THE LINDESTRÖM-MADDEN METHOD FOR SERIES SYSTEMS WITH
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THE LINDSTROM-MADDEN METHOD FOR SERIES SYSTEMS WITH REPEATED COMPONENTS

Andrew P. Soms

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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The Lindstrom-Madden method of computing lower confidence limits for series systems with unlike components is extended to series systems with repeated components utilizing the results of Harris and Soms (1983). An exact solution is given for no failures and key test results, together with an approximation for the general case. Numerical examples are also provided.

AMS (MOS) Subject Classifications: 62N05, 90B25

Key Words: Lindstrom-Madden approximation; Optimal confidence limits; Reliability; Repeated components; Series system

Work Unit Number 4 (Statistics and Probability)
SIGNIFICANCE AND EXPLANATION

Series systems with repeated components arise often in engineering and physics. It is therefore important to utilize data obtained on individual components in an efficient manner when assessing the reliability of the combined system. This paper gives one method for doing so.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.
THE LINDSTROM-MADDEN METHOD FOR SERIES SYSTEMS WITH REPEATED COMPONENTS

Andrew P. Soma*

1. INTRODUCTION AND SUMMARY

A problem of substantial importance to practitioners in reliability is the statistical estimation of the reliability of a series system of stochastically independent components when some components are repeated, using experimental data collected on the individual components. In the situations discussed in this paper, the component data consist of a sequence of Bernoulli trials. Thus, for component \( i, i = 1, 2, \ldots, k \), the data is the pair \((n_i, Y_i)\), where \( n_i \) is the number of trials and \( Y_i \) is the number of observations for which the component functions. \( Y_1, Y_2, \ldots, Y_k \) are assumed to be mutually independent random variables. We assume that there are \( y_i \) components of type \( i \), \( 1 \leq i \leq k \). Then the parameter of interest is \( h(p_1, p_2, \ldots, p_k) = h(\hat{p}) \), the reliability of the system, where

\[
h(\hat{p}) = \prod_{i=1}^{k} \frac{Y_i}{n_i}.
\]

More specifically, it is desired to obtain a Buehler (1957) optimal lower 1 - \( \alpha \) confidence limit on \( h(\hat{p}) \).

The case of \( Y_1 = Y_2 = \ldots = Y_k = 1 \) has been treated in Sudakov (1974), Winterbottom (1974), and Harris and Soma (1983).

In Section 2 we summarize the general theory of Harris and Soma (1983) applicable here. In Section 3 the exact solutions to no failures and key test results are given. Lindstrom-Madden type approximations are given in Section 4. Section 5 contains numerical examples.

*Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, WI 53201.

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2. BUEHLER'S METHOD FOR OPTIMAL CONFIDENCE LIMITS

We now specialize the general results of Harris and Soms (1983) on optimal confidence limits for system reliability to a series system with independent and repeated components. As in Section 1, let

\[ h(p) = \prod_{i=1}^{k} \frac{y_i}{p_i}, \]

\[ 0 < p_i < 1, \quad x_i = n_i - y_i, \quad x_i = n_i - y_i, \quad 1 < i \leq k \] and let \( g(x) = (x_1, x_2, \ldots, x_k) \) be an ordering function, i.e., for real \( x_i, \quad 0 < x_i \leq n_i \), \( g(x) \) is non-decreasing in each component. It is often convenient to normalize \( g(x) \) by letting \( g(0) = 1 \) and \( g(n) = 0 \). With such a normalization, \( g(x) \) is often selected to be a point estimator of \( h(p) \). Also let \( R = \{ r_1, r_2, \ldots, r_s, s > 2 \} \) be the range set of \( g(x) \). With no loss of generality we order \( R \) so that \( r_1 > r_2 > \ldots > r_s \) and let \( A_i = g(x) = r_i, \quad x \in S, \quad i = 1, 2, \ldots, s \). The sets \( A_i \) constitute a partition of \( S \) induced by \( g(x) \). We assume throughout that the data is distributed by

\[ f(x|p) = P_{p}(x = \tilde{x}) = \prod_{i=1}^{k} \binom{n_i}{x_i} \frac{y_i^{x_i}}{p_i^{x_i} q_i^{n_i-x_i}}, \]

\[ = \prod_{i=1}^{k} \binom{n_i}{y_i} \frac{y_i^{n_i-y_i}}{p_i^{n_i-y_i} q_i^{y_i}}. \]

where \( q_i = 1 - p_i, \quad i = 1, 2, \ldots, k \). With no loss of generality, we assume

\[ n_1 < n_2 < \ldots < n_k. \]

From these definitions, it follows that

\[ P_{p}(x \in \bigcup_{i=1}^{j} A_i) = P_{p}(g(\tilde{x}) > x_i). \]
\[ P_{\tilde{X}}(g(\tilde{X}) > r_j) = \frac{u_1}{i_1 = 0} \frac{u_2}{i_2 = 0} \cdots \frac{u_k}{i_k = 0} f(i; p), \quad (2.3) \]

where \( \tilde{1} = (i_1, i_2, \ldots, i_k) \) and \( u_2 = u_2(i_1), \ldots, u_k = u_k(i_1, i_2, \ldots, i_{k-1}) \) are integers determined by \( r_j \). Equivalently,

\[ P_{\tilde{X}}(g(\tilde{X}) > r_j) = \frac{t_1}{i_1 = 0} \frac{t_2}{i_2 = 0} \cdots \frac{t_k}{i_k = 0} f(i; p), \quad (2.4) \]

where \( t_2 = t_2(i_1), \ldots, t_k = t_k(i_1, i_2, \ldots, i_{k-1}) \), with \( t_1 = \sup(t|0 < t < n_1 \text{ and } g(t, 0, 0, \ldots, 0) > r_j) \) and \( t_k(i_1, i_2, \ldots, i_{k-1}) = \sup(t|0 < t < n_k \text{ and } g(i_1, i_2, \ldots, i_{k-1}, t, 0, \ldots, 0) > r_j), \ell = 2, 3, \ldots, k. \)

We now introduce the notion of Buehler optimal confidence limits. Let \( g(x) = r_j \).

Then define

\[ a_{g(x)} = \inf(h(p)|P_{\tilde{X}}(\tilde{1}|g(\tilde{1}) > g(x)) > a). \quad (2.5) \]

Equivalently, by (2.2), we can also write

\[ a_{g(x)} = \inf(h(p)|P_{\tilde{X}}(x \in \bigcup_{i=1}^j A_i) > a). \quad (2.6) \]

Then we have, from Harris and Soms (1983),

**Theorem 2.1.** \( a_{g(x)} \) is a \( 1 - a \) lower confidence limit for \( h(\tilde{p}) \). If \( b_{g(\tilde{x})} \) is any other \( 1 - a \) lower confidence limit for \( h(\tilde{p}) \) with \( b_{r_1} > b_{r_2} > \cdots > b_{r_j} \), then \( b_{g(x)} \leq a_{g(x)} \) for all \( \tilde{x} \in S. \)

Two possible choices of \( g(x) \) are

\[ g(x) = \frac{1}{k} \sum_{i=1}^k (x_i - \bar{x}) \frac{y_i}{x_i}, \quad (2.7) \]

or

-3-
Both reduce to the generally used \( g(x) \) for series systems with independent components when \( Y_1 = Y_2 = ... = Y_k = 1 \), i.e.,

\[
g(x) = \prod_{i=1}^{k} \frac{n_i - x_i}{n_i - 1}.
\]

Since (2.7) is the maximum likelihood estimator of \( h(p) \) we will use it here and from now on it will be understood that \( g(x) \) is given by (2.7). With this choice of \( g(x) \), we assume from now on that \( 0 < x_i < n_i \), \( i = 1, 2, ..., k \), since \( ag(x) = 0 \) if some \( x_i = n_i \). With this assumption, the \( t_i \) in (2.4) are given by

\[
t_i = n_i - \left( \prod_{j=1}^{i-1} (n_j - x_j) \right)^{Y_i} \left( \prod_{j=i+1}^{k} (n_j - i) \right)^{Y_i} \frac{1}{Y_i}.
\]

and

\[
t_k = n_k - \left( \prod_{i=1}^{k} (n_i - x_i) \right)^{Y_k} \left( \prod_{s=1}^{k-1} (n_s - i) \right)^{Y_k} \frac{1}{Y_k},
\]

\( i = 2, ..., k \), with \( \prod_{i=k+1}^{k} n_i = 1 \).

For the purpose of simplifying the calculation of \( ag(x) \) in special cases it is necessary to state additional results from Harris and Som (1983).

**Theorem 2.2.** Let \( g(x) = r_j \) and let

\[
f^*(x; a) = \sup_{h(p) = a} P_e(g(x) > r_j), \quad 0 < a < 1.
\]

Then
\[ \inf f^*(\tilde{x};a) = 0, \quad \sup f^*(\tilde{x};a) = 1 \]

and \( f^*(\tilde{x};a) \) is strictly increasing in \( a \).

**Theorem 2.3.** \( f^*(\tilde{x};a) = a \) has exactly one solution \( a_\alpha \) in \( a \) and \( a_\alpha = a(g(\tilde{x})) \).

### 3. Exact Solutions for Zero Failures and Key Test Results

We first assume that \( \tilde{x} = (0,0,...,0) = \tilde{0} \) and use Theorem 2.3 to obtain \( a_g(\tilde{0}) \).

**Theorem 3.1.** If \( \tilde{x} = \tilde{0} \), then

\[ f^*(\tilde{0};a) = \sup \frac{k}{\prod_{i=1}^{k} p_i^n_{j/a},} \quad (3.1) \]

where \( \frac{n_j}{y_j} = \min \frac{n_i}{y_i} \) and

\[ a_g(\tilde{0}) = \frac{y_1/n_j}{y_j}. \quad (3.2) \]

**Proof.**

\[ \frac{n_1}{p_1} = \frac{\prod_{i=1}^{k} p_i^n_{j/a,y_j}}{\prod_{i=1}^{k} p_i^n_{j/a,y_j}} \]

since \( n_j/y_j \) is \( n_j/y_j > 0 \) is equivalent to \( n_j/y_j > n_j/y_j \), which is true, and therefore

\[ \frac{\prod_{i=1}^{k} p_i^n_{j/a,y_j}}{\prod_{i=1}^{k} p_i^n_{j/a,y_j}} \leq 1. \ (3.1) \text{ follows by noting that the choice } \frac{p_j}{y_j} = a/n_j, \frac{p_i}{y_i} = 1, \frac{p_j}{y_j} = 1 \]

\[ \frac{\prod_{i=1}^{k} p_i^n_{j/a,y_j}}{\prod_{i=1}^{k} p_i^n_{j/a,y_j}} \leq 1. \]

Thus, using Theorem 2.3, we obtain (3.2), which reduces to the known series result if \( y_1 = y_2 = ... = y_k = 1 \).
We now turn to analogues of key test results (see, e.g., Winterbottom (1974) and Harris and Soms (1983)). We define a key test result if \( y_1 = \max_{1 \leq i \leq k} y_i \) (recall that \( n_1 = \min n_i \) and \( \bar{x} = (x_1, 0, ..., 0) \)).

**Theorem 3.2.** If \( \bar{x} \) is a key test result and

\[
\begin{align*}
\{ \bar{z} \} \sum_{i=1}^{k} (n_i - z_i) &> \begin{Bmatrix} \frac{k}{k} \\
\sum_{i=1}^{k} (n_i - x_i) \end{Bmatrix} = \{ \bar{z} \} \begin{Bmatrix} \frac{k}{k} \\
(n_1 - z_1) \end{Bmatrix}
\end{align*}
\]

then

\[
f^*(\bar{x}; a) = I_1(n - x_1, x_1 + 1),
\]

where \( I_1(a, b) \) is the incomplete beta function. Let \( b_\alpha \) denote the solution in \( \beta \) of

\[
a = I_b(n_1 - x_1, x_1 + 1).
\]

Then \( a_\alpha(\bar{x}) = b_\alpha \). Note that \( b_\alpha \) is the usual \( 1 - \alpha \) lower confidence limit on \( p \), given \( x_1 \) failures in \( n_1 \) trials.

**Proof.** Without loss of generality we can assume that \( n_1 = n_2 = ... = n_k \), for otherwise we can write (2.4) as

\[
\begin{align*}
p^*(g(\bar{x}) > x_j) &= \left\{ \begin{Bmatrix} \sum_{i=0}^{n_1-1} (i_1)!p_1^{i_1} q_1^{i_1+1} p_2^{i_2+1} q_2^{i_2+1} ... \\
\sum_{i_1=0}^{n_1-1} (i_1)!p_1^{i_1} q_1^{i_1+1} p_2^{i_2+1} q_2^{i_2+1} ... \\
\sum_{i_k-1=0}^{n_k-1} (i_k-1)!p_k^{i_k-1} q_k^{i_k-1+1} \end{Bmatrix} \right.
\end{align*}
\]
where $g(x) = r_j$, by the monotone likelihood ratio property of the beta distribution $(I_x(a,b)$ has a monotone likelihood ratio in $-a$ for fixed $b$, which implies that $I_x(a,b)$ is a decreasing function of $a$). A similar argument applies to the other indexes. Thus, if (3.4) is true for $n_1 = n_2 = ... = n_k$, by (3.5) it follows for $n_1 < n_2 < ... < n_k$.

So, assuming $\hat{n} = (n_1, n_1, ... , n_1)$, we seek to maximize

$$\prod_{i=1}^{k} \prod_{j=1}^{n_1} y_{ij} \geq \prod_{i=1}^{k} (n_1 - x_i) = \prod_{i=1}^{k} y_i$$

where $y_{ij}$ are independent Bernoulli random variables with parameter $p_i$ and $\prod_{i=1}^{k} p_i = a$. If $\prod_{i=1}^{k} p_i = a$, then $\prod_{i=1}^{k} p_i$ ranges from $a$ to $a^{-1}$.

In $a^{-1}$, $y_j = \min_{1<i<k} y_i$. This is seen as follows:

$$\prod_{i=1}^{k} p_i = \left( \prod_{i=1}^{k} p_i \right)^{1/y_1} \prod_{i=2}^{k} p_i$$

$$= a^{-1/y_1} \prod_{i=2}^{k} p_i$$

and

$$\prod_{i=1}^{k} p_i = \left( \prod_{i=1}^{k} p_i \right)^{1/y_j} \prod_{i=1}^{k} p_i$$

$$= a^{-1/y_j} \prod_{i=1}^{k} p_i$$

and the choices $p_1 = a$, $p_2 = ... = p_k = 1$, and $p_j = a^{-1/y_j}$, $p_i = 1$, $i \neq j$, attain these values. From the results of Pledger and Proschan (1971), for each $b = \prod_{i=1}^{k} p_i^{-1}$,
a < b < a', (3.6) is maximized by \( p_i = b_i, p_i = 1, 2 \leq i \leq k \). Further, the maximum over \( b, a \), of the maxima for each \( b \) is given by \( p_1 = a \), \( p_i = 1, 2 \leq i \leq k \), by the monotone likelihood ratio property of the binomial distribution, and \( p_1 = a \), \( p_i = 1, 2 \leq i \leq k \), satisfies \( \sum_{i=1}^k p_i = a \). This completes the proof.

If \( Y_1 = Y_2 = \ldots = Y_k = 1 \), some guidelines for the verification of (3.3) are given in Harris and Some (1983). In the present case (3.3) must be verified by trial and error by showing that \( \min_{\hat{x}} \sum_{i=1}^k (n_i - x_i)^{Y_i} = (n_i - x_i)^{Y_i} \) and that \( \max_{\hat{x}} \sum_{i=1}^k (n_i - x_i)^{Y_i} < (n_i - x_i)^{Y_i} \).

**Example 3.1.** Let \( k = 3, \hat{n} = (5, 5, 5), \hat{\gamma} = (3, 3, 2), a = .10 \) and \( \hat{x} = (1, 1, 1) \). Then
\[
\min_{\hat{x}} \sum_{i=1}^3 (n_i - x_i)^{Y_i} = 200000 \quad \text{and} \quad \max_{\hat{x}} \sum_{i=1}^3 (n_i - x_i)^{Y_i} = 1406.
\]
\( \hat{x} \) is a key test result and (3.3) is satisfied and hence
\[
a_g(\hat{x}) = .4161^3 = .0720 ,
\]
where .10 = \( \hat{I}, 4161(4.2) \). Further, it can also be verified that \( \hat{x} = (2, 0, 0) \) is a key test result for which (3.3) is satisfied, but that for \( \hat{x} = (3, 0, 0) \), (3.3) is violated.

Note that Theorem 3.2 asserts that \( a_g(\hat{x}) = b_0(\gamma) \) for \( 0 < a < 1 \). It is thus possible that (3.3) is not true but the conclusion still holds for \( a \) of practical importance.

This is taken up in Section 4.

4. THE LINDSTROM-MADDEN METHOD FOR SERIES SYSTEMS WITH REPEATED COMPONENTS

When \( Y_1 = Y_2 = \ldots = Y_k = 1 \), the Lindstrom-Madden method (henceforth abbreviated L-M) is an approximation \( b_g(\hat{x}) \) to \( a_g(\hat{x}) \) of the form
\[
b_g(\hat{x}) = \min_{\hat{x}} a_g(\hat{\gamma}) , \quad (4.1)
\]
where

\[ a = \sup_{b}(n_{i}(n_{i} - t_{0i}, t_{0i} + 1)), \quad (4.2) \]

with

\[ t_{0i} = n_{i}(1 - \frac{k}{i \in 1} (n_{i} - x_{i})/n_{i}), \quad (4.3) \]

i.e., \( t_{0i} \) is the maximum of the recursive indexes \( t_{i} \) defined by (2.4). For the usual levels of \( a, b_{g}(x) = b_{a}(n_{i}). \) Further, numerical evidence indicates (Harris and Soms (1983)) that for a levels of practical significance

\[ b_{g}(x) < a_{g}(x), \quad (4.4) \]

(4.4) was incorrectly claimed to be true for \( 0 < a < 1 \) in Sudakov (1974) and this is discussed at length in Harris and Soms (1983). However, (4.4) is known to hold for special cases (Winterbottom (1974) and Harris and Soms (1983)).

Motivated by the above, we now give an L-M approximation \( b_{g}(x) \) to \( a_{g}(x) \) for arbitrary \( Y_{i} \) by

\[ b_{g}(x) = \min_{k \leq k_{0}} a(n_{i})^{Y_{i}}, \quad (4.5) \]

where

\[ a = \sup_{b}(n_{i}(n_{i} - t_{0i}, t_{0i} + 1)), \quad (4.6) \]

with

\[ t_{0i} = n_{i}(1 - \frac{k}{j = 1} (n_{j} - x_{j})^{Y_{j}}/\frac{k}{j = 1} n_{j}^{Y_{j}/Y_{i}}), \quad (4.7) \]

i.e., \( t_{0i} \) is the maximum of the recursive indexes \( t_{i} \) defined by (2.4). However, in this case it is not clear which index \( i \) gives the minimum, except that the likely
candidate is the one for which \( \gamma_j \), \( 1 \leq j \leq k \), is a maximum. We might expect, by analogy, that for \( \alpha \) levels of practical interest

\[
b_g(\hat{x}) < a_g(\hat{x}).
\]

(4.8)

5. NUMERICAL EXAMPLES

For \( k = 2 \) and selected \( \hat{n}, \hat{Y}, \hat{x}, \alpha = .05 \) and \( .10 \), Table I gives \( b_g(\hat{x}), a_g(\hat{x}) \) and the best upper bound, \( u_g(\hat{x}) \),

\[
u_g(\hat{x}) = \min_{i \leq k} u_\alpha(n_i)\gamma_i.
\]

(5.1)

where

\[
a = I_\alpha(n_i)(n_i - [t_{0i}], [t_{0i}] + 1)
\]

(5.2)

and \( t_{0i} \) are defined as in (4.6).

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<th>((\gamma_1, \gamma_2))</th>
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Note that for all the cases in Table I, \( b_g(x) \) is a lower bound for \( a_g(x) \). The computations were done by a short FORTRAN program, a listing of which can be obtained from the author.

6. CONCLUDING REMARKS

In this paper we have extended the L-M method to series systems with repeated components. More work is needed to ascertain the region of validity of (4.8).
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Andrew P. Soms

Mathematics Research Center, University of
610 Walnut Street
Madison, Wisconsin 53706

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