

Rayleigh-Taylor Instability in Compressible Media

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<p>The theoretical work previously done on Rayleigh-Taylor instability in compressible fluids has been reviewed in a chapter written for the <i>Encyclopedia of Fluid Mechanics</i>, to be published by Gulf. The physical basis of the instability, the dependence of growth rate on adiabatic index γ, and the stability of uniform self-similar implosions and expansions are discussed.</p>				
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CONTENTS

INTRODUCTION	1
PHYSICAL BASIS OF GRAVITATIONAL INTERCHANGE INSTABILITIES	8
ENERGY PRINCIPLE	12
RAYLEIGH-TAYLOR INSTABILITY AT THE INTERFACE BETWEEN TWO ISOTHERMAL LAYERS	15
USE OF PIECEWISE ISOTHERMAL PROFILE TO APPROXIMATE AN ARBITRARY EQUILIBRIUM	24
UNIFORM SELF-SIMILAR IMPLOSIONS AND EXPANSIONS: FORMULATION	27
STABILITY OF UNIFORM SELF-SIMILAR IMPLOSIONS	31
STABILITY OF UNIFORM SELF-SIMILAR EXPANSIONS	34
CONCLUSIONS	41
ACKNOWLEDGMENT	42
REFERENCES	43

Rayleigh-Taylor Instability in Compressible Media

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Introduction

The Rayleigh-Taylor instability¹⁻² occurs when a fluid supports a denser fluid against gravity, whereupon the two tend to interchange positions. It is encountered frequently in nature and in the laboratory. For example, inertial confinement fusion experiments, in which an ablatively driven medium implodes, compressing the material ahead of the ablation front to high densities, can exhibit Rayleigh-Taylor instabilities in the ablative region, at the compression front, or (in the case of a layered target) at an interface between layers of different density.

When the time scale associated with the growth of the instability is short compared with the time $(kc_s)^{-1}$ for sound to traverse a wavelength $2\pi/k$, one should expect to have to include the finite compressibility of the fluid in calculating instability growth rates. It is not obvious a priori whether finite compressibility acts to increase or decrease growth rates. For example, compression absorbs some energy that might otherwise go into fluid motion. On the other hand, a compressible system exhibits more modes of propagation than an incompressible one, so the most unstable one might

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possibly have a faster growth rate than the most unstable mode in an incompressible medium. Moreover, Bernstein et al.³ have shown that in a broad class of general compressible hydromagnetic systems, the unstable modes with lowest threshold are associated with incompressible perturbations. It is thus conceivable that compressible and incompressible systems might display the same Rayleigh-Taylor growth rate.

Theoretical research into the Rayleigh-Taylor instability can be divided into analytical and computational approaches. Some of the early analytical work done on Rayleigh-Taylor instability in compressible fluids was inconclusive or erroneous. Vandervoort⁴ and Plesset and Hsieh⁵ both analyzed the instability at the interface between two polytropic media. Recently Shivamoggi⁶ pointed out that these two papers disagree with one another. Replying to his comment, Plesset and Prosperetti⁷ attributed the contradiction to an error in Vandervoort's analysis which invalidates the latter's treatment except when $\gamma = 1$. They went on to derive in a simple manner a general dispersion relation for an arbitrary equation of state. This derivation, however, itself makes use of an identity which is strictly true only for $\gamma = 1$, namely the statement that the ratio of the perturbed pressure to the perturbed density is equal to the square of the unperturbed sound speed.

Blake⁸ wrote down without derivation a dispersion relation for Rayleigh-Taylor instability in compressible fluids which in fact is correct for the interface between two isothermal (uniform-temperature) fluids satisfying an isothermal ($\gamma = 1$) equation of state, and argued that compressibility effects are negligible except in the long-wavelength limit.

Matthews and Blumenthal⁹ derived the same "isothermal-isothermal" formula with the inclusion of volume radiation forces ($\propto \rho^2$). [The formally identical dispersion relation for waves propagating in a medium consisting of two stably stratified isothermal layers is well known to atmospheric scientists; e.g., Tolstoy derives it in his review article¹⁰ for the case in which the fluids have an (identical) arbitrary γ and analyzes the different waves which arise.] McCrory et al.¹¹ made the physically plausible argument that pressure differences cannot be transmitted across the mode structure on a time scale shorter than $(kc_s)^{-1}$, so that compressibility effects must limit growth rates to values $\lesssim kc_s$. Takabe and Mima¹² wrote down a variational form for the growth rate in the presence of both compressibility and thermal conduction, but neglected to state clearly the assumptions they employed. Scannapieco¹³ investigated the stability of a slab of ideal polytropic gas with an exponentially increasing or decreasing density profile confined between two rigid horizontal walls separated by a distance d and found that growth rates are enhanced by compressibility. However, in analyzing the limit in which the scale height H satisfies $H \gg d$, he allowed the density to vary while treating the sound speed as a constant, which is inconsistent unless the density decreases in the upward direction. Baker¹⁴ solved the Blake⁸ dispersion relation numerically for the ratio of the square of the calculated growth rate to the incompressible value as a function of Atwood number $A = (\rho_2 - \rho_1)/(\rho_2 + \rho_1)$, the ratio c_1^2/c_2^2 of the squares of the sound speeds, and the nondimensionalized wavelength $g/2kc_1^2$. Because of the assumptions implicit in the "isothermal-isothermal" model, however, the first

two parameters are not independent: $c_1^2/c_2^2 = \rho_2/\rho_1 = (1 + A)/(1 - A)$. His conclusion that finite compressibility is sometimes stabilizing and sometimes destabilizing is therefore called into question.

On the whole, the problem with the analytical approach to studying compressibility effects on the Rayleigh-Taylor instability has most frequently been a failure to specify clearly the model being investigated. Many authors have attempted to derive model-independent dispersion relations, or at least formulas of wide applicability, which would not be restricted to a particular type of density profile. For a medium to behave compressibly with respect to a mode of wavenumber k , however, the dimensionless wavenumber must satisfy $kc_s^2/g = kH \lesssim 1$, where H is the equilibrium scale height. But h , the vertical scale of the perturbation, typically satisfies $h \sim k^{-1}$, so we must have $h \gtrsim H$. The mode samples the vertical variation of the equilibrium state and must therefore depend sensitively on it.

It is actually quite easy to show for ideal polytropic media that compressibility destabilizes the Rayleigh-Taylor instability.¹⁵ The energy principle of Bernstein et al.³ (an extension of the version given earlier by Chandrasekhar¹⁶) predicts that polytropic media with finite adiabatic index γ exhibit maximum growth rates which decrease with increasing γ . Incompressible fluids, which are obtained in the limit $\gamma \rightarrow \infty$, are thus more stable than compressible ones. This has been confirmed experimentally by Asay (see Ref. 14).

Most computational approaches to the problem strive so hard for realism that they treat too many physical processes simultaneously. When something happens in a calculation it is hard to say which process is responsible,

especially when all the parameters in the code have been chosen so as to simulate a particular laboratory experiment. While such simulations are a major reason for computation, they are worthless unless every effect included in the model has been carefully validated against analytical theory or reliable measurements. Failure to do this properly vitiated some early simulations of Rayleigh-Taylor instability in imploding pellets. Another difficulty arose from the necessity of performing well-resolved multidimensional calculations in place of the one-dimensional ones used for studying unperturbed implosions.

A way around this difficulty was found using so-called "piggyback" codes.¹⁷ The linearized equations of motion are analyzed into a superposition of angular (e.g., spherical) harmonics, and the equations for the radius-dependent amplitudes corresponding to one or more such modes are advanced in time together with the zeroth-order quantities. This technique, also successfully employed in connection with incompressible cylindrical liner implosions,¹⁸ eliminates the numerical resolution problem. The work of Shiau et al¹⁷ clearly showed for the first time that flow of plasma across an interface (a process which can only occur in compressible media), while stabilizing, is not by itself able to completely suppress the Rayleigh-Taylor instability. Subsequently, improvements in computational methods and techniques of code validation have enabled fully nonlinear two-dimensional calculations to be carried out which predict the linear and nonlinear evolution of the Rayleigh-Taylor instability with high accuracy.¹⁹⁻²⁰ Stimulated partly by experiment and partly by code results, quite comprehensive theories have now been developed which take into account such

diverse effects as vortex shedding, compressibility, thermal conduction, and ablation.²¹ In the remainder of this paper I will not be saying anything further about the use of numerical simulations to study the Rayleigh-Taylor instability.

A different question, which I will also not be considering here, involves the time required to establish a state as a result of an initial localized disturbance. This process, which is instantaneous in an incompressible fluid, lasts a time equal to that required for a sound wave to propagate a few times back and forth across the entire system. Instead, I will consider linear eigenmodes, which by definition are initiated in "prepared" states involving the entire system. Finding the eigenmodes in compressible fluids presents enough analytical difficulty to dissuade one from seeking the solution of the general initial value problem. Relatively few papers have been written on this topic.

This paper is organized as follows. A simple version of an argument originally employed by Schwarzschild (see, e.g., Landau and Lifshitz²²) in discussing hydrodynamic interchange is used to derive threshold criteria for the Rayleigh-Taylor and convective instabilities in arbitrary stratified media. Then the energy principle is used to show that compressibility is always destabilizing, and Newcomb's extension of this result to higher eigenmodes of the system is presented.²³ Next, the exact dispersion relation for the Rayleigh-Taylor instability at the interface between two ideal polytropic fluids with different adiabatic indices, each fluid having uniform temperature, is derived following Bernstein and Book,²⁴ and various limiting cases of this result are discussed. The extension to an arbitrary piecewise isothermal equilibrium is sketched, concluding the portion of the paper dealing with stability of static equilibria.

The only nonstationary fluid states in which the Rayleigh-Taylor instability can be treated analytically are self-similar expansions or contractions for which the velocity is proportional to the distance from the center of symmetry (uniform self-similar motions).²⁵ Following a summary of the formalism used to discuss stability of such states, examples are given, first for implosions and then for expansions. A final section summarizes the main results and attempts to draw them together by pointing out the common themes that run through all of these examples.

Physical Basis of Gravitational Interchange Instabilities

Suppose a fluid with vertical density and pressure profiles $\rho(y)$, $p(y)$ is in hydrostatic equilibrium:

$$\frac{\partial p}{\partial y} + \rho g = 0, \quad (1)$$

where g is the constant gravitational acceleration. Note that $p(y)$ must decrease monotonically as a function of y , but $\rho(y)$ need not. We assume that the adiabatic index (ratio of specific heats) γ is constant.

Now consider an element of fluid with differential volume ΔV at some arbitrary height y . It contains mass $\Delta m = \rho \Delta V$ and internal energy $p \Delta V / (\gamma - 1)$. Assume that it is displaced adiabatically to a new position y' , where it occupies a new volume $\Delta V'$ at a new density ρ' and pressure p' . By conservation of mass,

$$\rho' \Delta V' = \rho \Delta V = \Delta m; \quad (2)$$

by adiabatic invariance (entropy conservation),

$$p' (\Delta V')^\gamma = p (\Delta V)^\gamma. \quad (3)$$

To make room for the displaced parcel of fluid, a second differential volume $\overline{\Delta V}$ with initial density $\overline{\rho}$ and pressure \overline{p} is displaced from location $\overline{y} = y'$ to the first location $\overline{y}' = y$. We assume that $y - \overline{y} = h > 0$, i.e., the first location is above the second (possibly by a finite distance). Evidently the second parcel of fluid after displacement has density $\overline{\rho}'$ and pressure \overline{p}' satisfying

$$\overline{\rho}' \overline{\Delta V}' = \overline{\rho} \overline{\Delta V} = \overline{\Delta m}, \quad (4)$$

$$\overline{p}' (\overline{\Delta V}')^\gamma = \overline{p} (\overline{\Delta V})^\gamma. \quad (5)$$

Since the displacements are adiabatic, the total change in internal energy is

$$\delta W_I = (p' \Delta V' - p \Delta V + \bar{p}' \overline{\Delta V}' - \bar{p} \overline{\Delta V}) / (\gamma - 1). \quad (6)$$

This is accompanied by a total net change in gravitational energy given by

$$\delta W_G = \Delta m g(y' - y) + \overline{\Delta m} g(\bar{y}' - \bar{y}) = (\overline{\Delta m} - \Delta m)gh. \quad (7)$$

If after the interchange the displaced fluids are in equilibrium with their surroundings but the total change in energy is negative (i.e., energy is reduced),

$$\delta W = \delta W_I + \delta W_G < 0, \quad (8)$$

the interchange is energetically favored, and the configuration is therefore unstable.

There are two conditions for a displaced differential volume to be in equilibrium with its surroundings: a kinematic condition, that it "fill the hole" left by its counterpart, and a dynamic conditions, that it be in pressure balance. For the first parcel, these requirements imply

$$\Delta V' = \overline{\Delta V} \quad (9)$$

and

$$p' = \bar{p}; \quad (10)$$

for the second, they imply

$$\overline{\Delta V}' = \Delta V \quad (11)$$

and

$$\bar{p}' = p. \quad (12)$$

Equations (3) and (5) now both reduce to a relation connecting ΔV and $\overline{\Delta V}$:

$$\bar{p} (\overline{\Delta V})^\gamma = p (\Delta V)^\gamma. \quad (13)$$

If we calculate the energy in each parcel of fluid as it undergoes displacement we see that the work done on the first one by the surrounding fluid exactly equals the expansion work done by the second. Thus

$$\delta W_I = 0, \quad (14)$$

and so

$$\delta W = \delta W_G = \rho \Delta V g h \left[\frac{\bar{\rho}}{\rho} \left(\frac{p}{\bar{p}} \right)^{1/\gamma-1} \right]. \quad (15)$$

We consider three cases of interest.

Case I: Discontinuous change in density. Here we can take y and \bar{y} to be on opposite sides of the discontinuity, but contiguous, so that h is very small. Since the pressure is continuous, $\overline{\Delta V} = \Delta V$, and Eq. (15) reduces to

$$\delta W = (\bar{\rho} - \rho) \Delta V g h. \quad (16)$$

The system is thus unstable if $\rho > \bar{\rho}$. This is the usual criterion for the Rayleigh-Taylor instability.

Case II: Continuous density variation, $\rho > \bar{\rho}$. Now y and \bar{y} need not be contiguous. Since $\bar{p} > p$ always holds, δW_G is again negative if $\rho > \bar{\rho}$. The limit as $h \rightarrow 0$ yields the criterion for Rayleigh-Taylor instability in a smoothly stratified medium,

$$\nabla \rho \cdot \nabla p < 0. \quad (17)$$

Case III: Continuous density variation, $\rho < \bar{\rho}$.

Even if the density decreases monotonically in the upward direction, it is still possible to satisfy $\delta W < 0$, provided that

$$\bar{p}/\bar{\rho}^\gamma > p/\rho^\gamma. \quad (18)$$

This is the criterion for convective instability.²²

We thus see that in certain circumstances a fluid stratified under gravitational acceleration can be unstable to overturning, or interchange. When the instability is driven by a density inversion (dense fluid lying above less dense), the name "Rayleigh-Taylor instability" is used. By means of the energy principle, these ideas can be carried a step further to exhibit the manner in which the degree of compressibility affects stability.

Energy Principle

The equations describing the evolution of a small perturbation about a specific equilibrium state are linear in the perturbed fluid variables. They can be reduced to a single homogeneous differential equation in terms of one dependent variable, e.g., the perturbed pressure or displacement. If we assume time dependence $\sim \exp(-i\omega t)$, where ω is the frequency, an ordinary differential equation, usually of second order, results for the spatial dependence. The quantity ω (or ω^2) appears as an eigenvalue, determined by solving the equation subject to appropriate boundary conditions. If this eigenvalue problem is of Sturm-Liouville form, a number of rigorous theorems apply. The most important of these says that ω can be found from a variational principle, i.e., by looking for the extremal (usually minimum) value of some functional over a class of these functions which satisfy the boundary conditions and other physical constraints. In hydrodynamic stability problems, the variational principle has a natural interpretation in terms of energies.

Assuming an adiabatic equation of state, the potential energy W associated with a general small perturbation $\xi(x)$ about an allowed equilibrium of the ideal magnetohydrodynamic equations can be expressed in the form³

$$W = W_0 + \gamma \int d^3x p (\nabla \cdot \xi)^2, \quad (19)$$

where W_0 is a quadratic functional of ξ which is independent of γ . If the eigenvalues resulting from solution of the Sturm-Liouville problem determining ξ are ordered by magnitude according to

$$\omega_0^2 < \omega_1^2 < \omega_2^2 < \dots, \quad (20)$$

then the lowest (most unstable) eigenvalue is determined by a variational principle

$$\omega_0^2 = \min_{\xi} (W/K), \quad (21)$$

where K is a second (nonnegative) quadratic functional and the minimum is taken over all ξ satisfying the boundary conditions. By (19), W/K is an increasing function of γ for any fixed ξ , so that ω_0^2 is also an increasing function of the adiabatic index γ . An incompressible medium ($\gamma \rightarrow \infty$) is thus more stable than any with finite γ .

Newcomb²³ has shown how this result can be extended to the higher modes of the system. If Σ_n is the subspace spanned by $\xi_0, \xi_1, \dots, \xi_{n-1}$, the eigenvectors corresponding to $\omega_0^2, \omega_1^2, \dots, \omega_{n-1}^2$, then the energy principle for the next eigenvalue takes the form

$$\omega_n^2 = \min_{\xi \in C(\Sigma_n)} (W/K), \quad (22)$$

where $C(\Sigma)$ is the orthogonal complement of Σ . Let

$$F(\Sigma) = \min_{\xi \in C(\Sigma)} (W/K), \quad (23)$$

so that $\omega_n^2 = F(\Sigma_n)$. The subspace Σ_n is distinguished from all others of dimension n as that which maximizes the function F (because it is spanned by the eigenvectors with the n lowest eigenvalues). Hence

$$\omega_n^2 = \max_{\dim(\Sigma) = n} \min_{\xi \in c(\Sigma)} (W/K), \quad (24)$$

from which it follows that ω_n^2 is an increasing function of γ for all $n = 1, 2, \dots$

We saw (in the preceding section) that the degree of compressibility plays no role in determining the threshold for instability. The present result implies that, of two otherwise identical unstably stratified fluid systems, the more compressible one has the larger growth rate. In the sequel we illustrate this conclusion by calculating growth rates in some situations where analytic solutions are possible.

Rayleigh-Taylor Instability at the Interface between Two Isothermal Layers

An ideal polytropic fluid in a constant gravitational field evolves in time according to the system of equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underline{v} = 0; \quad (25)$$

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) + \nabla p + \rho \underline{g} = 0; \quad (26)$$

$$\frac{\partial p}{\partial t} + \underline{v} \cdot \nabla p + \gamma p \nabla \cdot \underline{v} = 0. \quad (27)$$

The condition for a stationary equilibrium ($\underline{v} = 0$) is expressed by Eq. (1).

Let us leave p and ρ otherwise unspecified for a moment and suppose that this state is subjected to an infinitesimal perturbation defined by the local displacement $\underline{\xi}(x,y,z,t)$ of an element of fluid. Using the subscript 1 to distinguish perturbed quantities, we have for the velocity

$$\underline{v}_1 = \frac{\partial \underline{\xi}}{\partial t}, \quad (28)$$

so that the perturbed density satisfies

$$\frac{\partial \rho_1}{\partial t} = - \nabla \cdot \rho \underline{v}_1 = - \frac{\partial}{\partial t} \nabla \cdot (\rho \underline{\xi}), \quad (29)$$

whence ρ_1 is given by

$$\rho_1 = - \nabla \cdot (\rho \underline{\xi}) \quad (30)$$

plus a time-independent quantity which can be set equal to zero. Likewise, the perturbed adiabatic law

$$\frac{\partial p_1}{\partial t} + \underline{v}_1 \cdot \nabla p + \gamma p \nabla \cdot \underline{v}_1 = 0 \quad (31)$$

has the solution

$$p_1 = - \gamma p \nabla \cdot \underline{\xi} - \underline{\xi} \cdot \nabla p. \quad (32)$$

Substitution of (30) and (32) in the perturbed momentum equation

$$\rho \frac{\partial^2 \underline{\xi}}{\partial t^2} + \nabla p_1 + \rho_1 \underline{g} = 0 \quad (33)$$

yields

$$\rho \frac{\partial^2 \underline{\xi}}{\partial t^2} - \nabla (\gamma p \nabla \cdot \underline{\xi} + \underline{\xi} \cdot \nabla p) - \underline{g} \nabla \cdot (\rho \underline{\xi}) = 0. \quad (34)$$

Assuming that the perturbed quantities vary sinusoidally with frequency ω as functions of time, we can rewrite (34) in the form

$$-\omega^2 \underline{\underline{\xi}} + (\gamma-1) \underline{\underline{g}} \sigma - \frac{\gamma P}{\rho} \nabla \sigma + \nabla(\underline{\underline{g}} \cdot \underline{\underline{\xi}}) = 0, \quad (35)$$

where $\sigma = \nabla \cdot \underline{\underline{\xi}}$. It is this equation which must be solved, subject to boundary conditions, in order to determine the eigenvalues ω^2 and the spatial dependence of the eigenvectors $\underline{\underline{\xi}}$.

Evidently the simplest choice of the basic state is isothermal,

$$p/\rho \equiv c^2 = \text{const}, \quad (36)$$

which leads to an equation for $\underline{\underline{\xi}}$ all of whose coefficients are constant. If $\underline{\underline{g}}$ is directed downward, i.e., in the negative y direction, we then have

$$\rho(y) = \rho_0 \exp(-gy/c^2). \quad (37)$$

As will be seen, the specification (36)-(37) for the basic state results in eigenfunctions which are likewise exponential in y , and yields an algebraic dispersion relation. Any other choice gives rise to transcendental functions which must be evaluated at ordinary (nonsingular) points when the boundary conditions are imposed, greatly complicating the form of the dispersion relation.

For our present purposes it suffices to consider a piecewise isothermal state with just two regions (Fig. 1). We take the interface separating them to be at $y = 0$, i.e., coinciding with the x - z plane. To distinguish between the regions we will label all quantities belonging to the lower one with bars. For the sake of generality we allow the adiabatic exponent to vary,

$$\bar{\gamma} \neq \gamma.$$

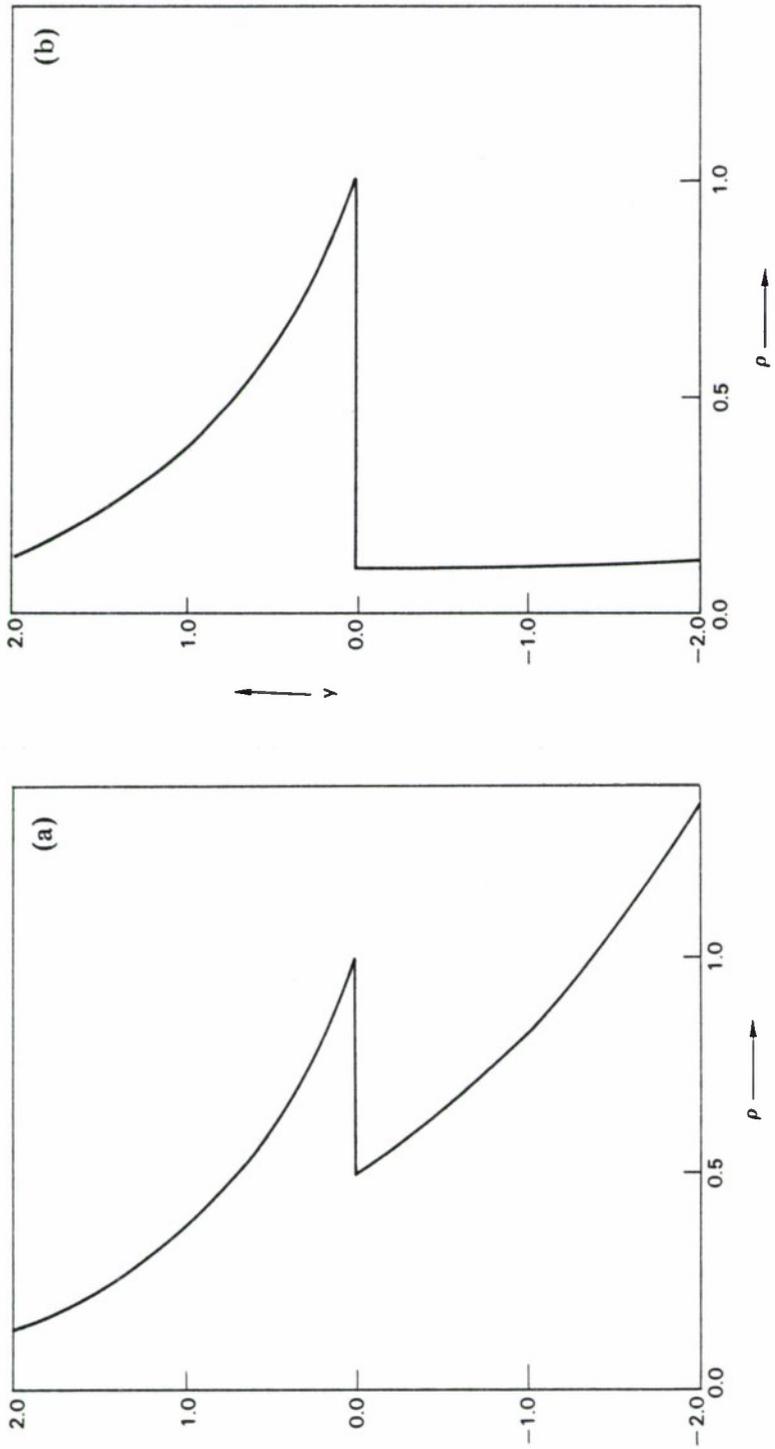


Figure 1

Density (horizontal axis) vs height for a system consisting of two constant-temperature media supported by pressure against gravity. The ratio of upper to lower density at the interface is (a) 2; (b) 10. The units are chosen to make the scale height in the upper region and the gravitational acceleration g both equal to unity.

With this choice, the coefficients of Eq. (35) become constants. Assuming harmonic dependence in the transverse plane, with the x axis chosen parallel to the wave vector k, we can look for solutions which are exponentials in y:

$$\xi(x,y,t) = (\underline{e}_x A + \underline{e}_y B) \exp[i(kx - \omega t) - \mu y], \quad (38)$$

A, B, and μ constant. Thus (35) becomes

$$-\omega^2 A - ikc^2(ikA - \mu B) + ikgB = 0, \quad (39)$$

$$-\omega^2 B + (\gamma - 1)g(ikA - \mu B) - \mu c^2(ikA - \mu B) - \mu gB = 0. \quad (40)$$

Setting the determinant of (39)-(40) equal to zero yields a quadratic for μ . The condition that the solution be well behaved as $y \rightarrow +\infty$ selects the root

$$\mu = \{-\gamma g + [\gamma^2 g^2 - 4\omega^2 \gamma c^2 + 4\gamma^2 k^2 c^4 - 4\gamma(\gamma - 1)g^2 k^2 c^2 / \omega^2]^{1/2}\} (2\gamma c^2)^{-1}. \quad (41)$$

Similarly we obtain a quadratic for $\bar{\mu}$ in the lower half-plane; the condition that the eigenfunctions vanish at $y = -\infty$ yields

$$\bar{\mu} = \{-\bar{\gamma} g - [\bar{\gamma}^2 g^2 - 4\omega^2 \bar{\gamma} c^2 + 4\bar{\gamma}^2 k^2 c^4 - 4\bar{\gamma}(\bar{\gamma} - 1)g^2 k^2 c^2 / \omega^2]^{1/2}\} (2\bar{\gamma} c^2)^{-1}. \quad (42)$$

The kinematic boundary condition at the interface reduces to continuity of the vertical component of $\underline{\xi}$, i.e.,

$$A = \bar{A}. \quad (43)$$

The dynamic boundary condition requires that $p_1 + \underline{\xi} \cdot \nabla p$ be continuous at $y = 0$, whence by Eq. (32)

$$\gamma p(0) \sigma = \bar{\gamma} \bar{p}(0) \bar{\sigma} \quad (44)$$

Since $p(0) = \bar{p}(0)$, we can combine (43) and (44) as

$$\gamma(ikA - \mu B)/A = \bar{\gamma}(ik\bar{A} - \bar{\mu}\bar{B})/\bar{A}. \quad (45)$$

Substituting B/A from (39) or (40) and writing μ in terms of ω^2 by means of (41), and performing the analogous operations with the barred counterparts of these equations, we find from (45) a relation determining ω^2 . When $\gamma = \bar{\gamma}$ it becomes formally identical with the wave dispersion relation given by Tolstoy¹⁰, and for the special case $\gamma = \bar{\gamma} = 1$ it reduces to the one given by Blake.⁸

The treatment can, however, be carried a step further. Using the interactive symbolic manipulation system MACSYMA, we can transform this equation into an algebraic equation in $Z = \omega^2/kg$. This is done by squaring the equation twice to eliminate the square roots appearing in Eqs. (41) and (42) and factoring the result. The physical root is found²⁴ to satisfy the quartic

$$D'^2 Z^4 - 2DD'SZ^3 + (D^2S^2 + 2DD' - D'^2)Z^2 - 2D^2(S - S')Z - D^4 = 0, \quad (46)$$

where

$$D = k(c^2 - \bar{c}^2)g^{-1}; \quad (47)$$

$$S = k(c^2 + \bar{c}^2)g^{-1}; \quad (48)$$

$$D' = k(c^2/\gamma - \bar{c}^2/\gamma)g^{-1}; \quad (49)$$

$$S' = k(c^2/\gamma - \bar{c}^2/\bar{\gamma})g^{-1}; \quad (50)$$

Evidently Eq. (46) always has one negative root, which for $D < 0$ is found numerically to satisfy Eq. (45). This solution can be exhibited by applying the general procedure for solving a quartic, but the result is far too cumbersome to be useful. Instead we look at some limits and special cases of physical interest.

First let $\bar{\gamma} = \gamma$, so that both regions contain fluids with the same compressibility properties. Then Eq. (46) becomes

$$Z^4 - 2\gamma SZ^3 + (\gamma^2 S^2 + 2\gamma - 1)Z^2 - 2\gamma(\gamma - 1)SZ - \gamma^2 D^2 = 0. \quad (51)$$

In the limit $\gamma \rightarrow \infty$, the solution of Eq. (51) associated with the instability satisfies

$$S^2 Z^2 - 2SZ - D^2 = 0, \quad (52)$$

whose negative root is given by

$$Z = [1 \pm (1 + D^2)^{1/2}]S^{-1}. \quad (53)$$

For negative values of D the lower sign in Eq. (53) yields a solution of Eq. (45), as confirmed by numerical evaluation. When we take $kc^2 \gg g$, $k\bar{c}^2 \gg g$ (wavelength short compared with both scale heights), we recover the usual dispersion relation for the Rayleigh-Taylor instability at an interface between two uniform incompressible media, viz.,

$$-\frac{\omega^2}{kg} = \frac{\rho_0 - \bar{\rho}_0}{\rho_0 + \bar{\rho}_0}. \quad (54)$$

At long wavelengths ($k \rightarrow 0$), Eq. (53) goes over to

$$\omega^2 = -k^2 \frac{(c^2 - \bar{c}^2)^2}{2(c^2 + \bar{c}^2)}, \quad (55)$$

displaying the effect of the spatial dependence of the unperturbed state. We thus recover the incompressible result, as expected. Figure 2 illustrates the approach to this limit. It shows plots of ω^2/kg as functions of k obtained by solving Eq. (46) numerically for various choices of γ between 1 and ∞ , assuming the unperturbed states shown in Fig. 1. As can be seen, finite compressibility increases the growth rates, the relative effect being

greatest at long wavelengths. When $k \rightarrow 0$ then S , D , S' and D' all become small and Eq. (46) reduces for finite γ , $\bar{\gamma}$ to

$$D'(2D - D')Z^2 - 2D^2(S - S')Z - D^4 = 0, \quad (56)$$

whence ω is proportional to k . For $\bar{\gamma} = \gamma$ Eq. (56) yields

$$Z = \frac{\gamma}{2\gamma-1} \{(\gamma-1)S - [(\gamma-1)^2 + (2\gamma-1)D^2]^{1/2}\}. \quad (57)$$

When $\gamma = 1$ this becomes

$$\omega^2 = k^2(c^2 - \bar{c}^2). \quad (58)$$

This is to be compared with the corresponding incompressible result given in Eq. (55). On the other hand, for short-wavelength perturbations ($k \rightarrow \infty$), Eq. (46) reduces to

$$S^2Z^2 - D^2 = 0, \quad (59)$$

whose solution is identical with Eq. (54).

Another interesting limit is that in which the density of the upper medium becomes infinite, so that $c = 0$. One of the extraneous roots factors out of Eq. (46), which then reduces to a cubic,

$$Z^3 + (2\bar{\gamma} - 1)Z - \frac{k\bar{\gamma}\bar{c}^2}{g}(Z^2 - 1) = 0. \quad (60)$$

Equation (60) holds even if $\gamma \rightarrow \infty$ in such a way that γc^2 (the square of the sound speed) remains finite. If instead we assume that the density in the lower region vanishes (i.e., $\bar{c}^2 \rightarrow \infty$), then the dispersion relation is even simpler, becoming

$$Z = -1, \quad (61)$$

which coincides with the incompressible result. This is consistent with the behavior shown in Fig. 2, which indicates that as $\bar{\rho}_0$ decreases (for fixed ρ_0), the difference between compressible and incompressible growth rates becomes less pronounced.

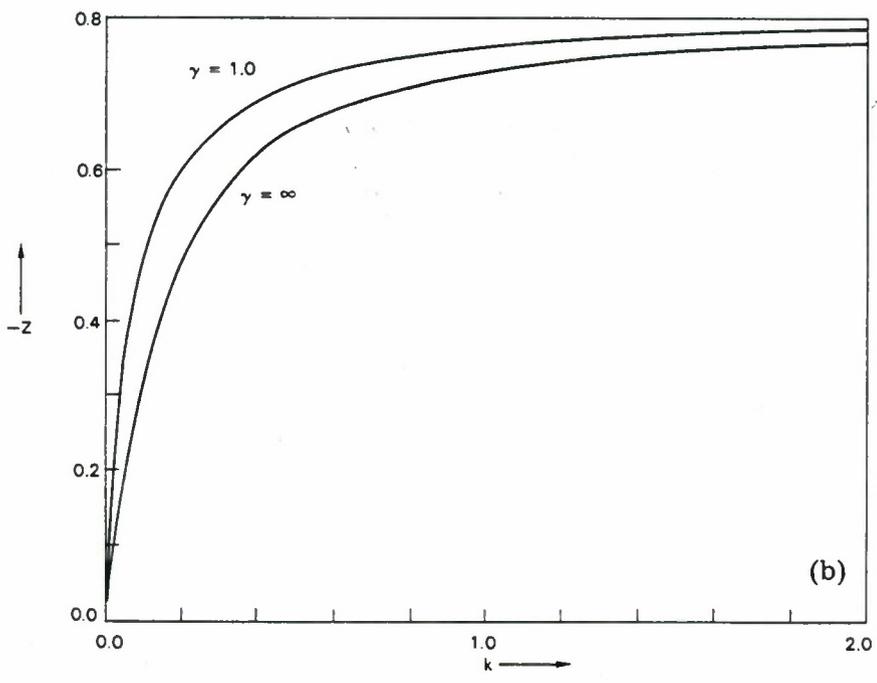
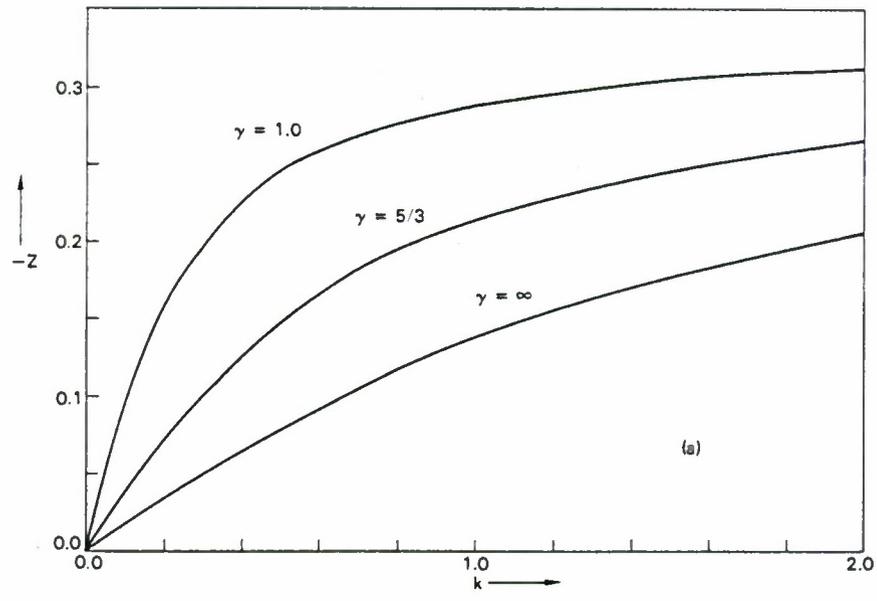


Figure 2

Dimensionless squared growth rate $-\omega^2/kg$ vs wavenumber k for the two basic states shown in Fig. 1, with the same choice of units. The adiabatic index γ in both regions is taken to be 1, 5/3 or ∞ , as indicated by the label. Note that the curves asymptotically approach the value $(\rho - \bar{\rho})/(\rho + \bar{\rho})$, equal to 0.333 and 0.818, respectively.

If we assume

$$\rho_0 \gamma = \bar{\rho}_0 \bar{\gamma}, \quad (62)$$

then D' vanishes identically and Eq. (46) simplifies to

$$S^2 Z^2 - 2(S - S')Z - D^2 = 0, \quad (63)$$

whence

$$Z = \frac{S - S' - [(S - S')^2 + D^2 S^2]^{1/2}}{2S^2}. \quad (64)$$

Finally, if

$$\rho_0 = \bar{\rho}_0, \quad (65)$$

so that $c^2 = \bar{c}^2$ and $D = 0$, then even for $\bar{\gamma} \neq \gamma$

$$Z = 0, \quad (66)$$

i.e., the perturbations are marginally stable.

Use of Piecewise Isothermal Profile to Approximate an Arbitrary Equilibrium

More generally, we can specify an equilibrium state consisting of $N-1$ slabs of finite thickness sandwiched between two semi-infinite regions, with density profiles in the various regions of the form

$$\rho_j(y) = \rho_0^j \exp(-gy/c_j^2), \quad (67)$$

pressure profiles given by

$$p_j(y) = \rho_j(y) c_j^2, \quad (68)$$

and adiabatic indices γ_j , $j = 0, 1, \dots, N$. We take the interface separating layer j from layer $j+1$ to be at $y = y_j$, $j = 0, 1, \dots, N-1$, with $y_0 = 0$. At each interface the changes discontinuously but the pressure is continuous, so that

$$\rho_0^j c_j^2 \exp(-gy_j/c_j^2) = \rho_0^{j+1} c_{j+1}^2 \exp(-gy_j/c_{j+1}^2), \quad (69)$$

$j = 0, 1, \dots, N-1$. Evidently such a piecewise isothermal state can be made to approximate an arbitrary ideal hydrostatic equilibrium state as closely as desired if the number of interfaces N is allowed to increase without bound. It is thus analogous to the piecewise isopycnic (constant-density) model used by Mikaelian²⁶ to approximate an arbitrary incompressible equilibrium state.

Following the treatment employed in the previous section, we seek a solution for the perturbed displacement in the form

$$\xi_0 = (e_x A_0^+ + e_y B_0^+) \exp[i(kx - \omega t) - \mu_0^+ y], \quad (70)$$

$$\xi_j = \sum_{\pm} (e_x A_j^{\pm} e^{-\mu_j^{\pm} y} + e_y B_j^{\pm} e^{-\mu_j^{\pm} y}) \exp[i(kx - \omega t)], \quad (71)$$

$j = 1, 2, \dots, N-1$, and

$$\xi_N = (\underbrace{e}_{\omega x} A_N^- + \underbrace{e}_{\omega y} B_N^-) \exp[i(kx - \omega t) - \mu_N^- y], \quad (72)$$

where

$$\mu_j^\pm = \{-\gamma_j g \pm [\gamma_j^2 g^2 - 4\omega^2 \gamma_j c_j^2 + 4\gamma_j^2 k^2 c_j^4 - 4\gamma_j(\gamma_j - 1)g^2 k^2 c_j^2 / \omega^2]^{1/2}\} (2\gamma_j c_j^2)^{-1}. \quad (73)$$

Evidently Eqs. (70)-(72) introduce $4N$ unknown quantities A_j^\pm, B_j^\pm . The boundary conditions

$$\sum_{\pm} A_j^\pm e^{-\mu_j^\pm y_j} = \sum_{\pm} A_{j+1}^\pm e^{-\mu_{j+1}^\pm y_j} \quad (74)$$

and

$$\gamma_j \sum_{\pm} (ikA_j^\pm - \mu_j^\pm B_j^\pm) e^{-\mu_j^\pm y_j} = \gamma_{j+1} \sum_{\pm} (ikA_{j+1}^\pm - \mu_{j+1}^\pm B_{j+1}^\pm) e^{-\mu_{j+1}^\pm y_j}, \quad (75)$$

$j = 0, 1, \dots, N-1$, provide $2N$ linear relations among these. (Note that $A_0^- = B_0^- = A_N^+ = B_N^+ = 0$.) We can eliminate the B_j^\pm in favor of the A_j^\pm using the analog of (39),

$$-\omega^2 A_j^\pm - ikc_j^2 (ikA_j^\pm - \mu_j^\pm B_j^\pm) + ikgB_j^\pm = 0, \quad (76)$$

leaving $2N$ linear homogeneous equations in the $2N$ quantities A_j^\pm . The

dispersion relation giving ω^2 as a function of k is then obtained by equating to zero the determinant of this system.

The resulting equation must be solved numerically. It is found that the number of roots associated with gravity modes equals $N-1$, the number of interfaces. For short wavelengths ($k \rightarrow \infty$), one mode is localized at each interface position y_j and is stable or unstable according as $\rho_0^{j+1} < \rho_0^j$ or $\rho_0^{j+1} > \rho_0^j$. At longer wavelengths the identity of various modes becomes obscure, and the distinction between Rayleigh-Taylor and convective instability may be blurred. Many interesting limiting cases can be distinguished, e.g., $\gamma_j \rightarrow \infty$, $j = 0, 1, \dots, N$ (incompressible fluid); $\rho_0^0 \rightarrow \infty$ (solid lower boundary); $\rho_0^N \rightarrow 0$ (free upper surface), etc. Of course, it might be argued that it is just as easy (in both incompressible and compressible cases) to approximate the differential equation for ξ by finite differences and obtain the spectrum using a standard eigenvalue-solving routine.

Uniform Self-Similar Implosions and Expansions: Formulation

Many of the problems in which compressible fluids are subject to Rayleigh-Taylor instability involve nonsteady basic states (e.g., laser pellet implosions, the outward motion of gas following a stellar explosion). Usually in such problems even the unperturbed motion can only be treated by solving the fluid equations numerically using a code with one spatial variable. To analyze perturbations with angular dependence requires a two- or three-dimensional code. A compromise approach involves linearizing the perturbed equations, expanding them in cylindrical or spherical harmonics, and advancing the perturbed radius-dependent amplitude functions in parallel with the variables describing the basic state.¹⁷

However, there exists a class of nontrivial ideal compressible flows which are sufficiently symmetric that both the unperturbed and perturbed equations can be solved analytically. These are the uniform self-similar solutions studied by Sedov.²⁵ They can readily be derived as follows.²⁷

We rewrite Eqs. (25)-(27) in the form

$$\dot{\rho} + \rho \nabla \cdot \underline{\underline{v}} = 0; \quad (77)$$

$$\rho \underline{\underline{v}} + \nabla p = 0; \quad (78)$$

$$(p \rho^{-\gamma})^{\cdot} = 0, \quad (79)$$

where the raised dot ($\dot{}$) denotes a total time derivative. In a system with planar, cylindrical, or spherical symmetry ($v = 1, 2, 3$, respectively), Eqs. (77)-(78) become

$$\dot{\rho} + \rho R^{-v} \frac{\partial}{\partial R} (R^v v), \quad (80)$$

$$\rho \underline{\underline{v}} + \frac{\partial p}{\partial R} = 0. \quad (81)$$

For uniform self-similar flow there is a function $f(t)$ such that for an arbitrary fluid element whose position at $t = 0$ was r , the position R at time t satisfies

$$R = rf(t), \quad (82)$$

with $f(0) = 1$ and $\dot{f}(0) = 0$. The continuity equation (80) then yields

$$\rho(r,t) = \rho_0(r) f^{-\nu}, \quad (83)$$

and hence from the adiabatic law (79)

$$p(r,t) = p_0(r) f^{-\nu\gamma} = s(r) \rho_0^\gamma f^{-\nu\gamma}, \quad (84)$$

where we have introduced the entropy function $s(r) = p_0 \rho_0^{-\gamma}$. If we specify an initial density profile $\rho_0(r)$ on some interval $r_1 < r < r_0$, then

substitution in Eq. (8) results in separation into a spatial part

$$\frac{dp_0}{dr} = \pm \rho_0 \tau^{-2} \quad (85)$$

which determines the pressure p_0 , and a time-dependent part

$$f^{\alpha+1} \ddot{f} = \mp \tau^{-2}, \quad (86)$$

where $\alpha = \nu(\gamma-1)$, determining f . In Eqs. (85)-(86) τ is a separation constant with units of time; the upper (lower) sign corresponds to implosion (expansion). If the pressures on the inner and outer surfaces, $p(r_1, t) = p_0(r_1) f^{-\nu\gamma}$ and $p(r_0, t) = p_0(r_0) f^{-\nu\gamma}$, are nonvanishing, they must be balanced by an equal pressure applied to the shell. It is difficult to imagine how this might be realized in practice, so we will assume $p_0(r_1) = 0$ for implosions and $p_0(r_0) = 0$ for expansions.

A quadrature can be performed on (86), with the result

$$\tau^2 \dot{f}^2 = \mp 2 \ln f \quad (87)$$

when $\gamma = 1$, and

$$\tau^2 \dot{f}^2 = \mp \frac{2}{\alpha} (1-f^{-\alpha}) \quad (88)$$

otherwise. When $\alpha = 2$ (corresponding to $\gamma = 3, 2, 5/3$ for $\nu = 1, 2, 3$, respectively), Eq. (88) can be integrated directly to give

$$f(t) = (1 \mp t^2/\tau^2)^{1/2}. \quad (89)$$

For other values of γ , f is most conveniently found by numerical means. In the case of implosions, f vanishes at a time t_0 given by

$$t_0/\tau = \left(\frac{\alpha}{2}\right)^{1/2} \int_0^1 \frac{df}{(f^{-\alpha} - 1)^{1/2}} = \left(\frac{\pi}{2\alpha}\right)^{1/2} \frac{\Gamma(1/\alpha + 1/2)}{\Gamma(1/\alpha + 1)}, \quad (90)$$

which decreases monotonically as a function of α .

To study the stability of uniform self-similar flows under small perturbations, we must solve the linearized form of (78) for the first-order displacement $\underline{\xi}$:

$$\rho \left(\ddot{\underline{\xi}} - \underline{\underline{\xi}} \cdot \underline{\underline{\nabla}} \underline{\underline{R}} \right) - \underline{\underline{R}} \underline{\underline{\nabla}} \cdot \rho \underline{\underline{\xi}} - \underline{\underline{\nabla}} \cdot (\gamma \rho \underline{\underline{\nabla}} \cdot \underline{\underline{\xi}} + \underline{\underline{\xi}} \cdot \underline{\underline{\nabla}} p) = 0. \quad (91)$$

Note that (91) is identical with (34), except that \underline{g} is replaced by $-\underline{\underline{R}}$. Like the unperturbed momentum equation, (91) is separable in Lagrangian variables.

Substituting

$$\underline{\underline{\xi}}(\underline{r}, t) = \underline{\underline{\xi}}(\underline{r}) T(t), \quad (92)$$

we have, on writing $\underline{\underline{\nabla}} = \underline{\underline{\nabla}}_{\underline{r}}$,

$$(\lambda - 1) \underline{\underline{\xi}} + (\gamma - 1) \underline{\underline{r}} \sigma \pm \tau^2 (\gamma \rho_0 / \rho_0) \underline{\underline{\nabla}} \sigma + \underline{\underline{\nabla}}(\underline{r} \cdot \underline{\underline{\xi}}) = 0 \quad (93)$$

[cf. Eq. (35)], and

$$\tau^2 \ddot{f}^{\alpha+2} T = \mp \lambda T, \quad (94)$$

where λ is the new separation constant, obtained as an eigenvalue in connection with the solution of Eq. (93). Once λ is known, (94) can be converted into a hypergeometric equation in the new variable $x = 1 - f^{-\alpha}$ and solved to give

$$T(t) = T(0)F(a, b; 1/2; x) + \dot{T}(0)(\tau 2x/\alpha)^{1/2}F(a + 1/2, b + 1/2; 3/2; x) \quad (95)$$

where F is the hypergeometric function.²⁸ Here

$$a = (\alpha + 2 + \Delta)/4\alpha, \quad (96a)$$

$$b = (\alpha + 2 - \Delta)/4\alpha, \quad (96b)$$

with $\Delta = [\alpha + 2]^2 - 8\alpha\lambda$ ^{1/2}. When $\gamma \rightarrow 1$, Eq. (95) goes over to

$$T(t) = T(0)\Phi(\lambda/2; 1/2; \alpha f) + \dot{T}(0)(\tau 2\alpha f)^{1/2}\Phi(\lambda/2 + 1/2; 3/2; \alpha f), \quad (97)$$

where $\Phi(a; b; x)$ is the Kummer function.²⁹

Since $T(t)$ is not exponential, we must decide what we mean by instability. A perturbation is defined to be unstable if the associated time-dependent factor satisfies

$$\lim_{t \rightarrow \infty} |T(t)|/f(t) = \infty, \quad (98)$$

and stable if the limit of this ratio is finite. This is equivalent to saying that a perturbation is unstable if and only if its amplitude eventually becomes infinitely larger than the radius of the unperturbed state.

Stability of Uniform Self-Similar Implosions

We begin by considering imploding systems. As $f \rightarrow 0$, both terms in Eq. (95) approach asymptotic forms containing terms proportional to $f^{(\alpha+2\pm\Delta)/4}$. With the lower sign this expression diverges whenever $\alpha + 2 < \Delta$, i.e., $\lambda < 0$. Since the condition for Δ to be real is that λ be no greater than $(\alpha + 2)^2/8\alpha$, whose minimum value as a function of α is unity, we see that $\lambda < 1$ is always sufficient to make T/f diverge. Thus $\lambda > 1$ is the stability criterion for uniform self-similar implosions. If Δ is imaginary, T/f still diverges when $\alpha < 2$, i.e., $\gamma < 1 + 2/\nu$. Elsewhere³⁰ I have presented a simple argument involving conservation of wave action to show that this describes sound wave amplification as a consequence of the geometric properties of the implosion.

Taking the scalar product of (93) with $\underline{\xi}$ and introducing notations for the transpose $\underline{\nabla\xi}^+$ and the curl $\underline{\omega} = \nabla \times \underline{\xi}$, we can multiply through by ρ_0 and integrate to obtain an energy principle:³¹

$$\tau^{-2} \lambda \int_V dV \rho_0 \xi^2 = \int_V dV \rho_0 [(\gamma-1) \sigma^2 + \underline{\nabla\xi} : \underline{\nabla\xi}^+ - \omega^2] + \gamma \int_S dS p_0 \sigma \underline{n} \cdot \underline{\xi}, \quad (99)$$

where V is the volume and S the surface of the shell, and \underline{n} is the unit vector defined so as to point away from the shell on both inner and outer surfaces. The expression multiplying λ and the first two terms in the volume integral on the right-hand side are manifestly nonnegative. From this we see that relative instability ($\lambda < 1$) can only result if $\underline{\omega} \neq 0$ or if the surface integral is nonvanishing. The latter is the case whenever a perturbation exists at a point where the density changes discontinuously, e.g., at $r = r_0$, $r = r_1$, or an internal density jump. The external pressure, which enters

the model as a boundary condition, produces an inward acceleration. There is thus an effective gravity in the outward direction. Hence we anticipate that a Rayleigh-Taylor instability should occur localized at the outer surface. This case was first studied by Kidder³², who assumed $\gamma = 5/3$. It turns out we can solve for $\underline{\xi}$ provided the perturbation wavelengths also expand or contract self-similarly in time, i.e., whenever they do not introduce a definite length scale into the problem. This means that in spherical geometry ($\nu = 3$) there is no restriction on the form of the perturbation; in cylindrical geometry ($\nu = 2$), however, we must have $k_z = 0$; and no general solution is possible in planar geometry ($\nu = 1$).

Operating on (93) with the divergence and with the curl, we get two equations:

$$[\lambda + \nu(\gamma-1) + 1]\sigma \pm \tau^2 \nabla \cdot \left(\frac{\gamma p_0}{\rho_0} \nabla \sigma \right) + \underline{r} \cdot \nabla \sigma = \underline{r} \cdot \nabla \times \underline{\omega} \quad (100)$$

and

$$(\lambda-1)\underline{\omega} + \underline{r} \times \nabla \sigma \pm \tau^2 \gamma p_0 \nabla(\rho_0^{-1}) \times \nabla \sigma = 0. \quad (101)$$

Let us suppose that $\omega = 0$. It follows from (100)-(101) that σ also vanishes.

If that happens,

$$\underline{\xi} = \nabla \psi, \quad (102)$$

where the potential ψ satisfies Laplace's equation

$$\nabla^2 \psi = 0. \quad (103)$$

The general solution of (103) is

$$\psi(r, \phi) = (\psi_+ r^\ell + \psi_- r^{-\ell}) e^{i\ell\phi} \quad (104)$$

in cylindrical coordinates (assuming $\frac{\partial}{\partial z} = 0$), and

$$\psi(r, \theta, \phi) = (\psi_+ r^\ell + \psi_- r^{-\ell-1}) Y_{\ell m}(\theta, \phi) \quad (105)$$

in spherical coordinates. The ψ_\pm are constants, and the $Y_{\ell m}(\theta, \phi)$ are

spherical harmonics. Evidently the first term in (104) and (105) corresponds to a solution localized at the outer surface of the shell, and the second to one localized at the inner surface, so we set $\psi_- = 0$. Substitution in Eq. (93) then yields

$$(\lambda-1)\xi + \nabla(\underline{r} \cdot \xi) = \nabla[(\lambda-1)\psi + \underline{r} \cdot \nabla\psi] = \nabla[(\lambda-1+l)\psi_+ r^l] = 0, \quad (106)$$

whence

$$\lambda = -l + 1. \quad (107)$$

For $l = 0$, $\lambda = 1$ and Eq. (94) reduces to Eq. (86), showing that the perturbations are marginally stable. For all $l > 0$, the limiting form of T/f diverges as

$$\frac{T}{f} \sim f \{ \alpha - 2 - [(\alpha - 2)^2 + 8\alpha l]^{1/2} \} \quad (108)$$

when $f \rightarrow 0$. Evidently the magnitude of the exponent in (108) increases with both l and $\alpha = \nu(\gamma-1)$. Thus, in contrast with the case of static equilibria considered previously, compressibility appears to be somewhat stabilizing. However, it must be noted from Eq. (86) that as α increases, the motion becomes increasingly stiff and so the effective gravity also increases, rendering comparisons difficult.

The problem of perturbations with $\omega \neq 0$ is treated elsewhere³⁰⁻³¹; it is completely analogous to that in the case of expanding solutions to be discussed shortly. Book and Bernstein³³ and Han and Suydam³⁴ have treated the stability of imploding cylindrical systems in detail.

Stability of Uniform Self-Similar Expansions

Here we consider expanding systems, in which $f \rightarrow \infty$. (Sedov distinguishes a third class of trajectories in which f varies between 0 and ∞ with no turning point; we do not treat them here.)

Bernstein and Book³⁵ and Han³⁶, who found the general solutions for the time dependence of arbitrary perturbations in spherical and in cylindrical geometry, respectively, both assumed that the unperturbed states were homentropic ($p\rho^{-\gamma}$ independent of radius). In both geometries the only instability was a Rayleigh-Taylor mode localized at the inner surface, where the driving pressure acts. Here we analyze a class of states, parametrized by the adiabatic index and a shape parameter, which relax the requirement of uniform entropy distribution.³⁷

Suppose the initial density profile is given by:

$$\rho_0(r) = \hat{\rho}(1-r^2/r_0^2)^\kappa, \quad (109)$$

μ and $\hat{\rho}$ constant. Then from Eq. (85),

$$p_0(r) = \hat{p}(1-r^2/r_0^2)^{\kappa+1}, \quad (110)$$

where

$$\hat{p} = \hat{\rho} r_0^2 / 2(\kappa+1) \tau^2, \quad (111)$$

and

$$s(r) = p_0 \rho_0^{-\gamma} = (\hat{p}/\hat{\rho}^\gamma)(1-r^2/r_0^2)^{\kappa+1-\kappa\gamma}. \quad (112)$$

The pressure vanishes at the outer radius of the shell, as does the density provided $\kappa > 0$. At the inner radius the imposed (driving) pressure must have the form

$$p_i = \hat{p}(1-r_i^2/r_0^2)^{\kappa+1} f^{-\nu\gamma}, \quad (113)$$

unless $r_i = 0$. In this model the temperature $\theta = p/\rho$ always decreases

quadratically as a function of radius:

$$\Theta(r,t) = \hat{\Theta} (1-r^2/r_0^2) f^{-\kappa(\gamma-1)}, \quad (114)$$

where $\hat{\Theta} = r_0^2/2(\kappa+1)\tau^2$

As in the implosion case, there is an incompressible irrotational perturbation mode satisfying Eqs. (102)-(103). The solution (which is independent of the shapes of the density and pressure profiles) is given again by (104) or (105). This time, though, it is the mode corresponding to ψ_- which is unstable, giving rise to eigenvalues $\lambda = \ell + 1$ (cylindrical geometry) or $\lambda = \ell + 2$ (spherical geometry). Using standard formulas²⁸ to evaluate hypergeometric functions of unit argument, we find from (95) that as $f \rightarrow \infty$,

$$\frac{T}{\bar{f}} \rightarrow \frac{\Gamma(1/2) \Gamma(1/\alpha) T(0)}{\Gamma(\frac{1}{4} + (2+\Delta)/4\alpha) \Gamma[\frac{1}{4} + (2-\Delta)/4\alpha]} + \frac{(2/\alpha)^{1/2} \Gamma(3/2) \Gamma(1/\alpha) \dot{T}(0)}{\Gamma[\frac{3}{4} + (2+\Delta)/4\alpha] \Gamma[\frac{3}{4} + (2-\Delta)/4\alpha]} \quad (115)$$

Although this limit is finite, for large values of λ (large ℓ) the constants are found from Stirling's formula to grow exponentially:

$$\frac{T}{\bar{f}} \sim \frac{\Gamma(1/\alpha) \exp[\pi(\lambda/2\alpha)^{1/2}]}{2\pi^{1/2} (\lambda/2\alpha)^{(2-\alpha)/4\alpha}} T(0) + \frac{\Gamma(1/\alpha) \exp[\pi(\lambda/2\alpha)^{1/2}]}{2(2\pi\alpha)^{1/2} (\lambda/2\alpha)^{(2+\alpha)/4\alpha}} \tau \dot{T}(0). \quad (116)$$

Figure 3 displays the late-time asymptotic behavior of these solutions as a function of the eigenvalue λ for $\alpha = 1, 2, 3$, and ∞ .

To study the modes for which $\underline{\omega} \neq 0$, we use Eqs. (100)-(101) with ρ_0 and p_0 given by (109)-(110). Using the lower sign, specializing to $\nu = 3$, and using r_0 to scale \underline{r} , we get

$$(\lambda + 3\gamma - 2) \sigma - \frac{\gamma}{2(\kappa+1)} [\nabla \cdot (1 - r^2) \nabla \sigma] + \gamma \underline{r} \cdot \nabla \sigma = \underline{r} \cdot \nabla \times \underline{\omega} \quad (117)$$

and

$$(\lambda - 1) \underline{\omega} = \left(\frac{\gamma\kappa}{\kappa + 1} - 1 \right) \underline{r} \times \nabla \sigma. \quad (118)$$

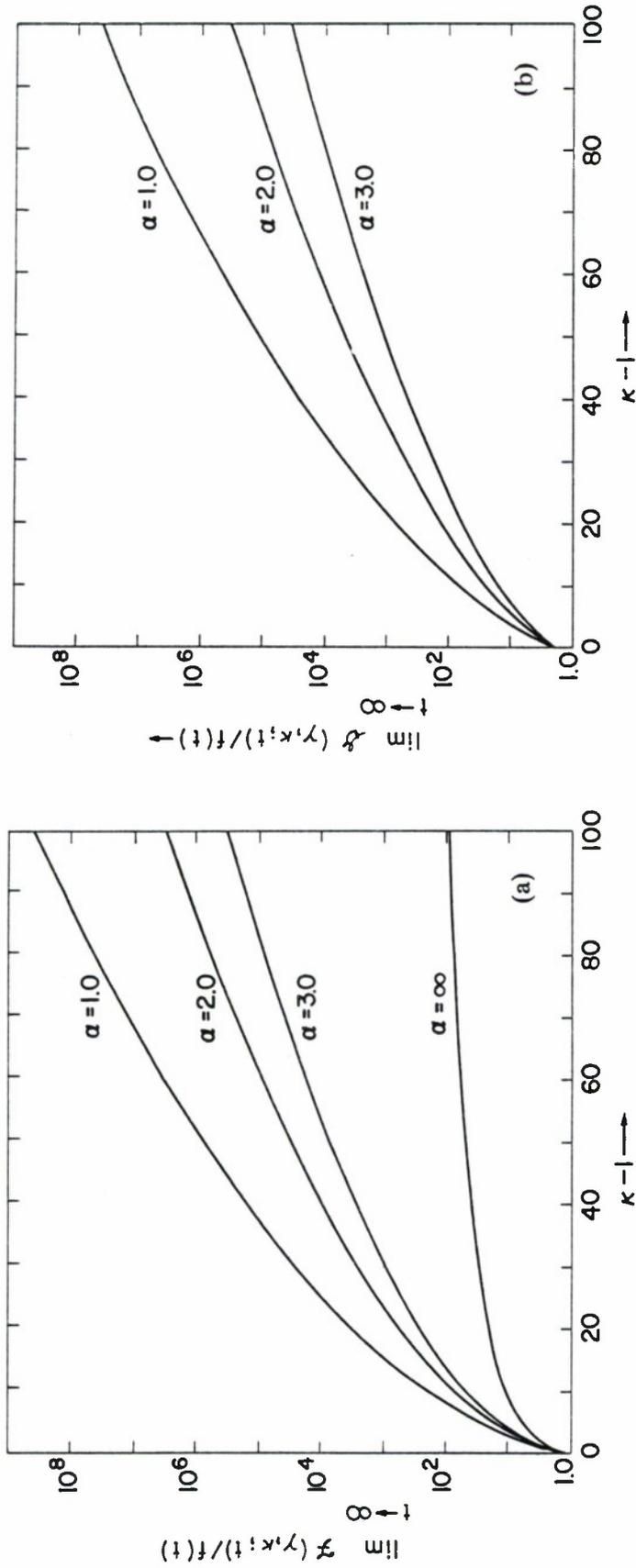


Figure 3

Limiting values [determined from Eq. (115)] as $t \rightarrow \infty$ of (a) \mathfrak{Z}/f , where

$$\mathfrak{Z}(\gamma, \lambda; t) = T(t) \text{ defined by Eq. (95) with } T(0) = 1, \dot{T}(0) = 0, \text{ and (b) } \mathfrak{D}/f,$$

where $\mathfrak{D}(\gamma, \lambda; t) = T(t)$ with $T(0) = 0, \dot{T}(0) = 1$, for $1 < \lambda < 100$. Shown are results for the cases $\alpha = 1, 2, 3$, and, in Fig. 3a, the limiting value

$$\lim_{\gamma, t \rightarrow \infty} \mathfrak{Z}(\gamma, \lambda; t) / f(t) = \lambda.$$

(The corresponding limit for \mathfrak{D} is infinite.)

Eliminating $\underline{\omega}$ between (117) and (118) and assuming separation of the angular and radial dependence of σ by setting

$$\underline{\sigma}(r) = \sigma(r) Y_{\ell m}(\theta, \phi), \quad (119)$$

we obtain a second-order equation for the radial factor $\underline{\sigma}(r)$,

$$\begin{aligned} \frac{\gamma}{2(\kappa+1)} \left[\frac{1-r^2}{r^2} \frac{d}{dr} \left(r \frac{d\sigma}{dr} \right) - \frac{1-r^2}{r^2} \ell(\ell+1)\sigma - 2r \frac{d\sigma}{dr} \right] \\ - \gamma r \frac{d\sigma}{dr} + \left[\frac{(\kappa+1-\kappa\gamma)\ell(\ell+1)}{(\gamma-1)(\kappa+1)} - 3\gamma - \lambda + 2 \right] \sigma = 0. \end{aligned} \quad (120)$$

Rewriting this by means of the substitution $\sigma = r^\ell y$ and $x = r^2$, we obtain the hypergeometric equation

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0, \quad (121)$$

where

$$\left. \begin{array}{l} a \\ b \end{array} \right\} = \frac{1}{2} \left\{ \kappa + \ell + \frac{5}{2} \pm \left[\left(\kappa + \ell + \frac{5}{2} \right)^2 - 4K \right]^{1/2} \right\}, \quad (122a,b)$$

$$c = \ell + 3/2. \quad (122c)$$

Here

$$K = \frac{(\kappa+2)\ell}{2} + \frac{\kappa+1}{2\gamma} \left[\lambda - 2 + 3\gamma - \frac{\ell(\ell+1)(\kappa+1-\kappa\gamma)}{(\lambda-1)(\kappa+1)} \right]. \quad (123)$$

The general solution of (121) is²⁸

$$y = C_1 F(a, b; c; x) + C_2 x^{1-c} F(a-c+1, b-c+1; 2-c; x), \quad (124)$$

where C_1, C_2 are constants. If $r_i \rightarrow 0$, the shell goes over to a gas sphere expanding under its own pressure. In this case only the first term in Eq. (124) is finite at the origin and we must set $C_2 = 0$.

The boundary conditions are found from the requirement that the perturbed pressure vanish at both surfaces of the shell. At the inner surface this implies $\sigma = 0$ or

$$C_1 F(a, b; c; r_i^2/r_o^2) + C_2 (r_i/r_o)^{1-c} \quad (125)$$

$$\cdot F(a - c + 1, b - c + 1; 2 - c; r_i^2/r_o^2) = 0.$$

Since the unperturbed pressure vanishes at the outer surface, it is only necessary that y be finite at $x = 1$. The linear connection formulas²⁸ of both terms of (124) contain a term that diverges as $(1 - x)^{-(\kappa+1)}$ unless a or b is a nonpositive integer. Thus we must have

$$\frac{1}{2} \{ \kappa + \ell + 5/2 - [(\kappa + \ell + 5/2)^2 - 4K]^{1/2} \} = -n, \quad (126)$$

$n = 0, 1, 2, \dots$. From (123) it follows that λ decreases with n . Hence the fastest growth (largest positive λ) corresponds to $n = 0$, which implies $K = 0$. Solving for λ , we finally obtain the dispersion relation

$$\lambda - 1 = - \frac{\gamma \ell (\kappa + 2) + (\kappa + 1)(3\gamma - 1)}{2(\kappa + 1)} \quad (127)$$

$$+ \frac{\{ [\gamma \ell (\kappa + 2) + (\kappa + 1)(3\gamma - 1)]^2 + 4\ell(\ell + 1)(\kappa + 1 - \kappa\gamma) \}^{1/2}}{2(\kappa + 1)}.$$

For the upper branch, $\lambda > 1$ for all $\ell > 0$, provided that

$$\kappa < 1/(\gamma - 1). \quad (128)$$

From (112), we see that this is just the condition that $s'(r) < 0$, i.e., that $s(r)$ decrease in the "upward" direction, that directed opposite to the effective gravitational acceleration $-\ddot{R}$. This is identical with the Schwarzschild criterion (18) derived for the convective instability in static media. Indeed, an interchange argument along the lines of that used to obtain (18) has been employed³⁷ to show that such a uniformly expanding fluid system should be convectively unstable whenever $s'(r) < 0$ holds.

For the $\gamma = 1$ case we can redo all the analysis in terms of confluent hypergeometric functions²⁹ instead of hypergeometric functions, or we can

reach the same results by formally letting $\gamma \rightarrow 1$ in the equations above. The solutions are qualitatively similar to those for $\gamma > 1$ except that now (128) always holds.

Motivated by experiments investigating the use of imploding cylindrical liquid metal liners to compress and heat plasma³⁸, Book and Bernstein³³ studied the Rayleigh-Taylor instability on the inner surface of a liner during both the implosion and rebound phases, assuming $\gamma = 1$. Since both terms in Eq. (97) diverge the same way at large t , it is useful to introduce in their place two new solutions which have asymptotically like Ψ , the standard confluent hypergeometric function of the second kind:²⁹

$$\left. \begin{array}{l} P^\ell(t) \\ Q^\ell(t) \end{array} \right\} = \Phi[(1+\ell)/2; 1/2; \ell n f] \mp \frac{\Gamma(1+\ell/2)}{\Gamma[(1+\ell)/2]} \mp f \Phi(1+\ell/2; 3/2; \ell n f). \quad (129a,b)$$

For large arguments ($t \rightarrow +\infty$), we have

$$P^\ell(t) \sim (\ell n f)^{-(\ell+1)/2} \quad (130a)$$

$$Q^\ell(t) \sim f(\ell n f)^{\ell/2}. \quad (130b)$$

Thus defined, $Q^\ell(t)$ has the property of increasing monotonically for all t , $-\infty < t < \infty$; at turnaround (the instant $t = 0$ when $f = 1$), $Q^\ell(0) = 1$ also.

The only perturbations which are unstable both before and after turnaround are those whose time dependence is proportional to $Q^\ell(t)$. Figure 4 compares the behavior of $Q^\ell(t)$ for $\ell = 1$ and $\ell = 10$ with that of f .

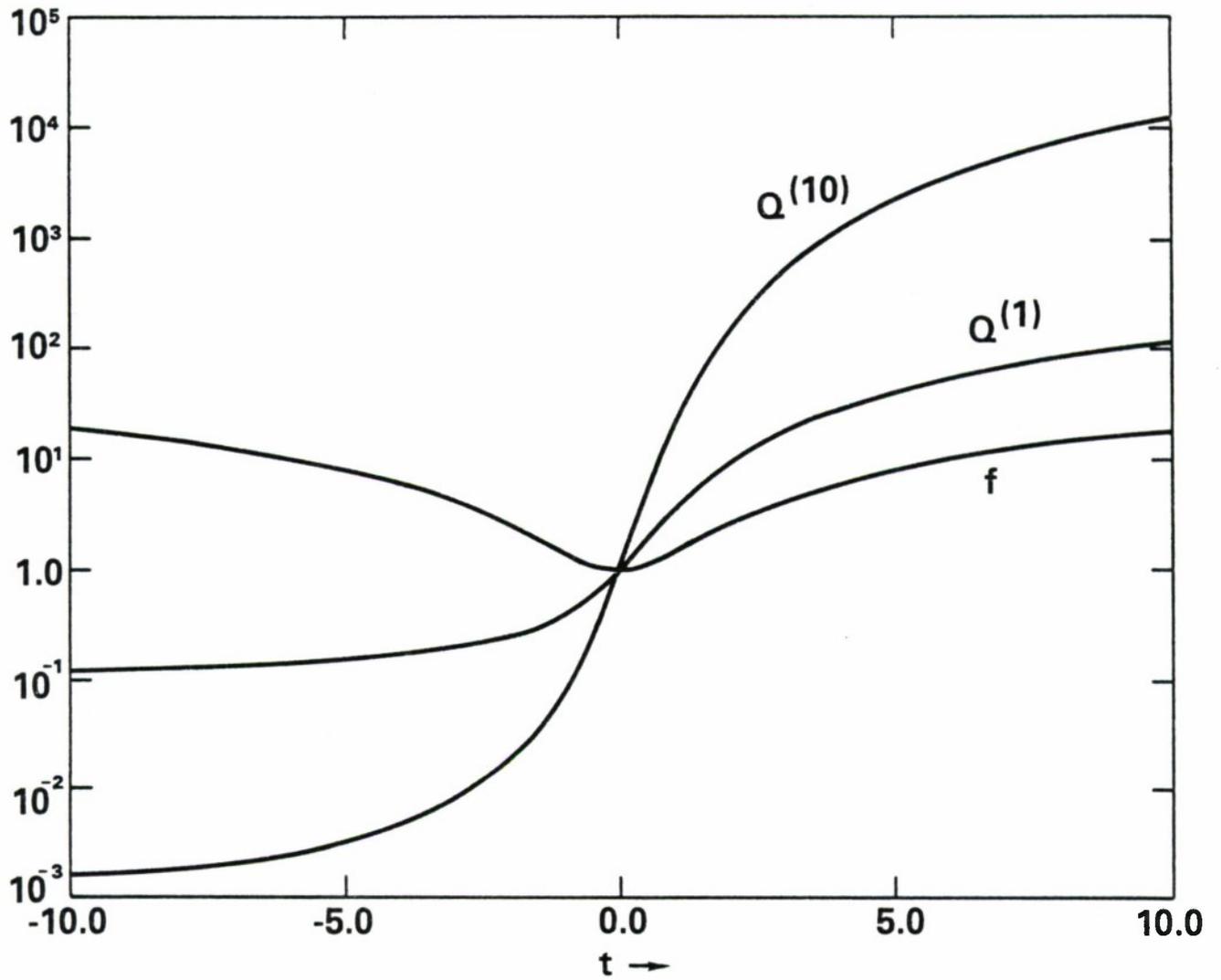


Figure 4

$Q^{\ell}(t)$ [defined by Eq. (129b)] for $\ell = 1$ and $\ell = 10$, obtained by numerical solution of Eq. (94) for $\alpha = 0$ and $\lambda = \ell + 1$, with f plotted for comparison.

Conclusions

In the theory of compressible fluids, the Rayleigh-Taylor instability at a density jump, the Rayleigh-Taylor instability in a continuously stratified medium, and the convective instability are close relatives. All are gravitational interchange modes. One can easily generate a series of examples which display a continuous transition from one mode to the next.

The energy principle applied to polytropic media shows that, by itself, compressibility increases instability. For only a handful of specific compressible states, however, is it possible to actually calculate growth rates in closed form. The only tractable equilibrium states involve contiguous isothermal layers of fluid satisfying the adiabatic law with constant γ . In the limit where the density of the lower layer vanishes, the growth rate reduces to the classical result found for incompressible fluids.

Closely related is the problem of the stability of uniformly imploding or expanding shells driven by pressure applied at a vacuum-material boundary. The unstable modes are incompressible and (if one allows for the nonexponential time dependence) the growth rates are given by the same classical incompressible fluid formula as in the static case.

Because the unstable eigenmodes are localized near a density jump within distances of order k^{-1} , one expects the growth rates not to change very much when the basic state is not isothermal, particularly at short wavelengths. Thus dispersion relation (46) is at least qualitatively correct most of the time in static situations.

To determine growth rates with precision for any but the simplest piecewise isothermal unperturbed states or to study the effects of thermal conduction, flow through the interface, nonlinearity, etc., one must resort to computational means. Nevertheless, numerical solutions present their own difficulties. Validation against nontrivial analytical solutions such as those discussed in this review is indispensable in the development of any code.

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