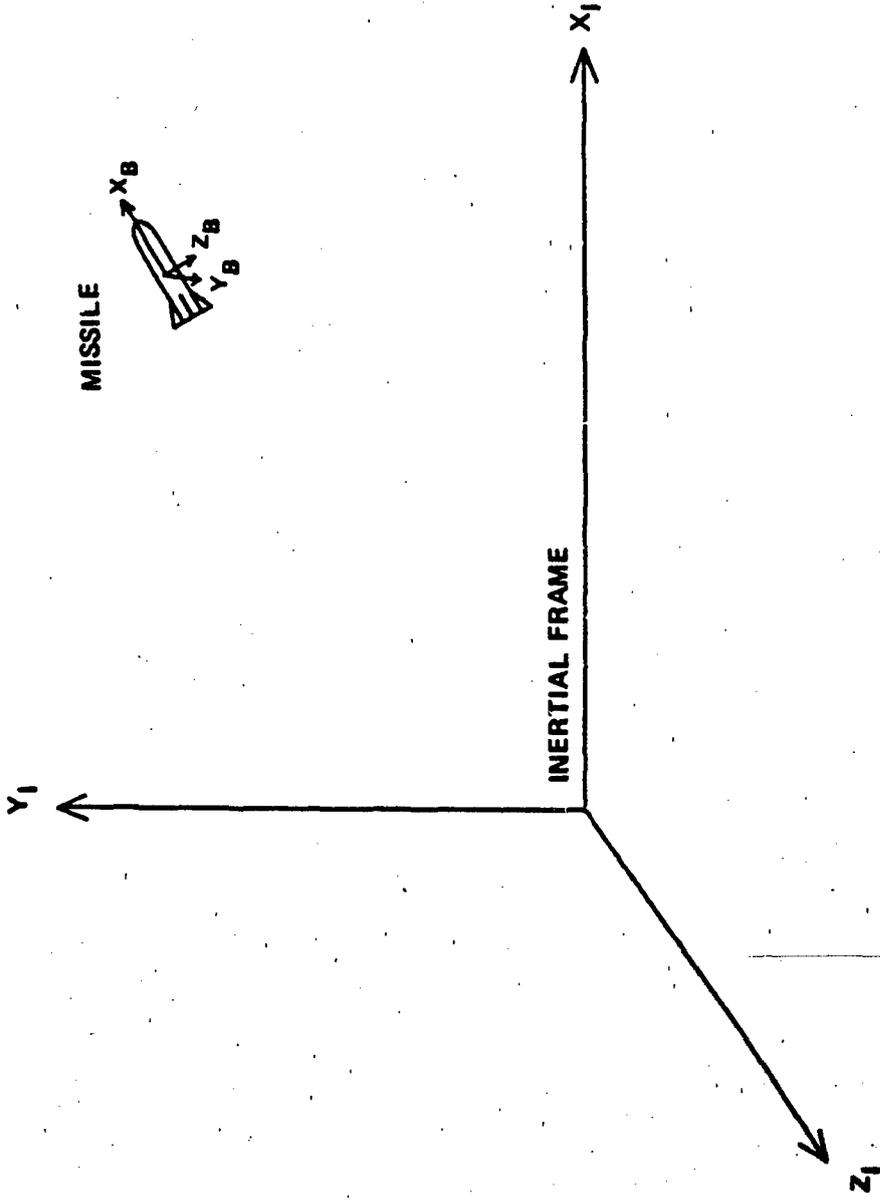


Figure 4. Relationship between missile body frame and inertial coordinate frame.

where  $\bar{M}$  is the moment or torque acting on the body and  $\bar{H}$  is the angular momentum. Note that the reference point about which the external moment and the angular momentum are calculated must be either at the center of mass (c.m.) of the body or fixed in the inertial frame. For our purposes we will assume that the reference point (origin of the body coordinate frame) is at the missile's center of mass. We should also note at this point that the c.m. and c.g. (center of gravity) of the missile can be taken to mean the same thing for our purposes due to the relatively small size of the missile when compared to the change in the gravitational field with respect to distance. The angular momentum " $\bar{H}$ " is given by:

$$\bar{H} = \bar{I}\bar{\omega} \quad (3)$$

where  $\bar{I}$  is the inertia tensor (3x3 matrix) and  $\bar{\omega}$  is the angular velocity of the body. Note that the elements of  $\bar{I}$  are not usually constant except when given in a body fixed coordinate system. Thus the use of a body coordinate system simplifies the application of this equation. The inertia tensor can be written in matrix form as:

$$\bar{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \quad (4)$$

and the angular velocity vector can be written in terms of the missile body rates (P, Q, and R) as:

$$\bar{\omega} = P\hat{i} + Q\hat{j} + R\hat{k} \quad (5)$$

At this point it is important to clarify what is meant by "referred to" and "relative to" or "with respect to". The term "referred to" a certain coordinate system means that the vector is expressed in terms of the unit vectors of that system while the terms "relative to" or "with respect to" a particular system mean as viewed by an observer fixed in that system and moving with it. Thus the same vector can be referred to either a moving or stationary coordinate system, but its rate of change with respect to time as viewed by observers in the two systems may appear quite different. This is of some importance when the derivative of a vector referred to a moving system is to be determined. Thus for a general vector " $\bar{B}$ " referred to a moving coordinate system (such as a body coordinate frame), the time derivative relative to an inertial coordinate system (the absolute time rate of change of  $\bar{B}$ ) is:

$$\dot{\bar{B}} = (\dot{\bar{B}})_r + \bar{\omega} \times \bar{B} \quad (6)$$

where  $(\dot{\bar{B}})_r$  is the rate of change (with respect to time) of  $\bar{B}$  as viewed by an observer in the rotating system, and  $\bar{\omega}$  is the angular velocity of the moving frame. It is important to note that  $\bar{B}$  is the same vector in both the inertial and moving coordinate frames; it is just expressed in the unit vectors of the moving system and since these unit vectors could be rotating, the time rate of change of  $\bar{B}$  as seen by an observer in the moving system is not necessarily the absolute time rate of change.

Equation 6 can be developed using the following analysis. Let

$$\bar{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k} \quad (7)$$

then

$$\dot{\bar{B}} = \dot{B}_x \hat{i} + \dot{B}_y \hat{j} + \dot{B}_z \hat{k} + B_x \dot{\hat{i}} + B_y \dot{\hat{j}} + B_z \dot{\hat{k}} \quad (8)$$

where  $\dot{\hat{i}}$ ,  $\dot{\hat{j}}$ , and  $\dot{\hat{k}}$  are simply the velocities of the tips of the unit vectors which are:

$$\dot{\hat{i}} = \bar{\omega} \times \hat{i}$$

$$\dot{\hat{j}} = \bar{\omega} \times \hat{j} \quad (9)$$

$$\dot{\hat{k}} = \bar{\omega} \times \hat{k}$$

or

$$\bar{\omega} \times \bar{B} = B_x \dot{\hat{i}} + B_y \dot{\hat{j}} + B_z \dot{\hat{k}} \quad (10)$$

and the derivative relative to the moving frame is:

$$(\dot{\bar{B}})_r = \dot{B}_x \hat{i} + \dot{B}_y \hat{j} + \dot{B}_z \hat{k} \quad (11)$$

The general equation for the absolute rate of change with respect to time of a vector referred to a rotating coordinate frame is then made up of the time rate of change of the vector relative to the rotating coordinate frame axes denoted by  $(\ )_r$  plus an  $(\bar{\omega} \times \bar{B})$  term to account for the rotation of these axes (unit vectors) relative to the inertial frame.

Now returning to the development of the rotational equations, equation 6 can be used to determine the derivative of the angular momentum " $\bar{H}$ " referred to the body coordinate frame.

$$\dot{\bar{H}} = (\dot{\bar{H}})_r + \bar{\omega} \times \bar{H} \quad (12)$$

where

$$(\dot{\bar{H}})_r = \dot{\bar{I}} \bar{\omega} \quad (13)$$

and

$$\bar{\omega} \times \bar{H} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ P & Q & R \\ H_x & H_y & H_z \end{bmatrix} \quad (14)$$

Note that equation 8 is simplified due to the choice of a body fixed coordinate frame (the derivative of the I matrix is zero since it is a constant in a body fixed frame). Equations 8 and 9 can be expanded as follows.

$$\dot{\bar{\omega}} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{P} \\ \dot{Q} \\ \dot{R} \end{bmatrix} \quad (15)$$

$$\begin{aligned} &= (I_{xx}\dot{P} + I_{xy}\dot{Q} + I_{xz}\dot{R}) \hat{i} \\ &+ (I_{xy}\dot{P} + I_{yy}\dot{Q} + I_{yz}\dot{R}) \hat{j} \\ &+ (I_{xz}\dot{P} + I_{yz}\dot{Q} + I_{zz}\dot{R}) \hat{k} \end{aligned} \quad (16)$$

and

$$\bar{\omega} \times \bar{H} = (H_z Q - H_y R) \hat{i} + (H_x R - H_z P) \hat{j} + (H_y P - H_x Q) \hat{k} \quad (17)$$

$$\begin{aligned} &= [(I_{xz}P + I_{yz}Q + I_{zz}R)Q - (I_{xy}P + I_{yy}Q + I_{yz}R)] \hat{i} \\ &[(I_{xx}P + I_{xy}Q + I_{xz}R)R - (I_{xz}P + I_{yz}Q + I_{zz}R)P] \hat{j} \\ &[(I_{xy}P + I_{yy}Q + I_{yz}R)P - (I_{xx}P + I_{xy}Q + I_{xz}R)Q] \hat{k} \end{aligned} \quad (18)$$

Thus from equation 2, the moment acting on the body is:

$$\begin{aligned} M_x &= I_{xx}\dot{P} + I_{xy}(\dot{Q} - PR) + I_{xz}(\dot{R} + PQ) \\ &+ (I_{zz} - I_{yy})QR + I_{yz}(Q^2 - R^2) \\ M_y &= I_{yy}\dot{Q} + I_{yz}(\dot{R} - PQ) + I_{xy}(\dot{P} + QR) \\ &+ (I_{xx} - I_{zz})PR + I_{xz}(R^2 - P^2) \\ M_z &= I_{zz}\dot{R} + I_{xz}(\dot{P} - QR) + I_{yz}(\dot{Q} + PR) \\ &+ (I_{yy} - I_{xx})PQ + I_{xy}(P^2 - Q^2) \end{aligned} \quad (19)$$

where  $M_x$ ,  $M_y$ , and  $M_z$  are the components of the moment about the x, y, and z body axes respectively and are governed by the right hand rule.

In order to gain more understanding of the possible simplifications of equations 19, some understanding of the inertia matrix is required. The inertia matrix can be derived by considering the rigid body to be a system of particles that do not move with respect to one another. The angular momentum of the rigid body will then be the sum of the angular momentum vectors for each of the particles. The angular momentum relative to the point 0 is then:

$$\vec{H}_0 = \sum_i (\vec{R}_i \times M_i \dot{\vec{R}}_i) \quad (20)$$

where  $\vec{R}_i$  is the position vector of the particle i (mass  $M_i$ ) relative to the reference point 0. If 0 is fixed in the body, then the magnitudes of all the  $\vec{R}_i$ 's are constant and thus:

$$\dot{\vec{R}}_i = \vec{\omega} \times \vec{R}_i \quad (21)$$

For a rigid body the mass distribution is constant, and thus the summation can be replaced by a volume integral with mass  $\rho dV$  where  $dV$  is a small volume element and  $\rho$  is the mass density at that point. The angular momentum  $H_0$  for a rigid body is then given by:

$$\vec{H}_0 = \int_V \rho \vec{R} \times (\vec{\omega} \times \vec{R}) dV \quad (22)$$

Now if the reference point 0 is considered to be at the origin of a cartesian system, with the volume element  $dV$  located at  $(x,y,z)$ , the position vector is given by:

$$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k} \quad (23)$$

and  $\vec{\omega}$  is given by

$$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} \quad (24)$$

$$\vec{R} \times (\vec{\omega} \times \vec{R}) = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ X & Y & Z \\ (Z\omega_y - Y\omega_z) & (X\omega_z - Z\omega_x) & (Y\omega_x - X\omega_y) \end{bmatrix} \quad (25)$$

$$= \begin{aligned} & [(Y^2 + Z^2)\omega_x - XY\omega_y - XZ\omega_z] \hat{i} \\ & + [-XY\omega_x + (X^2 + Z^2)\omega_y - ZY\omega_z] \hat{j} \\ & + [-XZ\omega_x - YZ\omega_y + (X^2 + Y^2)\omega_z] \hat{k} \end{aligned} \quad (26)$$

The moments of inertia are then defined as:

$$\begin{aligned}
 I_{xx} &= \int_V \rho(y^2 + z^2)dv \\
 I_{yy} &= \int_V \rho(x^2 + z^2)dv \\
 I_{zz} &= \int_V \rho(x^2 + y^2)dv
 \end{aligned}
 \tag{27}$$

and the products of inertia are defined to be:

$$\begin{aligned}
 I_{xy} = I_{yx} &= - \int_V \rho XYdv \\
 I_{xz} = I_{zx} &= - \int_V \rho XZdv \\
 I_{yz} = I_{zy} &= - \int_V \rho YZdv
 \end{aligned}
 \tag{28}$$

The angular momentum equation for a rigid body can then be written as:

$$\begin{aligned}
 \vec{H} &= (I_{xx}W_x + I_{xy}W_y + I_{xz}W_z)\hat{i} \\
 &+ (I_{xy}W_x + I_{yy}W_y + I_{yz}W_z)\hat{j} \\
 &+ (I_{xz}W_x + I_{yz}W_y + I_{zz}W_z)\hat{k}
 \end{aligned}
 \tag{29}$$

From equations 27 it can be seen that the moments of inertia are always positive and are actually just a second moment of the mass distribution with respect to a cartesian axis system. The products of inertia can be of either sign and can be zeroed by proper choice of the axis system. In general if a three dimensional body has a plane of symmetry such that the mass distribution is a mirror image of that on the other side, then the products of inertia involving an axis perpendicular to the plane of symmetry are zero if the other two axes lie in the plane of symmetry. An example of this is shown in Figure 5 where the missile shown is assumed to be of constant density and has two planes of symmetry such that the product of inertia terms are zero. Note that missiles are not in general of constant density, thus the mass distribution must be a mirror image on each side of the plane of symmetry for the product of inertia terms to be exactly zero. Assuming that the product of inertia terms are zero is a good assumption in many cases though, for missiles having two planes of symmetry such as shown in Figure 5.

Now returning to equations 19, it can be seen that considerable simplification results if the body frame is chosen such that the products of inertia are zero. It was noted above that a set of axes could always be found such that the products of inertia are all zero and thus the inertia matrix is a diagonal matrix. The three mutually orthogonal coordinate axes are known as the principal axes for this case, and the moments of inertia are known as the principal moments of inertia. From this it can be seen that the body coordinate frame for the missile in Figure 5 is the principal axes system. Equations 19 then become:

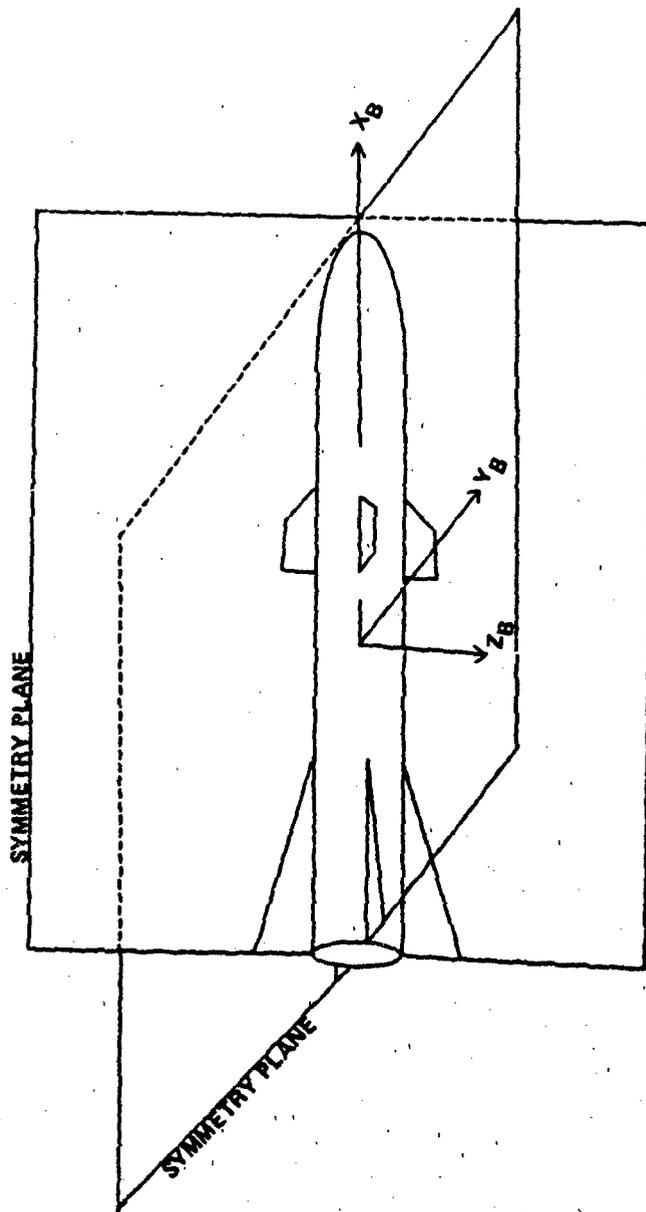


Figure 5. Missile with two planes of symmetry.

$$\begin{aligned}
 M_x &= I_{xx}\dot{P} + (I_{zz} - I_{yy})QR \\
 M_y &= I_{yy}\dot{Q} + (I_{xx} - I_{zz})PR \\
 M_z &= I_{zz}\dot{R} + (I_{yy} - I_{xx})PQ
 \end{aligned}
 \tag{30}$$

which are known as Euler's equations of motion. These are relatively simple and are widely used in solving the rotational motion of a missile. Note that as in the case of the external forces, the external moments are made up of aerodynamic and thrust related moments. These can also be estimated from standard design and test procedures.

The rotational rates of the missile (P, Q, & R) can be evaluated using equations 30 or equations 19 as appropriate. This, however; does not provide us with the angles relating the body coordinate frame to the inertial reference frame in a direct manner as the angular rates (P, Q, & R) are not the time derivatives of any angles which specify the orientation of the body. This problem can be solved by the use of Euler angles or quaternions, which use the angular rates to obtain coordinate transformations from the body coordinate frame to the inertial reference frame.

### C. Euler Angles

Euler angles can be used to define the angular orientation of a missile to an inertial reference frame. An Euler angle set consists of three angles and a specified sequence of rotation. In other words to arrive at any given angular orientation of the missile, the missile axes can be assumed to be initially aligned with the inertial reference frame axes; and then in a prescribed sequence, the missile is rotated through each Euler angle about a corresponding body axis. It should be noted that there are many different Euler angle sets in use and care should be taken to define which set is being used as different rotation sequences will usually give different results. An example of a common set for aerospace engineering is shown in Figures 6 through 8. This set consists of an initial rotation of  $\psi$  degrees about the missile z-body axis, followed by a rotation of  $\theta$  degrees about the missile y-body axis and completed by a rotation of  $\phi$  degrees about the missile x-body axis. This Euler angle set is also shown in Figures 9 through 11 in a pseudo three dimensional view.

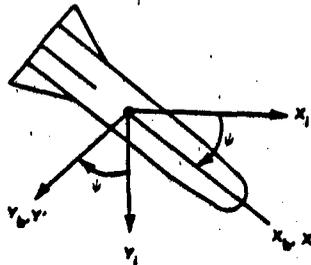


Figure 6. Rotation about the z-body axis (heading angle,  $\psi$ ).

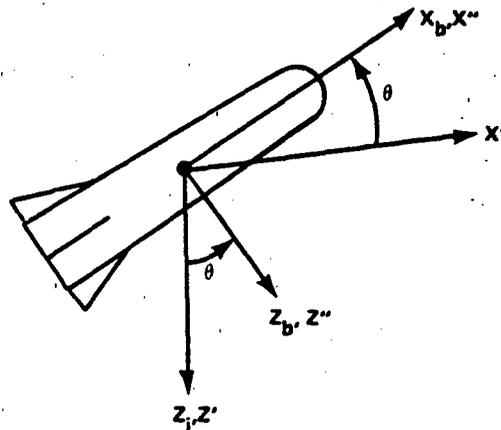


Figure 7. Rotation about the y-body axis (attitude angle,  $\theta$ ).

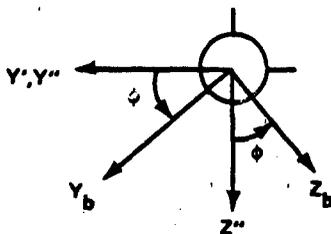


Figure 8. Rotation about the x-body axis (bank angle,  $\phi$ ).

In Figures 6-8 and 9-11, the body axes are aligned with intermediate axes systems denoted (') and (") after the  $\psi$  and  $\theta$  rotations respectively. These are labeled so as to facilitate the reader's understanding but are not used otherwise. Euler angles can be used to specify any desired angular orientation of the missile. Remember though that the sequence is important and for the case given another sequence such as a  $\theta$ - $\psi$ - $\phi$  sequence would in general give quite different results.

If the inertial frame shown in Figures 9-11 is an earth fixed coordinate frame such as an N-E-D (North-East-Down) coordinate frame, the Euler angles are sometimes referred to as the heading angle ( $\psi$ ), the attitude angle ( $\theta$ ), and the bank angle ( $\phi$ ). Thus letting the N-E-D axes correspond to the  $X_1, Y_1, Z_1$  axes respectively, the angle  $\psi$  would simply be the heading of the missile relative to north, the angle  $\theta$  would be the attitude of the missile relative to the north-east plane, and the angle  $\phi$  would be the lateral inclination or bank of the missile. Note that the  $\psi$ - $\theta$ - $\phi$  sequence used here could also be referred to as a yaw, pitch and roll sequence in missile terminology.

The Euler angle rotations shown in Figures 6-8 or 9-11 can now be used to develop a coordinate transformation matrix that will transform vectors from the inertial frame to the body frame. Referring to Figure 6, the coordinate transformation from the inertial to the intermediate (') system can be seen to be:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} \quad (31)$$

Note that in the preceding transformation, the rotation is about the  $Z_i$  axis and thus  $z'$  is collocated with  $Z_i$ . In a similar manner, the rotation from the intermediate (') system to the second intermediate (") system can be seen from Figure 7 to be:

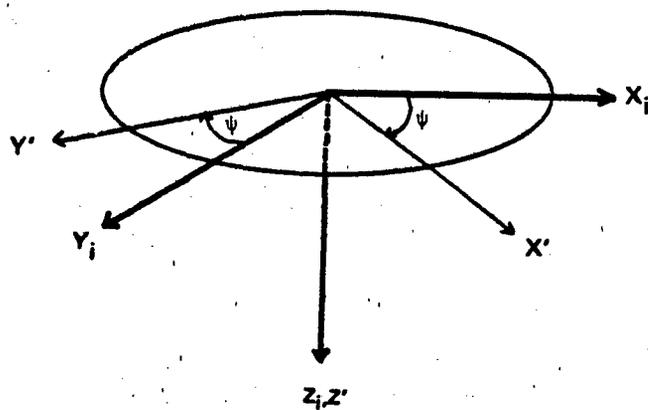


Figure 9. Rotation about the Z - inertial axis.

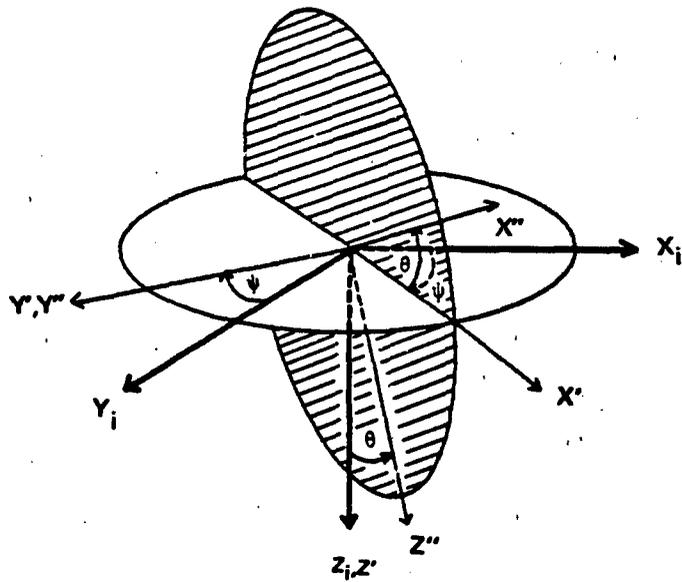


Figure 10. Rotation about the intermediate  $Y'$ ,  $Y''$  axis.

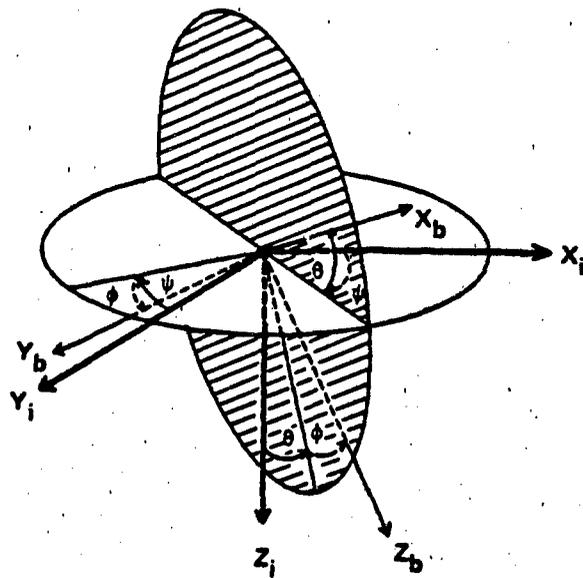


Figure 11. Rotation about the x-body axis.

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \quad (32)$$

where the rotation was about the  $y'$  axis and thus the  $y'$  and  $y''$  axes are the same. Then from Figure 8, the final transformation is:

$$\begin{bmatrix} X_b \\ Y_b \\ Z_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} \quad (33)$$

where the rotation was about the  $x''$  axis which was also the  $x$ -body axis. These transformations can be combined into one transformation as follows. In equation 33 substitute equation 32 for  $(x'', y'', z'')$ , then

$$\begin{bmatrix} X_b \\ Y_b \\ Z_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c \phi & s \phi \\ 0 & -s \phi & c \phi \end{bmatrix} \begin{bmatrix} c \theta & 0 & -s \theta \\ 0 & 1 & 0 \\ s \theta & 0 & c \theta \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \quad (34)$$

and

$$\begin{bmatrix} X_b \\ Y_b \\ Z_b \end{bmatrix} = \begin{bmatrix} c \theta & 0 & -s \theta \\ s \phi * s \theta & c \phi & s \phi * c \theta \\ c \phi * s \theta & -s \phi & c \phi * c \theta \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \quad (35)$$

where the sine and cosine terms have been written in a simplified notation. Now substitute equation 31 for  $(x', y', z')$

$$\begin{bmatrix} X_b \\ Y_b \\ Z_b \end{bmatrix} = \begin{bmatrix} c \theta & 0 & -s \theta \\ s \phi * s \theta & c \phi & s \phi * c \theta \\ c \phi * s \theta & -s \phi & c \phi * c \theta \end{bmatrix} \begin{bmatrix} c \psi & s \psi & 0 \\ -s \psi & c \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} \quad (36)$$

thus

$$\begin{bmatrix} X_b \\ Y_b \\ Z_b \end{bmatrix} = \begin{bmatrix} c \theta * c \psi & c \theta * s \psi & -s \theta \\ s \phi * s \theta * c \psi & s \phi * s \theta * s \psi & s \phi * c \theta \\ -c \phi * s \psi & +c \phi * c \psi & \\ c \phi * s \theta * c \psi & c \phi * s \theta * s \psi & c \phi * c \theta \\ +s \phi * s \psi & -s \phi * c \psi & \end{bmatrix} \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} \quad (37)$$

Equation 37 can now be used to transform vectors expressed in the inertial frame to the body frame. In other words vectors referred to the inertial frame (expressed in the inertial frame unit vectors) are transformed so that they are referred to the body frame (expressed in the body frame unit vectors). A transformation from the body frame to the inertial frame can also be developed from equation 37 by noting that a transformation matrix is an orthogonal matrix and thus its inverse is just its transpose. The inverse transformation can then be obtained by taking the transpose of the transformation matrix in equation 37. This can be better understood by first expressing equation 37 in matrix form as:

$$[B] = [T][I] \quad (38)$$

where [B] is the body vector matrix ( $X_b, Y_b, Z_b$ ), [I] is the inertial frame vector matrix ( $X_i, Y_i, Z_i$ ) and [T] is the transformation matrix. Now multiply by the inverse of the transformation matrix.

$$[T] [B] = [T] [T]^{-1} [I] = [I] \quad (39)$$

thus

$$[I] = [T]^{-1} [B] \quad (40)$$

or

$$\begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} = \begin{bmatrix} c \theta * c \psi & s \phi * s \theta * c \psi & c \phi * s \theta * c \psi \\ & - c \phi * s \psi & + s \phi * s \psi \\ c \theta * s \psi & s \phi * s \theta * s \psi & c \phi * s \theta * s \psi \\ & + c \phi * c \psi & - s \phi * c \psi \\ -s \theta & s \phi * c \theta & c \phi * c \theta \end{bmatrix} \begin{bmatrix} X_b \\ Y_b \\ Z_b \end{bmatrix} \quad (41)$$

Equation 41 can be used to transform vectors referred to the body system (such as missile acceleration, velocity or angular rate) to vectors referred to the inertial system.

Now that it has been shown how to obtain the transformation matrix when given the Euler angles, the next thing is to determine the Euler angles during the missile flight. This can be done by determining the missile's initial Euler angles at launch relative to the inertial frame and then updating these over the missile flight using the angular rates of the missile.

Initial Euler angles (before missile launch) can be determined using the definitions of the Euler angles. Thus the heading angle  $\psi$  is the angle between the missile pointing direction and the inertial X axis on the  $X_i$ - $Y_i$  plane, the attitude angle  $\theta$  is the angle between the  $X_i$ - $Y_i$  plane and the  $X_b$  axis, and the bank angle  $\phi$  is the rotation of the missile about the  $X_b$  axis with respect to the  $X_i$ - $Y_i$  plane. These angles can be used in the transformation matrix of equation 41 to provide the initial transformation from body coordinates to inertial coordinates. Once the missile is launched, the angular rates of the missile (P, Q, & R) can be determined using equation 19 or 30 and then related to the Euler angle rates ( $\dot{\psi}$ ,  $\dot{\theta}$ ,  $\dot{\phi}$ ) using:

$$\begin{aligned} P &= \dot{\phi} - \dot{\psi} * \sin \theta \\ Q &= \dot{\theta} * \cos \phi + \dot{\psi} * \cos \theta * \sin \phi \\ R &= \dot{\psi} * \cos \theta * \cos \phi - \dot{\theta} * \sin \phi \end{aligned} \quad (42)$$

Equations 42 were developed from inspection of Figure 12, which shows the relative orientation of P, Q, and R along with  $\dot{\psi}$ ,  $\dot{\theta}$ , and  $\dot{\phi}$ . Solving equations 42 for  $\dot{\psi}$ ,  $\dot{\theta}$ , and  $\dot{\phi}$  then yields:

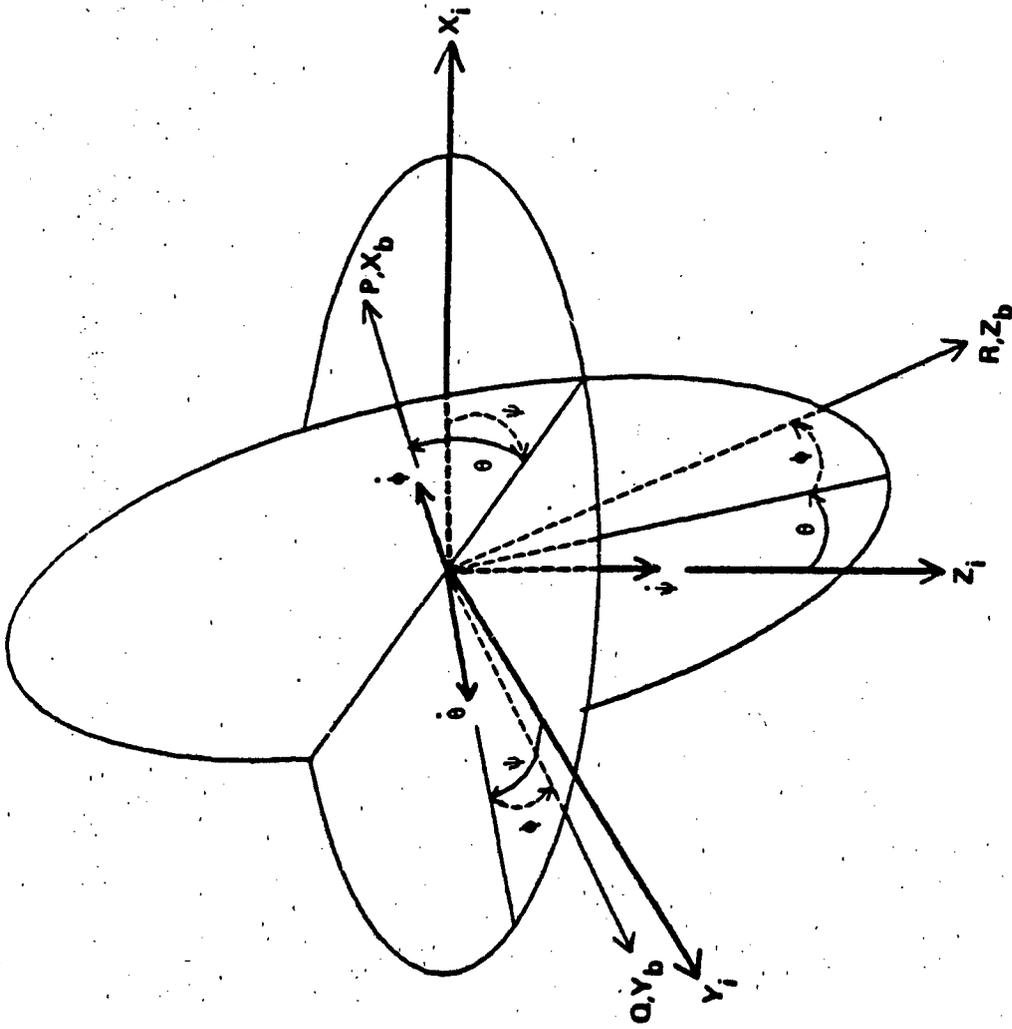


Figure 12. Relationship between missile body rates and euler angle rates.

$$\begin{aligned}
\dot{\psi} &= (Q \sin \phi + R \cos \phi) / \cos \theta \\
\dot{\theta} &= Q \cos \phi - R \sin \phi \\
\dot{\phi} &= P + \dot{\psi} \sin \theta \\
&= P + (Q \sin \phi + R \cos \phi) \tan \theta
\end{aligned}
\tag{43}$$

Equations 43 can be integrated to obtain  $\psi$ ,  $\theta$ , and  $\phi$  which can then be used to obtain the transformation matrix in equations 41 and 37. Note that the  $\dot{\psi}$  equation has a singularity at  $\theta = 90$  or  $-90$ .

#### D. Translational Equations

##### 1. Translational Equations in an Inertial Frame

Since we now have a way to relate vectors expressed in the body frame to the inertial frame, the translational equations of motion can be developed. Equation 1 can be rewritten as:

$$\bar{F}'' = \dot{v} \dot{m} + m \dot{v} \tag{44}$$

or

$$\bar{F}'' - \dot{v} \dot{m} = m \bar{a} \tag{45}$$

The term  $\dot{v} \dot{m}$  is due to mass ejection from the rocket motor and is included in the thrust force. Thus equation 45 can be written as:

$$\bar{F}' = m \bar{a} \tag{46}$$

where  $\bar{F}'$  is the total force on the missile and is the sum of the thrust, aerodynamic, and gravitational forces,  $\bar{a}$  is the acceleration of the missile relative to the inertial frame and  $m$  is the instantaneous mass of the missile. The acceleration of the missile referred to the body frame is:

$$\begin{aligned}
A_{bx} &= F'_x / m \\
A_{by} &= F'_y / m \\
A_{bz} &= F'_z / m
\end{aligned}
\tag{47}$$

Note that the components of acceleration in equation 47 are referred to the body frame, as  $F'$  is more easily evaluated in this frame. In other words  $A_{bx}$ ,  $A_{by}$ , and  $A_{bz}$  are referred to the body frame since  $F'_x$ ,  $F'_y$ , and  $F'_z$  are referred to the body frame. If an accelerometer triad was collocated with the missile axis system, these components would be equivalent to the accelerometer readings plus gravity. The velocity of the missile can then be determined using numerical integration, but care must be taken as integrating

equations 47 directly will not give the correct answer since the acceleration vector is referred to a rotating coordinate frame. In order to get the correct expression for the velocity derivative, equation 6 for the derivative of a vector in a rotating frame must be used. This then results in:

$$\bar{A}_b = (\dot{\bar{V}}_b)_r + \bar{\omega} \times \bar{V}_b \quad (48)$$

The angular rate ( $\bar{\omega}$ ) and the velocity ( $\bar{V}_b$ ) of the missile are referred to the body system and are given by:

$$\bar{\omega} = P\hat{i}_b + Q\hat{j}_b + R\hat{k}_b \quad (49)$$

and

$$\bar{V}_b = U\hat{i}_b + V\hat{j}_b + W\hat{k}_b \quad (50)$$

where  $\hat{i}_b, \hat{j}_b, \hat{k}_b$  are the unit vectors in the body frame. Thus from equations 49 and 50

$$\bar{\omega} \times \bar{V}_b = (QW - RV)\hat{i}_b + (RU - PW)\hat{j}_b + (PV - QU)\hat{k}_b \quad (51)$$

and equation 48 can be expanded and combined with equation 47 to give:

$$\dot{V}_{bx} = F'_x/m - (QW - RV)$$

$$\dot{V}_{by} = F'_y/m - (RU - PW) \quad (52)$$

$$\dot{V}_{bz} = F'_z/m - (PV - QU)$$

or

$$\dot{V}_{bx} = F_x/m + G_x - QW + RV$$

$$\dot{V}_{by} = F_y/m + G_y - RU + PW \quad (53)$$

$$\dot{V}_{bz} = F_z/m + G_z - PV + QU$$

where the  $F/m$  terms are the accelerations that would be indicated by an accelerometer triad collocated with the missile body axis system and "G" is the acceleration due to gravity (mass attraction). Equation 53 can be integrated to obtain the velocity of the missile referred to the body frame. The coordinate transformation of equation 41 can then be used to determine the velocity of the missile referred to the inertial frame, and as the inertial frame is not rotating, these velocity components can be integrated directly to obtain the displacements referred to the inertial frame.

Another method which could be used to determine the velocity and displacement of the missile is to use equation 41 to transform the accelerations in equation 47 to the inertial frame. The accelerations would then be referred to a non-rotating frame and could be integrated to obtain the velocity and displacement without the addition of any cross product terms. For missile simulation, the velocity of the missile referred to the body frame is usually needed for aerodynamic calculations. This can be determined by using equation 37 to transform the velocity from the inertial frame to the body frame.

It is important to remember that when we speak of transforming a vector, we mean that the vector is expressed using a new set of unit vectors, and the vector itself has not been changed. Thus transforming a vector from the body frame to the inertial frame is just using the components of the vector referred to the body frame and the transformation equation (equation 41) to find the components of the same vector referred to the inertial frame.

## 2. Translational Equations for a Moving Frame

In many cases it is desired to obtain the location and velocity of the missile with respect to a given point on the earth. This is especially important for inertially guided missiles, which need to know their location relative to the earth in order to arrive at the target point. For this case an earth fixed system is very convenient to work with, but as it is not an inertial system its movement must be considered. This can be done in the development of the missile equations of motion and is shown in the following.

In order to obtain the equations of motion for a missile in an earth fixed coordinate frame, equation 46 can be written in terms of the vector  $\bar{R}_1$  which gives the location of the missile with respect to the inertial frame. Thus

$$\bar{F}' = m\ddot{\bar{R}}_1 \quad (54)$$

where

$$\bar{R}_1 = \bar{R} + \bar{r} \quad (55)$$

and

$$\ddot{\bar{r}}_I = \ddot{\bar{R}} + \ddot{\bar{r}} \quad (56)$$

These vectors are shown in Figure 13. Since  $\bar{r}$  is referred to the rotating earth fixed frame, the second derivative of  $\bar{r}$  in equation 56 can be evaluated using equation 6.

$$\dot{\bar{r}} = (\dot{\bar{r}})_R + \bar{\omega} \times \bar{r} \quad (57)$$

and

$$\ddot{\bar{r}} = (\ddot{\bar{r}})_R + \dot{\bar{\omega}} \times \bar{r} + 2\bar{\omega} \times (\dot{\bar{r}})_R + \bar{\omega} \times (\bar{\omega} \times \bar{r}) \quad (58)$$

Now using equation 56 we have:

$$\ddot{\bar{r}}_I = (\ddot{\bar{r}})_R + \dot{\bar{\omega}} \times \bar{r} + 2\bar{\omega} \times (\dot{\bar{r}})_R + \bar{\omega} \times (\bar{\omega} \times \bar{r}) + \ddot{\bar{R}} \quad (59)$$

where  $\bar{\omega}$  is the angular velocity of the earth and  $\ddot{\bar{R}}$  is the acceleration of the earth fixed coordinate frame's origin. The equations of motion for an earth fixed coordinate frame can now be written as:

$$\bar{a} = \bar{F}'/m - \bar{\omega} \times (\bar{\omega} \times \bar{r}) - 2\bar{\omega} \times \bar{v} - \dot{\bar{\omega}} \times \bar{r} - \ddot{\bar{R}} \quad (60)$$

where  $\bar{v}$  is the velocity and  $\bar{a}$  is the acceleration of the missile referred to the earth fixed system. For an earth fixed frame,  $\bar{\omega}$  is constant and the acceleration of the coordinate frame's origin is only due to the earth's rotation. Thus equation 60 can be written as:

$$\bar{a} = \bar{F}'/m + \bar{\omega} \times (\bar{\omega} \times \bar{r}) - \bar{\omega} \times (\bar{\omega} \times \bar{r}) - 2\bar{\omega} \times \bar{v} \quad (61)$$

This is sometimes written as:

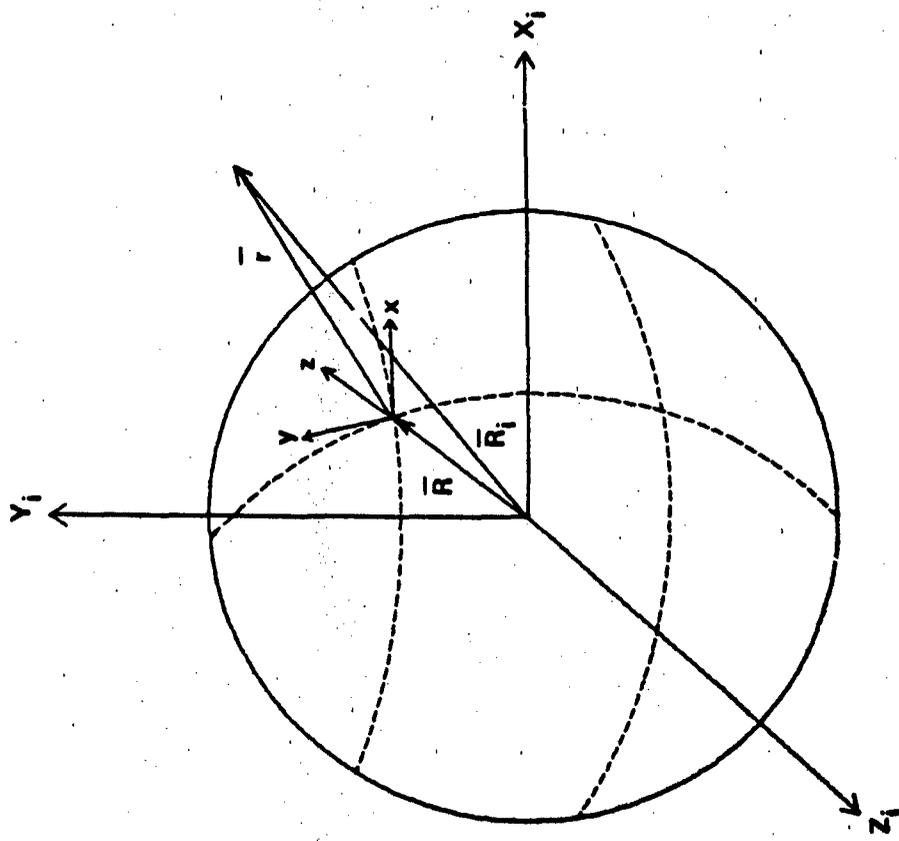


Figure 13. Relationship between inertial and moving coordinate frames.

$$\bar{a} = \bar{F}/m + \bar{g} + \bar{w} \times (\bar{w} \times \bar{R}) - \bar{w} \times (\bar{w} \times \bar{r}) - 2\bar{w} \times \bar{v} \quad (62)$$

where the forces have been split into gravitational (mass attraction) acceleration " $\bar{g}$ " and other external forces " $\bar{F}$ " (aero and thrust). For most missile simulations the distance flown is small enough that the term  $\bar{w} \times (\bar{w} \times \bar{r})$  in equation 62 is negligible and can be neglected without loss of accuracy.

The general equations of motion referred to an earth fixed coordinate frame are then given by:

$$\begin{aligned} A_x &= F_x/m + G_x - (W_y^2 + W_z^2)R_x + W_x W_y R_y + W_x W_z R_z - 2(W_y V_z - W_z V_y) \\ A_y &= F_y/m + G_y - (W_x^2 + W_z^2)R_y + W_x W_y R_x + W_z W_y R_z - 2(W_z V_x - W_x V_z) \\ A_z &= F_z/m + G_z - (W_x^2 + W_y^2)R_z + W_x W_z R_x + W_y W_z R_y - 2(W_x V_y - W_y V_x) \end{aligned} \quad (63)$$

where the subscripts x,y,z stand for the components along the x,y,z axes of the earth fixed frame. The coordinate transformation used should now relate the body and earth surface frames thus allowing the missile body forces or accelerations to be transformed to the earth fixed frame. Equations 63 can be integrated twice to obtain velocity and displacement of the missile referred to the earth fixed frame and the velocity of the missile referred to the missile body frame can be obtained using the inverse coordinate transformation.

A common earth fixed coordinate frame is the North-East-Down system shown in Figure 1. For this frame the earth radius is in the down or z direction thus  $R_x$  and  $R_y$  are zero and  $W_y$  is zero as the east direction is perpendicular to the rotation vector. Thus equation 63 can be reduced to:

$$\begin{aligned} A_x &= F_x/m + G_x + W_x W_z R_z + 2W_z V_y \\ A_y &= F_y/m + G_y - 2(W_z V_x - W_x V_z) \\ A_z &= F_z/m + G_z - W_x^2 R_z - 2W_x V_y \end{aligned} \quad (64)$$

Now noting that the earth's rotation rate is 0.000073 rad/sec and the earth's radius is approximately 6,371,000 m, the  $\bar{w} \times (\bar{w} \times \bar{R})$  (centrifugal acceleration) terms in equation 64 can be estimated as follows.

$$\begin{aligned}
W_x W_z R_z &= - W^2 \cos(\text{lat}) \sin(\text{lat}) * R \\
&= - (0.000073)^2 * \cos(\text{lat}) \sin(\text{lat}) * 6,371,000 \\
&= - .034 \cos(\text{lat}) \sin(\text{lat}) \text{ m/s}^2
\end{aligned}
\tag{65}$$

$$\begin{aligned}
W_x^2 R_z &= W^2 \cos^2(\text{lat}) * R \\
&= .034 \cos^2(\text{lat}) \text{ m/s}^2
\end{aligned}
\tag{66}$$

Thus the centrifugal acceleration terms are small and may be neglected for many cases depending upon the desired accuracy of the simulation. This also justifies neglecting the  $\bar{w} \times (\bar{w} \times \bar{r})$  term in equation 62 for cases where the range of the missile " $\bar{r}$ " is much less than the earth's radius " $\bar{R}$ ". The coriolis ( $2\bar{w} \times \bar{v}$ ) terms may also be estimated if the missile velocity is known. If we assume that the maximum missile velocity is Mach 2 or approximately 670 m/s, then the coriolis accelerations will be on the order of:

$$2WV = 2 * 0.000073 * 670 = .1 \text{ m/s}^2 \tag{67}$$

Thus coriolis acceleration can also be neglected for many cases, but becomes more important for the faster missiles. Earlier it was stated that one of the requirements for assuming the earth fixed frame to be an inertial frame was that the velocity of the missile must be much smaller than the velocity of escape from the earth. As the escape velocity is approximately 11,200 m/s, this can be seen to be a good approximation. From this simple analysis, it can be seen that equations 63 can be simplified to:

$$\begin{aligned}
A_x &= F_x/m + G_x \\
A_y &= F_y/m + G_y \\
A_z &= F_z/m + G_z
\end{aligned}
\tag{68}$$

for many cases involving short range, slower flying missiles which do not use inertial guidance.

From the above calculations, it might be thought that the coriolis and centrifugal acceleration can be neglected for all flights, but it must be remembered that even small errors in acceleration can become significant if they are constant over a long time. To get distance these terms are integrated twice which gives a time squared multiplier. Thus the distance curves for a constant error are parabolic as shown in Figures 14 and 15. The error in displacement due to the centrifugal acceleration at a latitude of 30 degrees (calculated from equation 66) is shown in Figure 14. In Figure 15 the error in displacement due to coriolis acceleration is shown for

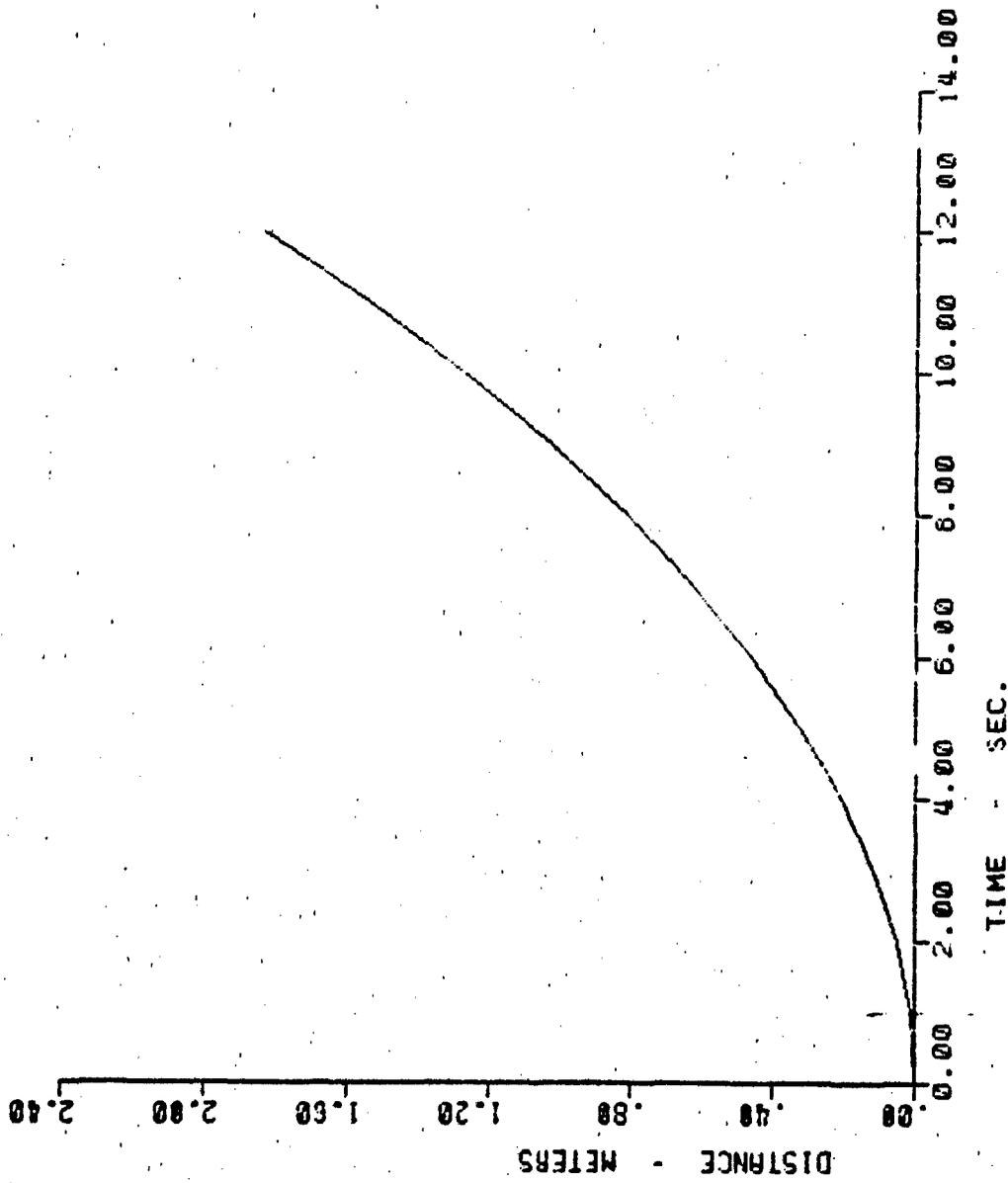


Figure 14. Displacement error due to neglecting centrifugal acceleration terms.

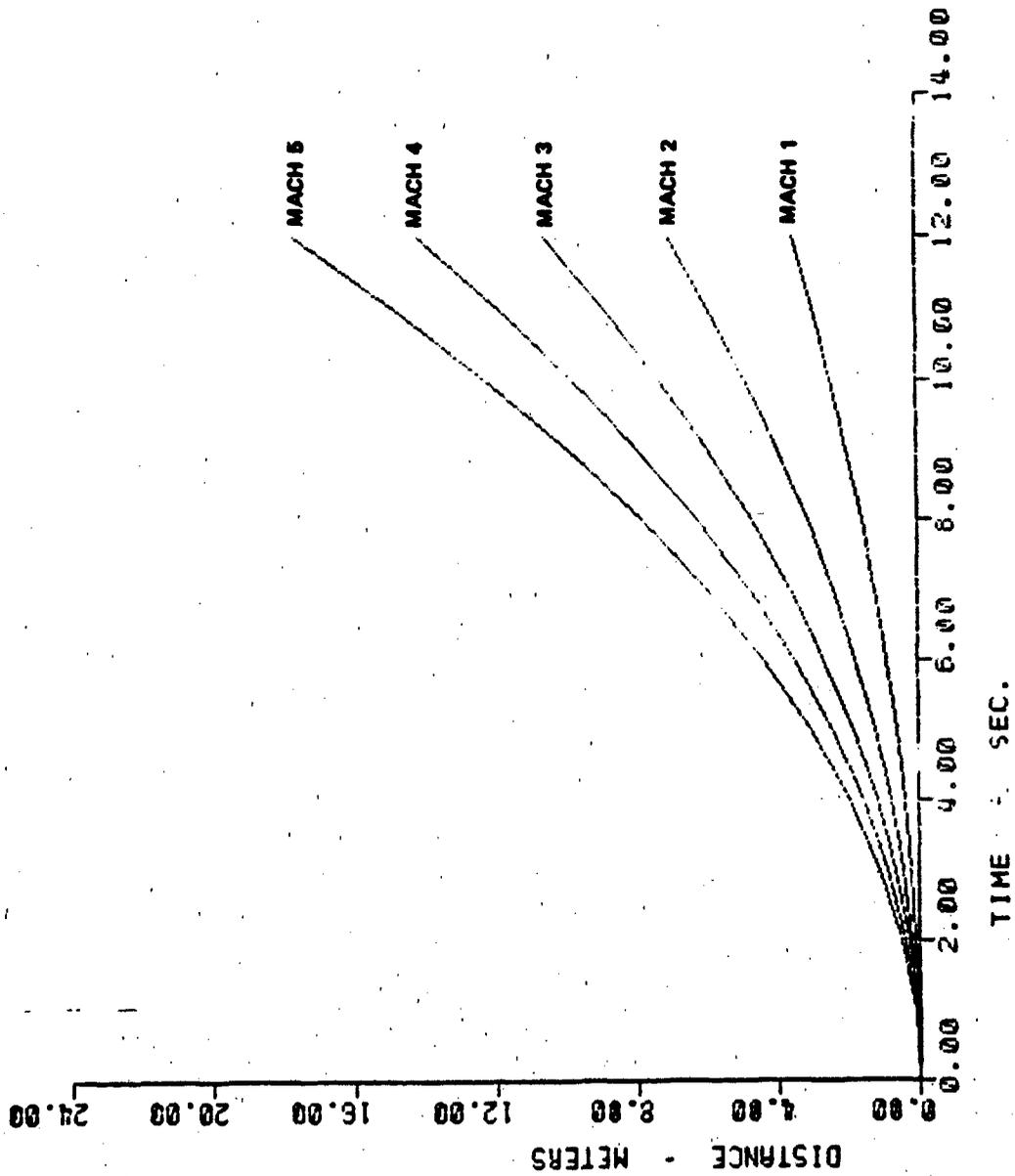


Figure 15. Displacement error due to neglecting coriolis acceleration terms.

missiles flying at constant velocities at 0 degrees latitude. Note that these are only approximations to give the reader an idea of the magnitude of the errors involved.

### III. SUMMARY

In order to clarify some of the preceding, a short summary of the possible choices of equations is presented below. Note that these are not the only options available, but they are some of the most popular. For each of these choices, a body axis system and an inertial or earth fixed coordinate system are needed and should be defined first along with their relative orientations. The moments and forces (or accelerations) along with the mass and moments of inertia are assumed to be known in the body system.

#### A. Rotational Equations

The rotational equations (equations 19 or 30) are the same for each choice and are used to obtain the missile's angular rates (P,Q,R) from which the transformation matrix relating the inertial and body coordinate systems can be developed. Thus

$$\begin{aligned}
 M_x &= I_{xx}\dot{P} + I_{xy}(\dot{Q} - PR) + I_{xz}(\dot{R} + PQ) \\
 &\quad + (I_{zz} - I_{yy})QR + I_{yz}(Q^2 - R^2) \\
 M_y &= I_{yy}\dot{Q} + I_{yz}(\dot{R} - PQ) + I_{xy}(\dot{P} + QR) \\
 &\quad + (I_{xx} - I_{zz})PR + I_{xz}(R^2 - P^2) \\
 M_z &= I_{zz}\dot{R} + I_{xz}(\dot{P} - QR) + I_{yz}(\dot{Q} + PR) \\
 &\quad + (I_{yy} - I_{xx})PQ + I_{xy}(P^2 - Q^2)
 \end{aligned} \tag{19}$$

or for a missile with a high degree of symmetry, Euler's equations of motion may be used:

$$\begin{aligned}
 M_x &= I_{xx}\dot{P} + (I_{zz} - I_{yy})QR \\
 M_y &= I_{yy}\dot{Q} + (I_{xx} - I_{zz})PR \\
 M_z &= I_{zz}\dot{R} + (I_{yy} - I_{xx})PQ
 \end{aligned} \tag{30}$$

There are several alternatives available for solving the translational equations. These are presented below.

## B. Translational Equations in the Inertial System

One of the simplest methods is to transform the missile acceleration components from the body coordinate frame to the inertial frame and then integrate to obtain velocity and displacement referred to the inertial axes system. The velocity referred to the missile body frame can then be obtained by using the inverse transformation. For this case the body accelerations to be transformed and integrated would be those sensed by the missile's accelerometers plus gravitational acceleration due to mass attraction.

$$\begin{aligned}A_x &= F_x/m + G_x \\A_y &= F_y/m + G_y \\A_z &= F_z/m + G_z\end{aligned}\tag{69}$$

The terms in equations 69 are referred to the inertial coordinate frame where the "F/m" terms are equivalent to the accelerations measured by the missile's accelerometers once they have been converted to the inertial frame and the "G" terms are gravitational acceleration (mass attraction).

## C. Translational Equations in the Body System

Another simple method is to use equations 53 to calculate the time derivative of velocity referred to the missile body axes system.

$$\begin{aligned}\dot{V}_{bx} &= F_x/m + G_x - QW + RV \\ \dot{V}_{by} &= F_y/m + G_y - RU + PW \\ \dot{V}_{bz} &= F_z/m + G_z - PV + QU\end{aligned}\tag{53}$$

These can then be integrated to obtain the missile velocity referred to the body frame. As displacement is usually desired in the inertial or an earth fixed frame, the velocity components obtained are generally transformed to the inertial frame and integrated directly to obtain the displacements referred to the inertial frame axes system.

Note that  $F_x/m$ ,  $F_y/m$ , and  $F_z/m$  would be the accelerations measured by an accelerometer triad collocated with the missile body coordinate frame and that  $G_x$ ,  $G_y$ , and  $G_z$  would be the acceleration due to gravity (mass attraction) along these axes.

## D. Translational Equations in an Earth Fixed System

In order to obtain the missile's motion with respect to an earth fixed frame, equations 63 can be used.

$$\begin{aligned}
A_x &= F_x/m + G_x - (W_y^2 + W_z^2)R_x + W_xW_yR_y + W_xW_zR_z - 2(W_yV_z - W_zV_y) \\
A_y &= F_y/m + G_y - (W_x^2 + W_z^2)R_y + W_xW_yR_x + W_zW_yR_z - 2(W_zV_x - W_xV_z) \\
A_z &= F_z/m + G_z - (W_x^2 + W_y^2)R_z + W_xW_zR_x + W_yW_zR_y - 2(W_xV_y - W_yV_x)
\end{aligned} \tag{63}$$

or for the specific case where the earth fixed frame is a North-East-Down coordinate frame, these can be reduced to:

$$\begin{aligned}
A_x &= F_x/m + G_x + W_xW_zR_z + 2W_zV_y \\
A_y &= F_y/m + G_y - 2(W_zV_x - W_xV_z) \\
A_z &= F_z/m + G_z - W_xR_z - 2W_xV_y
\end{aligned} \tag{64}$$

Note that all the terms in equations 63 and 64 are referred to the earth fixed system. Thus for this case  $F_x/m$ ,  $F_y/m$ , and  $F_z/m$  are the accelerations that would be obtained from the missile's accelerometers after transforming them to the earth fixed coordinate frame. Gravitational acceleration due to mass attraction is included as a separate term. These accelerations can be integrated twice to obtain the velocity and the displacement of the missile referred to the earth fixed frame.

#### E. Translational Equations with the Earth Fixed System Assumed to be an Inertial System

A somewhat simplified case that works for some of the shorter range, slower, non-inertially guided missiles is to reduce equations 63 to

$$\begin{aligned}
A_x &= F_x/m + G_x \\
A_y &= F_y/m + G_y \\
A_z &= F_z/m + G_z
\end{aligned} \tag{68}$$

by neglecting the centrifugal and coriolis acceleration terms. Note that again the terms in these equations are referred to the earth fixed system. Thus for this case  $F_x/m$ ,  $F_y/m$ , and  $F_z/m$  are once again the accelerations that would be obtained from the missile's accelerometers after transforming them to the earth fixed coordinate frame and "G" is gravitational acceleration due to mass attraction. This method is equivalent to alternative B with the earth fixed system taken as the inertial frame. Thus equations 68 can be integrated twice to obtain the velocity and displacement of the missile referred to the earth fixed frame.

F. Translational Equations in the Body System with the Earth Fixed System Assumed to be an Inertial System

Since alternatives B and E are equivalent if the earth fixed frame is assumed to be an inertial frame, an alternative equivalent to alternative C is also possible if the same simplifications used in alternative E are used. Thus the equations of alternative C are again obtained where this time the earth fixed frame is assumed to be an inertial frame.

$$\begin{aligned}\dot{V}_{bx} &= F_x/m + G_x - QW + RV \\ \dot{V}_{by} &= F_y/m + G_y - RU + PW \\ \dot{V}_{bz} &= F_z/m + G_z - PV + QU\end{aligned}\tag{70}$$

These can then be integrated to obtain the missile velocity referred to the body frame. The velocity obtained can then be transformed to the earth fixed frame and integrated to obtain the displacement referred to the earth fixed frame. Note that  $F_x/m$ ,  $F_y/m$ , and  $F_z/m$  are the accelerations that would be measured by an accelerometer triad collocated with the missile axis system and "G" is the acceleration due to gravity (mass attraction). The terms in equations 70 are referred to the missile body coordinate frame.

IV. CONCLUDING REMARKS

The missile dynamics equations have been developed using the basic laws in such a manner as to emphasize their similarity. Basically, the form of the equations is fixed by the choice of the coordinate system in which the integration is performed, with the critical factor being the movement (rotation) of that coordinate system relative to the inertial system. The use of the equation for the derivative of a vector referred to a rotating frame (equation 6) is then of fundamental importance, and must be understood in order to gain an intuitive knowledge of the dynamics involved. Many other key factors to the understanding of the missile dynamics equations have also been discussed and a thorough understanding of these will greatly enhance the practicing engineer's insight. This then forms the basis of this report which is intended to be both a primer for new engineers and a reference to experienced engineers involved in missile flight simulation.

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APPENDIX

The approach used in Section III, part B can be extended to obtain the displacement of the missile referred to the missile body frame. This is not usually done as the displacement is easier to visualize when related to an inertial or earth fixed frame, but will be done here in order to complete the discussion.

Equation 46 can be written in terms of the displacement vector " $\bar{R}$ " which is referred to the body frame by using equation 6 twice. Thus

$$\dot{\bar{R}} = (\dot{R})_r + \bar{\omega} \times \bar{R} \quad (71)$$

and

$$\ddot{\bar{R}} = (\ddot{R})_r + \dot{\bar{\omega}} \times \bar{R} + 2\bar{\omega} \times (\dot{R})_r + \bar{\omega} \times (\bar{\omega} \times \bar{R}) \quad (72)$$

Then rewriting equation 46 in terms of  $\bar{R}$  gives:

$$\bar{F}' = m\ddot{\bar{R}} \quad (73)$$

or

$$\bar{F}'/m = (\ddot{R})_r + \dot{\bar{\omega}} \times \bar{R} + 2\bar{\omega} \times (\dot{R})_r + \bar{\omega} \times (\bar{\omega} \times \bar{R}) \quad (74)$$

The acceleration referred to the missile body coordinate frame is then given by:

$$(\ddot{R})_r = \bar{F}'/m - \dot{\bar{\omega}} \times \bar{R} - 2\bar{\omega} \times (\dot{R})_r - \bar{\omega} \times (\bar{\omega} \times \bar{R}) \quad (75)$$

This equation can be integrated twice to obtain the displacement " $\bar{R}$ " referred to the missile body coordinate frame. It is interesting to note, however; that the first integration of equation 75 does not provide the velocity referred to the missile body frame. This can be better understood by examining the equation for the velocity derivative referred to the body frame which is:

$$(\dot{V})_r = \dot{\bar{V}}/m - \bar{\omega} \times \bar{V} \quad (76)$$

From equation 6,  $\bar{V}$  is found to be:

$$\bar{V} = (\dot{\bar{R}})_r + \bar{\omega} \times \bar{R} \quad (77)$$

Now writing equation 76 in the form:

$$\bar{F}'/m = (\ddot{\bar{V}})_r + \bar{\omega} \times \bar{V} \quad (78)$$

and substituting equation 77 for  $\bar{V}$  gives:

$$\bar{F}'/m = (\ddot{\bar{R}})_r + \dot{\bar{\omega}} \times \bar{R} + 2\bar{\omega} \times (\dot{\bar{R}})_r + \bar{\omega} \times (\bar{\omega} \times \bar{R}) \quad (79)$$

which is identical to equation 74. Thus equations 74 and 78 are equivalent and comparing terms in the equations reveals that:

$$(\dot{\bar{V}})_r = (\ddot{\bar{R}})_r + \dot{\bar{\omega}} \times \bar{R} + \bar{\omega} \times (\dot{\bar{R}})_r \quad (80)$$

and

$$\bar{\omega} \times \bar{V} = \bar{\omega} \times (\dot{\bar{R}})_r + \bar{\omega} \times (\bar{\omega} \times \bar{R}) \quad (81)$$

From equation 80, it is apparent that the derivative of  $\bar{V}$  does not equal the second derivative of  $\bar{R}$  when both are referred to the moving coordinate frame axis system. Thus integrating equations 75 and 76 will give different results for the velocity.

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