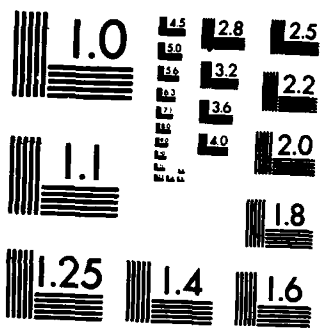
AD-A147 236 THE THEORY OF OPTIMAL CONFIDENCE LIMITS FOR SYSTEMS 1/1
RELIABILITY WITH COUN..(U) WISCONSIN UNIV-MADISON
MATHEMATICS RESEARCH CENTER B HARRIS ET AL. AUG 84
UNCLASSIFIED MRC-TSR-2726 DAAG29-80-C-0041 F/G 12/1 NL

END


1.0
2.8
2.5
3.2
2.2
3.6
1.1
4.0
2.0
1.8
1.25
1.4
1.6

3

MRC Technical Summary Report # 2726

THE THEORY OF OPTIMAL CONFIDENCE LIMITS FOR SYSTEMS RELIABILITY WITH COUNTEREXAMPLES FOR RESULTS ON OPTIMAL CONFIDENCE LIMITS FOR SERIES SYSTEMS

Bernard Harris
and
Andrew P. Soms

AD-A147 236

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

August 1984

(Received April 15, 1983)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

Office of Naval Research
Department of the Navy
Arlington, VA 22217

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

THE THEORY OF OPTIMAL CONFIDENCE LIMITS FOR SYSTEMS RELIABILITY WITH COUNTEREXAMPLES FOR RESULTS ON OPTIMAL CONFIDENCE LIMITS FOR SERIES SYSTEMS

Bernard Harris* and Andrew P. Soms**

Technical Summary Report #2726

August 1984

Justification

By

Distribution/

Availability Codes

Dist

A-1

ABSTRACT

This paper gives the general theory of optimal confidence limits for systems reliability introduced by Buehler (1957). This is specialized to series systems. It is noted that some results previously given are false. In particular, counterexamples for results of Sudakov (1974), Winterbottom (1974) and Harris and Soms (1980, 1981) are given. Numerical examples are provided, which suggest that despite the theoretical problems of the results, they are nevertheless valid for significance levels likely to be used in practice. Originator supplied keywords include:

AMS (MOS) Subject Classifications: 62N05, 90B25

Key Words: Optimal confidence bounds, Reliability; Series system

Work Unit Number 4 - Statistics and Probability

*University of Wisconsin-Madison

**University of Wisconsin-Milwaukee

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and the Office of Naval Research under Contract No. N00014-79-C-0321.

SIGNIFICANCE AND EXPLANATION

Systems with independent components arise naturally in engineering practice. Therefore it is of importance to efficiently utilize data obtained on individual components to obtain an assessment of the reliability of the system.

This paper presents a unified theory for doing so and points out errors in previous results on series systems.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

THE THEORY OF OPTIMAL CONFIDENCE LIMITS FOR SYSTEMS RELIABILITY WITH COUNTEREXAMPLES FOR RESULTS ON OPTIMAL CONFIDENCE LIMITS FOR SERIES SYSTEMS

Bernard Harris* and Andrew P. Soms**

1. Introduction and Summary

A problem of substantial importance to practitioners in reliability is the statistical estimation of the reliability of a system of stochastically independent components using experimental data collected on the individual components. In the situations discussed in this paper, the component data consist of a sequence of Bernoulli trials. Thus, for component i, $i=1,2,\ldots,k$, the data is the pair (n_i,Y_i), where n_i is the number of trials and Y_i is the number of observations for which the component functions. $Y_1,Y_2,\ldots,Y_k$ are assumed to be mutually independent random variables.

This problem was treated in Sudakov (1974), Winterbottom (1974), and Harris and Soms (1980,1981); one purpose of the present paper is to exhibit counterexamples to theorems in the above papers.

In Section 2 we discuss the general theory of optimal confidence limits for system reliability so that the notation and definitions to be employed in the balance of the paper have been prescribed.

In Section 3 the counterexamples previously mentioned are exhibited and the specific errors in the proofs of the theorems are indicated.

*University of Wisconsin-Madison

**University of Wisconsin-Milwaukee

Sponsored by the United State Army under Contract No. DAAG29-80-C-0041 and the Office of Naval Research under Contract No. N00014-79-C-0321.

Section 4 presents the proof of a special case of the key test theorem (Winterbottom (1974)), the general form of which was invalidated by a counterexample in Section 3.

The consequences for reliability applications are discussed in Section 5.

2. Buehler's Method for Optimal Lower Confidence Limits for System Reliability

We now introduce the notation, definitions, and assumptions that will be us^d throughout the balance of this paper.

1. Let p_i, $i=1,2,\ldots,k$ denote the probability that the i^{th} component functions. The components will be assumed to be stochastically independent. The reliability of the system will be denoted by $h(\tilde{p})$, where $\tilde{p} = (p_1,p_2,\ldots,p_k)$, $0 \leq p_i \leq 1$. It is assumed that $h(0,0,\ldots,0) = 0$, $h(1,1,\ldots,1) = 1$, and that $h(\tilde{p})$ is non-decreasing in each p_i, $i=1,2,\ldots,k$. Further, $h(\tilde{p})$ is continuous on $\{\tilde{p}|0 \leq p_i \leq 1\}$, which follows readily from the assumption of independence. These properties hold for coherent systems (see Barlow and Proschan (1975)).

2. Let $S = \{\tilde{x}|x_i = 0, 1,\ldots,n_i, i=1,2,\ldots,k\}$ be the failure set. $g(\tilde{x})$ is said to be an ordering function if for $x_1 \leq z_1$, $x_2 \leq z_2,\ldots,x_k \leq z_k$, $\tilde{x}$, $\tilde{z} \in S$, $g(\tilde{x}) \geq g(\tilde{z})$. (It is often convenient to normalize $g(\tilde{x})$ by letting $g(\tilde{0}) = 1$ and $g(\tilde{n}) = 0$. With such a normalization, $g(\tilde{x})$ is often selected to be a point estimator of $h(\tilde{p})$.)

3. Let $R = \{r_1,r_2,\ldots,r_s, s \geq 2\}$ be the range set of $g(\tilde{x})$. With no loss of generality we order R so that $r_1 > r_2 > \ldots > r_s$.

4. Let $A_i = \{\tilde{x} | g(\tilde{x}) = r_i, \tilde{x} \in S, i=1,2,\ldots,s\}$. The sets A_i constitute a partition of S induced by $g(\tilde{x})$.

5. We assume throughout that the data is distributed by

$$f(\tilde{x};\tilde{p}) = p_{\tilde{p}}(\tilde{X}=\tilde{x}) = \prod_{i=1}^{k} \binom{n_i}{x_i} p_i^{n_i-x_i} q_i^{x_i} = \prod_{i=1}^{k} \binom{n_i}{y_i} p_i^{y_i} q_i^{n_i-y_i}, \quad (2.1)$$

where $q_i = 1-p_i$, $x_i = n_i-y_i$, $i=1,2,\ldots,k$. With no loss of generality, we assume $n_1 \leq n_2 \leq \ldots \leq n_k$.

From these definitions, it follows that

$$P_{\tilde{p}}\left\{X \in \bigcup_{i=1}^{j} A_i\right\} = P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_j\right\} . \quad (2.2)$$

From (2.1) and (2.2), we have

$$P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_j\right\} = \sum_{i_1=0}^{u_1} \sum_{i_2=0}^{u_2} \cdots \sum_{i_k=0}^{u_k} f(\tilde{I};\tilde{p}) , \quad (2.3)$$

where $\tilde{I} = (i_1,i_2,\ldots,i_k)$ and $u_2 = u_2(i_1),\ldots,u_k = u_k(i_1,i_2,\ldots,i_{k-1})$ are integers determined by r_j.

6. Subsequently we will need to extend the definitions of S and $g(\tilde{x})$ to real values. We denote this as follows. Let

$$S^* = \left\{\tilde{x} | 0 \leq x_i \leq n_i, i=1,2,\ldots,k\right\} .$$

We assume that $g(\tilde{x})$ is nonincreasing on S^*. This requirement is satisfied by all ordering functions used in practice.

Then

$$P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_j\right\} = \sum_{i_1=0}^{[t_1]} \sum_{i_2=0}^{[t_2]} \cdots \sum_{i_k=0}^{[t_k]} f(\tilde{I};\tilde{p}) , \quad (2.4)$$

where $t_2 = t_2(i_1),\ldots,t_k = t_k(i_1,i_2,\ldots,i_{k-1})$, with $t_1 = \sup\left\{t | t \in S^* \text{ and } g(t,0,0,\ldots,0) \geq r_j\right\}$ and $t_\ell(i_1,i_2,\ldots,i_{\ell-1}) = \sup\left\{t | t \in S^* \text{ and } g(i_1,i_2,\ldots,i_{\ell-1},t,0,\ldots 0) \geq r_j\right\}$, $\ell=2,3,\ldots,k$.

We now introduce the notion of Buehler optimal confidence limits. Let $g(x) = r_j$. Then define

$$a_{g(\tilde{x})} = \inf\left\{h(\tilde{p}) \,\middle|\, P_{\tilde{p}}\left\{\tilde{i} \,|\, g(\tilde{i}) \geq g(\tilde{x})\right\} \geq \alpha\right\} . \tag{2.5}$$

Equivalently, by (2.2), we can also write

$$a_{g(\tilde{x})} = \inf\left\{h(\tilde{p}) \,\middle|\, P_{\tilde{p}}\left\{X \in \bigcup_{i=1}^{j} A_i\right\} \geq \alpha\right\} . \tag{2.6}$$

We now establish the following theorem.

<u>Theorem 2.1</u>. Let assumptions 1-5 be satisfied. Then, for $\tilde{x} \in S$, $a_{g(\tilde{x})}$ is a 1-α lower confidence limit for $h(\tilde{p})$. If $b_{g(\tilde{X})}$ is any other 1-α lower confidence limit for $h(\tilde{p})$ with $b_{r_1} \geq b_{r_2} \geq \ldots \geq b_{r_j}$, then $b_{g(\tilde{x})} \leq a_{g(\tilde{x})}$ for all $\tilde{x} \in S$.

<u>Proof</u>. Fix $\tilde{p}$ and let $m(\tilde{p})$ be the smallest integer such that

$$P_{\tilde{p}}\left\{\tilde{X} \in \bigcup_{i=1}^{m(\tilde{p})} A_i\right\} \geq \alpha .$$

Then

$$P_{\tilde{p}}\left\{\tilde{X} \in \bigcup_{i=m(\tilde{p})}^{s} A_i\right\} \geq 1-\alpha .$$

Let

$$D_{r_m} = \left\{\tilde{p} \,\middle|\, P_{\tilde{p}}\left\{\tilde{X} \in \bigcup_{i=1}^{m} A_i\right\} \geq \alpha\right\} .$$

Then $D_{g(\tilde{X})}$ is a 1-α confidence set for $\tilde{p}$, since

$$P_{\tilde{p}}\left\{\tilde{p} \in D_{g(\tilde{X})}\right\} = P_{\tilde{p}}\left\{g(\tilde{X}) \leq r_{m(\tilde{p})}\right\} \geq 1-\alpha .$$

By assumption 1, $h(\tilde{p})$ is continuous and the set of parameter points satisfying (2.5) is compact; therefore the infimum in (2.5) and (2.6) is attained.

Assume that there is an integer j, $1 \leq j \leq s-1$, such that $b_{r_j} > a_{r_j}$. Then there exists a $\tilde{p}_0$ such that

$$b_{r_j} > a_{r_j} = \inf\left\{h(\tilde{p}) \middle| P_{\tilde{p}}\left\{\tilde{X} \varepsilon \bigcup_{i=1}^{j} A_i\right\} \geq \alpha\right\} = h(\tilde{p}_0) \ . \qquad (2.7)$$

In addition, there exists a $\tilde{p}_1$ such that

$$P_{\tilde{p}_1}\left\{\tilde{X} \varepsilon \bigcup_{i=1}^{j} A_i\right\} > \alpha \ , \quad h(\tilde{p}_1) < b_{r_j} \ . \qquad (2.8)$$

Since $b_{r_1} \geq b_{r_2} \geq \dots \geq b_{r_s}$, from (2.7) we have

$$h(\tilde{p}_1) < b_{r_\ell} \ , \quad \ell = 1,2,\dots,j \ . \qquad (2.9)$$

Therefore

$$\alpha < P_{\tilde{p}_1}\left\{\tilde{X} \varepsilon \bigcup_{i=1}^{j} A_i\right\} \leq P_{\tilde{p}_1}\left\{h(\tilde{p}_1) < b_{g(\tilde{X})}\right\}, \qquad (2.10)$$

which is a contradiction. Consequently, there is no integer j, $i \leq j \leq s-1$, for which $b_{r_j} > a_{r_j}$.

From (2.6), it follows that $a_{r_s} = 0$ and b_{r_s} is also necessarily zero. Note further that in (2.7) it is possible that the infimum is attained at a point for which $P_{\tilde{p}}\left\{\tilde{X} \varepsilon \bigcup_{i=1}^{j} A_i\right\} > \alpha$. To see this consider the following example.

Let $k = 2$, $n_1 = 5$, $n_2 = 10{,}000$, $x_1 = 0$, $x_2 = 5$, $g(\tilde{x}) = n_1+n_2-x_1-x_2$, $h(\tilde{p}) = p_1p_2$. It is easily seen that the hypotheses of Theorem 2.1 are satisfied. Thus, for the data given, $g(\tilde{x}) = 10{,}000 = r_6$. The set $\bigcup_{i=1}^{6} A_i$ consists of all points (x_1,x_2) for which $x_1+x_2 \leq 5$, that is, $A_1 = \{(0,0)\}$, $A_2 = \{(1,0),(0,1)\}$, and so on. Consequently,

$$D_{r_6} = \left\{\tilde{p} \,\middle|\, P_{\tilde{p}}\left\{\tilde{X} \in \bigcup_{i=1}^{6} A_i\right\} \geq \alpha\right\}$$

includes the parameter points $(0, p_{2\alpha})$ where $p_{2\alpha}$ satisfies $P_{p_{2\alpha}}\{X_2=0\} \geq \alpha$, since $P_{\tilde{p}_1}\{X_1 \leq 5\} = 1$ when $p_1 = 0$. Thus inf $h(\tilde{p}) = 0$ for all $0 < \alpha < 1$.

We note that the monotonicity of $h(\tilde{p})$ is not utilized in the proof, which is valid whenever $h(\tilde{p})$ is continuous.

It is easy to see that $a_{g(\tilde{x})}$ is monotone, i.e., $a_{r_1} \geq a_{r_2} \geq \ldots \geq a_{r_s}$. This follows from (2.7) upon noting that as j increases, the set of $\tilde{p}$ satisfying (2.7) increases and the infimum is taken over a larger set.

Corollary. For a series system $h(\tilde{p}) = \prod_{i=1}^{k} p_i$. Then if $g(\tilde{x}) = \prod_{i=1}^{k} (n_i-x_i)/n_i = \prod_{i=1}^{k} y_i/n_i$, the hypotheses of Theorem 2.1 are satisfied and the conclusion follows.

Note that $g(\tilde{x}) = \prod_{i=1}^{k} (n_i-x_i)/n_i$ is the maximum likelihood estimator as well as the minimum variance unbiased estimator of $\prod_{i=1}^{k} p_i$ and is therefore a reasonable choice of an ordering function.

We now establish the following theorem.

Theorem 2.2. Let $g(\tilde{x}) = r_j$ and let

$$f^*(x;a) = \sup_{h(\tilde{p})=a} P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_j\right\}, \quad 0 < a < 1 . \qquad (2.10)$$

Then

$$\sup_{0<a<1} f^*(\tilde{x};a) = 1$$

and $f^*(\tilde{x};a)$ is non-decreasing in a.

Proof. Since $h(\tilde{p})$ is continuous and $h(\tilde{1}) = 1$,

$$\lim_{a\to 1} \sup_{h(\tilde{p})=a} P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_j\right\} = 1 .$$

Now choose a and b such that $0 < a < b < 1$,

$$P_{\tilde{p}_a}\left\{g(\tilde{X}) \geq r_j\right\} = f^*(\tilde{x};a)$$

and

$$P_{\tilde{p}_b}\left\{g(\tilde{X}) \geq r_j\right\} = f^*(\tilde{x};b) .$$

Let I_a be the set of indices i such that $p_{ia} < 1$. Then it is possible to replace p_{ia} by p'_{ib}, $i \in I_a$, where $p_{ia} < p'_{ib} < 1$, so that $h(\tilde{p}'_b) = b$, where $p'_{ib} = p_{ia}$, $i \in I_a^c$. This follows since $h(\tilde{1}) = 1 > a$ and $h(\tilde{p})$ is continuous. The conclusion follows from the monotone likelihood ratio property of the binomial distribution.

Note again that only the continuity of $h(\tilde{p})$ was used in the proof of Theorem 2.2.

For the case of series systems, it is possible to strengthen Theorem 2.2 and to exhibit the above construction. This is done below.

Corollary. Let $g(\tilde{x}) = r_j$. If $h(\tilde{p}) = \prod_{i=1}^{k} p_i$, then $\inf_{0<a<1} f^*(\tilde{x};a) = 0$ and $f^*(\tilde{x};a)$ is strictly increasing in a whenever all $u_j < n_j$ (see (2.3) for the definition of u_j), $j=1,2,\ldots k$.

Proof. From the hypotheses,

$$P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_j\right\} \leq 1-q_i^{\,n_i} \ , \quad i=1,2,\ldots,k \ ,$$

and since $\prod_{i=1}^{k} p_i \to 0$ implies at least one $p_i \to 0$, this gives

$$\inf_{0<a<1} f^*(\tilde{x};a) = 0 \ .$$

To show that $f^*(\tilde{x};a)$ is strictly increasing in a, consider $0<a<b<1$ and let $\tilde{p}_a = (p_{a1},\ldots,p_{ak})$ satisfy $f^*(\tilde{x};a) = P_{\tilde{p}_a}\left\{g(\tilde{X}) \geq r_j\right\}$. Similarly, let $\tilde{p}_b$ satisfy $f^*(\tilde{x};b) = P_{\tilde{p}_b}\left\{g(\tilde{X}) \geq r_j\right\}$. Let $I_a = \left\{i_1,i_2,\ldots,i_r\right\}$ be any non-empty set of indices such that $p_{ai_j}(\frac{b}{a})^{1/r} < 1$ (non-empty because otherwise multiplying the components would give $b > 1$, a contradiction) and let I_a^c be the remaining indices. Then

$$\Big(\prod_{j\varepsilon I_a} p_{ai_j}\big(\tfrac{b}{a}\big)^{1/r}\Big)\prod_{j\varepsilon I_a^c} p_{ai_j} = b \ . \tag{2.11}$$

From the monotone likelihood ratio property of the binomial distribution,

$$P_{\tilde{p}_a}\left\{g(\tilde{X}) \geq r_j\right\} < P_{\tilde{p}^*}\left\{g(\tilde{X}) \geq r_j\right\} \ ,$$

where the components of p^* are given by (2.11). This gives

$$f^*(\tilde{x};a) < f^*(\tilde{x};b) \ ,$$

which is the desired conclusion.

Note that if at least one $u_j = n_j$, it follows immediately from (2.5) that $a_{g(\tilde{x})} = 0$. For $g(\tilde{x}) = \prod_{i=1}^{k} (n_i-x_i)/n_i$ the condition $u_j < n_j$ is equivalent to $x_j < n_j$, $j=1,2,\ldots,k$.

We now establish a result which will prove useful in some of the subsequent material.

Theorem 2.3. If $f^*(\tilde{x};a) = \alpha$, $0<\alpha<1$, has at least one solution in a, then

$$a_{g(\tilde{x})} = \inf\left\{a \,\middle|\, f^*(\tilde{x};a) = \alpha\right\}. \tag{2.12}$$

If $f^*(\tilde{x};a) > \alpha$ for all a, then $a_{g(\tilde{x})} = 0$.

Proof. Let

$$c = \inf\left\{a \,\middle|\, f^*(\tilde{x};a) \geq \alpha\right\} . \tag{2.13}$$

The infimum in (2.13) is attained. Thus, there exists a $\tilde{p}_o$ such that $c = h(\tilde{p}_o)$. If $f^*(\tilde{x};a) > \alpha$ for all a, let $p_i \to 0$, $i=1,2,\ldots,k$. Then $h(\tilde{p}) \to 0$, since $h(\tilde{0}) = 0$ and $h(\tilde{p})$ is continuous, and $a_{g(\tilde{x})} = 0$.

Now assume there is at least one a with $f^*(\tilde{x};a) = \alpha$. Then $f^*(\tilde{x};a_{g(\tilde{x})}) \geq \alpha$ and therefore $c \leq a_{g(\tilde{x})}$. If $c < a_{g(\tilde{x})}$, then $c = h(\tilde{p}_o)$ and $f^*(\tilde{x};c) = \alpha$, which is a contradiction.

Again, only the continuity of $h(\tilde{p})$ was used in the proof of Theorem 2.3. Under the hypotheses of the Corollary to Theorem 2.2, for a series system, $a_{g(\tilde{x})}$ is the solution in a of

$$f^*(\tilde{x};a) = \alpha . \tag{2.14}$$

The general theory described in this section applies as well to what is known as systems with repeated components (see, e.g., Harris and Soms (1973]). For such systems, there are $1 \leq m \leq k$ unknown parameters $p_1,p_2,\ldots,p_m$, since the "repeated components" are assumed to have identical failure probabilities. This assumption permits the experimenter to regard the data as (n_i, Y_i), $i=1,2,\ldots,m$, and employ the previous results.

For example, if a series system of k components has α_1 of one type, α_2 of a second, ..., α_m of an m^{th} type, then

$$h(\tilde{p}) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}, \qquad \sum_{i=1}^{k} \alpha_i = k \ .$$

3. Counterexamples

In this section we restrict attention to series systems and employ the ordering function

$$g(\tilde{x}) = \prod_{i=1}^{k} (n_i - x_i)/n_i \ ,$$

introduced following Theorem 2.1. As noted previously, in this case the reliability function $h(\tilde{p}) = \prod_{i=1}^{k} p_i$. With this specialization we have for (2.4)

$$t_1 = n_1(1-r_m) \tag{3.1}$$

and for each fixed $0 \le i_1 \le t_1$, $0 \le i_2 \le t_2$, ..., $0 \le i_{j-1} \le t_{j-1}$,

$$t_j = n_j(1-r_m/[\prod_{\ell=1}^{j-1} (n_\ell - i_\ell)/n_\ell]), \qquad 2 \le j \le k \ , \tag{3.2}$$

whenever $g(\tilde{x}) = r_m$, $1 \le m < s$. If $m = s$, then $r_s = 0$ and $a_o = 0$.

For $\kappa > 0$, $\lambda > 0$, let

$$I_p(\kappa,\lambda) = \frac{1}{\beta(\kappa,\lambda)} \int_0^p t^{\kappa-1}(1-t)^{\lambda-1} dt \ , \qquad 0 \le p \le 1 \ , \tag{3.3}$$

the incomplete beta function.

It is well-known that if t is an integer, t < n, we have

$$\sum_{i=0}^{t} \binom{n}{i} p^{n-i} q^{i} = I_p(n-t, t+1) \ . \tag{3.4}$$

Sudakov (1974) published the inequality

$$P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_j\right\} \leq I_{\prod_{i=1}^{k} p_i}(n_1-t_1, t_1+1) \ . \tag{3.5}$$

This inequality and generalizations of it were further studied in Harris and Soms (1980,1981). (3.5) implies

$$f^*(\tilde{x};a) \leq I_a(n_1-t_1, t_1+1) \ ,$$

hence its usefulness. However, as we now establish, (3.5) is not universally valid, as was claimed in Sudakov (1974).

Let $(x_1,x_2) = (x_1,0)$ and let $(n_1,n_2) = (n_1,2n_1)$. Then $g(\tilde{x}) = (n_1-x_1)/n_1$ and $t_1 = x_1$. Consider $P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_m\right\}$. If $\tilde{p} = (1,a)$, $0 < a < 1$, we have

$$P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_m\right\} = P_a\left\{(n_2-X_2)/n_2 \geq r_m\right\} ,$$

since $P\left\{X_1=0\right\} = 1$, by (2.1). Consequently,

$$\begin{aligned} P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_m\right\} &= P_a\left\{X_2 \leq n_2(1-r_m)\right\} \\ &= P_a\left\{X_2 \leq 2n_1(1-r_n)\right\} . \end{aligned}$$

Since $r_m = (n_1-x_1)/n_1$,

$$P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_m\right\} = P_a\left\{X_2 \leq 2x_1\right\} .$$

Thus from (3.4),

$$P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_m\right\} = I_a(2(n_1-x_1),\ 2x_1+1)\ .$$

The Sudakov inequality implies that

$$I_a(2(n_1-x_1),\ 2x_1+1) \leq I_a(n_1-x_1,\ x_1+1)$$

or

$$I_a(2n_1r_m,\ 2n_1(1-r_m)+1) \leq I_a(n_1r_m,\ n_1(1-r_m)+1)\ . \quad (3.6)$$

Let $h_2(t;n_2,r_m)$ and $h_1(t;n_1,r_m)$ denote the beta density functions corresponding to the left and right hand side of (3.6), respectively. Then, provided $n_1r_m > 1$, there is an $\varepsilon > 0$ such that

$$h_2(t;n_2,r_m) < h_1(t;n_1,r_m) \quad 0 < t < \varepsilon,\ 1-\varepsilon < t < 1\ .$$

This implies that $h_1(t;n_2,r_m)$ and $h_2(t;n_2,r_m)$ intersect in at least two points. If t^* is such an intersection, setting $h_1(t;n_1,r_m)/h_2(t;n_2,r_m) = 1$ gives

$$t^{n_1r_m}(1-t)^{n_1(1-r_m)} = c(n_1,r_m) > 0\ .$$

Thus, for $1 \leq m < s$, there are exactly two such intersections. Therefore there is a z_o such that

$$I_{z_o}(n_1r_m,\ n_1(1-r_m)+1) = I_{z_o}(n_2r_m,\ n_2(1-r_m)+1)\quad ,$$

for $z > z_o$,

$$I_z(n_1r_m,\ n_1(1-r_m)+1) < I_z(n_2r_m,\ n_2(1-r_m)+1)$$

and for $z < z_o$,

$$I_z(n_1r_m,n_1(1-r_m)+1) > I_z(n_2r_m,\ n_2(1-r_m)+1)\ .$$

Thus for $z > z_0$, (3.6) is violated. (3.6) was used as a lemma by Sudakov (1974) to prove the inequality (3.5). This lemma was also employed in Harris and Soms (1980, 1981). It is the falsity of this lemma which invalidates (3.5).

Table 1 provides some illustrations of the violation of (3.5) for $k = 2$ and selected values of (n_1,n_2), (x_1,x_2). The smallest value of p_1p_2 for which this violation occurs is also given in the table, where it is denoted by a^*. In addition, $f^*(\tilde{x};a^*)$ is tabulated. Thus for $\alpha < f^*(\tilde{x};a^*)$, (3.5) is valid. The calculations were made by means of a FORTRAN program. Note the for $(n_1,n_2) = (5,5)$ and $(x_1,x_2) = (1,1)$ the inequality was not violated.

Table 1. The Smallest a, a^*, and $f^*(\tilde{x};a^*)$

(n_1,n_2)	(x_1,x_2)	a^*	$f^*(\tilde{x};a^*)$
(5,5)	(1,1)	1.0000	1.0000
(5,5)	(3,3)	.7454	.9998
(5,10)	(1,0)	.8798	.8909
(5,15)	(0,3)	.8698	.8791
(5,30)	(1,0)	.8498	.8467

4. The Theory of Key Test Results

If for $n_1 \leq n_2 \leq \dots \leq n_k$, $(x_1,x_2,\dots,x_k) = (x_1,0,\dots,0)$, $k \geq 2$, then $\tilde{x}$ is called a key test result. Winterbottom (1974) asserted that subject to $x_1 < f(k,n_1)$, where $f(k,n_1)$ is the solution in f of

$$n_1k-f-1 = k[(n_1-f)n_1^{k-1}]^{1/k} , \qquad (4.1)$$

we have $a_{g(\tilde{x})}$ is the solution in a of

$$I_a(n_1-x_1,\ x_1+1) = \alpha \quad , \quad 0 < \alpha < 1 \ . \tag{4.2}$$

This would imply the inequality (3.5), which we have disproved in Section 3.

As we subsequently establish, the error in Winterbottom's (1974) result is a consequence of falsely concluding that $f(k,n_1)$ depends only on n_1. It is easy to be led to this conclusion on intuitive grounds, since $(n_1-x_1,\ n_1,\ldots,\ n_1)$ would seem to be a less favorable experimental result than $(n_1-x_1,n_2,\ldots,n_k)$, whenever $n_i > n_1$ for at least one index i, $2 \le i \le k$. We now establish a modified key test result that holds for $x_1 < f(k,\tilde{n})$, where $f(k,\tilde{n})$ is the smallest solution in f of

$$\sum_{i=1}^{k} n_i - f - 1 = k[(n_1-f) \prod_{i=2}^{k} n_i]^{1/k} \ . \tag{4.3}$$

<u>Theorem 4.1</u>. If $n_1 \le n_2 \le \ldots \le n_k$ and $\tilde{x} = (x_1,0,\ldots,0)$, with $x_1 < f(k,\tilde{n})$ where $f(k,\tilde{n})$ is given by (4.3), then

$$P_{\tilde{p}}\left\{g(\tilde{X}) \ge r_j\right\} \le I_{\prod_{i=1}^{k} p_i}(n_1-x_1,x_1+1) \ , \tag{4.4}$$

where $g(x_1,0,\ldots,0] = r_j$.

Proof. The proof consists of finding a necessary and sufficient condition under which

$$\left\{\tilde{z}\,\middle|\,\prod_{i=1}^{k}(n_i-z_i) \geq \prod_{i=1}^{k}(n_i-x_i)\right\} = \left\{\tilde{z}\,\middle|\,\sum_{i=1}^{k}(n_i-z_i) \geq \sum_{i=1}^{k}(n_i-x_i)\right\} ,\quad (4.5)$$

and then applying the results of Pledger and Proschan (1971) to find the supremum of the right hand side of (4.5) subject to $\prod_{i=1}^{k} p_i = a$. Clearly, we must have $\sum_{i=1}^{k} x_i < n_1$. For fixed $\sum_{i=1}^{k} x_i = f$, or equivalently, for fixed $\sum_{i=1}^{k}(n_i-x_i) = \sum_{i=1}^{k} n_i - f$, $\prod_{i=1}^{k}(n_i-x_i)$ is minimized by $(n_1-f)\prod_{i=2}^{k} n_i$. This follows from the strict Schur-concavity of $\prod_{i=1}^{k} x_i$, $x_i > 0$, $1 \leq i \leq k$ (see, e.g., Marshall and Olkin (1979, p. 78)). Therefore we must have $\tilde{x} = (f,0,\ldots,0)$. Then a necessary and sufficient condition for (4.5) to hold is that $\tilde{x} = (x,0,\ldots,0)$, where $x < f'(k,n)$ and $f'(k,n)$ is the first positive integer f for which

$$(n_1-f)\prod_{i=2}^{k} n_i \leq \max_{\sum_{i=1}^{k} x_i = f+1} \prod_{i=1}^{k}(n_i-x_i) \quad . \qquad (4.6)$$

The Schur-concavity of $\prod_{i=1}^{k} x_i$ then gives (4.3) as a sufficient condition and the subsequent corollary gives a simple method of calculating f' exactly.

We now assume that (4.5) is satisfied and hence that

$$f^*(x;a) = \sup_{\prod_{i=1}^{k} p_i = a} P\left\{\sum_{i=1}^{k} Y_i \geq n_1 - x_1 + \sum_{i=2}^{k} n_i\right\} \quad . \qquad (4.7)$$

Writing (4.7) as an iterated sum and noting that $I_t(n-x,x+1)$ is a decreasing function of n for fixed x, we have

$$\sup_{\prod_{i=1}^{k} p_i = a} P\left\{\sum_{i=1}^{k} Y_i \geq n_1 - x_1 + \sum_{i=2}^{k} n_i\right\} \leq \sup_{\prod_{i=1}^{k} p_i = a} P\left\{Y_1 + \sum_{i=2}^{k} U_i \geq n_1 - x_1 + (k-1)n_1\right\},$$

where the U_i are independent binomial random variables with parameters (n_1,p_i), $i=2,\ldots,k$. Writing

$$Y_1 + \sum_{i=2}^{k} U_i = \sum_{i=1}^{k} \sum_{j=1}^{n_1} Y_{ij} ,$$

where the Y_{ij} are independent Bernoulli random variables with parameter p_i, a result of Pledger and Proschan (1971) may be employed to show that the upper tail of $\sum_{i=1}^{k} \sum_{j=1}^{n} Y_{ij}$ is a Schur-convex function of $(-\ln p_1, -\ln p_1, \ldots, -\ln p_1, -\ln p_2, \ldots, -\ln p_2, \ldots, -\ln p_k, \ldots, -\ln p_k)$ and therefore $f^*(\tilde{x};a) = I_a(n_1-x_1, x_1+1)$, as required.

Corollary. For each f, form the vector $\tilde{z} = (z_1,z_2,\ldots,z_k)$ from $\tilde{n} = (n_1,n_2,\ldots,n_k)$ by continually reducing the maximum (s) until the subtractions total $f+1$, $f \geq 0$. Denote by $f'(k,n_1,n_2,\ldots,n_k)$ the first f for which

$$\prod_{i=1}^{k} z_i \geq (n_1-f) \prod_{i=2}^{k} n_i .$$

Then a necessary and sufficient condition for (4.5) to hold is that $x_1 < f'(k,\tilde{n})$.

Proof. The proof proceeds exactly as for Theorem 4.1 by noting that $\tilde{z}$ maximizes $\prod_{i=1}^{k} r_i$ subject to $0 < r_i \leq n_i$ and $\sum_{i=1}^{k} r_i = \sum_{i=1}^{k} n_i-f-1$. This follows since $\tilde{z}$ is majorized by $\tilde{r}$ and the product is strictly Schur-concave.

If $n_1 = n_2 = \ldots = n_k$, (4.3) reduces to (4.1) which is Winterbottom's (1974) condition. However, s should be replaced by s+1 in his formula, which also has a sign error. As an example, for $k = 2$, $n_1 = n_2 = 50$, from Winterbottom (1974), (4.4) is stated to hold for $x_1 \leq 17$ or $n_1 - x_1 \geq 33$. However, $33 \cdot 50 < 41 \cdot 41$, and therefore (4.4) only holds for $x_1 \leq 13$ or $n_1 - x_1 \geq 37$, as the Corollary to Theorem 4.1 shows, or the solution of (4.3), which gives $f(2,50,50) = 13.14$.

The dependence of f on $\tilde{n}$ may be seen by considering an example. Let $k = 2$, $n_1 = 5$, $n_2 = 10$. Then from the Corollary following Theorem 4.1, (4.4) only holds for $x_1 = 0$, whereas for $n_1 = n_2 = 5$, it holds for $x_1 = 0,1,2$, and 3. Thus the case of equal n_i, $i=1,2,\ldots,k$, does not give the minimal f. In fact, it may be seen that if $n_k \geq 2n_1$, then (4.4) holds only for $x_1 = 0$.

5. Concluding Remarks

From Table 1, it seems reasonable to conjecture that (3.5) is valid for those values of $\alpha,k,\tilde{n}$ likely to arise in practice. The authors are continuing to investigate the problem and hope to report more precise conditions for the validity of (3.5) in subsequent work.

References

Barlow, R. E. and Proschan, F. (1975), Statistical Theory of Reliability and Life Testing, Probability Models, New York: Holt, Rinehart and Winston.

Buehler, R. J. (1957), "Confidence Limits for the Product of Two Binomial Parameters," Journal of the American Statistical Association, 52, 482-93.

Harris, B. and Soms, A. P. (1973), "The Reliability of Systems of Independent Parallel Components When Some Components Are Repeated," Journal of the American Statistical Association, 68,894-8.

Harris, B. and Soms, A. P. (1980), "Bounds for Optimal Confidence Limits for Series Systems," Technical Summary Report #2093, Mathematics Research Center, University of Wisconsin-Madison.

Harris, B. and Soms, A. P. (1981), "Improved Sudakov-Type Bounds for Optimal Confidence Limits on the Reliability of Series Systems," Technical Summary Report #2294, Mathematics Research Center, University of Wisconsin-Madison.

Marshall, A. W. and Olkin, I. (1979), Inequalities: Theory of Majorization and Its Applications, New York: Academic Press.

Pledger, G. and Proschan, F. (1971), "Comparison of Order Statistics and of Spacings from Heterogeneous Distributions," Optimizing Methods in Statistics, New York: Academic Press, 89-113.

Sudakov, R. S. (1974), "On the Question of Interval Estimation of the Index of Reliability of a Sequential System," Engineering Cybernetics, 12, 55-63.

Winterbottom, A. (1974), "Lower Limits for Series System Reliability from Binomial Data," _Journal of the American Statistical Association_, 69, 782-8.

BH/APS/jvs

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2726	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) The Theory of Optimal Confidence Limits for Systems Reliability with Counterexamples for Results on Optimal Confidence Limits for Series Systems		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Bernard Harris and Andrew P. Soms		8. CONTRACT OR GRANT NUMBER(s) N00014-79-C-0321 DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 4 - Statistics & Probability
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below		12. REPORT DATE August 1984
		13. NUMBER OF PAGES 19
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

Office of Naval Research
Department of the Navy
Arlington, VA 22217

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Optimal confidence bounds; Reliability; Series system

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The paper gives the general theory of optimal confidence limits for systems reliability introduced by Buehler (1957). This is specialized to series systems. It is noted that some results previously given are false. In particular, counterexamples for results of Sudakov (1974), Winterbottom (1974) and Harris and Soms (1980, 1981) are given. Numerical examples are provided, which suggest that despite the theoretical problems of the results, they are nevertheless valid for significance levels likely to be used in practice.

END

FILMED

12-84

DTIC