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AXIOMATIC CHARACTERIZATIONS OF CONTINUUM STRUCTURE FUNCTIONS*

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A continuum structure function is a nondecreasing mapping from the unit hypercube to the unit interval. Axiomatic characterizations of the continuum structure functions based on the Barlow-Wu and Natvig multistate structure functions are derived.
ABSTRACT

A continuum structure function is a nondecreasing mapping from the unit hypercube to the unit interval. Axiomatic characterizations of the continuum structure functions based on the Barlow-Wu and Natvig multistate structure functions are derived.

KEYWORDS: Reliability; continuum structure function; multistate structure function.

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1. INTRODUCTION

Let \( C = \{1, 2, \ldots, n\} \) denote a set of components and let \( \Delta = [0,1]^n \).

A nondecreasing mapping \( \gamma: \Delta \mapsto [0,1] \) with \( \gamma(0) = 0 \) and \( \gamma(1) = 1 \) is said to be a continuum structure function (CSF). If \( \sup_{X \in \Delta} [\gamma(1_X) - \gamma(0_X)] > 0 \) for each \( i \in C \), where \( (\delta, X) \) denotes \( (X_1, \ldots, X_{i-1}, \delta, X_{i+1}, \ldots, X_n) \), \( \gamma \) is said to be weakly coherent.

Definition

Let \( P_1, \ldots, P_r \) denote the \( r \) minimal path sets of a binary coherent structure function. If

\[
\gamma(X) = \max_{1 \leq j \leq r} \min_{i \in P_j} X_i \quad (X \in \Delta),
\]

\( \gamma \) is said to be a Barlow-Wu CSF [2].

Definition

Let \( \{ \phi_\alpha, 0 < \alpha < 1 \} \) be a class of binary coherent structure functions such that \( \phi_\alpha(Y_\alpha) \) is a left-continuous and non-increasing function of \( \alpha \) for fixed \( \tilde{X} \) where \( Y_\alpha \) is the indicator of \( \{ X_i \geq \alpha \}, i = 1, 2, \ldots, n \). If

\[
\gamma(X) > \alpha \ \text{iff} \ \phi_\alpha(Y_\alpha) = 1 \quad (X \in \Delta, 0 < \alpha < 1),
\]

\( \gamma \) is said to be a Natvig CSF [3].

In this paper, we present axiomatic characterizations of the Barlow-Wu and Natvig CSFs. In particular, we show that \( \gamma \) is a Barlow-Wu CSF if and only if it satisfies the following conditions:
2. SOME DEDUCTIONS FROM THE AXIOMS

Let $U_\alpha = \{x \in \Delta | \gamma(x) \geq \alpha\}$ and $L_\alpha = \{x \in \Delta | \gamma(x) < \alpha\}$, $0 < \alpha < 1$. Further, define $K_\alpha = \{x \in \Delta | \gamma(x) < \alpha\}$ whereas $\gamma(y) > \alpha$ for all $y < x$, $0 < \alpha < 1$.

**Proposition 2.1**

Let $\gamma$ be a CSF.

(i) $\gamma$ is right (left)-continuous if and only if each $U_\alpha$ ($L_\alpha$) is closed.

(ii) If $\gamma$ is right (left)-continuous, then each $P_\alpha(K_\alpha)$ is nonempty and $x \in U_\alpha$ ($L_\alpha$) if and only if $x \geq (\leq) y \in P_\alpha(K_\alpha)$.

(iii) If $\gamma$ is continuous, then $\gamma(P_\alpha) = \{\alpha\}$, $0 < \alpha < 1$, and $\gamma(K_\alpha) = \{0\}$, $0 < \alpha < 1$. 

C1 $\gamma$ is continuous

C2 $P_\alpha \neq \emptyset$ and $P_\alpha \subset [0, \alpha]^n$, $0 < \alpha < 1$

C3 There is no nonempty open set $A \subset \Delta$ such that $\gamma$ is constant on $A$

C4 $\gamma$ is weakly coherent

where $P_\alpha = \{x \in \Delta | \gamma(x) > \alpha\}$ whereas $\gamma(y) < \alpha$ for all $y < x$ and where $y < x$ means that $y \leq x$ but that $y \neq x$.

Some consequences of these axioms are deduced in Section 2, and in Section 3 we present our main results: an axiomatic characterization of the Barlow-Wu CSF and an analogous characterization of the Natvig CSF.

Our approach was suggested by the Borges-Rodrigues characterizations of the Barlow-Wu and Natvig multistate structure functions [5] though, as we show in Section 4, their characterizations are incorrect.
Proof: The proofs of (i) and (iii) are straightforward; see [4] for the
proof of (ii).

**Proposition 2.2**

If \( \gamma \) is a continuous CSF, conditions C2 and

\[
C2', \quad K_\alpha \neq \emptyset \text{ and } K_\alpha \subset \{\alpha,1\}^n, \quad 0 \leq \alpha < 1
\]

are equivalent.

Proof: Since \( \gamma \) is continuous, each \( K_\alpha \) is nonempty. We show that, if C2
holds, then \( K_\alpha \subset \{\alpha,1\}^n \) for all \( \alpha \in [0,1) \).

Suppose, conversely, that for some \( \alpha \in [0,1) \) there exists a vector
\( \gamma \in K_\alpha \) such that \( \gamma \notin \{\alpha,1\}^n \). Then there exists at least one component, \( k \) say, such that \( \gamma_k \notin \{\alpha,1\} \). Either \( 0 \leq \gamma_k < \alpha < 1 \) or \( 0 < \alpha < \gamma_k < 1 \); we
consider these two cases separately.

Suppose, firstly, that \( 0 \leq \gamma_k < \alpha < 1 \). By Proposition 2.1, \( \gamma\gamma) = \alpha \)
and \( \gamma(\delta_k,\gamma) > \alpha \) if \( \gamma_k < \delta < \alpha \). Let \( \gamma(\delta_k,\gamma) = \xi \); then \( (\delta_k,\gamma) \in U_\xi \). Since
\( U_\xi \) is closed there exists, by Proposition 2.1, an \( X \in (\delta_k,\gamma) \) such that
\( X \in P_\xi \). Now \( \gamma \notin U_\xi \) and so \( \gamma_k < X_k \leq \delta \). Thus \( 0 \leq \gamma_k < X_k \leq \delta < \alpha < \xi \) and
so \( \gamma \notin \{0,\xi\}^n \), in contradiction to C2.

Suppose, now, that \( 0 < \alpha < \gamma_k < 1 \). Again \( \gamma\gamma) = \alpha \). Let \( \gamma(1_k,\gamma) = \delta > \alpha \). Since \( \gamma(x_k,\gamma) \) is a continuous, nondecreasing function of \( x \) for
fixed \( (1_k,\gamma) \), it follows from the intermediate value theorem that, for
given \( \xi \) with \( \alpha < \xi < \gamma_k \wedge \delta \), there exists a \( w \in (Y_k,1) \) such that \( \gamma(w_k,\gamma) = \xi \).
Thus \( (w_k,\gamma) \in U_\xi \) and hence there exists an \( X \in (w_k,\gamma) \) such that \( X \in P_\xi \).
Now \( \gamma \notin U_\xi \) and so \( \gamma_k < X_k \leq w \). It follows that \( 0 \leq \alpha < \xi < \gamma_k < X_k < w \)
and hence \( X \notin \{0,\xi\}^n \), in contradiction to C2.
Thus, a continuous CSF satisfying C2 also satisfies C2'. A similar argument verifies the converse.

Proposition 2.3

If \( \gamma \) is a CSF which satisfies Cl, C2 and C3, then \( \gamma((0,\alpha)^n) = \{0,\alpha\} \) for all \( \alpha \in [0,1] \).

Proof: If \( \alpha = 0 \) there is nothing to prove, so suppose that, for some \( \alpha \in (0,1] \), there exists a vector \( \bar{X} \in \{0,\alpha\}^n \) such that \( \beta = \gamma(\bar{X}) \notin \{0,\alpha\} \).

It is easily seen that \( 0 < \beta < \alpha \) and that \( \bar{X} \neq \bar{0} \) or \( \bar{g} \), and hence we can write

\[
X_i = \begin{cases} 
0 & \text{for } j=1,2,\ldots,k \\
\alpha & \text{for } j=k+1,\ldots,n 
\end{cases}
\]

for some \( k \) with \( 1 \leq k \leq n-1 \).

Since \( \bar{X} \in U_\beta \cap L_\beta \), and both are closed, it follows from Proposition 2.1 that there exist a \( Z \in P_\beta \) and a \( \bar{W} \in K_\beta \) such that \( Z \leq \bar{X} \leq \bar{W} \). This ordering will only hold if \( Z \in (0,\beta)^n - \{0\} \) satisfies \( Z_i = 0 \) for \( j=1,2,\ldots,k \) and if \( \bar{W} \in (\beta,1)^n - \{1\} \) satisfies \( W_j = 1 \) for \( j=k+1,\ldots,n \) and so \( A = (Z_1,W_1)\times\cdots\times(Z_n,W_n) \subset \Delta \) is open. Further, since \( Z \in P_\beta \) and \( \bar{W} \in K_\beta \), it follows that \( \gamma(\bar{X}) = \beta \) for all \( \bar{X} \in A \), in contradiction to C3. Thus \( \gamma(X) \in \{0,\alpha\} \) as claimed.

Proposition 2.4

If \( \gamma \) is a CSF which satisfies Cl, C2 and C3, then \( P_\alpha = \alpha P_1 \) for all \( \alpha \in (0,1] \).
Proof: Suppose that \( \alpha < 1 \), otherwise there is nothing to prove, and let \( X \in P_\alpha \) so that \( \gamma(X) = \alpha \). Then \( X < \frac{1}{\alpha} X \) and so \( \gamma(X) \leq \gamma(\frac{1}{\alpha} X) \). Since \( \frac{1}{\alpha} X \in \{0, 1\}^n \), it follows from Proposition 2.3 that \( \gamma(\frac{1}{\alpha} X) = 1 \). We claim that \( \frac{1}{\alpha} X \in P_1 \).

Suppose, conversely, that \( \frac{1}{\alpha} X \notin P_1 \). Since \( U_1 \) is closed, it follows from Proposition 2.1 that there exists a \( \tilde{W} < \frac{1}{\alpha} X \) such that \( \tilde{W} \in P_1 \). Consider the vector \( \alpha \tilde{W} \in \{0, \alpha\}^n \); it is easily seen that \( \gamma(\alpha \tilde{W}) = \alpha \) and thus there exists a vector \( \omega \tilde{X} < X \) such that \( \omega \tilde{W} \in U_\alpha \). This contradicts the assumption that \( \tilde{X} \notin P_\alpha \) and hence \( \frac{1}{\alpha} X \in P_1 \) as claimed. This holds for all \( X \in P_\alpha \) and so \( P_\alpha \subset \alpha P_1 \).

Similarly, it can be shown that \( \alpha P_1 \subset P_\alpha \). \( \Box \)

3. THE CHARACTERIZATION THEOREMS

Theorem 3.1

A CSF \( \gamma \) is of the Barlow-Wu type if and only if it satisfies conditions C1, C2, C3 and C4.

Proof: It is easily verified that the Barlow-Wu CSF satisfies C1, C2, C3 and C4. To prove the converse, observe that

\[
\gamma(X) \geq \alpha \iff X \geq \tilde{X} \in P_\alpha \\
\begin{align*}
\iff \min_{\{i \mid Y_i = \alpha\}} X_i &\geq \alpha \text{ for some } \tilde{X} \in P_\alpha \\
\iff \max_{\tilde{X} \in P_\alpha} \min_{\{i \mid Y_i = \alpha\}} X_i &\geq \alpha
\end{align*}
\]
\[ \max_{Y \in \gamma \mathcal{P}_1} \min \left\{ i \mid Y_i = \alpha \right\} \]

\[ \max_{Z \in \mathcal{P}_1} \min \left\{ i \mid Z_i = 1 \right\} \]

This holds for all \( \bar{x} \in \Delta \) and \( \alpha \in (0,1] \) and so

\[ \gamma(\bar{x}) = \max_{Z \in \mathcal{P}_1} \min \left\{ i \mid Z_i = 1 \right\} \]

Write \( \mathcal{P}_1 = \{X^{(1)}, \ldots, X^{(N)}\} \) and let \( T_j = \{i \in \mathcal{C} \mid X^{(j)}_i = 1\} \). By the definition of \( \mathcal{P}_1 \), it is clear that each \( T_j \) is nonempty and that \( T_j \neq T_k \) for all \( j, k = 1, 2, \ldots, N \) with \( j \neq k \). Thus

\[ \gamma(\bar{x}) = \max_{1 \leq j \leq N} \min_{i \in T_j} X_i \]

where each \( T_j \subset \mathcal{C} \). Condition \( C_4 \) ensures that \( \bigcup_{j=1}^{N} T_j = \mathcal{C} \), completing the proof. \( \square \)

**Theorem 3.2**

A CSF \( \gamma \) is of the Natvig type if and only if it satisfies \( C_2 \) and

\( C_1' \) \( \gamma \) is right-continuous

\( C_4' \) For each \( i \in \mathcal{C} \) and all \( \alpha \in (0,1] \), there exists an \( \bar{x} \in \Delta \) such that \( \gamma(\alpha, \bar{x}) > \alpha \) whereas \( \gamma(\beta, \bar{x}) < \alpha \) for all \( \beta < \alpha \).

**Proof:** Baxter [3] proves that Natvig CSFs are right-continuous, and it is readily seen that such functions satisfy \( C_2 \) and \( C_4' \). Conversely, from the preceding proof,
\[ y(X) > \alpha \iff \max_{\gamma \in \mathcal{P}_\alpha} \min_{1 \leq i \leq n(\alpha)} Z_{\alpha i} = 1 \]

where \( Z_{\alpha i} \) is the indicator of \( \{X_i \geq \alpha\} \) \((0 < \alpha < 1, \mathcal{X}_\alpha)\). Write \( \mathcal{P}_\alpha = \{X(\alpha, 1), \ldots, X(\alpha, N(\alpha))\} \) and let \( T^\alpha_j = \{i \in C\mid X_i(\alpha, j) = \alpha\}, j = 1, 2, \ldots, N(\alpha) \).

Then \( y(X) > \alpha \) if and only if \( \phi_\alpha(Z_{\alpha}) = 1 \) where

\[ \phi_\alpha(Z_{\alpha}) = \max_{1 \leq i \leq N(\alpha)} \min_{i \in T^\alpha_j} Z_{\alpha i}. \]

We claim that the binary functions \( \{\phi_\alpha, 0 < \alpha < 1\} \) satisfy the conditions of the definition of the Natvig CSF.

It is clear that \( \phi_\alpha \) is nondecreasing in each argument for all \( \alpha \in (0, 1] \) and that \( \phi_\alpha(Z_{\alpha}) \) is nonincreasing in \( \alpha \) for fixed \( X \).

To verify left-continuity, it is sufficient to consider the point at which the function decreases. Thus, suppose that \( y(X) = \alpha \) \((0 < \alpha < 1)\); then there exists an \( X' \leq X \) such that \( X' \in \mathcal{P}_\alpha \). Clearly, \( y(X') = \alpha \) and hence \( \phi_\alpha(Z'_{\alpha}) = 1 \) whereas, if \( \beta > \alpha \), \( y(X') < \beta \) and so \( \phi_\beta(Z'_{\alpha}) = 0 \). Thus \( \phi_\alpha(Z_{\alpha}) \) is left-continuous as claimed.

Lastly, observe that, by C4', for each \( i \in C \) and all \( \alpha \in (0, 1] \), there exists an \( X \in \Delta \) such that \( \phi_\alpha(1_i, Z_{\alpha}) = 1 \) whereas \( \phi_\alpha(0_i, Z_{\alpha}) = 0 \) and so each \( \phi_\alpha \) is coherent.

This completes the proof. \( \Box \)
4. SOME REMARKS ON THE BORGES-RODRIGUES CHARACTERIZATION

Let $S = \{0,1,\ldots,M\}$, $M \geq 1$. A nondecreasing mapping $\phi: S^n \mapsto S$ with $\phi(0) = 0$ and $\phi(M) = M$ is said to be a multistate structure function (MSF). It is weakly coherent if $\max_{\tilde{X} \in S^n} \left[ \phi(M, \tilde{X}) - \phi(0, \tilde{X}) \right] > 1$ for each $i \in \mathbb{C}$.

If

$$\phi(\tilde{X}) = \max_{1 \leq j \leq r} \min_{\tilde{Y} \in P_j} \left[ \min_{\tilde{X} \in S^n} \tilde{X}_j \right]$$

where $P_1, \ldots, P_r$ are the $r$ minimal path sets of a binary coherent structure function, then $\phi$ is said to be a Barlow-Wu MSF [1]. If $\phi(\tilde{X}) \geq j$ if and only if $\phi_j(\tilde{Y}_j) = 1$ ($\tilde{X}_j \in S^n$, $j=1,2,\ldots,M$) where $\{\phi_1, \ldots, \phi_M\}$ is a collection of binary coherent structure functions such that $\phi_j(\tilde{Y}_j)$ is nonincreasing in $j$ for fixed $\tilde{X}$, and where $\tilde{Y}_j$ is the indicator of $\{X_j \geq j\}$, then $\phi$ is said to be a Natvig MSF [6].

Borges and Rodrigues [5] present axiomatic characterizations of the Barlow-Wu and Natvig MSFs in terms of the following conditions:

B1 For every $\tilde{X} \in S^n$ with $\phi(\tilde{X}) \geq k > 1$, there exists a $\tilde{Y} \in \{0,k\}^n$ such that $\tilde{Y} \leq \tilde{X}$ and $\phi(\tilde{Y}) \geq k$

B2 $\phi(\{0,M\}^n) = \{0,M\}$

B3 $\phi$ is weakly coherent.


(1) $\phi$ is a Barlow-Wu MSF if and only if it satisfies B1, B2 and B3

(2) $\phi$ is a Natvig MSF if and only if it satisfies B1 and B3.

Both claims are false as the following examples attest.
**Example 4.1**

Consider the MSF $\Phi_1: \{0,1,2\}^2 \mapsto \{0,1,2\}$ defined as follows:

$\Phi_1(0,0) = 0$ \quad $\Phi_1(0,1) = 0$ \quad $\Phi_1(0,2) = 2$

$\Phi_1(1,0) = 0$ \quad $\Phi_1(1,1) = 1$ \quad $\Phi_1(1,2) = 2$

$\Phi_1(2,0) = 2$ \quad $\Phi_1(2,1) = 2$ \quad $\Phi_1(2,2) = 2$.

This satisfies B1, B2 and B3 and yet is clearly not of the Barlow-Wu type since the only Barlow-Wu MSFs of size two are $X_1 \land X_2$ and $X_1 \lor X_2$. Notice in particular that $\Phi_1$ provides a counter-example to Lemma 4 of [5].

**Example 4.2**

Let $\phi_1(Y_{11}, Y_{12}) = Y_{11}$ and $\phi_2(Y_{21}, Y_{22}) = Y_{21} \land Y_{22}$ and define the MSF $\phi_2: \{0,1,2\}^2 \mapsto \{0,1,2\}$ as the function which satisfies $\phi_2(X_1, X_2) \geq j$

if and only if $\phi_j(Y_{1j}, Y_{2j}) = 1$ where $Y_{ij}$ is the indicator of $\{X_i \geq j\}$ ($i, j = 1, 2$). This is clearly not a Natvig MSF since the binary function $\phi_1$ is not coherent, but it is easily verified that $\phi_2$ satisfies B1 and B3.
REFERENCES


