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Solitons and SEASAT
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Solitons and SEASAT

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ABSTRACT

It has been suggested that Soliton formation might be relevant to SeaSat observations. It has also been said that there are no Solitons in more than one space dimension. The second statement is demonstrated to be false. The Kadomtsev-Petviashvile equation relevant to Internal Waves is shown not to have Soliton solutions. This lends support to the view that Solitons and SeaSat have little in common.
1.0 **INTRODUCTION**

The persistence of long V-shaped wakes observed by SeaSat has led to suggestions that the phenomena may be related to Internal Wave Solitons. Most observations were made under conditions for which one would have little or no reason to expect Solitons to be relevant. However, there is one case (Rev. 407) for which one might think otherwise. This was a big ship in shallow water with a strong thermidine.

A priori there is little likelihood that the equations describing internal waves generated by ships will admit Soliton solutions. (By a Soliton we mean a non-singular disturbance localized in space at any time which retains its integrity on interactions with similar disturbances.) While there are many equations in one space dimension which have Soliton solutions, little is known about spaces of higher dimensions. (Indeed it is sometimes said that there are no true solitons in case the number of space dimensions is greater than one. Below we will see this is not true.)

Since the paradigm of an equation describing Solitons is the Korteweg-deVries (KdV) equation it is natural to take as our starting point an equation closely related to this which does
describe fully three dimensional internal waves. (The class of solutions described is not exactly those we would expect to be ship produced. However, it is hoped that the results obtained will give support to our conviction that Internal Wave Solitons are not relevant for the SeaSat photographs.)

The equation we have in mind is

\[ u_t + \frac{1}{2} \left( \frac{\partial}{\partial x} \right)^2 u_{yy} + \frac{1}{2} \left( \frac{\partial}{\partial x} \right) (3u^2 + u_{xx}) = 0 \]  

(1)

where

\[ \frac{1}{2} \left( \frac{\partial}{\partial x} \right)^2 = \frac{1}{2} \left[ \int_{-\infty}^{x} - \int_{-\infty}^{x'} \right] \phi(x') \; dx' \]

Closely related to Eq. (1) is the equation

\[ u_t - \frac{1}{2} \left( \frac{\partial}{\partial x} \right)^2 u_{yy} + \frac{1}{2} \left( \frac{\partial}{\partial x} \right) (3u^2 + u_{xx}) = 0 \]  

(2)

The Eqs. (1) and (2) are known as the Kadomtsev–Petviashvili (1) equations. A derivation of Eq. (1) in the case of internal waves is given in reference (2). An important point is that for internal waves the sign of the term in Eq. (1) involving \( u_{yy} \) is unambiguously required to be plus. (There are physical situations...
where Eq. (2) can hold. An example is when capillarity is important.

Our main result is the following: The extension of the N-Soliton solutions of K-deV to solutions of Eqs. (1) and (2) generally break up asymptotically into K-deV Solitons moving in arbitrary directions in the x-y plane. (These are then plane waves.) They are physically unacceptable as being non-localized. However, for special values of the parameters these solutions are localized in x,y. However, for Eq. (1) these localized solutions are singular. Hence these solutions again are not physical. (This is not so for Eq. (2). True localized, non-singular, non-interacting "lumps" result. We include these primarily to show that multi-dimensional Solitons do indeed exist.)

In Section 2 the formalism used by Zakharov and Shabat (3) to integrate the K-P equations is summarized. The generalization of the N-Soliton KdeV solution is given in Section 3. In Section 4 the "lumps" which result when special relations exist between the N-Soliton parameters is presented.
2.0 FORMALISM

Zakharov and Shabat (3) have introduced a method to integrate Eqs. (1) and (2). We summarize this here.

Let \( F(x,z;y,t) \) satisfy the two equations

\[
\frac{1}{4} \frac{3F}{3t} + \frac{3F}{3x} + \frac{3F}{3z} = 0 ,
\]

(3)

and

\[
\frac{1}{\sqrt{3}} \frac{3F}{3y} - \frac{3F}{3x^2} - \frac{3F}{3z^2} = 0 .
\]

(4)

Determine \( K(x,z;y,t) \) from the Volterra equation

\[
F(x,z) + K(x,z) + \int_x^\infty K(x,s) F(s,z) ds = 0
\]

(5)

(Here we have suppressed the parametric arguments \( y \) and \( t \).)
Then they show that

$$u = 2 \frac{d}{dx} K(x, x)$$  \hspace{1cm} (6)

satisfies Eq. (1).

To find solutions of Eq. (2) we merely note that from any solution of Eq. (1) we can obtain a solution of Eq. (2) by the replacement $y \rightarrow iy$. 
3.0 GENERALIZATION OF THE N-SOLITON K-deV SOLUTIONS

We follow reference (3).

A. Suppose

\[ F = e^{-\kappa x - \eta z} M(t, y) \]  

(7)

From Eq. (3) we find

\[ M = C(y) e^{4(\kappa^3 + \eta^3) t} \]  

(8)

Then Eq. (4) gives

\[ C = M_0 e^{-\sqrt{3}(\kappa^2 - \eta^2) y} \]  

(9)

i.e.

\[ M = M_0 e^{-\sqrt{3}(\kappa^2 - \eta^2) y + 4(\kappa^3 + \eta^3) t} \]  

(10)

Our Gelfand-Levitan Equation (5) becomes

\[ K(x, z) + M_0 e^{-\kappa x - \eta z} + \int_{x}^{\infty} K(x, s) M_0 e^{-\kappa s - \eta z} ds = 0, \]  

(11)
Clearly $K(x,z) = K(x) e^{-\eta z}$. Inserting in Eq. (11) yields

$$K(x) + M e^{-\kappa x} + \frac{M e^{-(\kappa + \eta)x}}{\kappa + \eta} = 0 \quad (12)$$

Solving gives

$$K(x,z) = \frac{-M e^{-(\kappa x + \eta z)}}{1 + \frac{M}{\kappa + \eta} e^{-(\kappa + \eta)x}} \quad (13)$$

Then

$$K(x,x) = \frac{-M e^{-(\kappa + \eta)x}}{1 + \frac{M}{\kappa + \eta} e^{-(\kappa + \eta)x}}$$

$$\equiv \frac{\partial}{\partial x} \ln \left[ 1 + \frac{M}{\kappa + \eta} e^{-(\kappa + \eta)x} \right]$$

and so

$$u = 2 \frac{\partial^2}{\partial x^2} \ln \left[ 1 + \frac{N}{\kappa + \eta} e^{-(\kappa + \eta)x} \right] \quad (14)$$

Defining $x_o$ by

$$(\kappa + \eta) x_o = \ln \frac{M}{\kappa + \eta}$$
gives

\[ x_0 = \frac{1}{\kappa + n} \ln \frac{M_0}{\kappa + n} + \sqrt{3} (n - \kappa) y + 4(\kappa^2 - \kappa n + n^2) t \]  

(15)

Then Eq. (14) becomes

\[ u = \frac{1/2 (\kappa + n)^2}{\cosh^2 (\kappa + n)(x - x_0)} \]  

(16)

To interpret this consider \( \kappa = n \) then

\[ U = \frac{2\kappa^2}{\cosh^2 \kappa(x - x_0)} \]  

(17)

with

\[ x_0 = \text{constant} + 4\kappa^2 t \]

This is just a K-deV Soliton. For \( \kappa \neq n \) the solution (16) is then a K-deV type soliton propagating at an arbitrary angle with respect to the x-axis. This is a plane wave - it is constant on the line

\[ x - \sqrt{3} (n^2 - \kappa^2) y = \text{constant} , \]
and hence physically really not acceptable.

B) The above is readily extended to get the 2-dimensional extension of the N-Soliton K-deV solution. Thus we generalize Eq. (10) by choosing

$$F = \sum_{n} M_n(t,y)e^{\kappa_n x - \eta_n z}$$  \hspace{1cm} (18)

Since Eq. (3) and (4) are linear we have in analogy to Eq. (10)

$$M_n = M_n(0) e^{[\sqrt{3}(\eta_n^2 - \kappa_n^2)y + 4(\kappa_n^3 + \eta_n^3)t]}$$  \hspace{1cm} (19)

The degenerate integral equation for K(x,z) is then satisfied by

$$K(x,z) = \sum_{n} K_n(x) e^{\eta_n z}$$  \hspace{1cm} (20)

where

$$K_n(x) + M_n e^{\kappa_n x} + M_n \sum_{m} e^{-(\kappa_m + \eta_m)x} K_m(x) = 0$$  \hspace{1cm} (21)

This is readily solved using Cramer's rule. Thus
\[ K_n(x) = \frac{A_n(x)}{\Delta(x)} \]  \hspace{1cm} (22)

where

\[ \Delta = \det \left( \delta_{nm} + \frac{M_n e^{-(\kappa_n + n_m)x}}{\kappa_n + n_m} \right) \]  \hspace{1cm} (23)

and \( A_n \) is obtained from \( \Delta \) by replacing the \( n' \)th column of the matrix by the vector

\[ (-M_1 e^{-\kappa_1 x}, -M_2 e^{-\kappa_2 x}, \ldots) \].

Suggested by the form of Eq. (14) we look at \( \Sigma A_n(x) e^{-nx} \) and verify that

\[ \Sigma A_n(x) e^{-nx} = \frac{\partial}{\partial x} \Delta(x) \]

therefore \( K(x,x) = \frac{\partial}{\partial x} \ln \Delta \) and so

\[ u(x) = 2 \frac{\partial^2}{\partial x^2} \ln \Delta \]  \hspace{1cm} (24)

To interpret this result we consider the limit as \( y, t \) go to infinity. Assuming no special relations between the
various pairs \((\kappa_n, \eta_n)\). Then we note that for large \(|y|\), \(|t|\) there will be regions where one of the \(M_n\) (say \(M_j\)) is much larger than all others. Hence then

\[
\Delta = 1 + \frac{M_j e^{-(\kappa_j + \eta_j)x}}{\kappa_j + \eta_j},
\]

(25)

which is just the single soliton result of Eq. (14). Thus asymptotically the solution breaks up into a sum of the simple "plane soliton" solutions.
4.0 DEGENERATE CASES

It is well known that the K-deV equation has in addition to solutions like those of Eq. (24) solutions which are rational functions of the coordinates. These can be obtained by making a suitable ansatz for the form of solution. This is

\[ U = \sum c_n \frac{1}{n} \left[ x - a_n(t) \right]^\alpha \]

(26)

Inserting in the K-deV shows that this will be a solution if \( \alpha = 2 \), the \( c_n \) are constants and the \( a_n \) satisfy simple coupled ordinary differential equations. An alternative approach is to take the general solution of Eq. (24), specify relations between the parameters, \( \kappa_n, \eta_n \), and pass to limits. This is the procedure given in reference (4) - and the one we will follow.

We have seen that a solution is obtained from Eq. (24) with \( \Delta \) given by Eq. (23). Introduce \( \lambda_n, \gamma_n \) by

\[ \kappa_n = \frac{\lambda_n + \gamma_n}{2}, \quad \eta_n = \frac{\lambda_n - \gamma_n}{2} \]

(27)

then choosing \( \gamma_n(0) = -a_n \lambda_n \) we have \( \Delta = \det \Gamma \) where
\[ \Gamma_{nn} = \delta_{nn} + \frac{2 M_n}{\gamma_n - \gamma_m + \lambda_n + \lambda_m} \quad (28) \]

\[ M_n = -a_n \lambda_n e_n^\dagger \quad (29) \]

\[ [L_n] = [-\sqrt{3} \gamma_n y + (\lambda_n^2 + 3\gamma_n^2) \tau] \quad (30) \]

Now look at the limit as all \( \lambda_n \rightarrow 0 \).

Assuming \( a_n \sim 1 + \zeta_n \lambda_n \) we have on expanding in \( \lambda_n \) and keeping only matrix elements of first order in the \( \lambda_n \)

\[ \Gamma_{nn} = \lambda_n d_n \quad (28) \]

\[ n \neq m \quad \Gamma_{nm} = \frac{-2\lambda_n}{\gamma_n - \gamma_m} \quad (31) \]

where

\[ d_n = x - \zeta_n + \sqrt{3} \gamma_n y - 3\gamma_n^2 \tau \quad (32) \]

Then
\[
\det \Gamma = (h \lambda_n) \det \Gamma'
\]  
(33)

with \(\Gamma_{nn}' = d_n\)

\[
n \neq m \quad \Gamma_{nm}' = \frac{-2}{\gamma_n - \gamma_m}.
\]  
(34)

We can then write

\[
u = 2 \frac{\partial^2}{\partial x^2} \ln \det \Gamma'
\]  
(35)

The essential points are seen by considering the case of \(N = 2\). Then

\[
\Delta' = \det \Gamma' = d_1 d_2 + \frac{4}{(\gamma_1 - \gamma_2)^2}
\]  
(36)

\[
= [x - \zeta_1 + \sqrt{3}\gamma_1 y - 3\gamma_1^2 t][x - \zeta_2 + \sqrt{3}\gamma_2 y - 3\gamma_2^2 t]
\]

\[
+ \frac{4}{(\gamma_1 - \gamma_2)^2}
\]  
(37)

Clearly \(u\) is then a rational function of \(x, y, t\). Consider various possibilities for \(\zeta_n, \gamma_n\).

\((1) \quad \zeta_n, \gamma_n \text{ real. The solution is clearly singular.}\)
(There are points where $\Delta' = 0$.)

(ii) $\zeta_n, \gamma_n$ complex. For the solution to be real we must have

\[
\zeta_1 = \zeta, \quad \zeta_2 = \zeta^* \\
\gamma_1 = \gamma, \quad \gamma_2 = \gamma^*
\]

Then

\[
\Delta' = -\frac{1}{\gamma_1^2} + \left| x - \zeta + \sqrt{3} \gamma y - 3\gamma^2 t \right|^2 \quad (38)
\]

The second term can vary from 0 to $\infty$ and therefore $\Delta'$ vanishes at least once.

These rational solutions of Eq. (1) are physically unacceptable.

On the other hand we noted that a solution of Eq. (2) is obtained from one of Eq. (1) by the replacement $y + iy$. In such a case

\[
\Delta' + \left[ x - \zeta_1 + i\sqrt{3} \gamma_1 y - 3\gamma_1^2 t \right] \left[ x - \zeta_2 + i\sqrt{3} \gamma_2 y - 3\gamma_2^2 t \right]
\]
Again if the $\gamma_r$, $\gamma_r$ are real the solution is singular. If these are complex we must require in order that the solution be real that $\gamma_1 \equiv \gamma$, $\gamma_2 = -\gamma^*$. In this case

$$\Delta^* = \frac{1}{\gamma_r^2} + \frac{1}{|x - \zeta + r\sqrt{3} \gamma y - 3y^2 t|^2}.$$  

(40)

If $\gamma_r$ (real part of $\gamma$) is non-zero this obviously gives a non-singular solution. It is also well-behaved at infinity. As $|x|$ (\(|y|\)) go infinity $U \sim \frac{1}{x} \left( \frac{1}{y} \right)$. \)
5.0 CONCLUSIONS

Two dimensional Solitons exist. However, some equations describing internal waves which might be strongly suspected of having Soliton solutions do not.

While not definitive this lends significant support to the view that Solitons have no connection with the phenomena observed by SeaSat.
REFERENCES


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