AXIOMATIC CHARACTERIZATIONS OF CONTINUUM STRUCTURE FUNCTIONS*

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KEYWORDS: Reliability; continuum structure function; multistate structure function.

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1. INTRODUCTION

Let $C = \{1, 2, \ldots, n\}$ denote a set of components and let $\Delta = [0,1]^n$. A nondecreasing mapping $\gamma: \Delta \to [0,1]$ with $\gamma(0) = 0$ and $\gamma(1) = 1$ is said to be a continuum structure function (CSF). If $\sup_{X \in \Delta} [\gamma(1, X) - \gamma(O, X)] > 0$ for each $i \in C$, where $(\delta, X)$ denotes $(X_1, \ldots, X_{i-1}, \delta, X_{i+1}, \ldots, X_n)$, $\gamma$ is said to be weakly coherent.

Definition

Let $P_1, \ldots, P_r$ denote the $r$ minimal path sets of a binary coherent structure function. If

$$\gamma(X) = \max_{1 \leq j \leq r} \min_{i \in P_j} X_i \quad (X \in \Delta),$$

$\gamma$ is said to be a Barlow-Wu CSF [2].

Definition

Let $\{\phi_\alpha, 0 < \alpha < 1\}$ be a class of binary coherent structure functions such that $\phi_\alpha(Y_{\alpha})$ is a left-continuous and non-increasing function of $\alpha$ for fixed $\bar{X}$ where $Y_{\alpha_1}$ is the indicator of $\{X_i > \alpha\}$, $i=1,2,\ldots,n$. If

$$\gamma(X) \geq \alpha \text{ iff } \phi_\alpha(Y_{\alpha}) = 1 \quad (X \in \Delta, 0 < \alpha < 1),$$

$\gamma$ is said to be a Natvig CSF [3].

In this paper, we present axiomatic characterizations of the Barlow-Wu and Natvig CSFs. In particular, we show that $\gamma$ is a Barlow-Wu CSF if and only if it satisfies the following conditions:
C1  \( \gamma \) is continuous

C2  \( P_\alpha \neq \emptyset \) and \( P_\alpha \subseteq (0, \alpha]^n, 0 < \alpha \leq 1 \)

C3  There is no nonempty open set \( A \subseteq \Delta \) such that \( \gamma \) is constant on \( A \)

C4  \( \gamma \) is weakly coherent

where \( P_\alpha = \{ X \in \Delta | \gamma(X) > \alpha \} \) whereas \( \gamma(Y) < \alpha \) for all \( Y < X \}) \) and where \( Y < X \) means that \( Y \leq X \) but that \( Y \neq X \).

Some consequences of these axioms are deduced in Section 2, and in Section 3 we present our main results: an axiomatic characterization of the Barlow-Wu CSF and an analogous characterization of the Natvig CSF.

Our approach was suggested by the Borges-Rodrigues characterizations of the Barlow-Wu and Natvig multistate structure functions [5] though, as we show in Section 4, their characterizations are incorrect.

2. SOME DEDUCTIONS FROM THE AXIOMS

Let \( U_\alpha = \{ X \in \Delta | \gamma(X) > \alpha \} \) and \( L_\alpha = \{ X \in \Delta | \gamma(X) < \alpha \}, 0 < \alpha \leq 1 \). Further, define \( K_\alpha = \{ X \in \Delta | \gamma(X) < \alpha \} \) whereas \( \gamma(Y) > \alpha \) for all \( Y > X \}, 0 < \alpha < 1 \).

**Proposition 2.1**

Let \( \gamma \) be a CSF.

(i)  \( \gamma \) is right (left)-continuous if and only if each \( U_\alpha (L_\alpha) \) is closed.

(ii) If \( \gamma \) is right (left)-continuous, then each \( P_\alpha (K_\alpha) \) is nonempty and \( X \in U_\alpha (L_\alpha) \) if and only if \( X > (\leq) Y \in P_\alpha (K_\alpha) \).

(iii) If \( \gamma \) is continuous, then \( \gamma(P_\alpha) = \{ \alpha \}, 0 < \alpha \leq 1 \), and \( \gamma(K_\alpha) = \{ \alpha \}, 0 < \alpha < 1 \).
Proof: The proofs of (i) and (iii) are straightforward; see [4] for the proof of (ii).

Proposition 2.2

If $\gamma$ is a continuous CSF, conditions C2 and

$$\tag{C2'} K_{\alpha} \neq \emptyset \text{ and } K_{\alpha} \subseteq \{\alpha, 1\}^n, \quad 0 \leq \alpha < 1$$

are equivalent.

Proof: Since $\gamma$ is continuous, each $K_{\alpha}$ is nonempty. We show that, if C2 holds, then $K_{\alpha} \subseteq \{\alpha, 1\}^n$ for all $\alpha \in [0, 1)$.

Suppose, conversely, that for some $\alpha \in [0, 1)$ there exists a vector $y \in K_{\alpha}$ such that $y \notin \{\alpha, 1\}^n$. Then there exists at least one component, $k$, say, such that $y_k \notin \{\alpha, 1\}$. Either $0 < y_k < \alpha < 1$ or $0 < \alpha < y_k < 1$; we consider these two cases separately.

Suppose, firstly, that $0 \leq y_k < \alpha < 1$. By Proposition 2.1, $\gamma(y) = \alpha$ and $\gamma(\delta, y) > \alpha$ if $y_k < \delta < \alpha$. Let $\gamma(\delta, y) = \xi$; then $(\delta, y) \in U_{\xi}$. Since $U_{\xi}$ is closed there exists, by Proposition 2.1, an $x \leq (\delta, y)$ such that $x \in P_{\xi}$. Now $y \notin U_{\xi}$ and so $y_k < x_k \leq \delta$. Thus $0 < y_k < x_k \leq \delta < \alpha < \xi$ and so $x \notin \{0, \xi\}^n$, in contradiction to C2.

Suppose, now, that $0 < \alpha < y_k < 1$. Again $\gamma(y) = \alpha$. Let $\gamma(1, y) = \delta > \alpha$. Since $\gamma(x, y)$ is a continuous, nondecreasing function of $x$ for fixed $(x, y)$, it follows from the intermediate value theorem that, for given $\xi$ with $\alpha < \xi < y_k \wedge \delta$, there exists a $w \in (y_k, 1)$ such that $\gamma(w, y) = \xi$. Thus $(w, y) \in U_{\xi}$ and hence there exists an $x \leq (w, y)$ such that $x \in P_{\xi}$. Now $y \notin U_{\xi}$ and so $y_k < x_k \leq w$. It follows that $0 \leq \alpha < \xi < y_k < x_k \leq w$ and hence $x \notin \{0, \xi\}^n$, in contradiction to C2.
Thus, a continuous CSF satisfying C2 also satisfies C2'. A similar argument verifies the converse. □

Proposition 2.3

If \( \gamma \) is a CSF which satisfies Cl, C2 and C3, then \( \gamma([0,\alpha]^n) = \{0,\alpha\} \) for all \( \alpha \in [0,1] \).

Proof: If \( \alpha = 0 \) there is nothing to prove, so suppose that, for some \( \alpha \in (0,1] \), there exists a vector \( \mathbf{x} \in \{0,\alpha\}^n \) such that \( \beta = \gamma(\mathbf{x}) \notin \{0,\alpha\} \).

It is easily seen that \( 0 < \beta < \alpha \) and that \( \mathbf{x} \neq \mathbf{0} \) or \( \mathbf{g} \), and hence we can write

\[
X_{i,j} = \begin{cases} 
0 & \text{for } j = 1, 2, \ldots, k \\
\alpha & \text{for } j = k+1, \ldots, n
\end{cases}
\]

for some \( k \) with \( 1 \leq k \leq n-1 \).

Since \( \mathbf{x} \in U_{\beta} \cap L_{\beta} \), and both are closed, it follows from Proposition 2.1 that there exist a \( Z \in P_{\beta} \) and a \( W \in K_{\beta} \) such that \( Z \leq \mathbf{x} \leq W \). This ordering will only hold if \( Z \in (0,\beta)^n - \{0\} \) satisfies \( z_{i,j} = 0 \) for \( j = 1, 2, \ldots, k \) and if \( W \in (\beta,1)^n - \{1\} \) satisfies \( w_{i,j} = 1 \) for \( j = k+1, \ldots, n \) and so \( A = (z_1,w_1) \times \cdots \times (z_n,w_n) \subset \Delta \) is open. Further, since \( Z \in P_{\beta} \) and \( W \in K_{\beta} \), it follows that \( \gamma(\mathbf{x}) = \beta \) for all \( \mathbf{x} \in A \), in contradiction to C3. Thus \( \gamma(\mathbf{x}) \in \{0,\alpha\} \) as claimed. □

Proposition 2.4

If \( \gamma \) is a CSF which satisfies Cl, C2 and C3, then \( P_\alpha = \alpha P_1 \) for all \( \alpha \in (0,1] \).
Proof: Suppose that \( \alpha < 1 \), otherwise there is nothing to prove, and let \( \bar{X} \in P_\alpha \) so that \( \gamma(\bar{X}) = \alpha \). Then \( \bar{X} < \frac{1}{\alpha} \bar{X} \) and so \( \gamma(\bar{X}) < \gamma(\frac{1}{\alpha} \bar{X}) \). Since \( \frac{1}{\alpha} \bar{X} \in \{0,1\}^n \), it follows from Proposition 2.3 that \( \gamma(\frac{1}{\alpha} \bar{X}) = 1 \). We claim that \( \frac{1}{\alpha} \bar{X} \in P_1 \).

Suppose, conversely, that \( \frac{1}{\alpha} \bar{X} \notin P_1 \). Since \( U_1 \) is closed, it follows from Proposition 2.1 that there exists a \( \bar{W} < \frac{1}{\alpha} \bar{X} \) such that \( \bar{W} \in P_1 \). Consider the vector \( \alpha \bar{W} \in \{0,\alpha\}^n \); it is easily seen that \( \gamma(\alpha \bar{W}) = \alpha \) and thus there exists a vector \( \alpha \bar{W} < \bar{X} \) such that \( \alpha \bar{W} \in U_\alpha \). This contradicts the assumption that \( \bar{X} \in P_\alpha \) and hence \( \frac{1}{\alpha} \bar{X} \in P_1 \) as claimed. This holds for all \( \bar{X} \in P_\alpha \) and so \( P_\alpha \subset \alpha P_1 \).

Similarly, it can be shown that \( \alpha P_1 \subset P_\alpha \). \( \square \)

3. THE CHARACTERIZATION THEOREMS

Theorem 3.1

A CSF \( \gamma \) is of the Barlow-Wu type if and only if it satisfies conditions C1, C2, C3 and C4.

Proof: It is easily verified that the Barlow-Wu CSF satisfies C1, C2, C3 and C4. To prove the converse, observe that

\[
\gamma(\bar{X}) \geq \alpha \iff \bar{X} \geq \bar{Y} \in P_\alpha \\
\iff \min_{\{i|Y_i=\alpha\}} X_i \geq \alpha \text{ for some } \bar{Y} \in P_\alpha \\
\iff \max_{\bar{Y} \in P_\alpha} \min_{\{i|Y_i=\alpha\}} X_i \geq \alpha.
\]
max \min_{X \in \mathcal{P}_1 \{i|Y_i = \alpha\}} X_i \geq \alpha \text{ by Proposition 2.4}

\max \min_{Z \in \mathcal{P}_1 \{i|Z_i = 1\}} X_i \geq \alpha \text{ where } Z = \frac{1}{\alpha} Y_i.

This holds for all \( X \in \Delta \) and \( \alpha \in (0,1] \) and so

\gamma(X) = \max \min_{Z \in \mathcal{P}_1 \{i|Z_i = 1\}} X_i.

Write \( P_1 = \{X_1^{(1)}, \ldots, X_1^{(N)}\} \) and let \( T_j = \{i \in \mathcal{C}|X_i^{(j)} = 1\} \). By the definition of \( P_1 \), it is clear that each \( T_j \) is nonempty and that \( T_j \neq T_k \) for all \( j, k = 1, 2, \ldots, N \) with \( j \neq k \). Thus

\gamma(X) = \max_{1 \leq j \leq N} \min_{i \in T_j} X_i

where each \( T_j \subset \mathcal{C} \). Condition C4 ensures that \( \bigcup_{j=1}^{N} T_j = \mathcal{C} \), completing the proof. \( \square \)

Theorem 3.2

A CSF \( \gamma \) is of the Natvig type if and only if it satisfies C2 and

C1' \( \gamma \) is right-continuous

C4' For each \( i \in \mathcal{C} \) and all \( \alpha \in (0,1] \), there exists an \( X \in \Delta \) such that \( \gamma(\alpha, X) \geq \alpha \) whereas \( \gamma(\beta, X) < \alpha \) for all \( \beta < \alpha \).

Proof: Baxter [3] proves that Natvig CSFs are right-continuous, and it is readily seen that such functions satisfy C2 and C4'. Conversely, from the preceding proof,
\[
\gamma(X) > \alpha \iff \max_{Y \in P_{\alpha}} \min_{i \mid Y_i = \alpha} Z_{\alpha i} = 1
\]

where \(Z_{\alpha i}\) is the indicator of \(\{X_i \geq \alpha\}\) \((0 < \alpha < 1, X \in \Delta)\). Write \(P_{\alpha} = \{X_{(\alpha,1)}, \ldots, X_{(\alpha,N(\alpha))}\}\) and let \(T_{\alpha j} = \{i \in C \mid X_{i j}^{(\alpha)} = \alpha\}\), \(j = 1, 2, \ldots, N(\alpha)\).

Then \(\gamma(X) > \alpha\) if and only if \(\phi_{\alpha}(Z_{\alpha}) = 1\) where

\[
\phi_{\alpha}(Z_{\alpha}) = \max_{1 < i < N(\alpha)} \min_{i \in T_{\alpha j}} Z_{\alpha i}.
\]

We claim that the binary functions \(\{\phi_{\alpha}, 0 < \alpha < 1\}\) satisfy the conditions of the definition of the Natvig CSF.

It is clear that \(\phi_{\alpha}\) is nondecreasing in each argument for all \(\alpha \in (0,1]\) and that \(\phi_{\alpha}(Z_{\alpha})\) is nonincreasing in \(\alpha\) for fixed \(X\).

To verify left-continuity, it is sufficient to consider the point at which the function decreases. Thus, suppose that \(\gamma(X) = \alpha \ (0 < \alpha < 1)\); then there exists an \(X' < X\) such that \(X' \in P_{\alpha}\). Clearly, \(\gamma(X') = \alpha\) and hence \(\phi_{\alpha}(Z_{\alpha'}) = 1\) whereas, if \(\beta > \alpha\), \(\gamma(X') < \beta\) and so \(\phi_{\beta}(Z_{\beta'}) = 0\). Thus \(\phi_{\alpha}(Z_{\alpha})\) is left-continuous as claimed.

Lastly, observe that, by C4', for each \(i \in C\) and all \(\alpha \in (0,1]\), there exists an \(X \in \Delta\) such that \(\phi_{\alpha}(1_i, Z_{\alpha}) = 1\) whereas \(\phi_{\alpha}(0_i, Z_{\alpha}) = 0\) and so each \(\phi_{\alpha}\) is coherent.

This completes the proof. \(\square\)
4. SOME REMARKS ON THE BORGES-RODRIGUES CHARACTERIZATION

Let $S = \{0, 1, \ldots, M\}$, $M \geq 1$. A nondecreasing mapping $\phi: S^n \mapsto S$ with $\phi(0) = 0$ and $\phi(M) = M$ is said to be a multistate structure function (MSF). It is weakly coherent if $\max \{\phi(M_i, X) - \phi(0, X)\} \geq 1$ for each $i \in C$. If

$$\phi(X) = \max_{1 \leq j \leq r} \min_{i \in P_j} X_i \quad (X \in S^n)$$

where $P_1, \ldots, P_r$ are the $r$ minimal path sets of a binary coherent structure function, then $\phi$ is said to be a Barlow-Wu MSF [1]. If $\phi(Y) \geq j$ if and only if $\phi_j(Y_j) = 1$ ($X \in S^n$, $j=1, 2, \ldots, M$) where $\{\phi_1, \ldots, \phi_M\}$ is a collection of binary coherent structure functions such that $\phi_j(Y_j)$ is nonincreasing in $j$ for fixed $X$, and where $Y_{ji}$ is the indicator of $\{X_i \geq j\}$, then $\phi$ is said to be a Natvig MSF [6].

Borges and Rodrigues [5] present axiomatic characterizations of the Barlow-Wu and Natvig MSFs in terms of the following conditions:

B1 For every $X \in S^n$ with $\phi(X) \geq k \geq 1$, there exists a $Y \in \{0, k\}^n$ such that $Y < X$ and $\phi(Y) \geq k$

B2 $\phi((0,M)^n) = \{0, M\}$

B3 $\phi$ is weakly coherent.


(1) $\phi$ is a Barlow-Wu MSF if and only if it satisfies B1, B2 and B3

(2) $\phi$ is a Natvig MSF if and only if it satisfies B1 and B3.

Both claims are false as the following examples attest.
Example 4.1

Consider the MSF $\Phi_1: \{0,1,2\}^2 \to \{0,1,2\}$ defined as follows:

\[
\begin{align*}
\Phi_1(0,0) &= 0 & \Phi_1(0,1) &= 0 & \Phi_1(0,2) &= 2 \\
\Phi_1(1,0) &= 0 & \Phi_1(1,1) &= 1 & \Phi_1(1,2) &= 2 \\
\Phi_1(2,0) &= 2 & \Phi_1(2,1) &= 2 & \Phi_1(2,2) &= 2.
\end{align*}
\]

This satisfies B1, B2 and B3 and yet is clearly not of the Barlow-Wu type since the only Barlow-Wu MSFs of size two are $X_1 \land X_2$ and $X_1 \lor X_2$. Notice in particular that $\Phi_1$ provides a counter-example to Lemma 4 of [5].

Example 4.2

Let $\phi_1(Y_{11}, Y_{12}) = Y_{11}$ and $\phi_2(Y_{21}, Y_{22}) = Y_{21} \land Y_{22}$ and define the MSF $\Phi_2: \{0,1,2\}^2 \to \{0,1,2\}$ as the function which satisfies $\Phi_2(X_1, X_2) \geq j$ if and only if $\phi_j(Y_{j1}, Y_{j2}) = 1$ where $Y_{ji}$ is the indicator of $\{X_i \geq j\}$ ($i,j=1,2$). This is clearly not a Natvig MSF since the binary function $\phi_1$ is not coherent, but it is easily verified that $\Phi_2$ satisfies B1 and B3.
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