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19. ABSTRACT (Continue on reverse if necessary and identify by block number) The mathematical modelling of the two dimensional compressible flow over an open rectangular cavity is considered. The flow field is divided into three regions: (1) the two-dimensional compressible flow outside the cavity which is governed by a convective wave equation; (2) the two dimensional flow inside the cavity for which the mean flow is neglected and thus is governed by the wave equation; and, (3) the shear layer spanning the cavity which is the major focus of the investigation. The work of Tam and Block (1) is followed and a method for determining the discrete frequencies of shear layer oscillation is determined including shear layer thickness effects and assuming that the shear layer grows downstream. A line source placed at the trailing edge is used to model the trailing edge interaction process and the analysis of a shear layer excited by a reflected wave of arbitrary frequency is considered. The mean flow in the shear layer is nonuniform, thus standard stability procedures do not work. The nonparallel effects are assumed to be weak. The method of multiple scales is			
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used to analyze the nonparallel stability of the shear layer. The stability characteristics of the shear layer are used in an equation developed by Tam and Block to predict discrete oscillation frequencies. The numerical methods for solving the equations are discussed.

nonparallel shear layer scales.
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1. Tam, C. D. and Block, P.J.W., "On the Tones and Pressure Oscillations Induced by Flows over Rectangular Cavities", Journal of Fluid Mechanics, 69, 373-399.

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AN INVESTIGATION OF COMPRESSIBLE FLOWS OVER OPEN
CAVITIES INCLUDING SHEAR LAYER THICKNESS EFFECTS

by

S. Graham Kelly

The University of Akron
Department of Mechanical Engineering
Akron, OH 44325

Approved for public release;
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TABLE OF CONTENTS

	<u>Page</u>
Nomenclature	2
I. Introduction	4
II. Problem Formulation	9
III. Shear Layer Modelling	13
IV. Numerical Solution of Shear Layer Equation	31
V. Discussion	35
References	39

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFOSR)
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NOMENCLATURE

ϵ = measure of non-parallelism

Independent Coordinates

t = time

x = horizontal spatial coordinate

y = vertical spatial coordinate

$x_1 = \epsilon x$

Mean Flow Variables

$U(x_1, y) = U_0(x_1, y) + \epsilon U_1(x_1, y)$ = component of velocity

$V(x_1, y) = \epsilon V_1(x_1, y)$ = y component of velocity

$P(x_1, y) = P_0(x_1) + \epsilon P_1(x_1, y)$ = pressure

$R(x_1, y) = R_0(x_1, y) + \epsilon R_1(x_1, y)$ = density

$a(x_1, y)$ = speed of sound

c_p = specific heat

Perturbation Variables

$\tilde{u}(x_1, y, t) = (a_0(x_1, y) + \epsilon a_1(x_1, y))e^{i\theta}$ = x component of velocity

$\tilde{v}(x_1, y, t) = (b_0(x_1, y) + \epsilon b_1(x_1, y))e^{i\theta}$ = y component of velocity

$\tilde{\rho}(x_1, y, t) = (c_0(x_1, y) + \epsilon c_1(x_1, y))e^{i\theta}$ = density

$\tilde{p}(x_1, y, t) = (d_0(x_1, y) + \epsilon d_1(x_1, y))e^{i\theta}$ = pressure

$\tilde{s}(x_1, y, t) = (f_0(x_1, y) + \epsilon f_1(x_1, y))e^{i\theta}$ = entropy

Perturbation Variables (Cont'd)

$$\theta = k(x_1) - \omega t$$

$$\omega = \text{frequency}$$

$$k(x_1) = \text{wave function}$$

Others

$$[L] = \text{matrix operator}$$

$$[L]^* = \text{adjoint matrix operator}$$

$$H(\) = \text{Hankel function}$$

$$i = \sqrt{-1}$$

$$\beta, \Delta = \text{intermediate functions of } k$$

$$\phi = \text{vector of perturbation components}$$

$$\eta = \text{shear layer deflection}$$

$$\mu(x_1), \nu(x_1) = \text{intermediate functions of } x_1$$

$$(\)^* = \text{adjoint function}$$

$$(\tilde{\ }) = \text{perturbation quantity}$$

$$(\hat{\ }) = \text{perturbation amplitude}$$

$$(\)^T = \text{matrix transpose}$$

$$(\)_+ = \text{variable for flow outside cavity}$$

$$(\)_- = \text{variable for flow inside cavity}$$

$$L = \text{cavity length}$$

$$D = \text{cavity depth}$$

$$M = \text{free stream Mach number}$$

I. INTRODUCTION

The problem of a compressible flow over an open cavity is one of great practical importance. Examples of such flows are those over aircraft wheel wells and weapons bays. Bartel and McAvoy (1981) experimentally measured sound pressure levels as high as 170 dB in a weapons bay environment. Levels this high can lead to structural failure and will cause extreme personal discomfort. Thus, methods of reducing sound pressure levels demand immediate attention. The first step in this reduction is to obtain a useful analytical model of the compressible flow over an open cavity.

Open cavity flows have been investigated analytically and experimentally. However, many analytical models have severe limitations on their range of application or do not match experimental data. Some of the best analytical models are, in fact, semi-empirical.

Karamcheti (1955) performed one of the first experimental investigations on a simulated weapons bay. He obtained discrete frequency radiation for both subsonic and supersonic mean flows. This result has been verified by other investigators including Gibson (1958), Spee (1966), East (1966), and Smith and Shaw (1974), among others.

The first significant analytical study of open cavity flows was that of Plumblee, Gibson, and Lassiter (1962). They identified acoustic resonances of the cavity and speculated that the observed discrete frequencies are identical to the resonant cavity frequencies. They also suggested that the driving mechanism of cavity oscillations is the turbulent shear layer spanning the cavity. Their result is in conflict with the experimental observations of Karamcheti (1955) and Rossiter (1964) whose experiments indicate that larger pressure oscillations are obtained for laminar boundary layers.

Based upon his own experimental observations, Rossiter (1964) proposed

a vortex shedding model to explain the generation of cavity tones. He was then able to derive a semi-empirical formula for the tone frequencies. In addition to Rossiter, Smith and Shaw (1974) and Bartel and McAvoy (1981) have developed semi-empirical models of open cavity flows based upon their experimental observations. These models do reflect the data obtained by the respective investigators, but are limited in application. For example, the validity of Rossiter's model appears to be limited to the Mach number range $0.4 < M < 1.2$.

Analytical efforts at modelling open cavity flows have increased within the past 15 years. Covert (1970) classified cavities as shallow or deep depending upon their length to depth ratio. He argued that vorticity tends to excite the modes in the direction of the greatest physical dimension.

Bilanin and Covert (1973) developed an analytical model of a shallow cavity using an acoustic monopole to model the trailing edge behavior. Their model is based upon a feedback mechanism first proposed by Rossiter (1964) where the pressure field of the monopole drives the shear layer. The cavity floor influences the shear layer by changing the amplification rate of the shear layer motion and thus, changing the excitation frequency. The model predicts excitation frequencies for $M > 1$. Block (1976) extended the work of Bilanin and Covert to include effects of length to depth ratio.

The appropriate trailing edge behavior has created some controversy. Heller and Bliss (1975) analyzed the wave motion of a thin shear layer over a boundary. They superimposed solutions for waves moving between a shear layer and solid boundary to approximate an oscillating cavity. They modelled the trailing edge behavior with a moving piston to simulate an entrainment process. Heller and Bliss argue that while the feedback mechanism model of Bilanin and Covert will predict excitation frequencies it will not predict

whether they will occur. They believe that an entrainment model is superior as mass addition and subtraction must occur at the trailing edge as the shear layer deflects. Results from their model compare favorably with results from their water table tests, but they are unable to predict discrete frequencies. Heller and Bliss also examined methods of oscillation suppression using vortex spoilers.

Tam and Block (1978), in a study directed toward open wheel well noise, analyzed the open cavity problem using a feedback model significantly different from that of Bilanin and Covert. However, their model is consistent with the water table observations of Heller and Bliss. The work of Tam and Block is important because it was, until this study, the only analytical cavity study to include shear layer thickness effects. They used a mean velocity profile for the shear layer the same as that of a two-dimensional free turbulent mixing layer near the trailing edge of a thin flat plate and assumed the shear layer to be of constant momentum thickness. Their results compare well with the experimental data of Rossiter. Tam and Block also show that shear layer instabilities could be the origin of acoustic energy which produces cavity resonances.

The above survey of literature led the author to believe that a unified analytical theory of open cavity flows is lacking. Some theories apply only over certain ranges of significant parameters while others ignore shear layer thickness effects which have been found to be extremely important. In view of the work of Tam and Block the author also decided that a more detailed analysis of the shear layer including thickness effects is necessary. Indeed, Bartel and McAvoy recommend a thorough mathematical description of the unsteady pressure distribution in the shear layer. Thus, the author proposed [Kelly (1982)] an investigation of open cavity flows including shear layer

thickness effects. This report is to present the results of this investigation.

The author decided that the work of Tam and Block was a significant contribution to the analysis of the shear layer spanning an open cavity. Their results are valid for a wide range of Mach numbers and length to depth ratios. In addition, their work was the first to include shear layer thickness effects. They analyzed a shear layer with a constant momentum thickness, but noted that due to entrainment the thickness of the shear layer increases in the downstream direction. They also used their shear layer instability analysis to predict excitation frequencies from an equation derived assuming a thin vortex sheet. These equations are summarized in the next section.

From his preliminary research the author realized that a mathematical investigation of the shear layer including all desired effects would be very involved and would exceed the scope of the project. Thus, it was decided to focus on a specific aspect of the shear layer based upon Tam's analysis. The author recognized that there are two major drawbacks to the work of Tam and Block: (1) The shear layer thickness was assumed constant across the length of the cavity and, (2) The eigenvalue relation they use should be modified to include thickness effects. Actually, the modification of their eigenvalue relation depends upon reanalyzing the flow above and below the shear layer assuming a shear layer of finite thickness. Thus, it was decided to modify the work of Tam and Block to handle a shear layer whose thickness increases in the downstream direction.

Section II presents a statement of the problem under consideration as well as an overview of the work of Tam and Block.

Section III presents the mathematical analysis of a compressible inviscid shear layer whose thickness increases in the downstream direction. The

analysis is based upon non-parallel stability theory using the method of multiple scales.

Section IV presents a scheme for numerical implementation of the analysis in Section III.

Section V discusses the method and presents recommendations for further study.

II. Problem Formulation

The objective of the research is to provide a mathematical formulation of the shear layer over an open cavity including thickness effects. A side objective is to evaluate the feasibility of using a mathematical analysis of the shear layer in a finite element formulation to analyze the flow in all regions around the cavity.

A literature survey indicated that the only study including shear layer thickness effects was made by Tam and Block (1978). They first analyzed a thin shear layer and obtained a frequency-wave number relationship. This relationship was used to predict, from a condition derived from a trailing edge boundary condition, the discrete shear layer oscillation frequencies. They then analyzed a shear layer of finite thickness in the same manner. Their assumptions were:

1. The flow is inviscid.
2. No mean flow exists within the cavity.
3. The shear layer is of constant momentum thickness.
4. The mean flow in the shear layer is parallel.
5. A feedback mechanism occurs due to the deflection of the shear layer into the cavity. Reflected waves excite the shear layer.

Tam and Block were able to predict, with favorable comparison to experimental data, discrete oscillation frequencies. The author decided that a first step toward a thorough mathematical analysis of the shear layer would be to improve the analysis of Tam and Block by eliminating assumptions (3) and (4). Indeed the shear layer thickness increases downstream due to entrainment. Thus a shear layer whose thickness increases downstream was analyzed using the work of Tam and Block as a guide. The essential features of their work is presented below. The current research is described in Section III.

The flow around the cavity is divided into three regions: (1) The flow inside the cavity; (2) The flow outside the cavity; and, (3) The shear layer between these two regions. The equations governing the flow inside and outside the cavities are

$$\frac{1}{a_+^2} \left(\frac{\partial}{\partial t} + \bar{U}_\infty \frac{\partial}{\partial x} \right)^2 p_+ - \nabla^2 p_+ = 0 \quad (2.1)$$

$$\rho_+ \left(\frac{\partial v_+}{\partial t} + \bar{U}_\infty \frac{\partial v_+}{\partial x} \right) = - \frac{\partial p_+}{\partial y} \quad (2.2)$$

$$\frac{1}{a_-^2} \frac{\partial^2 p_-}{\partial t^2} - \nabla^2 p_- = 0 \quad (2.3)$$

$$\rho_- \frac{\partial v_-}{\partial t} = - \frac{\partial p_-}{\partial y} \quad (2.4)$$

Where a + subscript refers to quantities outside the cavity and a - subscript refers to quantities inside the cavity.

The boundary conditions at $y = 0$ are

$$\left(\frac{\partial}{\partial t} + \bar{U}_\infty \right) \eta = v_+ \quad (2.5)$$

$$\frac{\partial \eta}{\partial t} = v_- + A \delta(x-\xi) e^{-i\omega t} \quad (2.6)$$

$$p_+ = p_- + B \delta(x-\xi) e^{-i\omega t} \quad (2.7)$$

Where η is the displacement of the shear layer, A and B are constants and ξ is the point where the shear layer is excited. Tam and Block use a periodic line source at the trailing edge to simulate the trailing edge interaction process. They calculated the pressure field inside and outside of the cavity due to the line source and the resulting loading on the shear layer due to the differences in pressure and velocity above and below the shear layer. They are then able to obtain a relationship between the shear layer deflection at the trailing edge and the source strength. The pressure attains a maximum

value when the shear layer deflection is most negative. Using this they derive a condition which defines the shear layer frequencies. That is

$$\text{Phase } \psi = 2\pi n \quad n = 1, 2, 3, \dots$$

where

$$\psi = \frac{1}{2\pi} \frac{\beta_+ \beta_-}{\frac{\partial \Delta}{\partial k}} e^{ikL} \int_0^L f(\xi) e^{-ik\xi} d\xi \quad (2.8)$$

where

$$f(\xi) = H_0^{(1)} \left[\frac{\omega}{a_-} (L-\xi) \right] - H_0^{(1)} \left[\frac{\omega}{a_+} \frac{L-\xi}{1-M^2} \right]$$

$$e^{i\omega \frac{M(L-\xi)}{a_+(1-M^2)}} + H_0^{(1)} \left[\frac{\omega}{a_+} (L+\xi) \right]$$

$$+ H_0^{(1)} \left[\frac{\omega}{a_-} ((L-\xi)^2 + 4D^2)^{1/2} \right] - \frac{\omega}{a_- B_-} \frac{2D}{((L-\xi)^2 + 4D^2)^{1/2}}$$

$$H_1^{(1)} \left[\frac{\omega}{a_-} ((L-\xi)^2 + 4D^2)^{1/2} \right]$$

$$\beta_+ = \left[k^2 - \frac{1}{a_+^2} (\omega - U_\infty k)^2 \right]^{1/2}$$

$$\beta_- = \left[k^2 - \omega^2 / a_-^2 \right]^{1/2}$$

$$\Delta = \rho_+ a_+^2 (\beta_- + \beta_+) (k^2 - \beta_+ \beta_-)$$

The frequency-wave number analysis is determined from hydrodynamic stability theory. Equation (2.8) is then used to determine the discrete frequencies.

Simply stated, the problem at hand is to develop a technique to predict the discrete shear layer oscillation frequencies for a shear layer whose thickness varies with x . The analysis is based upon hydrodynamic stability theory. A weakly non-parallel mean flow is perturbed. The method of multiple scales is used to predict an eigenvalue problem which governs the wave number

frequency relationship. The wave number is a function of x , $k(x)$. The eigenvalue problem is solved numerically by finite-difference methods. Thus a numerical relationship between $k(L)$ and ω is determined. Equation (2.8) is used to determine the discrete shear layer oscillation frequencies. The method provides a correction to the technique of Tam and Block to account for varying shear layer thickness effects. A discussion follows the analysis to indicate modifications that should be made to the procedure.

The assumptions made in the analysis are:

- (1) The flow is inviscid.
- (2) The flow is two-dimensional.
- (3) No mean flow exists in the cavity.
- (4) Equation (2.8) is valid for this problem.
- (5) The specific heat is constant across the thickness of the shear layer.

The shear layer spanning the cavity is modelled by a two dimensional inviscid compressible layer which grows in the downstream direction. The stability characteristics of the shear layer are considered when it is excited by a reflected wave of frequency ω .

The equations governing a two dimensional compressible inviscid flow are:

Mass

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial y} + v \frac{\partial \rho}{\partial y} = 0 \quad (3.1)$$

x-Momentum

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + \frac{\partial p}{\partial x} = 0 \quad (3.2)$$

y-Momentum

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) + \frac{\partial p}{\partial y} = 0 \quad (3.3)$$

Entropy Production

$$\frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} + v \frac{\partial s}{\partial y} = 0 \quad (3.4)$$

Thermodynamics Relation for an Ideal Gas

$$d\rho = a^2 d\rho + \frac{\rho a^2}{c_p} ds \quad (3.5)$$

The mean flow is assumed to be a steady two-dimensional flow.

Thus, the mean flow quantities (U, V, P, R, S) satisfy

$$R \frac{\partial U}{\partial x} + U \frac{\partial R}{\partial x} + R \frac{\partial V}{\partial y} + V \frac{\partial R}{\partial y} = 0 \quad (3.6)$$

$$R U \frac{\partial U}{\partial x} + R V \frac{\partial V}{\partial y} + \frac{\partial P}{\partial x} = 0 \quad (3.7)$$

$$R V \frac{\partial V}{\partial x} + R V \frac{\partial V}{\partial y} + \frac{\partial P}{\partial y} = 0 \quad (3.8)$$

$$U \frac{\partial S}{\partial x} + V \frac{\partial S}{\partial y} = 0 \quad (3.9)$$

$$dP = a^2 dR + \frac{Ra^2}{c_p} dS \quad (3.10)$$

The flow variables in the shear layer are assumed to be the mean flow component plus a perturbation quantity. Thus

$$u(x, y, t) = U(x, y) + \tilde{u}(x, y, t) \quad (3.11)$$

$$v(x, y, t) = V(x, y) + \tilde{v}(x, y, t) \quad (3.12)$$

$$\rho(x, y, t) = R(x, y) + \tilde{\rho}(x, y, t) \quad (3.13)$$

$$p(x, y, t) = P(x, y) + \tilde{p}(x, y, t) \quad (3.14)$$

$$s(x, y, t) = S(x, y) + \tilde{s}(x, y, t) \quad (3.15)$$

Where a quantity with a tilde is a perturbation quantity and is assumed to be small in comparison with the mean flow quantities. Equations (3.11) - (3.15) are substituted into equations (3.1) - (3.5). Equations (3.6) - (3.10) are used to subtract out the basic state. The resulting equations are linearized by neglecting all terms which are nonlinear in the perturbation quantities. The resulting equations are:

Mass

$$\begin{aligned} \frac{\partial \tilde{\rho}}{\partial t} + R \frac{\partial \tilde{u}}{\partial x} + \tilde{\rho} \frac{\partial U}{\partial x} + U \frac{\partial \tilde{\rho}}{\partial x} + \tilde{u} \frac{\partial R}{\partial x} + R \frac{\partial \tilde{v}}{\partial y} + \tilde{\rho} \frac{\partial V}{\partial x} \\ + V \frac{\partial \tilde{\rho}}{\partial y} + \tilde{v} \frac{\partial R}{\partial y} = 0 \end{aligned} \quad (3.16)$$

x-Momentum

$$\begin{aligned} R \frac{\partial \tilde{u}}{\partial t} + R U \frac{\partial \tilde{u}}{\partial x} + R \tilde{u} \frac{\partial U}{\partial x} + \tilde{\rho} U \frac{\partial U}{\partial x} + R V \frac{\partial \tilde{u}}{\partial y} + R \tilde{v} \frac{\partial U}{\partial y} \\ + \tilde{\rho} V \frac{\partial U}{\partial y} + \frac{\partial \tilde{p}}{\partial x} = 0 \end{aligned} \quad (3.17)$$

y- Momentum

$$\begin{aligned}
 R \frac{\partial \tilde{v}}{\partial t} + R U \frac{\partial \tilde{v}}{\partial x} + R \tilde{u} \frac{\partial V}{\partial x} + \tilde{\rho} U \frac{\partial V}{\partial x} + R V \frac{\partial \tilde{v}}{\partial y} + R \tilde{v} \frac{\partial V}{\partial y} \\
 + \tilde{\rho} V \frac{\partial V}{\partial y} + \frac{\partial \tilde{p}}{\partial y} = 0
 \end{aligned}
 \tag{3.18}$$

Entropy Generation

$$\frac{\partial \tilde{s}}{\partial t} + U \frac{\partial \tilde{s}}{\partial x} + \tilde{u} \frac{\partial S}{\partial x} + V \frac{\partial \tilde{s}}{\partial y} + \tilde{v} \frac{\partial S}{\partial y} = 0
 \tag{3.19}$$

Thermodynamics Relation

$$\tilde{p} = R a^2 \left(\frac{\tilde{p}}{R} + \frac{\tilde{s}}{c_p} \right)
 \tag{3.20}$$

Equations (3.16) - (3.20) form a set of five equations for the five unknown perturbation quantities. Note that the mean flow is non-parallel. Thus the coefficients in the differential equations (3.16) - (3.20) are functions of both spatial variables, x and y . Standard parallel stability theory would suggest assuming

$$\tilde{u} = \hat{u}(y)e^{i(kx - \omega t)} \quad (3.21)$$

where k is the wave number, ω is the excitation frequency and $\hat{u}(y)$ is the amplitude function. However, an assumption of the form (3.21) is not appropriate in this problem as the coefficients are dependent on x .

Instead, an approximation to the solution of (3.16) - (3.20) is obtained by following the work of Saric and Nayfeh (1975). The non-parallel effects are assumed to be weak and can be represented by a small parameter ϵ . That is, whenever x appears in the mean flow quantities it appears in combination with ϵ as ϵx . In addition the non-parallel component of velocity V must be of order ϵ while all other mean flow quantities are of order 1. Thus it is consistent to introduce a slow scale defined by

$$x_1 = \epsilon x \quad (3.22)$$

Then all mean flow quantities are written as functions of x_1 and y . Substitution of (3.22) into (3.6) - (3.9) yields

$$\epsilon \left(R \frac{\partial U}{\partial x_1} + U \frac{\partial R}{\partial x_1} \right) + R \frac{\partial V}{\partial y} + V \frac{\partial R}{\partial y} = 0 \quad (3.23)$$

$$\epsilon \left(U \frac{\partial U}{\partial x_1} + \frac{\partial P}{\partial x_1} \right) + RV \frac{\partial U}{\partial y} = 0 \quad (3.24)$$

$$\epsilon RU \frac{\partial V}{\partial x_1} + RV \frac{\partial V}{\partial y} + \frac{\partial P}{\partial y} = 0 \quad (3.25)$$

$$\epsilon U \frac{\partial S}{\partial x_1} + V \frac{\partial S}{\partial y} = 0 \quad (3.26)$$

From equations (3.23) - (3.26) it is indeed apparent that

$$U = O(1), R = O(1), V = O(\epsilon), \frac{\partial P}{\partial y} = O(\epsilon), S = O(1)$$

Thus it is also convenient to expand the mean flow quantities in power series in ϵ . To this end

$$U(x_1, y) = U_0(x_1, y) + \epsilon U_1(x_1, y) + \dots \quad (3.27)$$

$$V(x_1, y) = \epsilon V_1(x_1, y) + \dots \quad (3.28)$$

$$R(x_1, y) = R_0(x_1, y) + \epsilon R_1(x_1, y) + \dots \quad (3.29)$$

$$P(x_1, y) = P_0(x_1) + \epsilon P_1(x_1, y) + \dots \quad (3.30)$$

$$S(x_1, y) = S_0(x_1, y) + \epsilon S_1(x_1, y) + \dots \quad (3.31)$$

The method of multiple scales [Nayfeh (1982)] is used to modify the assumed forms of the perturbation quantities from those of standard stability theory.

To this end let

$$\tilde{u}(x_1, y, t) = a(x_1, y) e^{i\theta(x_1, t)} \quad (3.32)$$

$$\tilde{v}(x_1, y, t) = b(x_1, y) e^{i\theta(x_1, t)} \quad (3.33)$$

$$\tilde{r}(x_1, y, t) = c(x_1, y) e^{i\theta(x_1, t)} \quad (3.34)$$

$$\tilde{p}(x_1, y, t) = d(x_1, y) e^{i\theta(x_1, t)} \quad (3.35)$$

$$\tilde{s}(x_1, y, t) = f(x_1, y) e^{i\theta(x_1, t)} \quad (3.36)$$

where

$$\frac{\partial \theta}{\partial t} = -\omega$$

$$\frac{\partial \theta}{\partial x_1} = k(x_1)$$

or

$$\theta = k(x_1) - \omega t \quad (3.37)$$

It is noted that both the amplitude and the wave number are assumed to be functions of the slow scale x_1 . It is also noted that the chain rule can be used to express derivatives with respect to x as

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_1} \frac{dx_1}{dx} + \frac{\partial}{\partial \theta} \frac{\partial \theta}{\partial x}$$

or

$$\frac{\partial}{\partial x} = \epsilon \frac{\partial}{\partial x_1} + k(x_1) \frac{\partial}{\partial \theta} \quad (3.38)$$

For example

$$\frac{\partial \tilde{u}}{\partial x} = \epsilon \frac{\partial}{\partial x_1} [a(x_1, y) e^{i\theta}] + k(x_1) \frac{\partial}{\partial \theta} [a(x_1, y) e^{i\theta}] = [\epsilon \frac{\partial a}{\partial x_1} + ika] e^{i\theta}$$

and

$$\frac{\partial U}{\partial x} = \epsilon \frac{\partial U}{\partial x_1}$$

It is also convenient to expand the amplitude functions in power series of ϵ . Thus

$$a(x_1, y) = a_0(x_1, y) + \epsilon a_1(x_1, y) + \dots \quad (3.39)$$

$$b(x_1, y) = b_0(x_1, y) + \epsilon b_1(x_1, y) + \dots \quad (3.40)$$

$$c(x_1, y) = c_0(x_1, y) + \epsilon c_1(x_1, y) + \dots \quad (3.41)$$

$$d(x_1, y) = d_0(x_1, y) + \epsilon d_1(x_1, y) + \dots \quad (3.42)$$

$$f(x_1, y) = f_0(x_1, y) + \epsilon f_1(x_1, y) + \dots \quad (3.43)$$

Equations (3.39) - (3.43) are substituted into equations (3.31) - (3.36).

The resulting equations are substituted into equations (3.16) - (3.20) making use of equation (3.28). Coefficients of like powers of ϵ are collected in each equation and set equal to zero independently. The results are summarized below

Order ϵ^0 :

Mass

$$i(kU_0 - \omega)c_0 + b_0 \frac{\partial R_0}{\partial y} + iR_0 ka_0 + R_0 \frac{\partial b_0}{\partial y} = 0 \quad (3.44)$$

x-Momentum

$$i(kU_0 - \omega)R_0 a_0 + R_0 b_0 \frac{\partial U_0}{\partial y} + ikd_0 = 0 \quad (3.45)$$

y-Momentum

$$i(kU_0 - \omega)R_0 b_0 + \frac{\partial d_0}{\partial y} = 0 \quad (3.46)$$

Entropy

$$i(kU_0 - \omega)f_0 + b_0 \frac{\partial S_0}{\partial y} = 0 \quad (3.47)$$

Thermodynamic Relation

$$d_0 = R_0 a^2 \left(\frac{c_0}{R_0} + \frac{f_0}{c_p} \right) \quad (3.48)$$

Order ϵ^1 :

Mass

$$\begin{aligned} & i(kU_0 - \omega)c_1 + b_1 \frac{\partial R_0}{\partial y} + ikR_0 a_1 + R_0 \frac{\partial b_1}{\partial y} \\ & = -a_0 \frac{\partial R_0}{\partial x_1} - U_0 \frac{\partial c_0}{\partial x_1} - ikc_0 U_1 - V_1 \frac{\partial c_0}{\partial y} \\ & - c_0 \frac{\partial U_0}{\partial x_1} - R_0 \frac{\partial a_0}{\partial x_1} - R_1 ika_0 - R_1 \frac{\partial b_0}{\partial y} - c_0 \frac{\partial V_1}{\partial y} \end{aligned} \quad (3.49)$$

x-Momentum

$$i(kU_0 - \omega)R_{01}a_1 + R_{01}b_1 \frac{\partial U_0}{\partial y} + ikd_1 = \quad (3.50)$$

$$i\omega R_{10}a_0 - R_{00}U \frac{\partial a_0}{\partial x_1} - R_{10}U_0ika_0 - R_{00}a_0$$

$$\frac{\partial U_0}{\partial x_1} - c_{00}U \frac{\partial U_0}{\partial x_1} - R_{00}U_0ika_0 - R_{01}V \frac{\partial a_0}{\partial y}$$

$$- R_{00}b_0 \frac{\partial U_1}{\partial y} - R_{10}b_0 \frac{\partial U_0}{\partial y} - c_{01}V \frac{\partial U_0}{\partial y} - \frac{\partial d_0}{\partial x_1}$$

y-Momentum

$$i(kU_0 - \omega)R_{01}b_1 + \frac{\partial d_1}{\partial y} = i\omega R_{10}b_0 - R_{00}U \frac{\partial b_0}{\partial x_1} \quad (3.51)$$

$$- R_{01}U_0ikb_0 - R_{10}U_0ikb_0 - R_{01}V \frac{\partial b_0}{\partial y} - R_{00}b_0 \frac{\partial V_1}{\partial y}$$

Entropy

$$i(kU_0 - \omega)f_1 + b_1 \frac{\partial S_0}{\partial y} = -U_0 \frac{\partial f_0}{\partial x_1} - U_1 ikf_0 \quad (3.52)$$

$$- a_0 \frac{\partial S_0}{\partial x_1} - V_1 \frac{\partial f_0}{\partial y} - b_0 \frac{\partial S_1}{\partial y}$$

Thermodynamic relation

$$d_1 = a^2 \left(c_1 + \frac{R_0 f_1}{cp} + \frac{R_1 f_0}{cp} \right) \quad (3.53)$$

Equations (3.44)-(3.48) represent a set of equations to solve for the zeroth order perturbation amplitudes a_0, b_0, c_0, d_0 , and f_0 . The boundary conditions in the y direction are simply that the flow quantities must be bounded as $y \rightarrow \pm\infty$. The equations can be written in the following form

$$[L] [\phi_0] = [0] \quad (3.54)$$

where

$$[\phi_0] = \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \\ e_0 \end{bmatrix}$$

and $[L]$ is a matrix of linear operators given by

$$[L] = \begin{bmatrix} i R_0 & \frac{\partial R_0}{\partial y} + R_0 \frac{\partial}{\partial y} & i(kU_0 - \omega) & 0 & 0 \\ i(kU_0 - \omega)R_0 & R_0 \frac{\partial U_0}{\partial y} & 0 & i k & 0 \\ 0 & i(kU_0 - \omega) & 0 & \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial S_0}{\partial y} & 0 & 0 & i(kU_0 - \omega) \\ 0 & 0 & -a^2 & 1 & -\frac{R_0 a^2}{c_p} \end{bmatrix}$$

One notes that the operations involved in $[L]$ are simply multiplications by known functions and differentiations with respect to y . Thus it is possible to consider equation (3.54) at a given spanwise location independent of every other spanwise location. This is a result of the weakly non-parallel approximation. Arbitrary functions of x_1 will be involved in the solution of equation (3.54). Indeed it is convenient to write

$$[\phi_0] = A(x_1) [\hat{\phi}_0(x_1, y)] \quad (3.55)$$

where $A(x_1)$ is at this level, an arbitrary function of x . However, since differentiations in $[L]$ are with respect to y only

$$[L][\phi_0] = [L]\{A[\hat{\phi}_0]\} = A[L][\hat{\phi}_0]$$

Thus equation (3.54) yields

$$[L][\hat{\phi}_0(x_1, y)] = 0 \quad (3.56)$$

Equation (3.56) is a homogeneous system of equations to solve for the components of $[\hat{\phi}_0(x_1, y)]$. The matrix $[L]$ is a matrix of linear operators that acts on the components of a five dimensional vector whose components are functions of y and transforms it into a new five dimensional vector. Thus the vector space \mathcal{V} is defined as all five dimensional vectors whose components are complex functions of a real variable y whose range is from $-\infty$ to $+\infty$. An inner product can be defined on this vector space according to

$$([u], [v]) = \int_{-\infty}^{\infty} [u]^T [\bar{v}] dy \quad (3.57)$$

for all $[u], [v] \in \mathcal{V}$. The bar in equation (3.57) denotes a complex conjugate.

For a given excitation frequency ω it is desired to find the amplitudes of each variable as well as the spatial wave function $k(x_1)$. Equation (3.5) is a homogeneous system to solve for these amplitudes at each spatial location with the local value of $k(x_1)$ as a parameter. However this system will have a non-trivial solution only for certain values of $k(x_1)$. The values of $k(x_1)$ for which this non-trivial solution occurs can be determined numerically.

Equations (3.49)-(3.55) represent a set of equations to solve for the first order perturbation amplitudes a_1, b_1, c_1, d_1, e_1 , and f_1 . These equations can be written in the form

$$[L][\phi_1(x_1, y)] = [g(x_1, y)] \quad (3.58)$$

where

$$[\phi_1(x_1, y)] = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ f_1 \end{bmatrix}$$

and

$$[g(x_1, y)] = \begin{bmatrix} g_1(x_1, y) \\ g_2(x_1, y) \\ g_3(x_1, y) \\ g_4(x_1, y) \\ g_5(x_1, y) \end{bmatrix}$$

where

$$\begin{aligned}
 g_1(x_1, y) = & -a_0 \frac{\partial R_0}{\partial x_1} - U_0 \frac{\partial c_0}{\partial x_1} - ikc_0 U_1 \\
 & -V_1 \frac{\partial c_0}{\partial y} - c_0 \frac{\partial U_0}{\partial x_1} - R_0 \frac{\partial a_0}{\partial x_1} - ikR_1 a_0 \\
 & -R_1 \frac{\partial b_0}{\partial y} - c_0 \frac{\partial V_1}{\partial y}
 \end{aligned} \tag{3.59}$$

$$\begin{aligned}
 g_2(x_1, y) = & -i\omega R_1 a_0 - R_0 U_0 \frac{\partial a_0}{\partial x_1} - R_1 U_0 ika_0 \\
 & -R_0 a_0 \frac{\partial U_0}{\partial x_1} - c_0 U_0 \frac{\partial U_0}{\partial x_1} - R_0 U_0 ika_0 \\
 & -R_0 V_1 \frac{\partial a_0}{\partial y} - R_0 b_0 \frac{\partial U_1}{\partial y} - R_1 b_0 \frac{\partial U_0}{\partial y} - c_0 V_1 \frac{\partial U_0}{\partial y} \\
 & - \frac{\partial d_0}{\partial x_1}
 \end{aligned} \tag{3.60}$$

$$\begin{aligned}
 g_3(x_1, y) = & i\omega R_1 b_0 - R_0 U_0 \frac{\partial b_0}{\partial x_1} - R_0 U_1 ikb_0 \\
 & -R_1 U_0 ikb_0 - R_0 V_1 \frac{\partial b_0}{\partial y} - R_0 b_0 \frac{\partial V_1}{\partial y}
 \end{aligned} \tag{3.61}$$

$$\begin{aligned}
 g_4(x_1, y) = & -U_0 \frac{\partial f_0}{\partial x_1} - U_1 ikf_0 - a_0 \frac{\partial S_0}{\partial x_1} \\
 & -V_1 \frac{\partial f_0}{\partial y} - b_0 \frac{\partial S_1}{\partial y}
 \end{aligned} \tag{3.62}$$

$$g_5(x_1, y) = - \frac{R_1 f_0}{c_p} \tag{3.63}$$

The operator $[L]$ in equation (3.58) is the same operator that appears in equation (3.54). Equation (3.58) is a non-homogeneous system, whereas equation (3.54) is a homogeneous system. Furthermore non-trivial solutions of equation (3.54) exist. Thus the Fredholm Alternative implies that a solution to equation (3.58) exists if and only if the non-homogeneous terms satisfy a solvability condition. In particular one can show that the non-homogeneous terms must be orthogonal to all non-trivial solutions of the corresponding adjoint problem.

The adjoint of an operator $[L]$ with respect to a given inner product is the operator $[L^*]$ that

$$([L][u],[v]) = ([u],[L^*][v]) \quad (3.64)$$

for all $[u],[v] \in \infty([L])$. Equation (3.64) applied to the inner product given in equation (3.57) becomes

$$\int_{-\infty}^{\infty} ([L][u])^T [\bar{v}] dy = \int_{-\infty}^{\infty} [u]^T ([L^*][v]) dy$$

Let

$$[u] = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} \quad [v] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$$

be arbitrary elements of \mathcal{V} . Then using the definition of $[L]$ from equation (3.55) the left hand side of (3.65) becomes

$$\begin{aligned}
\int_{-\infty}^{\infty} ([L][u])^T [\bar{v}] dy = & \int_{-\infty}^{\infty} \left\{ [iR_0 u_1 + \frac{\partial R_0}{\partial y} u_2 \right. \\
& + R_0 \frac{\partial u_2}{\partial y} + i(kU_0 - \omega) u_3] \bar{v}_1 + [i(kU_0 - \omega) R_0 u_1 \\
& + R_0 \frac{\partial U_0}{\partial y} u_2 + iku_4] \bar{v}_2 + [i(kU_0 - \omega) u_2 \\
& + \frac{\partial u_4}{\partial y}] \bar{v}_3 + [\frac{\partial S_0}{\partial y} u_2 + i(kU_0 - \omega) u_5] \bar{v}_4 \\
& \left. + [-a^2 u_3 + u_4 - \frac{R_0 a^2}{C_p} u_5] \bar{v}_5 \right\} dy
\end{aligned} \tag{3.66}$$

Integrating by parts and rearranging terms in equation (3.66) yields

$$\begin{aligned}
\int_{-\infty}^{\infty} ([L][u])^T [\bar{v}] dy = & \int_{-\infty}^{\infty} \left\{ [iR_0 \bar{v}_1 \right. \\
& + i(kU_0 - \omega) R_0 \bar{v}_2] u_1 + [\frac{\partial R_0}{\partial y} \bar{v}_1 \\
& - \frac{\partial}{\partial y} (R_0 \bar{v}_1) + R_0 \frac{\partial U_0}{\partial y} \bar{v}_2 + i(kU_0 - \omega) \bar{v}_3 \\
& + \frac{\partial S_0}{\partial y} \bar{v}_4] u_2 + [i(kU_0 - \omega) \bar{v}_1 - a^2 \bar{v}_5] u_3 \\
& + [ik\bar{v}_2 - \frac{\partial \bar{v}_3}{\partial y} + \bar{v}_5] u_4 \\
& \left. + [i(kU_0 - \omega) \bar{v}_4 - \frac{R_0 a^2}{C_p} \bar{v}_5] u_5 \right\} dy
\end{aligned} \tag{3.67}$$

The right hand side of equation (3.67) can be written in the same form as the right hand side of equation (3.65) with

$$[L^*] = \begin{bmatrix} -iR_0 & -i(\bar{k}U_0 - \omega)R_0 & 0 & 0 & 0 \\ \frac{\partial R_0}{\partial y} - \frac{\partial(R_0)}{\partial y} & R_0 \frac{\partial U_0}{\partial y} & -i(\bar{k}U_0 - \omega)R_0 & \frac{\partial S_0}{\partial y} & 0 \\ -i(\bar{k}U_0 - \omega) & 0 & 0 & 0 & -a^2 \\ 0 & -i\bar{k} & -\frac{\partial}{\partial y} & 0 & 1 \\ 0 & 0 & 0 & -i(\bar{k}U_0 - \omega) & -\frac{R_0 a^2}{C_p} \end{bmatrix}$$

Let $[\hat{\phi}_0^*]$ be the solution to

$$[L^*][\hat{\phi}_0^*] = 0 \quad (3.69)$$

subject to the requirement that all components of $[\hat{\phi}_0^*]$ are bounded as $y \pm \infty$.

The Fredholm Alternative states that for a solution of equation (3.58) to exist $[g]$ must be orthogonal to $[\hat{\phi}_0^*]$. That is

$$([g], [\hat{\phi}_0^*]) = 0$$

or

$$\int_{-\infty}^{\infty} [g]^T [\hat{\phi}_0^*] dy = 0 \quad (3.70)$$

Equation (3.70) becomes, upon substitution of all appropriate quantities,

$$\begin{aligned}
A(x_1) = & \int_{-\infty}^{\infty} \left\{ \left[-\hat{a}_0 \frac{\partial R_0}{\partial x_1} - U_0 \frac{\partial \hat{c}_0}{\partial x_1} - ik\hat{c}_0 U_1 - v_1 \frac{\partial \hat{c}_0}{\partial y} \right. \right. \\
& - \hat{c}_0 \frac{\partial U_0}{\partial x_1} - R_0 \frac{\partial \hat{a}_0}{\partial x_1} - ikR_1 \hat{a}_0 - R_1 \frac{\partial \hat{b}_0}{\partial y} - \hat{c}_0 \frac{\partial V_1}{\partial y} \left. \right] \hat{a}_1^* \\
& + \left[i\omega R_1 \hat{a}_0 - R_0 U_0 \frac{\partial \hat{a}_0}{\partial x_1} - R_1 U_0 ik\hat{a}_0 \right. \\
& - R_0 \hat{a}_0 \frac{\partial U_0}{\partial x_1} - \hat{c}_0 U_0 \frac{\partial U_0}{\partial x_1} - R_0 U_0 ik\hat{a}_0 - R_0 V_1 \frac{\partial \hat{a}_0}{\partial y} \\
& - R_0 \hat{b}_0 \frac{\partial U_1}{\partial y} - R_1 \hat{b}_0 \frac{\partial U_0}{\partial y} - \hat{c}_0 V_1 \frac{\partial U_0}{\partial x} - \frac{\partial d_0}{\partial x_1} \left. \right] \hat{b}_1^* \\
& + \left[i\omega R_1 \hat{b}_0 - R_0 U_0 \frac{\partial \hat{b}_0}{\partial x_1} - R_0 U_1 ik\hat{b}_0 - R_1 U_0 ik\hat{b}_0 \right. \\
& - R_0 V_1 \frac{\partial \hat{b}_0}{\partial y} - R_0 \hat{b}_0 \frac{\partial V_1}{\partial y} \left. \right] \hat{c}_0^* + \left[-U_0 \frac{\partial \hat{f}_0}{\partial x_1} - U_1 ik\hat{f}_0 \right. \\
& - \hat{a}_0 \frac{\partial S_0}{\partial x_1} - V_1 \frac{\partial \hat{f}_0}{\partial y} - \hat{b}_0 \frac{\partial S_1}{\partial y} \left. \right] \hat{d}_0^* - \frac{R_1 \hat{f}_0}{C_p} \hat{f}_0^* \left. \right\} dy \\
& + \frac{dA}{dx_1} \int_{-\infty}^{\infty} \left[(-U_0 \hat{c}_0 - R_0 \hat{a}_0) \hat{a}_0 + (-R_0 U_0 \hat{a}_0 \right. \\
& - \hat{d}_0) \hat{b}_0^* - R_0 U_0 \hat{b}_0 \hat{c}_0^* - U_0 \hat{f}_0 \hat{d}_0 \left. \right] dy = 0
\end{aligned} \tag{3.71}$$

Equation (3.71) can be written as

$$\mu_1(x_1) \frac{dA}{dx_1} - i\mu_2(x_1) A(x_1) = 0 \quad (3.72)$$

where $\mu_1(x_1)$ and $\mu_2(x_1)$ can be determined by comparing equation (3.72) to equation (3.71). The solution of equation (3.72) is

$$A(x_1) = A_0 e^{+i \int \frac{\mu_2(x_1)}{\mu_1(x_1)} dx_1} = A_0 e^{+iv(x_1)} \quad (3.73)$$

where A_0 is an arbitrary constant of integration. The solution for the zeroth order perturbation quantities can be written as

$$\begin{aligned} \tilde{u}(x_1, y) = A_0 e^{i(k(x_1) + v(x_1) - \omega t)} a_0(x_1, y) \\ + \dots \end{aligned} \quad (3.74)$$

$$\begin{aligned} \tilde{v}(x_1, y) = A_0 e^{i(k(x_1) + v(x_1) - \omega t)} b_0(x_1, y) \\ + \dots \end{aligned} \quad (3.75)$$

$$\begin{aligned} \tilde{p}(x_1, y) = A_0 e^{i(k(x_1) + v(x_1) - \omega t)} c_0(x_1, y) \\ + \dots \end{aligned} \quad (3.76)$$

$$\begin{aligned} \tilde{p}(x_1, y) = A_0 e^{i(k(x_1) + v(x_1) - \omega t)} d_0(x_1, y) \\ + \dots \end{aligned} \quad (3.77)$$

$$\begin{aligned} \tilde{s}(x_1, y) = A_0 e^{i(k(x_1) + v(x_1) - \omega t)} f_0(x_1, y) \\ + \dots \end{aligned} \quad (3.78)$$

The sum

$$\lambda(x_1) = k(x_1) + v(x_1) \quad (3.79)$$

represents a corrected value for the wave number to include the effects of the shear layer thickness increasing downstream. The value of $\lambda(L)$ is dependent upon the frequency at which the shear layer is excited. For a given ω the procedure described in the next section can be used to calculate $\lambda(L)$. A numerically defined function $Q(\omega) = \lambda(L)$ for a given ω is defined. This function can then be used in equation (2.8) as a correction to the $k(\omega)$ used by Tam and Block.

IV. NUMERICAL SOLUTION OF SHEAR LAYER EQUATION

The use of the theory described in Section III depends upon accurate numerical evaluation of (a) the eigenvalue $k(x_1)$; (b) the eigenfunction $[\hat{\phi}_0(x_1)]$; (c) the adjoint eigenfunction $[\hat{\phi}_0^*(x_1)]$; and, (d) the wave function correction $v(x_1)$. Each of these quantities must be evaluated at each spatial location along the length of the cavity. This section details the numerical solution for each of these quantities.

First the determination of the eigenvalue $k(x_1)$ and the eigenfunction $[\hat{\phi}_0(x_1, y)]$ is considered. These quantities are obtained by solving equation (3.54) with appropriate boundary conditions at each spatial location such that a non-trivial solution is obtained. All mean flow quantities are assumed to be known before the solution begins. The system, when written in matrix form, leads to a very complicated numerical solution. It is convenient to manipulate the component equations of equation (3.54) such that a single equation is obtained in terms of one variable. The component equations of equation (3.54) are

$$i(kU_0 - \omega)\hat{c}_0 + \hat{b}_0 \frac{\partial R_0}{\partial y} + iR_0 k \hat{a}_0 + R_0 \frac{\partial \hat{b}_0}{\partial y} = 0 \quad (4.1)$$

$$i(kU_0 - \omega)R_0 \hat{a}_0 + R_0 \hat{b}_0 \frac{\partial U_0}{\partial y} + ik \hat{d}_0 = 0 \quad (4.2)$$

$$i(kU_0 - \omega)R_0 \hat{b}_0 + \frac{\partial \hat{d}_0}{\partial y} = 0 \quad (4.3)$$

$$i(kU_0 - \omega)\hat{f}_0 + \hat{b}_0 \frac{\partial S_0}{\partial y} = 0 \quad (4.4)$$

$$\hat{d}_0 = R_0 a_1^2 \left(\frac{\hat{c}_0}{R_0} + \frac{\hat{f}_0}{C_p} \right) \quad (4.5)$$

Equation (4.3) is solved for \hat{b}_0 to yield

$$\hat{b}_0 = \frac{1}{(kU_0 - \omega)} \frac{\partial \hat{d}_0}{\partial y} \quad (4.6)$$

Equation (4.6) is substituted into equation (4.2) and \hat{a}_0 is obtained as

$$\hat{a}_0 = - \frac{1}{R_0(kU_0 - \omega)} \left[ik\hat{d}_0 + \frac{1}{kU_0 - \omega} \frac{\partial U_0}{\partial y} \frac{\partial \hat{d}_0}{\partial y} \right] \quad (4.7)$$

Equation (4.6) is substituted into equation (4.4) to yield

$$\hat{f}_0 = - \frac{1}{R_0(kU_0 - \omega)^2} \frac{\partial S_0}{\partial y} \frac{\partial \hat{d}_0}{\partial y} \quad (4.8)$$

Equation (4.8) is substituted into equation (4.5) to yield

$$\hat{c}_0 = \frac{\hat{d}_0}{a^2} + \frac{1}{C_p(kU_0 - \omega)^2} \frac{\partial S_0}{\partial y} \frac{\partial \hat{d}_0}{\partial y} \quad (4.9)$$

Equations (4.6), (4.7), and (4.9) are substituted into equation (4.1) to yield the following differential equation for \hat{d}_0

$$\frac{\partial^2 \hat{d}_0}{\partial y^2} - \left(\frac{2k}{kU_0 - \omega} \frac{\partial U_0}{\partial y} + \frac{1}{R_0} \frac{\partial R_0}{\partial y} \right) \frac{\partial \hat{d}_0}{\partial y} + \left(k^2 - \frac{(kU_0 - \omega)^2}{a^2} \right) \hat{d}_0 = 0 \quad (4.10)$$

A central finite difference method is used to solve equation (4.10) for the eigenvalue $k(x_1)$ and the function $\hat{d}_0(x_1, y)$ at discrete values of x_1 . The range in the y direction is discretized from $-H \leq y \leq H$ where H is sufficiently large. Convergence tests are made to check the value of H . Central finite difference formulas are used to approximate the derivatives and a system of equations is derived for the values of \hat{d}_0 at the discrete points. The system is tridiagonal thus the Thomas algorithm [Carnahan, Luther, and Wilkes (19)] is used. Let Δy be the distance between points in the y direction and define.

$$\hat{d}_{0\ell} = \hat{d}_0(x_1, -H + \ell \Delta y) \quad (4.11)$$

A recurrence relation of the form

$$\hat{d}_{0\ell} = G_\ell + H_\ell \hat{d}_{0\ell+1} \quad (4.12)$$

is assumed where additional recurrence relations are derived, in terms of differential equation coefficients, for G_ℓ and H_ℓ . The evaluation of the G's and H's occurs in a forward manner, $\ell = 0, \dots, N$. Then the \hat{d}_ℓ 's are evaluated recursively in a backward fashion $\ell = N, \dots, 0$ using $\hat{d}_{N+1} = 0$. However the G's and H's are in terms of the unknown eigenvalue k , for which an initial guess must be made. The correct solution is the value of k for which the differential equation and its boundary conditions has a non-trivial solution. Values of G_0 and H_0 are arbitrarily selected and the process begins. The value of \hat{d}_{00} is checked against zero. If it is not close enough to zero then the guess for k must be changed. A Newton-Raphson iteration is used to iterate the values of k .

The remaining perturbation amplitudes are determined using equations (4.6) - (4.9) and numerical differentiation.

The next step is to obtain the solution of the adjoint problem given the eigenvalue k . The equations defining the components of the adjoint vector are obtained from equation (3.68) as

$$-iR_0 a^* - i(\bar{k}U_0 - \omega)R_0 b^* = 0 \quad (4.13)$$

$$a^* \frac{\partial R_0}{\partial y} - \frac{\partial}{\partial y} (R_0 a^*) + R_0 \frac{\partial U_0}{\partial y} b^* - i(\bar{k}U_0 - \omega)c^* + \frac{\partial S_0}{\partial y} d^* = 0 \quad (4.14)$$

$$-i(\bar{k}U_0 - \omega)a^* - a^2 f^* = 0 \quad (4.15)$$

$$-i\bar{k}b^* - \frac{\partial c^*}{\partial y} + f^* = 0 \quad (4.16)$$

$$-i(\bar{k}U_0 - \omega)d^* - \frac{R_0 a^2}{C_p} f^* = 0 \quad (4.17)$$

Equation (4.17) is used to solve for f^* in terms of d^* as

$$f^* = - \frac{iC_p(\bar{k}U_0 - \omega)}{R_0 a^2} d^* \quad (4.18)$$

Equations (4.15) and (4.18) are used to obtain

$$a^* = \frac{C_p}{R_0} d^* \quad (4.19)$$

Equations (4.13) and (4.19) are used to obtain

$$b^* = \frac{C_p}{R_0(\bar{k}U_0 - \omega)} d^* \quad (4.20)$$

Equations (4.16), (4.18), and (4.20) are used to obtain

$$\frac{\partial c^*}{\partial y} = \frac{iC_p}{R_0} \left[\frac{\bar{k}}{\bar{k}U_0 - \omega} - \frac{\bar{k}U_0 - \omega}{a^2} \right] d^* \quad (4.21)$$

Equation (4.14) is divided by $(\bar{k}U_0 - \omega)$ and differentiated with respect to y . Equations (4.19) - (4.21) are substituted into the resulting equation to yield the following differential equation for d^*

$$\begin{aligned} \frac{\partial^2 d^*}{\partial y^2} - \left[\frac{(\bar{k}+1)}{(\bar{k}U_0 - \omega)} \frac{\partial U_0}{\partial y} \right] \frac{\partial d^*}{\partial y} - \left[\frac{\partial^2 U_0}{\partial y^2} - \frac{2\bar{k}}{(\bar{k}U_0 - \omega)^2} \left(\frac{\partial U_0}{\partial y} \right)^2 \right. \\ \left. + \frac{\bar{k}}{R_0} - \frac{(\bar{k}U_0 - \omega)^2}{R_0 a^2} \right] d^* = 0 \end{aligned} \quad (4.22)$$

Equation (4.22) can be solved by a central finite difference technique once k is known. Then equations (4.18) - (4.20) are used to solve for a^* , b^* , and f^* . Equation (4.21) along with a numerical integration scheme is used to solve for c^* .

The process outlined above is continued at discrete intervals over the entire length of the cavity.

Numerical integrations in the y direction at each x_1 location are performed to calculate $\mu_1(x_1)$ and $\mu_2(x_1)$ as defined from equation (3.71). A numerical integration in the x_1 direction is then performed to calculate $v(x_1)$.

V. DISCUSSION

A theory has been presented for the analysis of a shear layer over an open cavity. The shear layer is of a finite thickness which increases downstream. The analysis is based upon non-parallel stability theory using the method of multiple scales. The theory is used to predict a wave number for a given excitation frequency. The wave number-frequency relationship is then used to predict the discrete shear layer excitation frequencies using equations developed by Tam and Block (1978).

A numerical scheme for implementing the theory has also been presented. Numerical calculations have not been completed, but are under consideration, and will be available in papers published from this work. The wave numbers obtained will be corrections to the wave numbers obtained by Tam and Block and will be compared with their results.

The analysis presented is not intended as an ultimate analysis of the shear layer over an open cavity. The author recognizes that several problems exist with the analysis and many questions remain. The ultimate goal is to attain a complete mathematical analysis of the shear layer over an open cavity.

One question concerning the analysis presented regards the mean flow. Tam and Block (1978) note that the mean flow in the shear layer has never been measured such that the results are useful for calculation purposes. Indeed they use a standard hyperbolic tangent velocity profile in the shear layer. The mean flow assumed in the described research is a two dimensional non-parallel mean flow. The non-parallelism is due to the fact that the shear layer grows downstream due to entrainment. However, the rate of growth of the shear layer thickness is related to the amplitude of the cavity oscillation and is not predicted by the analysis. It is believed that a thorough analysis

of the problem will yield this information. Preliminary ideas are presented below.

The stability characteristics of the shear layer are only required over a finite distance ($0 \leq x \leq L$, $0 < x_1 < \epsilon L$). Thus the mean flow variables can be expanded in a Taylor Series in x_1 about $x_1 = 0$. To this end

$$U_0(x_1, y) = U_0(0, y) + x_1 \frac{\partial U_0}{\partial x_1}(0, y) + \dots \quad (5.1)$$

or

$$U_0(x_1, y) = A_1(y) + x_1 A_2(y) \quad (5.2)$$

The nonlinear terms have been truncated in equation (5.2). The other mean flow variables can be written as

$$R_0(x_1, y) = B_1(y) + x_1 B_2(y) \quad (5.3)$$

$$S_0(x_1, y) = C_0(y) + x_1 C_2(y) \quad (5.4)$$

$$V_1(x_1, y) = D_1(y) + x_1 D_2(y) \quad (5.5)$$

$$U_1(x_1, y) = E_1(y) + x_1 E_2(y) \quad (5.6)$$

$$R_1(x_1, y) = F_1(y) + x_1 F_2(y) \quad (5.7)$$

$$S_1(x_1, y) = G_1(y) + x_1 G_2(y) \quad (5.8)$$

Equations (5.2) - (5.8) are substituted into equations (3.27) - (3.30) and the resulting equations substituted into equations (3.23) - (3.26).

A simpler version of the mean flow analysis is to assume a hyperbolic tangent velocity profile whose thickness varies with x_1 . To this end

$$U(x_1, y) = U_\infty \tanh\left(1 + \frac{y}{\theta(x_1)}\right) \quad (5.9)$$

where

$$\theta(x_1) = \theta_0 + x_1 \theta_1 \quad (5.10)$$

The result of the method described is the shear layer stability characteristics for a given frequency (i.e., a frequency-wavenumber relationship). This relationship is then used to predict the discrete shear layer frequencies. The author feels that the method and technique are good but several modifications should be considered before the method is applied for design analysis. The eigenvalue prediction equation (2.8) derived by Tam and Block is based upon a vortex sheet between parallel streams. This prediction equation should be modified to include shear layer thickness effects, and the effect of shear layer growth. This will involve a more detailed mathematical analysis of both the matching between the shear layer and the external flow and the fluid behavior at the trailing edge of the cavity. The entire fluid mechanics problem must be considered with the flow field divided into three regions. The non-parallelism of the shear layer flow will have some effects on the external flows which are not included in the eigenvalue prediction equation. Additionally, this equation is based upon the deflection of the shear layer at the trailing edge of the cavity. This concept must be extended to handle shear layer thickness effects.

The boundary condition at the trailing edge must be considered. The shear layer, while being excited by a wave of frequency ω may either reattach at some point part the trailing edge of the cavity or deflect into the cavity at the trailing edge. When the latter occurs due to entrainment waves are reflected back into the cavity and excite the shear layer. Thus a feedback

process is born. The analysis of this process is heavily dependent upon the boundary condition at the trailing edge. An analysis of the shear layer with this effect should be performed. Perhaps it will be discovered that a multiple deck like analysis is necessary where higher decks are needed due to the trailing edge behavior. Thus a complete asymptotic analysis of the shear layer should be performed.

Some investigators have attempted to solve for the flow variables around an open cavity using finite element techniques. It has been suggested [Wolfe (1982)] that a mathematical analysis of the shear layer could be used in a finite element formulation of the problem to predict the pressure inside the cavity. This analysis is the first step toward such a goal. A further step in this process is to conduct an analysis of the shear layer with this goal in mind. That is, the flow variables outside the shear layer will be solved for by a finite element technique while the variables in the shear layer will be provided by an analytical formulation. However, these flow regions are intimately connected due to the trailing edge interaction process. It is possible to use the current shear layer formulation at a discrete frequency. However, the amplitude of the shear layer variables is not currently known and must be solved for as part of the finite element routine.

In summary, the research presented is just a first step toward understanding the shear layer over an open cavity. Much remains to be done, but it is believed that future work is manageable and will lead to great benefits to reducing sound pressure levels in cavities.

REFERENCES

1. Bartel and McAvoy (1981), "Cavity Oscillation in Cruise Missile Aircraft", Air Force Wright Patterson Aeronautical Laboratories Report, AFWAL-TR-81-3036.
2. Betchov and Criminale (1967), Stability of Parallel Flows, Academic Press, New York.
3. Bilanin, A. J. and Covert, E.E. (1973), "Estimation of Possible Excitation Frequencies for Shallow Cavities", AIAA Journal, 11, p. 347-351.
4. Block, R.J.W., (1976), "Noise Response of Varying Dimensions of Subsonic Speeds", NASA Tech. Note D-8351.
5. Covert, E.E., (1970), "An Approximate Calculation of the Onset Velocity of Cavity Oscillations", AIAA Journal, 8, p. 2189-2194.
6. East, L.F. (1966), "Aerodynamically Induced Resonance In Rectangular Cavities", Journal of Sound and Vibration, 3, p. 277-287.
7. Gibson, J.F. (1958), "An Analysis of Supersonic Cavity Flow", S.M. Thesis, M.I.T. Department of Aeronautics and Astronautics, Cambridge, Massachusetts.
8. Heller, H.H, and Bliss, D.B., (1975), "Aerodynamically Induced Pressure Oscillations in Cavities--Physical Mechanisms and Suppression Concepts", Air Force Flight Dynamics Lab Report AFFDL-TR-74-133.
9. Karamcheti, K., (1955), "Acoustic Radiation From Two-Dimensional Rectangular Cut Outs in Aerodynamic Surfaces", NASA TN-3487.
10. Ling and Reynolds (1973), "Non-Parallel Flow Corrections for the Stability of Shear Flows", Journal of Fluid Mechanics, 59, p. 571-591.

11. Michalke (1965), "On Spatially Growing Disturbances in an Inviscid Shear Layer", *Journal of Fluid Mechanics*, 23, p. 521-544.
12. Nayfeh (1982), An Introduction to Perturbation Methods, John Wiley and Sons, New York.
13. Plumblee, H.E., Gibson, J.S., and Lassiter, L.W. (1962), "A Theoretical and Experimental Investigation of the Acoustical Response of Cavities in an Aerodynamic Flow", United States Air Force, WADD-TR-61-75.
14. Rossiter, J.E. (1964), "Wind Tunnel Experiments on the Flow Over Rectangular Cavities at Subsonic and Transonic Speeds", ARCR7M No. 3438.
15. Saric and Nayfeh (1975), "Non-parallel Stability of Boundary Layer Flows", *The Physics of Fluids*, 18, p. 945-950.
16. Smith, D.L., and Shaw, L.L. (1975), "Prediction of the Pressure Oscillations in Cavities Exposed to Aerodynamic Flow", Air Force Flight Dynamics Lab Report, AFFDL-TR-75-34.
17. Spee, B.M., (1966), "Wind Tunnel Experiments on Unsteady Cavity Flow at High Subsonic Speeds, Separated Flows", GARD CP No. 4, p. 947-974.
18. Tam, C.K.W., and Block, P.J.W., (1978), "On the Tones and Pressure Oscillations Induced by Flow Over Rectangular Cavities", Journal of Fluid Mechanics, 89, p. 373-399.
19. Wolfe, H.W., (1982), Private Communication.

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