Range-residuated Mappings

M. F. Janowitz

University of Massachusetts
Amherst, MA 01003

August, 1984

19

Digital imagery, residuated mappings, cluster analysis

Range-residuated mappings are introduced and their order theoretic and semigroup properties studied. Applications are given to cluster analysis as well as to the simplification of digital images.
RANGE RESIDUATED MAPPINGS

M. F. Janowitz

1. Introduction. A digital picture may be thought of as a mapping $d: X \rightarrow L$ where $X$ is a finite set and $L$ a finite chain or the cartesian product of finitely many such chains. The idea is that $X$ is of the form $S \times T$, where $S$ is the set consisting of the first $s$, and $T$ the set consisting of the first $t$ positive integers, while $L$ represents the numerical coding of the brightness settings of the color guns that produce the picture. For a monochromatic picture, there would be only a single gun, so that $L$ would be a chain. Thus $d(x)$ yields the color or intensity level at site $x$. The mapping $d$ produces a clustering of $X$ into disjoint subsets by the rule

$$A_h = \{x \in X: d(x) = h\} \quad (h, L).$$

It is sometimes convenient to think instead of the clusters

$$B_h = \{x \in X: d(x) \leq h\} \quad (h, L)$$

and note that this produces a situation quite analogous to the model for cluster analysis that was described in [2]. In order to demonstrate an essential difference between the two situations, it turns out to be useful to examine in some detail the nature of the earlier model. One is given a finite (nonempty) set $X$ and a dissimilarity measure on $X$. This is a mapping $d: X \times X \rightarrow L$, where $L$ denotes the nonnegative reals and $d$ satisfies

*Research supported by ONR Contract N-00014-79-C-0629
2.

\[(DC1)\quad d(a,b) = d(b,a)\]

\[(DC2)\quad d(a,a) = 0\]

for all \(a, b \in X\). One associates with \(d\) a numerically stratified clustering \(T_d : L \to P(X \times X)\) defined by the rule

\[T_d(h) = \{(a,b) : d(a,b) \leq h\} \quad (h \in L).\]

The mapping \(T_d : L \to P(X \times X)\) turns out to be residual in the sense of [1], p. 11. This situation may then be generalized by taking \(L\) to be a join semilattice with 0, replacing \(P(X \times X)\) with a bounded poset \(M\), and defining an \(L\)-stratified clustering to be a residual mapping \(C : L \to M\) as in [2], p. 61. It is useful to recall here that \(C : L \to M\) is residual if \(C\) is isotone and there exists an isotone mapping \(C^* : M \to L\) such that

\[(1)\quad C^*C(h) \leq h \quad (2)\quad CC^*(m) \geq m\]

for all \(m \in M\), \(h \in L\). The mapping \(C^*\) is called the residuated mapping associated with \(C\), and the reader is referred to [1] for further details. One often wishes to take a residual mapping \(C : L \to M\) and shift the output levels by means of a mapping \(\alpha : L \to L\). The only reasonable choice for such \(\alpha\) is to take \(\alpha\) to be residual since one is then guaranteed that \(C \circ \alpha : L \to M\) is residual. Now this treats the 0 element of \(L\) as a distinguished element, since \(\alpha^*(0) = 0\) for every residuated mapping \(\alpha^*\) on \(L\). This makes sense in the cluster analysis context, since \(d(a,b) = 0\) is generally taken to mean that \(a, b\) cannot be distinguished in terms of the given input data.
In the context of digital images, one does not wish to distinguish the 0 element of \( L \) in the above manner. In order to avoid this, it becomes necessary to modify the notion of an \( L \)-stratified clustering. Specifically, we shall drop the requirement that \( M \) have a least element and consider mappings \( C^*:M \rightarrow L \) that are residuated when considered as mappings from \( M \) into the order filter generated by their range. Thus there exists an isotone mapping \( C:F \rightarrow M \), where \( F \) denotes the aforementioned order filter, and \( C,C^* \) are linked by the requirement that

\[
\begin{align*}
(3) & \quad CC^*(m) \geq m \quad \text{for all } m \in M \\
(4) & \quad C^*C(h) \leq h \quad \text{provided } h \geq \text{some } C^*(m) \quad \text{for } m \in M.
\end{align*}
\]

By [1], Theorem 2.5, p. 10, this amounts to saying that the preimage under \( C^* \) of a principal ideal of \( L \) is either empty or itself a principal ideal of \( L \). To be more specific, if we are to work with a digital picture, we are given a finite nonempty set \( X \) and a mapping \( d:X \rightarrow L \). If \( P'(X) \) denotes the semilattice formed by the nonempty subsets of \( X \), then \( d \) may be extended to a mapping \( d^*:P'(X) \rightarrow L \) by the rule

\[
(5) \quad d^*(A) = \vee \{d(x) : x \in A\}
\]

for every nonempty subset \( A \) of \( X \). It is then easy to see that \( d^* \) is residuated on the order filter generated by its range. Such mappings will henceforth be called range-residuated. They have already been used in [3] in connection with an investigation of ordinal filters in digital imagery, and in [4] in connection with a characterization of the semilattice of weak orders on a finite set. We agree to let \( RR(P,Q) \) denote the collection of range-residuated mappings of the poset \( P \) into the poset \( Q \), and
\[ RR^+(Q,P) \] the associated collection of residual mappings from order filters of \( Q \) into \( P \). In case \( P = Q \), we shall use \( RR(P) \) and \( RR^+(P) \) in place of \( RR(P,P) \) or \( RR^+(P,P) \). If \( P \) is a finite chain then \( RR(P) \) is nothing more than the set of all isotone mappings on \( P \), while if \( P \) is a finite join semilattice, then \( RR(P) \) consists of the join endomorphisms of \( P \). If digital pictures are thought of as elements \( C \) of \( RR^+(L,M) \), and if \( L \) is a finite chain, this shows that the levels of \( C \) may be shifted by means of any isotone mapping \( \alpha \) on \( L \) to produce a new picture \( C \alpha \). \( RR^+(L,M) \). In view of all this, we now embark on an investigation into order theoretic properties of these mappings.

2. Range-Residuated Mappings. Let \( P, Q \) be posets each having a largest element \( 1 \). For each \( q \in Q \), the constant mapping \( \kappa_q : P \to \Omega \) defined by \( \kappa_q(x) = q \) for all \( x \in P \) is range-residuated, with \( \kappa^+_q \) given by \( \kappa^+_q(y) = 1_P \) for all \( y > q \). If \( Q \) happens to be a join semilattice, then the join translation \( \tau_q(x) = x \vee q \) is in \( RR(Q) \) with \( \tau^+_q(y) = y \) for all \( y \geq q \). Before proceeding, let us develop some elementary properties of range-residuated mappings. They are basically generalizations of results on residuated mappings, but are included here for completeness.

**THEOREM 1** (see [1], Theorem 2.8, p.14). Let \( P, Q, S \) be posets. Let \( RR(P,Q) \) and \( RR(Q,S) \). Then \( \phi : P \to R \) is range-residuated with \( (\phi)^+ = \phi^+ \circ \phi^+ \).

Proof: Evidently \( \phi : P \to R \) is isotone. If \( p \in P \), then \( \phi(p) \) is in the domain of \( \phi^+ \), so that \( \phi^+ \phi(p) > \phi(p) \) and we have
\( \phi^+ \psi \phi(p) \geq \phi^+ \phi(p) \geq p \). On the other hand, if \( s \geq \psi \phi(p) \), then \( \psi^+ (s) \geq \psi(p) \) puts \( \psi^+ (s) \) in the domain of \( \phi^+ \). Thus \( \phi^+ \psi^+(s) \) can be formed and \( \psi \phi^+ \psi^+ \leq \psi \phi^+(r) \leq r \). In that the domain of \( \phi^+ \circ \psi^+ \) is precisely the order filter generated by the range of \( \psi \phi \), this completes the proof.

**COROLLARY 2.** \( RR(P) \) forms a semigroup with identity.

**Proof:** The identity map acts as a multiplicative identity element for \( RR(P) \).

Assuming that mappings are written on the left, we also have

**COROLLARY 3.** \( RR(P) \) has a left (but not right) zero element.

**Proof:** Let \( x : p \) and \( \phi : RR(P) \). One simply notes that

\[
\phi \kappa_x = \kappa_{\phi(x)} \quad \text{and} \quad \kappa_x \phi = \kappa_x,
\]

so that \( \kappa_x \) is a left (but not right) zero element for \( RR(P) \).

It is easy to show that any left zero element of \( RR(P) \) is of the form \( \kappa_x \) for some \( x \in P \). Of special interest is the case where \( P \) is bounded and one works with \( \kappa_0 \).

If \( \phi : P \rightarrow Q \) is a residuated mapping with associated residual mapping \( \phi^+ : Q \rightarrow P \), and if both \( P \) and \( Q \) are equipped with their dual orderings, then \( \phi^+ \) becomes residuated with \( \phi \) its associated residual mapping. This leads to an obvious duality between residuated and residual mappings. This duality does not carry over to range-residuated mappings since \( \cdot : RR(P,Q) \)
has an associated residual mapping whose domain is an order filter of $Q$ rather than being all of $Q$. Bearing this in mind, we agree to say (as in [4]) that $\phi \in \text{RR}(P,Q)$ is range-closed if $\phi(a) \leq q \leq \phi(p)$ implies $q$, range $\phi$; to say that $\phi$ is dually range-closed will be to say that the range of $\phi^+$ is an order filter of $P$. An obvious modification of the proof of [1], Theorem 13.1, p. 119 now produces

**THEOREM 4.** Let $P,Q$ be bounded posets. For $\phi \in \text{RR}(P,Q)$, the following are equivalent:

1. $\phi$ is range-closed.
2. The restriction of $\phi$ to $[\phi^+(0), 1]$ is a surjection onto $[\phi(0), \phi(1)]$.
3. In the interval $[\phi(0), 1]$ of $Q$, $q \land \phi(1)$ exists and equals $\phi^+(q)$.
4. $\phi^+$ is injective.

Similarly, an obvious modification of the proof of [1], Theorem 13.1*, p. 119 would produce

**THEOREM 5.** Let $P,Q$ be bounded posets. For $\phi \in \text{RR}(P,Q)$, the following are equivalent:

1. $\phi$ is dually range-closed.
2. $\phi^+$ is a surjection onto $[\phi^+(0), 1]$.
3. For all $p \in P$, $p \lor \phi^+(0)$ exists and equals $\phi^+(p)$.
4. The restriction of $\phi$ to $[\phi^+(0), 1]$ is injective.
As in [1], p. 120, we also agree to call $\phi \in RR(P,Q)$ weakly regular in case $\phi$ is both range-closed and dually range-closed. Examples of such mappings are provided by the constant mappings $\kappa_x$ as well as by the join translations $i_x$. The analog of [1], Theorem 13.2, p. 121 may now be stated as

**THEOREM 6.** Let $P, Q$ be bounded posets.

1. If $\phi \in RR(P,Q)$ is weakly regular, then its restriction to $[\phi(0), 1]$ is an isomorphism onto $[\phi(0), \phi(1)]$; furthermore, for $p \in P$ and $q \geq \phi(0)$, we have that $p \lor \phi^+(0)$ exists and is given by $\phi^+(p)$, and that $q \land \phi(1)$ exists in $[\phi(0), 1]$ and is given by $\phi^+(q)$.

2. Let $a, P$ and $b, c \in Q$ with $b < c$. Suppose that $p \lor a$ exists for all $p \in P$, that $q \land c$ exists for all $q \geq b$ in $Q$, and that $i$ is an isomorphism of $[a, 1]$ onto $[b, c]$. If $\phi: P \to Q$ is defined by $\phi(p) = i(p \lor a)$, then $\phi \in RR(P,Q)$, $\phi$ is weakly regular, and $\phi^+$ is given by $\phi^+(q) = i^{-1}(q \land c)$ for $q \geq b$.

Recall now that a pair $(a, b)$ of elements of a lattice is modular and denoted $M(a, b)$ if $x \leq b$ implies that $x \lor (a \land b) = (x \lor a) \land b$; dually, a dual modular pair is denoted $M^*(a, b)$ and signifies that $x \geq b$ implies $x \land (a \lor b) = (x \land a) \lor b$. We then have

**THEOREM 7.** Let $P$ be a bounded lattice and $\phi \in RR(P)$ a range-closed idempotent. Then $M(\phi^+(0), \phi(1))$ holds.

**Proof:** Let $a = \phi^+(0)$ and $b = \phi(1)$. If $a \land b \leq x \leq b$, then $x = \phi(y)$ for some $y \leq a$ by Theorem 4. Hence
shows $x = (x \lor a) \land b$. In general, if $x \leq b$, then $a \land b \leq x \lor (a \land b) \leq b$ shows that

$$x \lor (a \land b) = [x \lor (a \land b) \lor a] \land b = (x \lor a) \land b,$$

whence $M(a, b)$.

Dually, we have

**THEOREM 8.** Let $P$ be a bounded lattice and $\phi \in RR(P, Q)$ a dual range-closed idempotent. Then $M^*(\phi(1), \phi^+(0))$, and $1 = \phi(1) \lor \phi^+(0)$.

Combining Theorems 7 and 8, we generalize [1], Theorem 13.4, p. 123.

**THEOREM 9.** Let $P$ be a lattice and $\phi \in RR(P)$. The following are necessary and sufficient conditions for $\phi$ to be a weakly regular idempotent:

1. $\phi^+(0) \lor \phi(1) = 1$
2. $M(\phi^+(0), \phi(1))$ and $M^*(\phi(1), \phi^+(0))$
3. $\phi(x) = [x \lor \phi^+(0)] \land \phi(1)$.

**Proof:** Let $a \lor b = 1$, $M(a, b)$ and $M^*(b, a)$. Define $\phi$ and $\psi$ by

$$\phi(x) = (x \lor a) \land b \quad (x \in P)$$
$$\psi(x) = (x \land b) \lor a \quad (x \geq a \land b).$$

Then

$$\phi(x) = [(x \lor a) \land b] \lor a = x \lor a \geq x$$
and for \( x \geq a \wedge b \),

\[
\phi\psi(x) = [(x \wedge b) \vee a] \wedge b \\
= (x \wedge b) \vee (a \wedge b) = x \wedge b \leq x.
\]

Thus \( \phi \in \text{RR}(P) \) with \( \psi = \phi^+ \). The fact that \( \phi \) is a weakly regular idempotent is now also clear. For the converse, apply Theorems 7 and 8.

Continuing along these lines, we say that a range-residuated mapping \( \phi : \text{RR}(P,Q) \) is totally range-closed if the image under \( \phi \) of a principal ideal of \( P \) is necessarily a convex subset of \( Q \). We then have

**THEOREM 10** (See [1], Theorem 13.5, p. 124). Let \( P \) be a bounded lattice. The following conditions on a element \( \phi \) of \( \text{RR}(P) \) are then equivalent:

1. \( \phi \) is totally range-closed.
2. \( \phi \) range-closed implies \( \psi \) range-closed for every \( \psi \in \text{RR}(P) \).
3. For \( x \geq \phi(0), \ y \in L, \ [\phi^+(x) \wedge y] = x \wedge \phi(y) \).

**Proof:**

(1) \( \rightarrow \) (2) is clear.

(2) \( \rightarrow \) (3) If \( x \geq \phi(0) \), choose a residuated mapping \( \psi \) on \( P \) so that \( \psi(1) = y \). Then \( \phi\psi \) is range-closed, and we note that

\[
\phi[\phi^+(x) \wedge y] = \phi\psi\phi^+(x) = (\phi\psi)(\phi\psi)^+(x) = x \wedge \phi(1) = x \wedge \phi(y).
\]

The fact that \( \psi(0) = 0 \) was used to guarantee that \( \psi\phi^+(x) \) could be formed.

(3) \( \rightarrow \) (1) Let \( b \in P \). We are to show that \( \phi([0,b]) = [\phi(0), \phi(b)] \).

But if \( \phi(0) \leq x \leq \phi(b) \), then by (3),

\[
x = \phi(b) \wedge x = \phi[b \wedge \phi^+(x)].
\]
If we agree to call $\phi \in RR(P, Q)$ dual totally range-closed in case the image under $\phi^+$ of a principal filter of the domain of $\phi^+$ is a principal filter of $P$, we then have

**Theorem 11.** Let $P$ be a bounded lattice, and $\phi \in RR(P)$. The following are then equivalent:

1. $\phi$ is dual totally range-closed.
2. $\psi$ dual range-closed implies $\psi$ dual range-closed.
3. For $y \geq \phi(0)$, $x \in L$, $\phi^+[\phi(x) \vee y] = x \vee \phi^+(y)$.

The above is the obvious generalization of [1], Theorem 13.6, p. 124, and its proof will be omitted.

As in the case of residuated mappings, there is a strong tie between the notions of range-closed and modularity. A further discussion of this topic will be covered in a later paper.

3. **Annihilator Properties of Range-Residuated Mappings.** In this section, it will be assumed that we are working in a fixed bounded poset $P$. Recall that $RR(P)$ is a semigroup with identity element 1 and left zero elements $\{\kappa_x : x \in P\}$. The left zero element $\kappa_0$ will be of special interest. For $\phi \in RR(P)$, we define the right annihilator of $\phi$ by the rule

$$R(\phi) = \{\psi : \phi \psi = \kappa_\phi(0)\};$$

similarly, the left annihilator of $\phi$ is defined by

$$L(\phi) = \{\cdot : \phi = \kappa_\phi(0)\}.$$

We shall make strong use of the fact that
(5) \( \phi \psi = \kappa_\phi(0) \iff \psi(1) \leq \phi^+\phi(0) \).

The idea now is to relate order properties of the poset \( P \) to annihilator properties of the semigroup \( RR(P) \). To show that there is some hope in doing this, we let

\[
R = \{ R(\phi) : \phi \in RR(P) \} \\
L = \{ L(\phi) : \phi \in RR(P) \}
\]

with both sets partially ordered by set inclusion. We may then define mappings \( F : R \to P, \ G : L \to P \) by the rules

\[
F(R(\phi)) = \phi^+\phi(0) \\
G(L(\phi)) = \phi(1)
\]

and note that \( F \) is an isomorphism of \( R \) onto \( P \), and \( G \) is a dual isomorphism of \( L \) onto \( P \). To see this, note first that if \( R(\phi) \subseteq R(\alpha) \), then

\[
\phi^+\phi(0) = \kappa_\phi(0) = \kappa_\alpha(0) \Rightarrow \omega\phi^+\phi(0) = \kappa_\alpha(0)
\]

so that by (5), \( \phi^+\phi(0) \leq \alpha^+\alpha(0) \). If conversely, \( \phi^+\phi(0) \leq \alpha^+\alpha(0) \), then

\[
\phi \psi = \kappa_\phi(0) \Rightarrow \psi(1) \leq \phi^+\phi(0) \leq \alpha^+\alpha(0) \Rightarrow \alpha \psi = \kappa_\alpha(0) \Rightarrow \psi(1) \leq \phi^+\phi(0) = \kappa_\alpha(0). \]

So \( R(\phi) \subseteq R(\alpha) \).

We would be done if we could show \( F \) to be onto. But this follows from the observation that if \( \beta_X \) is defined by \( \beta_X(p) = 0 \) if \( p \leq x \) and 1 otherwise, then \( \beta_X \) is residuated with \( \beta^+_X(0) = x \). A similar argument works for \( G \). We now have

**THEOREM 12.** Let \( P \) be a bounded poset. Then:

(1) \( P \) is a meet semilattice if and only if the right annihilator of each element of \( RR(P) \) is a principal right ideal generated by an idempotent.
(2) $P$ is a join semilattice if and only if the left annihilator of each element of $RR(P)$ is a principal left ideal generated by an idempotent.

Proof: (1) Assume $P$ to be a meet semilattice. Then for $p \in P$, we may define $\theta_p$ by the rule $\theta_p(x) = x$ ($x \leq p$) and $p$ otherwise. Noting that $\theta_p$ is a range-closed idempotent residuated mapping, it follows from (5) that $\phi \psi = \kappa \phi(0) \iff \psi = \theta \phi \psi(0) \psi$. The converse follows from Theorem 4.

(2) If $P$ is a join semilattice, then by (5), $\psi \phi = \kappa \psi \phi(0) \iff \psi = \theta \phi \psi(0)$. The converse follows from Theorem 5.

4. Baer LZ-semigroups. Let $S$ be a semigroup with a two-sided zero element 0. For a given $x \in S$, define the left and right annihilators of $x$ by the rules

$L(x) = \{y \in S : yx = 0\}$

$R(x) = \{y \in S : xy = 0\}$.

To say that $S$ is a Baer semigroup ([1], p. 104) is to say that for each $x \in S$ there correspond idempotents $e_x, f_x$ such that

$L(x) = \{y \in S : y = yf_x\} = Sf_x$

$R(x) = \{y \in S : y = e_xy\} = e_xS$.

An introduction to these semigroups is contained in [1], and an attempt is made there to relate properties of bounded posets to properties of suitable associated semigroups. For further details, the reader is referred to [1]. The link between Baer semigroups and lattices is made by means of certain residuated mappings. In order to develop a similar theory for
range-residuated mappings, one needs an analog of a Baer semigroup that
only has a one-sided zero element. This we now proceed to introduce.

**DEFINITION.** A semigroup $S$ is said to be a **Baer LZ-semigroup** if
   (1) $S$ has a distinguished left zero element $z$, and
   (2) For each $x \in S$, there correspond idempotents $e_x$, $f_x$ such
       that
       
       \[
       L(x) = \{y \in S : yx = yz\} = \{y \in S : y = yf_x\},
       R(x) = \{w \in S : xw = xz\} = \{w \in S : w = e_xw\}.
       \]

Unless otherwise specified, $S$ will denote such a semigroup, and

\[
L(S) = \{L(x) : x \in S\}
\]

\[
R(S) = \{R(x) : x \in S\}
\]

with both $L(S)$ and $R(S)$ partially ordered by set inclusion. To say
that a poset $P$ can be **coordinatized** by such an $S$ will be to say that
$P$ is isomorphic to $R(S)$. Note that if $z$ is a two-sided 0, then $S$
becomes a Baer semigroup in the sense of [1], p. 104. Note also that the
left zero elements of $S$ correspond to the elements of the form $xz$ ($x \in S$).

**THEOREM 13.** $S$ has a multiplicative identity.

**Proof:** Let $L(z) = Se$ and $R(z) = fS$ with $e$, $f$ idempotent. Then
$R(z) = \{y \in S : yz = zz\} = S$ shows $f$ to be a right identity for $S$, while
$L(z) = \{y \in S : yz = yz\} = S$ shows $e$ to be a left identity.
If we agree to let PRI(S), PLI(S) denote the set of principal right, left ideals of S with both sets partially ordered by set inclusion, we also have

**Theorem 14.** (1) The mappings $\hat{L}:\text{PRI}(S) \rightarrow \text{PLI}(S)$, $\hat{R}:\text{PLI}(S) \rightarrow \text{PRI}(S)$ defined by $\hat{L}(xS) = L(x)$, $\hat{R}(Sx) = R(x)$ set up a galois connection in the sense of [1], p. 18.

(2) $\hat{L} = \hat{L} \circ \hat{R} \circ \hat{L}$ and $\hat{R} = \hat{R} \circ \hat{L} \circ \hat{R}$.

(3) $xS \subseteq R(S) \iff xS = (\hat{R} \circ \hat{L})(x)$, and $Sx \subseteq L(S) \iff Sx = (\hat{L} \circ \hat{R})(x)$.

(4) The restriction of $\hat{L}$ to $R(S)$ is a dual isomorphism of $R(S)$ onto $L(S)$ whose inverse is the restriction of $\hat{R}$ to $L(S)$.

**Proof:** In view of the similarity of this result to [1], Theorem 11.1, p. 95, we restrict our attention to the proof of (1).

If $xS \subseteq yS$, then $x = yw$ for some $w \in S$. Then $a \in L(y)$ implies $ay = az$, so $ax = ayw = azw = ax$. Thus

$$xS \subseteq yS \implies L(y) \subseteq L(x).$$

Similarly, if $Sx \subseteq Sy$, then $x = wy$, so $a \in R(y)$ implies $xa = wya = wyz = xz$, thereby putting $a \in R(x)$. In other words,

$$Sx \subseteq Sy \implies R(y) \subseteq R(x).$$

The fact that $a \in L(x)$ implies $ax = az$ also puts $x \in R(a)$, so

$$xS \subseteq (R \circ L)(xS);$$

similarly, $Sx \subseteq (L \circ R)(Sx)$, thus completing the proof.
We shall frequently need

**Lemma 15.** If \( eS \in R(S) \) with \( e = e^2 \), then \( z = ez \).

**Proof:** Let \( eS = R(x) \). Since \( z \in R(x) \), it follows that \( z = ez \).

For \( M \) a subset of \( S \), we agree to let \( R(M) = \{ x: mx = mz \text{ for all } m \in M \} \) and note that if \( R(M) = eS \) with \( e = e^2 \), then \( eS = \bigwedge \{ R(m): m \in M \} \) in \( R(S) \). For each fixed \( x \in S \), we define mappings \( \phi_x, \eta_x : R \to R \) by the rules

\[
\phi_x(eS) = (\hat{R} \circ L)(xe) \\
\eta_x(eS) = R(e^#x)
\]

where \( Se^# = L(e) \), and \( e^# \) is idempotent. The domain of \( \eta_x \) is taken to be \( \{ eS \in R(S): \phi_x(zS) \subseteq eS \} \). From here on in, the elements \( e, f, g, h \) (with or without superscripts) will, unless otherwise specified, denote idempotents. We agree further to let \( R = R(S) \) and \( L = L(S) \). We then have

**Theorem 16.** For each \( x \in S \), \( \phi_x \in RR(R) \), with \( \phi_x^+ = \eta_x \).

**Proof:** We begin by showing \( \phi_x, \eta_x \) to be well defined and isotone. Accordingly, let \( eS \subseteq fS \) in \( R \). Then \( e = fe \) and \( y \in L(xf) \) implies

\[
 yxe = yxfe = yze = yz
\]

thus showing \( y \in L(xe) \). It follows that \( \phi_x \) is well defined and isotone.
Now let $\phi_x(zS) \subseteq eS \subseteq fS$ in $R$, with $Se^\# = L(e)$ and $Sf^\# = L(f)$. Then $L(f) \subseteq L(e)$, so $f^\# = f^#e^\#$. If $y \in R(e^\#x)$, then $e^\#xy = e^#xz$, and then

$$f^\#xy = f^#e^\#xy = f^#e^#xz = f^#xz,$$

thus putting $y \in R(f^\#x)$. Consequently, $\eta_x$ is well defined and isotone.

Suppose now that $\phi_x(eS) \subseteq fS$ in $R$. Then $\phi_x(zS) \subseteq fS$, so $xz = fxz$, and $f^\#xz = f^#fxz = f^#z$. It follows that

$$f^\#xe = f^#fxe = f^#z = f^#xz,$$

whence $eS \subseteq R(f^\#x)$. On the other hand, if $\phi_x(zS) \subseteq fS$, and $eS \subseteq R(f^\#x)$, then

$$f^\#xe = f^#xz = f^#z$$

puts $xe$ in $R(Sf^\#) = (\hat{R} \circ \hat{L})(fS)$, so $\phi_x(eS) = (\hat{R} \circ \hat{L})(xe) \subseteq fS$. This shows that $\eta_x = \phi_x^+$, as claimed.

Actually as is seen by the next result, $L = R(S)$ is in fact a bounded lattice. The proof is similar to that of (1), Theorem 12.2, p. 107.

**Lemma 17.** $L = R(S)$ is a bounded lattice.

**Proof:** Let $eS, fS \subseteq L$ with $Se^\# = L(e)$, and $Sf^\# = L(f)$. If $gS = R(f^\#e)$, then

$$(f^\#e)(eg) = f^#eg = f^#ez$$

shows $eg \in R(f^\#e) = gS$, so $eg = geg$ and $eg$ is idempotent. Now let $x \in R((e^\#, f^\#))$. Then
17.

\[ e^x = e^z \implies x = ex, \]

so

\[ f^ex = f^x = f^z = f^ez \]

puts \( x \in R(f^e) = gS, \) and \( x = gx = egx. \)

If conversely, \( x = egx, \) then

\[ e^x = e^egx = e^z \]
\[ f^x = f^egx = f^ez = f^z \]

puts \( x \in R(\{e^#, f^#\}). \) It is immediate that \( eS \cap fS = egS \in L, \) and this shows \( L \) to be a meet semilattice.

In order to show that \( L \) is a join semilattice, it suffices by Theorem 14 to show that \( L(S) \) is a meet semilattice. Accordingly, we let \( Se, Sf \in L(S) \) with \( e'S = R(e), f'S = R(f), \) and \( Sg = L(ef'). \) We shall show that \( Sf \cap Se = Sg \cap Se = Sge. \) Note first that

\[ (ge)(ef') = gef' = gz. \]

By Lemma 15,

\[ gez = gef'z = gz, \]

so \( (ge)(ef') = gz = gez, \) and \( ge \in L(ef') = Sg. \) It follows that \( ge = geg, \) so \( ge \) is idempotent.

If \( x \in L(\{e', f'\}) \) then \( xe' = xz, \) so \( x = xe. \) It follows that \( xef' = xf' = xz, \) and \( x = xg. \) Consequently, \( x = xg = xge. \) On the other hand, if \( x = xge, \) then
\[ xe' = xgee' = xgez = xz, \]

so \( x \in L(e') \). Also, a second application of Lemma 15 produces

\[ xf' = xgef' = xgz = xgez = xz \]

thus showing that \( x \in L(f') \).

An immediate consequence of Theorem 12 and Lemma 17 is

**THEOREM 18.** For a bounded poset \( P \), the following conditions are equivalent:

1. \( P \) is a lattice.
2. \( RR(P) \) is a Baer LZ-semigroup.
3. \( P \) can be coordinatized by a Baer LZ-semigroup.

The question of what it means for the mapping \( x \rightarrow \phi_x \) to be a semi-group homomorphism of \( S \) into \( RR(R(S)) \) is settled by

**THEOREM 19.** Let \( S \) be a Baer LZ-semigroup, and \( L = R(S) \). The following conditions are then equivalent:

1. The mapping \( x \rightarrow \phi_x \) is a semigroup homomorphism of \( S \) into \( RR(L) \).
2. \( \phi_x(zS) \leq \phi_{xy}(zS) \) for every \( x,y \) in \( S \).
3. \( a \in L(xyz) \Rightarrow ax \in L(yz) \) for all \( x,y \) in \( S \).

**Proof:** (1) \( \Rightarrow \) (2) is clear.

(2) \( \Rightarrow \) (3). Let \( a \in L(xyz) \). By hypothesis, \( \phi_x(zS) \leq \phi_{xy}(zS) \), so \( L(xyz) \subseteq L(xz) \). Thus \( a \in L(xyz) \Rightarrow a \in L(xz) \), whence \( axz = az \). But then \( ayz = az = axz \) puts \( ax \in L(yz) \), as claimed.
(3) $\Rightarrow$ (1). For $eS \in L$, $\phi_x \phi_y (eS) = (\hat{R} \circ L)(xg)$, where $gS = (\hat{R} \circ L)(ye)$, and $\phi_x \phi_y (eS) = (\hat{R} \circ L)(xye)$. We would be done if we could show that $L(xg) = L(xye)$. To see this, note that

$$a \in L(xg) \Rightarrow ax \in L(g) = L(ye).$$

Thus

$$az = axz = axg = axye,$$

and this puts $a \in L(xye)$. The reverse inclusion is established in a similar manner.

REFERENCES


