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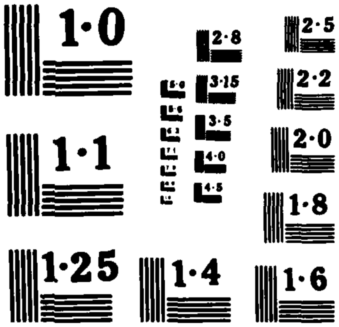
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CONVERGENCE OF DIFFERENCE APPROXIMATIONS OF QUASILINEAR EVOLUTION EQUATIONS

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ABSTRACT

A very successful theory of quasilinear evolution equations, which applies to many problems of mathematical physics, has been developed by T. Kato. The theory obtains solutions of quasilinear problems via contraction mappings which are defined by means of a theory of linear evolution equations also developed by Kato. In the current work we show how the existence and continuous dependence theorems obtained by Kato can be proved by discretization in time. As opposed to earlier work in this direction, the current results are much sharper concerning the continuity properties of the solutions of the discretized problem and the strength of the *norms in which* they converge.

AMS (MOS) Subject Classifications: 35G20, 47H20, 47H06

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CONVERGENCE OF DIFFERENCE APPROXIMATIONS OF

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We are interested in the quasilinear initial-value problem

$$\text{(1)} \qquad \begin{aligned} &\frac{du}{dt} + A(u)u = 0, \\ &u(0) = \varphi, \end{aligned}$$

in which $A(u)$ is a linear operator in a Banach space X for each u belonging to a subset W of X. T. Kato has studied (1) in [8] and [9]. He obtained the existence of a classical solution under assumptions detailed in Section 1 and showed the relevance of these assumptions by applying his theory to a wide variety of problems from mathematical physics. The main goal of this paper is to show that, under these assumptions, the existence theory for (1) can be obtained very directly by showing that the simple difference approximation of (1) given by

$$\text{(2)} \qquad \begin{aligned} &\frac{u_\lambda(t)-u_\lambda(t-\lambda)}{\lambda} + A(u_\lambda(t-\lambda))u_\lambda(t) = 0 \quad \text{for} \quad 0 < t \leq T, \\ &u_\lambda(t) = \varphi \quad \text{for} \quad t \leq 0, \end{aligned}$$

is solvable for $u_\lambda(t)$, $0 < t \leq T$ (for appropriate λ and T), that

$$\text{(3)} \qquad \lim_{\lambda \downarrow 0} u_\lambda(t) = u(t)$$

exists uniformly on $0 \leq t \leq T$ and the u so obtained satisfies (1) in the classical sense.

Results in this general direction were obtained in [5] (which is not going to appear in the periodical literature). See also [7]. The current work sharpens the results of [5] as applied to (1) in several ways: By restricting attention to (1), the presentation

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is clearer. We give a simpler proof of the convergence (3) and the proof of the existence of u_λ solving (2) is given under different assumptions than in [5]. Finally, and this is the point we emphasize, the convergence in (3) is shown to be better than in [5]. This is of numerical interest and the proof allows the current line of attack to obtain the sharpness of some of Kato's results that was not previously matched by this method.

Kato's approach to (1) and its generalizations involves obtaining sharp results for linear problems of the form

$$\begin{aligned} &\frac{du}{dt} + B(t)u = 0 \\ &u(0) = \varphi \end{aligned} \tag{4}$$

and then using these with a contraction mapping argument to solve (1). (For a current account of the state of Kato's theory and more references to other approaches, we refer the reader to [10] and its bibliography.) Our approach to solving (1) does not require a preliminary linear theory - not even the Hille-Yosida theorem. Indeed, the solvability of (2) under hypotheses of Kato's type is proved in a straighforward fashion and the convergence (3) follows from standard *elementary estimates of* "nonlinear semigroup theory". We will rely on the form given these standard estimates in [3], but other approaches work as well (e.g., [11], [13]). This direct attack on (1) is carried out in Section 3. However, there is ample reason to study (4) by our methods in any case, and this is done in Section 2. It is also a simpler matter to show the optimal convergence of the u_λ if one has appropriate results for (4) in hand, and the arguments in the case of (4) exhibit clearly several main points which can then be briefly treated in the case of (1). Hence we have organized the presentation by discussing (4) before (1), as is the common practice. The interested reader can take up Section 3 before Section 2, and if he does so he will quickly obtain an existence result for (1) which asserts a little less than both optimal regularity of u and optimal convergence in (3). To obtain these sharper results we have relied on Section 2. The main results concerning (4) are given in Section 2 and state that, under hypotheses of Kato's type, (4) has a unique solution which may be computed as the limit of solutions of simple difference approximations to (4), and these approximate solutions converge in as strong a sense as is possible. Section 1 collects some preliminaries, notations and precise formulations of the results. Of course, there are many variants and generalizations possible, and we comment on some of these following the proof of Theorem 2 in Section 3. In the final Section 4 we briefly sketch how one would prove (known) results on continuous dependence in this setting.

The authors are grateful to R. Pego for useful discussions about this work.

Section 1. Preliminaries and Statements of Results

Let Z be a Banach space. We use $\| \ \|_Z$ to denote the norm of Z, as well as the norm of elements of $B(Z)$ (the bounded linear selfmaps of Z). If β is a real number, we denote by $N(Z,\beta)$ the set of densely defined linear operators C in Z such that if $\lambda > 0$ and $\lambda\beta < 1$, then $(I + \lambda C)$ is one to one with a bounded inverse defined everywhere on Z and

$$\|(I + \lambda C)^{-1}\|_Z \le (1 - \lambda\beta)^{-1}.$$

Here and below we use "I" to denote various identity operators depending on the context. The Hille-Yosida Theorem - which we will not need in this work - states that $C \in N(Z,\beta)$ exactly when $-C$ is the infinitesimal generator of a strongly continuous semigroup e^{-tC}, $0 \le t$, on Z satisfying $\|e^{-tC}\|_Z \le e^{\beta t}$ for $0 \le t$.

More generally, if C is a (possibly) nonlinear operator C from its domain $D(C) \subseteq Z$ into Z with the property that $I + \lambda C$ has a well defined inverse $(I + \lambda C)^{-1}$ on the range of $I + \lambda C$ with $(1 - \lambda\theta)^{-1}$ as a Lipschitz constant provided that $\lambda > 0$ and $\lambda\theta < 1$, then we say that $C + \theta I$ is *accretive*. We recall a simple lemma about accretive operators that we will have occassion to use. A proof can be found in [3] or [11].

Lemma 1. *Let $\theta \in \mathbb{R}$, C be an operator in a Banach space Z and $C + \theta I$ be accretive. If $\gamma, \delta > 0$ and $\gamma\theta, \delta\theta < 1$, and $z, \hat{z}, w, \hat{w} \in D(C)$, $f, g \in Z$ satisfy*

$$\frac{z - \hat{z}}{\gamma} + Cz = f, \qquad \frac{w - \hat{w}}{\delta} + Cw = g$$

then

$$\left(1 - \theta\frac{\delta\gamma}{\gamma+\delta}\right)\|z - w\|_Z \le \frac{\delta}{\gamma+\delta}\|\hat{z} - w\|_Z + \frac{\gamma}{\gamma+\delta}\|z - \hat{w}\|_Z + \frac{\gamma\delta}{\gamma+\delta}\|f - g\|_Z.$$

Throughout this paper, X and Y are Banach spaces which have properties we call (X):

(X) $\quad$ *X and Y are reflexive and Y is continuously and densely imbedded in X.*

The operator norm of a bounded linear mapping $C: Y \to X$ will be denoted by $\|C\|_{YX}$. If

$T > 0$, the set of continuously differentiable mappings $f:[0,T] \to X$ will be written $C^1[0,T:X]$ and $C[0,T:Y]$ denotes the continuous maps into Y, etc..

In most of this paper X and Y will be related via a linear isometric isomorphism $S:Y \to X$. We denote this condition by (S):

(S) $\qquad S:Y \to X$ is a linear isometric isomorphism.

We next formulate our results in the case of the equation $u' + A(u)u = 0$. Concerning the operators $A(u)$ we assume:

(A1) $\qquad$ There is a $\beta > 0$, an open subset W of Y and a mapping $A:W \to N(X,\beta)$.

The next assumption restricts the domain $D(A(w))$ of $A(w)$ and the joint continuity of "$A(u)v$".

(A2) $\qquad$ For every $w \in W$, $Y \subseteq D(A(w))$. Moreover, there are constants μ_A, γ_A such that for u, $\hat{u} \in W$ and $v \in Y$

$$\|(A(u) - A(\hat{u}))v\|_X \le \mu_A \|u-\hat{u}\|_X \|v\|_Y \text{ and } \|A(u)v\|_X \le \gamma_A \|v\|_Y.$$

The next assumption is more subtle:

(A3) $\qquad$ There is a mapping $P:W \to B(X)$ and a constant γ_P such that

(i) $SA(w) = A(w)S + P(w)S$ for $w \in W$,

and

(ii) $\|P(w)\|_X \le \gamma_P$ for $w \in W$.

The assumptions (A1) - (A3) will suffice to guarantee the solvability and convergence of the scheme (2) to the classical solution of (1). However, we will obtain sharper convergence results under the further restriction

(A4) There is a μ_P such that $\|P(u) - P(\hat{u})\|_X \leq \mu_P \|u - \hat{u}\|_Y$ for $u, \hat{u} \in W$.

Kato [8] has shown the relevance of these assumptions by exhibiting many important examples which enjoy these properties. We will prove:

Theorem 1. Let (X), (S), (A1), (A2) and (A3) hold and $\varphi \in W$. Then there are $T, \lambda_0 > 0$ such that there is a unique finite sequence x_i, $i = 0, \ldots, N$, in W which satisfies

$$\text{(1.1)} \qquad \begin{aligned} &\frac{x_i - x_{i-1}}{\lambda} + A(x_{i-1})x_i = 0\,, \qquad i = 1, \ldots, N \\ &x_0 = \varphi, \end{aligned}$$

provided that $0 < \lambda \leq \lambda_0$ and $T \leq N\lambda \leq T + \lambda$. Moreover, if $u_\lambda(t)$ is defined by

$$\text{(1.2)} \qquad \begin{aligned} &u_\lambda(0) = \varphi \text{ and} \\ &u_\lambda(t) = x_i \quad \text{for} \quad (i-1)\lambda < t \leq i\lambda \text{ and } i = 1, \ldots, N \end{aligned}$$

then

$$\text{(1.3)} \qquad \lim_{\lambda \downarrow 0} u_\lambda(t) = u(t)$$

exists in X uniformly on $[0,T]$ and the function u so defined is continuously differentiable into X, continuous into Y, satisfies $u([0,T]) \subseteq W$ and

$$\text{(1.4)} \qquad u'(t) + A(u(t))u(t) = 0 \quad \text{for} \quad 0 \leq t \leq T.$$

If (A4) also holds, then the convergence in (1.3) holds in Y uniformly on $[0,T]$.

Remarks. The description (1.2) of u_λ coincides with the scheme (2) (which produces piecewise constant functions). The assumptions (A1) - (A4) are an amalgam of conditions used by Kato in [8] and [9]. (A4) was used by Kato to establish strong results concerning the dependence of the solution of (1) on A and φ, and its role in our work is related to this. In [8] Kato imposed an extra condition which was also used by us in [5] to obtain the existence of u_λ. This was dropped in [9] and is now dropped here. (However, one can relax (A3) if this extra condition is imposed - see [2, Section 4] for a simple account.

This work obtains the existence of $u \in C^1[0,T:X] \cap C[0,T:Y]$ solving $u' + A(u)u = 0$, $u(0) = \varphi$ via the scheme (1.1). This sharpens the result of [5] which, under somewhat different assumptions, produced only a Lipschitz continuous function. If one takes the existence as given via Kato, then our main result is the fact that the solutions of (1.1) converge so nicely to Kato's solution.

As was mentioned in the introduction, we will first study the associated linear problem $u' + B(t)u = 0$. The assumptions on $B(t)$ parallel (A1) - (A3) above.

(B1) $T > 0$ and there is a $\beta > 0$, such that $B(t) \in N(X,\beta)$ for $0 \le t \le T$.

(B2) $Y \subseteq D(B(t))$ for $0 \le t \le T$ and the mapping $[0,T] \ni t \to B(t)|_Y$ (the restriction of $B(t)$ to Y) is continuous into $B(Y,X)$.

(B3) There is a strongly measurable mapping $D:[0,T] \to B(X)$ and a constant γ_D such that

$$SB(t) = B(t)S + D(t)S \text{ and } \|D(t)\|_X \le \gamma_D \text{ for } 0 \le t \le T.$$

Before formulating the result in this case, we recall a standard lemma which is often used in the sequel.

<u>Lemma 2</u>. Let (S) hold, $C \in N(X,\beta)$, $Y \subseteq D(C)$, $P \in B(X)$, and $SC = CS + PS$. Set $\theta = \beta + \|P\|_X$. Then for every $y \in X$ and $\lambda > 0$ such that $\lambda\theta < 1$, the problems

$$x + \lambda Cx = y \tag{1.5}$$

and

$$\hat{x} + \lambda(C\hat{x} + P\hat{x}) = y \tag{1.6}$$

have unique solutions x and $\hat{x}$ in X. Moreover

$$\|x\|_X \le (1 - \lambda\theta)^{-1}\|y\|_X \text{ and } \|\hat{x}\|_X \le (1-\lambda\theta)^{-1}\|y\|_X \tag{1.7}$$

and if $y \in Y$, then $x \in Y$ and

$$\|x\|_Y \le (1-\lambda\theta)^{-1}\|y\|_Y \quad . \tag{1.8}$$

Proof. The unique solvability of (1.5) and the estimate

$$\|x\|_X \le (1 - \lambda\beta)^{-1}\|y\|_X \tag{1.9}$$

are by definition of $N(X,\beta)$. We have weakened (1.9) to the first estimate of (1.7) for later convenience in writing. The assertions concerning (1.6) are standard perturbation remarks, and can be deduced easily and directly from the unique solvability of (1.5) with the estimate (1.9). (We leave it as an exercise for the reader who may not be familiar with the perturbation results.) If $y \in Y$ in (1.5), write $\hat{y} = Sy$, apply S to (1.5) and use the assumptions to arrive at the equivalent problem $\hat{x} + \lambda(C\hat{x} + P\hat{x}) = \hat{y}$ for $\hat{x} = Sx$. The auxiliary assertions in the case $y \in Y$ then follow at once from the case just discussed and the assumption (S).

We will abbreviate the information contained in Lemma 1 when it applies by writing

$$\|(I+\lambda C)^{-1}\|_Z \le (1-\lambda\theta)^{-1} \text{for } Z = X \text{ or } Y \text{ and } \quad \|(I+\lambda(C+P))^{-1}\|_X \le (1-\lambda\theta)^{-1}$$

with appropriate choices of C and P.

Let

$$\mathbb{P} = \{0 = t_0 < t_1 < \dots < t_N = T\}$$

be a partition of $[0,T]$. The mesh size $m(\mathbb{P})$ of $\mathbb{P}$ is the largest step $t_i - t_{i-1}$, $i = 1, \dots, N$. If (B1) - (B3) hold, $\theta = \beta + \gamma_D$, $m(\mathbb{P})\theta < 1$ and $\varphi \in X$, then Lemma 2 guarantees that the scheme

$$\begin{aligned} &\frac{x_i - x_{i-1}}{t_i - t_{i-1}} + B(t_i)x_i = 0, \quad i = 1,\dots,N, \\ &x_0 = \varphi, \end{aligned} \tag{1.10}$$

is uniquely solvable. Indeed, the solution is given by iterating

$$x_i = (I + (t_i-t_{i-1})B(t_i))^{-1}x_{i-1}$$

to find

$$x_i = \prod_{j=1}^{i} (I + (t_j-t_{j-1})B(t_j))^{-1}\varphi$$

where the product (and all others in this paper) is "time-ordered". More generally, given a partition $\mathbb{P}$ as above, with a sufficiently fine mesh, we set

(1.11)
$$U_{\mathbb{P}}(t,s) = \prod_{j=m}^{n} (I + (t_j - t_{j-1})B(t_j))^{-1}$$

$$\text{for } t_{m-1} \leq s < t_m \text{ and } t_{n-1} < t \leq t_n$$

with the understanding that $U_{\mathbb{P}}(t,t) = I$ for $0 \leq t \leq T$.

Theorem 2. Let (X), (S) and (B1) - (B3) hold and $x \in X$. Then the limit

(1.12)
$$\lim_{m(\mathbb{P})\to 0} U_{\mathbb{P}}(t,s)x = U(t,s)x$$

exists uniformly in X on $\Delta = \{0 \leq s \leq t \leq T\}$ and defines a strongly continuous mappin[g] $U(t,s)$ from Δ to $B(X)$ with the property that if $\varphi \in Y$ and $u(t) = U(t,s)\varphi$ on $s \leq t \leq T$ then $u \in C^1[s,T:X] \cap C[s,T:Y]$, $u(s) = \varphi$ and $u'(t) + B(t)u(t) = 0$ for $s \leq t \leq T$. If, moreover, $D(t)$ in (B3) is strongly continuous into $B(X)$ and $x \in Y$, then the limit (1.[12]) is uniform in Y.

The proof of Theorem 2 is given in Section 2. Here we will be interested in the following corollary of Theorem 2.

Corollary 1. Let $\mathbb{P}(n) = \{0 = t_0^n < t_1^n < \ldots < t_{N(n)}^n = T\}$ be a partition of $[0,T]$ for $n = 1,2, \ldots\ldots$. Let x_i^n, $f_i^n \in X$ for $i = 1,\ldots,N(n)$ and $f \in L^1[0,T:X]$ (the strongly integrable functions from $[0,T]$ to X). Assume that

$$\lim_{n\to\infty} m(\mathbb{P}(n)) = 0 \text{ and } \lim_{n\to\infty} \sum_{i=1}^{N(n)} \int_{t_{i-1}^n}^{t_i^n} \| f_i - f(t) \| dt = 0.$$

Let

(1.13)
$$\frac{x_i^n - x_{i-1}^n}{t_i^n - t_{i-1}^n} + B(t_i^n)x_i^n = f_i^n \quad \text{for } i = 1,\ldots\ldots,N(n),$$

$$x_0 = \varphi,$$

and $u^n(t) = x_i^n$ for $t_{i-1}^n < t \leq t_i^n$. Then

(1.14) $$\lim_{n\to\infty} u^n(t) = U(t,0)\varphi + \int_0^t U(t,s)f(s)ds.$$

in X uniformly on [0,T].

Proof of Corollary 1. If (1.13) is solved explicitly, one finds

(1.15) $$u^n(t) = U_{P(n)}(t_i^n,0)\varphi + \int_0^{t_i^n} U_{P(n)}(t_i^n,s)f^n(s)ds \text{ for } t_{i-1}^n < t < t_i^n,$$

where $f^n(s) = f_i^n$ on $(t_{i-1}^n, t_i^n]$. It is an elementary matter to use the convergence asserted for U_P in Theorem 2 and the assumed convergence of f^n to f in L^1 to pass to the limit in (1.15) to find (1.14), and we leave it to the reader to supply details as desired.

SECTION 2. The Linear Case

We begin the proof of Theorem 2. The proof is broken into four steps. Step 1 establishes the convergence of U_P to a limit U. Step 2 establishes properties of U. Step 3 proves that $u(t) = U(t,0)\varphi$ is continuous into Y for $\varphi \in Y$ and the final Step 4 proves that the convergence holds in Y.

Step 1: The convergence of U_P

Let us begin by remarking that this step involves only routine arguments and could be deduced from various references, but we give it here for completeness and later convenience. We assume that (B1) - (B3) are satisfied and let

$$\theta = \beta + \gamma_D. \tag{2.1}$$

When $\lambda > 0$ and $\lambda\theta < 1$, Lemma 2 and the assumptions imply that the operator

$$J_\lambda(t) = (I + \lambda B(t))^{-1} \tag{2.2}$$

satisfies

$$\|J_\lambda(t)\|_Z \le (1 - \lambda\theta)^{-1} \ \text{for } Z \in \{X,Y\}. \tag{2.3}$$

Hereafter we will always assume that whenever we use an operator $J_\lambda(t)$ then λ is positive and satisfies $\lambda\theta < 1/2$, in which case the elementary inequality $(1 - \lambda\theta)^{-1} \le e^{2\lambda\theta}$ holds and (2.3) implies

$$\|J_\lambda(t)\|_Z \le e^{2\lambda\theta} \ \text{for } Z \in \{X,Y\}. \tag{2.4}$$

In particular, with this implicit restriction on $m(P)$, it follows that

$$\|U_P(t,s)\|_Z = \Big\| \prod_{j=m}^{n} J_{t_j - t_{j-1}}(t_j)\Big\|_Z \le \prod_{j=m}^{n} e^{2(t_i - t_{i-1})\theta} \le e^{2(t-s+m(P))\theta} \tag{2.5}$$

where the notation is that of (1.11) and Z is either X or Y. We will also assume the mesh of every partition we deal with is at most 1.

Let

$$P = \{0 = t_0 < t_1 \ldots < t_N = T\}, \ \hat{P} = \{0 = s_0 < s_1 < \ldots < s_M = T\}$$

be a pair of partitions. Fix $s \in [0,T)$ and choose i_0, j_0 according to

$$s_{i_0 - 1} \le s < s_{i_0} \ \text{and} \ t_{j_0 - 1} \le s < t_{j_0}. \tag{2.6}$$

Next choose $\varphi \in Y$ and put

(2.7) $\quad y_i = U_{\hat{P}}(s_i,s)\varphi,\ x_j = U_P(t_j,s)\varphi,\ \gamma_i = s_i - s_{i-1}$ and $\delta_j = t_j - t_{j-1}$.

The the proof will proceed by estimating the numbers

(2.8) $$a_{i,j} = \|y_i - x_j\|_X = \|U_{\hat{P}}(s_i,s)y - U_P(t_j,s)y\|_X$$

for $i_0 < i$ and $j_0 < j$. Indeed, the $a_{i,j}$ satisfy certain inequalities which allow us to estimate them in a standard way.

First, observe that by (2.5)

(2.9) $$\|x_i\|_Y,\ \|y_j\|_Y \le K_1\|\varphi\|_Y$$

where we introduce the practice of denoting by K_i a constant which may be estimated in terms of the "data"

(2.10) $$T,\ \beta,\ \gamma_D,\ \text{and}\ \gamma_B,$$

which includes the constant

(2.11) $$\gamma_B = \max_{0\le t\le T} \|B(t)\|_{YX},$$

which is well defined by (B2). For example, in this case we may take $K_1 = e^{2(T+1)\theta}$.

We begin by estimating a_{i,j_0} for $i_0 < i$. To this end, first observe that for $y \in Y$

(2.12) $$\|J_\lambda(t)y-y\|_X = \|J_\lambda(t)(y-(y+\lambda B(t)y))\|_X \le \|J_\lambda(t)\|_X\lambda\|B(t)y\|_X \le K_2\lambda\|y\|_Y.$$

Now, by definition,

$$a_{i,j_0} = \|\prod_{l=i_0}^{i} J_{\gamma_l}(s_l)\varphi - J_{\delta_{j_0}}(t_{j_0})\varphi\|_X$$

so, using the triangle inequality, (2.5) and (2.12) we first find

$$a_{i,j_0} \le \|\varphi - J_{\delta_{j_0}}(t_{j_0})\varphi\|_X + \sum_{k=i_0}^{i} \|\prod_{p=k+1}^{i} J_{\gamma_p}(s_p)(\varphi - J_{\gamma_k}(s_k)\varphi)\|_X$$

and then

(2.13) $$a_{i,j_0} \le K_3(\delta_{j_0} + \sum_{k=i_0}^{i} \gamma_k)\|\varphi\|_Y = K_3(\delta_{j_0} + s_i - s_{i_0-1})\|\varphi\|_Y.$$

Similarly, if $j_0 < j$

(2.14) $$a_{i_0,j} \le K_3(\gamma_{i_0} + t_j - t_{j_0-1})\|\varphi\|_Y.$$

Next observe that, by definition,

$$y_i + \gamma_i B(s_i) y_i = y_{i-1}$$

and, writing it in a complicated way,

$$x_j + \delta_j B(s_i) x_j = x_{j-1} + \delta_j (B(s_i) - B(t_j)) x_j.$$

Since $B(s_i) + \theta I$ is accretive, the above relations and Lemma 1 imply that

$$\left(1 - \theta \frac{\gamma_i \delta_j}{\gamma_i + \delta_j}\right) \| y_i - x_j \|_X \le \frac{\delta_j}{\gamma_i + \delta_j} \| y_{i-1} - x_j \|_X +$$

$$+ \frac{\gamma_i}{\gamma_i + \delta_j} \| y_i - x_{j-1} \|_X + \frac{\gamma_i \delta_j}{\gamma_i + \delta_j} \| (B(s_i) - B(t_j)) x_j \|_X .$$

Moreover, using (2.9),

$$\| (B(s_i) - B(t_j)) x_j \|_X \le K_4 \| \varphi \|_Y \| B(s_i) - B(t_j) \|_{YX}$$

so we have

$$\left(1 - \theta \frac{\gamma_i \delta_j}{\gamma_i + \delta_j}\right) a_{i,j} \le \frac{\delta_j}{\gamma_i + \delta_j} a_{i-1,j} + \frac{\gamma_i}{\gamma_i + \delta_j} a_{i,j-1} +$$

(2.15)

$$+ \frac{\gamma_i \delta_j}{\gamma_i + \delta_j} K_4 \| \varphi \|_Y \| B(s_i) - B(t_j) \|_{YX}.$$

The results of [3] imply that for $\varepsilon > 0$ we can guarantee that

$$a_{i,j} \le w(\xi, \eta) + \varepsilon \text{ for } s_{i-1} \le \xi \le s_i,\ t_{j-1} \le \eta \le t_j$$

and $i_0 \le i$, $j_0 \le j$, as soon as $m(\mathbb{P})$ and $m(\hat{\mathbb{P}})$ are small, where w is the solution of the simple boundary-value problem

$$w_\xi + w_\eta - \theta w = K_4 \| \varphi \|_Y \| B(\xi) - B(\eta) \|_{YX} \text{ for } s \le \xi, \eta \le T$$

and

$$w(\xi, \eta) = K_2((\xi - s) + (\eta - s)) \| \varphi \|_Y \text{ if } \xi = s \text{ or } \eta = s,$$

given by integration along characteristics. While we could write the formula for w, it is enough to know that w is continuous and $w(\xi,\xi) = 0$ for $s \le \xi \le T$. In particular

$$\| U_{\mathbb{P}}(t,s)\varphi - U_{\hat{\mathbb{P}}}(t,s)\varphi \|_X \le w(t,t) + \varepsilon = \varepsilon$$

as soon as $m(\mathbb{P})$ and $m(\hat{\mathbb{P}})$ are sufficiently small. We conclude that

(2.16) $$\lim_{m(P)\to 0} U_P(t,s)\varphi = U(t,s)\varphi$$

exists in X uniformly on Δ. Since $U_P(t,s)$ is bounded in $B(X)$ and Y is a dense in X, the limit in (2.16) exists uniformly if $\varphi \in X$ as well, and $U:\Delta \to B(X)$.

Step 2. Properties of U

We now establish simple properties of U. If $0 \le r \le s \le t \le T$, we may choose any partition of $[0,T]$ with each of r, s and t as partition points and see that

$$U_P(t,r) = U_P(t,s)U_P(s,r)$$

and, in the limit,

(2.17) $$U(t,r) = U(t,s)U(s,r).$$

We next establish continuity of U in (t,s). Let P be a partition and $t = t_j$ be a point of P. As in the proof of (2.13) one sees that

$$\|U_P(t,0)\varphi - \varphi\|_X = \|\prod_{k=0}^{j} J_{\delta_k}(t_k)\varphi - \varphi\|_X \le K_5(\delta_1+\ldots+\delta_j)\|w\|_Y = K_5 t\|\varphi\|_Y$$

so, in the limit,

(2.18) $$\|U(t,0)\varphi - \varphi\|_X \le K_5 t\|\varphi\|_Y.$$

The relation $U(t+h,s) = U(t+h,t)U(t,s)$ for $0 \le s \le t \le t+h \le T$ and the above estimate leads to

$$\|U(t+h,s)\varphi - U(t,s)\varphi\|_X = \|U(t+h,t)U(t,s)\varphi - U(t,s)\varphi\|_X \le K_5 h\|U(t,s)\varphi\|_Y \le K_6 h\|\varphi\|_Y$$

since the restriction of U to Y is bounded in $B(Y)$ by (2.5). In a similar way we see that $U(t,s)\varphi$ is Lipschitz continuous into X as a function of s for $\varphi \in Y$. Since Y is dense in X, we obtain that $U(t,s)x$ is continuous in (t,s) into X for arbitrary $x \in X$.

Let $\varphi \in Y$ and consider $u(t) = U(t,0)\varphi$. We want to argue that $u([0,T]) \subseteq Y$ and u is weakly continuous as a Y-valued function. But this is obvious, since u is the uniform (in X) limit of the functions $U_P(t,0)\varphi$ which remain bounded in the reflexive space Y. It is also clear that any function which is bounded in Y and continuous into X is weakly continuous into Y, and $U_P(t,0)\varphi$ converges weakly to $u(t)$ in Y as $m(P) \to 0$. It now follows from (B2) that the function $B(t)u(t)$ is weakly continuous into X and hence strongly integrable. If the points of $\mathbb{P}$ are t_j, the relations

$$x_j = x_{j-1} - (t_j - t_{j-1})B(t_j)x_j$$

satisfied by $x_j = U_{\mathbb{P}}(t_j,0)\varphi$ imply, upon summing,

$$U_{\mathbb{P}}(t_j,0)\varphi = \varphi - \int_0^{t_j} B(g_{\mathbb{P}}(s))U_{\mathbb{P}}(s,0)\varphi ds$$

where $g_{\mathbb{P}}(s) = t_k$ on $t_{k-1} < t \le t_k$. Choosing partitions $\mathbb{P}$ which have $t = t_j$ as a partition point, one easily verifies - using (B2) and the above remarks - that the right-hand side of the above relation converges weakly (in X) to the right-hand side of

$$U(t,0)\varphi = \varphi - \int_0^t B(s)U(s,0)\varphi ds \tag{2.19}$$

and it follows that (2.19) holds and $u(t)$ satisfies the equation $u'(t) + B(t)u(t) = 0$ almost everywhere. The weak continuity of $B(t)u(t)$ then implies that the equation holds weakly everywhere. Once we know that $u(t)$ is continuous into Y so that $B(t)u(t)$ is continuous into X, it will follow that $u \in C^1[0,T:X]$ and the equation holds classically.

<u>Step 3</u>. Continuity into Y

We wish to establish the strong continuity of $u(t) = U(t,0)\varphi$ into Y. It is equivalent, by (S), to show that $Su(t)$ is continuous into X. The above remarks show that $Su(t)$ is weakly continuous into X and thus it is strongly measurable. By (B3) we then have that $D(t)Su(t)$ is bounded and strongly measurable and therefore strongly integrable (in X), and then so is $s \to U(t,s)D(s)Su(s)$. The proof will proceed by showing that

$$Su(t) = U(t,0)S\varphi - \int_0^t U(t,s)D(s)Su(s)ds \tag{2.20}$$

from which it is obvious that Su is continuous into X.

Since $D(t)Su(t)$ is strongly integrable in X and $u(t)$ is strongly integrable in Y, there is a sequence of partitions

$$\mathbb{P}(n) = \{0 = t_0^n < t_1^n < \ldots\ldots < t_{N(n)}^n = T\}$$

such that $m(\mathbb{P}(n)) \to 0$ and

$$\lim_{n\to\infty} \sum_{j=1}^{N(n)} \int_{t_{j-1}^n}^{t_j^n} \|D(t)Su(t)-D(t_j^n)Su(t_j^n)\|_X dt = \lim_{n\to\infty} \sum_{j=1}^{N(n)} \int_{t_{j-1}^n}^{t_j^n} \|u(t)-u(t_j^n)\|_Y dt = 0. \tag{2.21}$$

The scheme

$$\frac{x_i^n - x_{i-1}^n}{t_i^n - t_{i-1}^n} + B(t_i^n)x_i^n = 0, \qquad i = 1,2,\dots,N(n), \tag{2.22}$$

$$x_0^n = \varphi$$

has the solution $u^n(t) = U_{P(n)}(t,0)\varphi$ where $u_n(t) = x_i^n$ on $(t_{i-1}^n, t_i^n]$. Consider the auxiliary scheme

$$\frac{z_i^n - z_{i-1}^n}{t_i^n - t_{i-1}^n} + B(t_i^n)z_i^n = -D(t_i^n)Su(t_i^n), \quad i = 1,\dots\dots, N(n), \tag{2.23}$$

$$z_0^n = S\varphi$$

which defines the values of the piecewise constant function $z^n(t)$. By Corollary 1 and (2.21)

$$\lim_{n\to\infty} z^n(t) = U(t,0)S\varphi - \int_0^t U(t,s)D(s)Su(s)ds$$

holds in X uniformly in t. Define $z(t) = \lim_{n\to\infty} z^n(t)$. Next we show that $z(t) = Su(t)$. To this end, set $v^n(t) = S^{-1}z^n(t)$. The values v_i^n of $v^n(t)$ satisfy - using (2.23) and (B3) -

$$\frac{v_i^n - v_{i-1}^n}{t_i^n - t_{i-1}^n} + B(t_i^n)v_i^n = S^{-1}D(t_i^n)Sv_i^n - S^{-1}D(t_i^n)Su(t_i^n) \tag{2.24}$$

$$v_0^n = \varphi.$$

Since z^n converges as above, v^n converges in Y uniformly in t to a continuous function v(t). We are done upon showing that $v = u$. Using (2.24), (2.22), (B1) and Lemma 2 we find

$$\|v_i^n - x_i^n\|_Y < e^{2(t_i^n - t_{i-1}^n)\theta}\Big(\|v_{i-1} - x_{i-1}\|_Y +$$

$$+ (t_i^n - t_{i-1}^n)\|S^{-1}(D(t_i^n)Su(t_i^n) - D(t_i^n)Sv_i^n)\|_Y\Big)$$

$$< e^{2(t_i^n - t_{i-1}^n)\theta}\Big(\|v_{i-1}^n - x_{i-1}^n\|_Y + (t_i^n - t_{i-1}^n)\gamma_D\|u(t_i^n) - v_i^n\|_Y\Big).$$

Iterating this yields

(2.25) $$\|v^n(t_i^n) - u_n(t_i^n)\|_Y \le e^{2T\theta}\gamma_D\Big(\sum_{k=1}^{i}\int_{t_{k-1}^n}^{t_k^n} \|u(t_i^n) - v^n(s)\|_Y ds\Big).$$

Since v^n converges in Y uniformly and u^n converges weakly in Y and (2.21) holds, we may take limits in (2.25) upon appealing to the lower semicontinuity of the Y-norm with respect to the weak topology to conclude that

(2.26) $$\|v(t) - u(t)\|_Y \le e^{2T\theta}\gamma_D \int_0^t \|u(s) - v(s)\|_Y ds \text{ for } 0 \le t \le T.$$

Since $\|v(t) - u(t)\|_Y$ is integrable, this implies that $u(t) = v(t)$ and we are done.

<u>Step 4</u>. Convergence in Y

We now impose the condition that D(t) is strongly continuous. Since we established above that $u(t) = U(t,0)\varphi \in C[0,T:Y]$, the relation (2.21) holds for an arbitrary sequence of partitions $P(n)$ satisfying $m(P(n)) \to 0$. By the analysis of Step 3 we conclude that if $P = \{0 = t_0 < \dots < t_N = T\}$ and z_P *is the piecewise constant function on* P whose values z_i are given by

(2.27) $$\frac{z_i - z_{i-1}}{t_i - t_{i-1}} + B(t_i)z_i + D(t_i)Su(t_i) = 0,$$
$$z_0 = S\varphi,$$

then $z_P \to Su$ uniformly in X as $m(P) \to 0$. To show that $U_P(t,0)\varphi$ converges in Y we need to show that $w_P(t) = SU_P(t,0)\varphi$ converges in X. The values w_i of w_P are given by

(2.28) $$\frac{w_i - w_{i-1}}{t_i - t_{i-1}} + B(t_i)w_i + D(t_i)w_i = 0$$
$$w_0 = S\varphi.$$

Rewriting the relations (2.27) as

$$\frac{z_i - z_{i-1}}{t_i - t_{i-1}} + B(t_i)z_i + D(t_i)z_i = D(t_i)(z_i - Su(t_i))$$

and using (2.28) and the accretivity of $B(t_i) + D(t_i) + \theta I$, we find

$$\|w_i - z_i\|_X \le e^{2\theta(t_i - t_{i-1})}\big(\|w_{i-1} - z_{i-1}\|_X + (t_i - t_{i-1})\|D(t_i)(z_i - Su(t_i))\|_X\big).$$

Iteration of this inequality yields

$$(2.29)\qquad \|w_i - z_i\|_X \le e^{2\theta T}\gamma_D\Big(\sum_{k=1}^{i}(t_k - t_{k-1})\|z_k - Su(t_k)\|_X\Big).$$

Since $z_P \to Su$ in X as $m(P) \to 0$ and $Su \in C[0,T:X]$, the right-hand side tends to zero as $m(P) \to 0$. We conclude that $w_P - z_P \to 0$ in X uniformly as $m(P) \to 0$ and so $w_P \to Su$ as cla'med.

Section 3. Proof of Theorem 1

The proof is again broken into four pieces, and (A4) is invoked only in the fourth part. In Step 1 we show that the scheme

$$\text{(3.1)}\qquad \begin{aligned} &\frac{x_i - x_{i-1}}{\lambda} + A(x_{i-1})\, x_i = 0, \quad i = 1,\ldots,N \\ &x_0 = \varphi \end{aligned}$$

is solvable and obtain some appropriate estimates along the way. In Step 2 we show that if

$$\text{(3.2)}\qquad u_\lambda(t) = x_i \text{ on } ((i-1)\lambda, i\lambda],\ u_\lambda(0) = \varphi,$$

then the limit

$$\text{(3.3)}\qquad \lim_{\lambda \downarrow 0} u_\lambda(t) = u(t)$$

exists in X uniformly. Moreover, in this step it will be proved that the limit is a solution of the evolution problem in a strong - but not quite classical - sense. Up to this point, the results of Section 2 will not be used. In Step 3 it is shown that the limit u in (3.3) lies in C[0,T:Y] and for this we will rely on the results of Section 2. In Step 4 we demonstrate that the limit in (3.3) exists in the topology of Y.

Step 1: Existence of u_λ

We will now discuss the solvability of (3.1). To this end let $\varphi \in W$ and

$$\text{(3.4)}\qquad d_\varphi = \inf\{\|\varphi - v\|_Y : v \in Y\backslash W\}.$$

be the distance in Y of φ to the boundary of W. We have:

Lemma 3. Let (A1) - (A3) hold and

$$\text{(3.5)}\qquad \theta = \beta + \gamma_P.$$

Let T > 0 satisfy

$$\text{(3.6)}\qquad \inf_{z \in Y} \left((1+e^{2\theta T})\|S\varphi - z\|_X + T(\gamma_A \|z\|_Y + \gamma_P \|z\|_X)\right) < d_\varphi.$$

Then there is a $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$ there is a unique sequence x_i, $i = 0,\dots, N$, in W satisfying (3.1) and $T < N\lambda < T + \lambda$.

<u>Proof</u>. The desired relation between x_i and x_{i-1} given by (3.1), can be, if $\lambda > 0$ and $\lambda\theta < 1$, rewritten as

(3.7) $$x_i = (I + \lambda A(x_{i-1}))^{-1}x_{i-1}.$$

By Lemma 2, $x_i \in Y$ is uniquely determined by (3.7) so long as $x_{i-1} \in W$. We thus seek to estimate $\|x_i-\varphi\|_Y$ and keep this below d_φ since the open Y ball centered at φ of radius d_φ lies in W. Assuming $x_{i-1} \in W$, we put $w_i = Sx_i$, $w_{i-1} = Sx_{i-1}$ and operate on (3.1) with S to obtain - via (A3) -

(3.8) $$w_i + \lambda(A(x_{i-1})w_i + P(x_{i-1})w_i) = w_{i-1} .$$

Choose $z \in Y$. We have

(3.9) $$\begin{aligned} w_i - z + \lambda(A(x_{i-1})(w_i- z) + P(x_{i-1})(w_i- z)) = \\ w_{i-1} - z - \lambda(A(x_{i-1})z + P(x_{i-1})z). \end{aligned}$$

Using Lemma 2 in conjunction with (3.9) we obtain

(3.10) $$\|w_i - z\|_X < (1-\lambda\theta)^{-1}(\|w_{i-1}-z\|_X+ \lambda(\gamma_A\|z\|_Y + \gamma_P\|z\|_X))$$

Again we assume that $\lambda\theta < 1/2$ so that $(1 - \lambda\theta)^{-1} < e^{2\lambda\theta}$ everywhere below. Then we can iterate (3.10) to obtain

$$\|w_i-z\|_X < e^{2i\lambda\theta}(\|S\varphi-z\|_X+i\lambda(\gamma_A\|z\|_Y+\gamma_P\|z\|_X)).$$

(recall $w_0 = Sx_0 = S\varphi$). This further implies

(3.11) $$\|w_i-S\varphi\|_X < (1+e^{2i\lambda\theta})(\|S\varphi-z\|_X+i\lambda(\gamma_A\|z\|_Y+\gamma_P\|z\|_X)).$$

By (3.5) and considerations of continuity we can choose $\alpha > 0$ and $z \in Y$ such that the right hand side of (3.11) is less than d_φ if $i\lambda < T + \alpha$. Set $\lambda_0 = \min(\alpha, \frac{1}{2}\theta)$. By what we have shown (3.5) implies the existence of an $r < d_\varphi$ so that for $0 < \lambda < \lambda_0$ and $T < N\lambda < T+\lambda$, one can solve (3.1) and

(3.12) $$x_i \in B_Y(r,\varphi) = \{v \in Y : \|v-\varphi\|_Y < r\} \text{ for } i = 1,\dots,N.$$

<u>Remark</u>. In contrast with [5] we have used the full force of (A3) here. This is because we do not assume any bounds on expressions like $\|A(w)y\|_Y$ in this case.

Step 2: Convergence of u_λ

For the rest of the discussion, r is fixed at the value above.

Lemma 4. If $x, \hat{x} \in B_Y(r,\varphi)$ and μ_A, θ be as in (A3) and (3.5) respectively. Let

$$\psi = \theta + \mu_A(\|\varphi\|_Y + r). \tag{3.13}$$

If

$$0 < \lambda, \ \lambda\psi < 1 \quad \text{and}$$

$$x + \lambda A(x)x = z, \qquad \hat{x} + \lambda A(\hat{x})\hat{x} = \hat{z} \tag{3.14}$$

then

$$\|x - \hat{x}\|_X \le (1-\lambda\psi)^{-1}\|z - \hat{z}\|_X. \tag{3.15}$$

Before we give the simple proof, let us explain why Lemma 4 and standard results establish Step 2. The conclusion of Lemma 4 is that the mapping $B_Y(r,\varphi) \ni x \to A(x)x + \psi x$ is accretive. That is, if $C(x) = A(x)x$ for $x \in D(C) = B_Y(r,\varphi)$, then $C + \psi I$ is accretive (in X) according to Lemma 3. It is known that if $C + \psi I$ is accretive in X for some ψ and for each *small* $\lambda > 0$, x_i*'s are given so that* $x_0 \in D(C)$ and

$$\begin{gathered} \frac{x_i - x_{i-1}}{\lambda} + C(x_i) = \varepsilon_i \ \text{for } i = 0,1,\dots,N \text{ with } T \le N\lambda < T + \lambda, \\ \lambda \sum_1^N \|\varepsilon_i\|_X \to 0 \quad \text{as} \quad \lambda \downarrow 0, \end{gathered} \tag{3.16}$$

then the $u_\lambda(t)$ given as x_i on $((i-1)\lambda, i\lambda]$ converge uniformly on $[0,T]$ in X to a Lipschitz continuous $u \in C[0,T:X]$. This is a basic result of [2] when $\varepsilon_i = 0$, $i = 1,\dots,N$. In our case we have, with the x_i's of Lemma 3, and $C(x) = A(x)x$,

$$\frac{x_i - x_{i-1}}{\lambda} + C(x_i) = (A(x_i) - A(x_{i-1}))x_i$$

so, by (A2),

$$\|\varepsilon_i\|_X = \|(A(x_i) - A(x_{i-1}))x_i\|_X \le \mu_A\|x_i - x_{i-1}\|_X\|x_i\|_Y .$$

By $x_i - x_{i-1} = \lambda A(x_{i-1})x_i$ and (A3)

$$\|x_i - x_{i-1}\|_X \le \lambda\gamma_A \|x_i\|_Y \le \lambda\gamma_A(\|\varphi\|_Y + r)$$

and thus $\|\varepsilon_i\|_X \le C\lambda$. It follows that for some constant C_1

$$\lambda \sum_1^N \|\varepsilon_i\| \le C_1(N\lambda)\lambda \le C_1(T+\lambda)\lambda \to 0 \quad \text{as} \quad \lambda \to 0.$$

The convergence of u_λ when $\|\varepsilon_i\| \to 0$ uniformly in i as $\lambda \to 0$ is a simple extension of [2]. More generally, the results of [1] or Kobayashi [11] or Takahashi [13], or Crandall and Evans [3] can (as we have already done in Section 1) be applied. Indeed, from these works one has an error estimate of the form

$$\|u_\lambda(t) - u(t)\|_X \le C(C_1(T + \lambda)\lambda + \lambda^{1/2}\|A(\varphi)\varphi\|_X)$$

where C depends on T and ψ.

<u>Proof of Lemma 4</u>. Forming the difference of the relations (3.14) and rearranging suitably yields

$$x - \hat{x} + \lambda A(x)(x - \hat{x}) = z - \hat{z} + \lambda(A(\hat{x}) - A(x))\hat{x}.$$

Since x, $\hat{x} \in B_Y(r,\varphi)$ and $A(x) \in N(X,\beta)$ this implies

$$(1 - \lambda\beta)\|x - \hat{x}\|_X \le \|z - \hat{z}\|_X + \lambda\|(A(\hat{x}) - A(x))\hat{x}\|_X$$

$$\le \|z - \hat{z}\|_X + \lambda\mu_A\|x - \hat{x}\|_X\|\hat{x}\|_Y \le \|z - \hat{z}\|_X + \lambda\mu_A\|x - \hat{x}\|_X(\|\varphi\|_Y + r)$$

and rearranging this proves Lemma 4.

By the above, the convergence (3.3) takes place in X uniformly in t and the limit u is Lipschitz continuous. Since the values of u_λ are bounded in Y (they lie in $B_Y(r,\varphi)$), and Y is reflexive, the limit u therefore takes its values in Y (in fact in $B_Y(r,\varphi)$). Since u is continuous into X it is weakly continuous into Y. Similarly, the convergence u_λ to u takes place weakly in Y. Iterating the relations (3.1) we find

$$u_\lambda(i\lambda) = \varphi - \int_0^{i\lambda} A(u_\lambda(s-\lambda))u_\lambda(s)ds \quad .$$

It is a simple matter, using the above remarks, (A1) and (A2), to see that as $i\lambda \to t$ and $\lambda \to 0$ (e.g., let $\lambda = t/i$ and $i \to \infty$) the right-hand side above tends to the right-hand side of

$$u(t) = \varphi - \int_0^t A(u(s))u(s)ds$$

weakly in X, thus establishing the validity of the integral relation. Observe that $A(u(t))u(t)$ is weakly continuous into X - and thus integrable - by the assumptions and the properties of u. Thus $u' + A(u)u = 0$ holds strongly a.e. and weakly everywhere. In the next step we will prove that u is continuous into Y. This will make $A(u(t))u(t)$ continuous into X and so $u \in C^1[0,T:X]$.

Step 3. Continuity into Y

Set

(3.17) $$B(t) = A(u(t)) \text{ for } 0 \le t \le T.$$

where A and u are as above. It follows from (A1) - (A3) that B(t) satisfies (B1) - (B3). We may take β of (A1) for β of (B1), (A2) and the continuity of u into X and $u([0,T]) \in W$ imply (B2) while $D(t) = P(u(t))$ works in (B3). We briefly recall Kato's reasoning concerning this latter point. From the assumptions on A and the properties of u it follows that

$$S^{-1}P(u(t))y = S^{-1}A(u(t))y - A(u(t))S^{-1}y \text{ for } y \in Y,$$

and the right hand side of this expression is continuous into X. Thus $S^{-1}P(u(t))y$ is continuous (in t) into X and bounded into Y, hence it is weakly continuous into Y and then $P(u(t))y$ is weakly continuous (and therefore strongly measurable) into X. Since Y is dense is X, $P(u(t))$ is strongly measurable.

We want to show next that the schemes (3.1) and

(3.18) $$\frac{y_i - y_{i-1}}{\lambda} + A(u(i\lambda))y_i = 0, \quad i = 1,\dots,N, \qquad y_0 = \varphi,$$

are equivalent. More generally, let us argue that if $(t,s) \in \Delta = \{(t,s): 0 \le s \le t \le T\}$ and

$$V_\lambda(t,s) = \prod_{k=m}^{n} (I + \lambda A(x_{k-1}))^{-1}$$

$$U_\lambda(t,s) = \prod_{k=m}^{n} (I + \lambda A(u(k\lambda)))^{-1}$$

when $(m-1)\lambda < s < m\lambda$, and $(n-1)\lambda < t < n\lambda$, then

$$\|V_\lambda(t,s)\|_Z \ , \ \|U_\lambda(t,s)\|_Z < C_1 \text{ for } Z \in \{X,Y\} \tag{3.19}$$

and

$$\lim_{\lambda\to 0} V_\lambda(t,s)x = U(t,s)x \text{ for } x \in X \tag{3.20}$$

in X uniformly on Δ where

$$U(t,s)x = \lim_{\lambda\to 0} U_\lambda(t,s)x \tag{3.21}$$

exists by Section 2 and the fact that $B(t)$ satisfies (B1)-(B3). We now adopt the convention that the C_i's are constants estimable in terms of the data. The first estimate of (3.19) is proved just like the second, and this is part of the proof of Theorem 2. It suffices to consider the case $s = 0$, as the general case is entirely similar. Assume that (3.1) and (3.18) hold, but allow $x_0 = y_0 = x$ to be an arbitrary element of Y (and not necessarily $u(0)$). Writing (3.18) as

$$\frac{y_i - y_{i-1}}{\lambda} + A(x_{i-1})y_i = (A(x_{i-1}) - A(u(t_i)))y_i$$

and using (3.1) and the accretivity of $A(x_{i-1}) + \beta I$ yields

$$\begin{aligned}
\|x_i - y_i\|_X &< (1-\lambda\beta)^{-1}\left(\|x_{i-1} - y_{i-1}\|_X + \lambda\|(A(x_{i-1}) - A(u(t_i))y_i\|_X\right) \\
&< e^{2\lambda\beta}\left(\|x_{i-1} - y_{i-1}\|_X + \lambda\mu_A\|x_{i-1} - u(t_i)\|_X\|y_i\|_Y\right) \\
&< e^{2\lambda\beta}\left(\|x_{i-1} - y_{i-1}\|_X + \lambda C_2\|x\|_Y\|x_{i-1} - u(t_i)\|_X\right).
\end{aligned}$$

Iteration yields

$$\|x_i - y_i\|_X < e^{2\beta(T+\lambda)}C_3T\|x\|_Y \max_{1<k<N} \|x_{k-1} - u(t_k)\|_X$$

and then, since u_λ converges uniformly to u, we conclude that $U_\lambda(t,0)x - V_\lambda(t,0)x \to 0$ in X uniformly as desired.

It is now established that $u(t) = U(t,0)\varphi$ is the solution of $u' + B(t)u = 0$ produced in Section 2, and so $u \in C[0,T:Y] \cap C^1[0,T:X]$ and $u'(t) + A(u(t))u(t) = 0$ by the results of Section 2.

Step 4: Convergence in Y

We now assume (A4). We will be considering four families of functions: The functions u_λ whose values x_i are given by (3.1), the functions $w_\lambda = Su_\lambda$ whose values w_i satisfy

$$\text{(3.22)} \qquad \begin{aligned} &\frac{w_i - w_{i-1}}{\lambda} + A(x_{i-1})w_i + P(x_{i-1})w_i = 0, \qquad i = 1,\ldots\ldots,N, \\ &w_0 = S\varphi, \end{aligned}$$

the functions z_λ whose values z_i satisfy

$$\text{(3.23)} \qquad \begin{aligned} &\frac{z_i - z_{i-1}}{\lambda} + A(x_{i-1})z_i + P(u(i\lambda))Su(i\lambda) = 0, \quad i = 1,\ldots\ldots,N, \\ &z_0 = S\varphi, \end{aligned}$$

and the functions $v_\lambda = S^{-1}z_\lambda$ whose values v_i satisfy

$$\text{(3.24)} \qquad \begin{aligned} &\frac{v_i - v_{i-1}}{\lambda} + A(x_{i-1})v_i + S^{-1}P(u(i\lambda))Su(i\lambda) - S^{-1}P(x_{i-1})Sv_i = 0 \text{ for } i=1,..,N, \\ &v_0 = \varphi. \end{aligned}$$

Concerning these we claim several things. First, it is obvious that u_λ converges in Y exactly when w_λ converges in X. Since we cannot show the convergence of the w_λ directly, we begin by observing that z_λ satisfies

$$\text{(3.25)} \qquad \lim_{\lambda \downarrow 0} z_\lambda(t) = U(t,0)S\varphi - \int_0^t U(t,s)P(u(s))Su(s)ds,$$

in X where $U(t,s)$, given by (3.21), is the evolution generated by $-B(t) = -A(u(t))$. The relation (3.25) holds in X uniformly in t, because of arguments like that sketched in the proof of Corollary 1 in Section 1 together with (3.20) and (3.23), the convergence of the function whose value on $((i-1)\lambda, i\lambda]$ is $P(u(i\lambda))Su(i\lambda)$ to $P(u(t))u(t)$ - which follows in turn from (A4) and the continuity of u into Y and of $Su(t)$ into X from Step 3. Second, (3.26) $$\lim_{\lambda\to 0} v_\lambda(t) = u(t)$$

holds in X. The reasoning here parallels the corresponding arguments which led to (2.26). Indeed, from (3.1) and (3.24) we deduce that

$$(1-\lambda\theta)\|v_i - x_i\|_Y \le \|v_{i-1} - x_{i-1}\|_Y + \lambda\|P(u(i\lambda))Su(i\lambda) - P(x_{i-1})Sv_i\|_X$$

$$\le \|v_{i-1} - x_{i-1}\|_Y + \lambda\|(P(u(i\lambda)) - P(x_{i-1}))Su(i\lambda) + P(x_{i-1})S(u(i\lambda) - v_i)\|_X$$

and then, using (A4) and letting C denote a bound on $\|u(t)\|_Y$ and $\|v_i\|_Y$, we find

$$(1-\lambda\theta)\|v_i - x_i\|_Y \le \|v_{i-1} - x_{i-1}\|_Y + \lambda C\mu_P(\|u(i\lambda) - v_{i-1}\|_Y + \|v_{i-1} - x_{i-1}\|_Y) +$$

$$+ \lambda C\gamma_P\|u(i\lambda) - v_i\|_Y.$$

The rest of the proof of (3.26) is essentially the same as that of (2.26) and is left to the reader. At this point we have identified $\lim_{\lambda\to 0} z_\lambda$ with Su(t). One can then show, using (3.22) and (3.23), that

$$(1 - \lambda\theta)\|z_i - w_i\|_X \le \|z_{i-1} - w_{i-1}\|_X + \lambda\gamma_P\|z_i - Su(i\lambda)\|_X +$$

$$+\lambda\mu_P C(\|w_{i-1} - z_{i-1}\|_X + \|z_{i-1} - Su(i\lambda)\|_X).$$

Iterating this inequality and using the uniform convergence of z_λ to Su establishes

$$\lim_{\lambda\to 0}(w_\lambda - z_\lambda)(t) = 0$$

in X uniformly in t in the same way as established (2.27) in Section 2, and the proof is complete.

Remarks on Generalizations.

The problem

$$\frac{du}{dt} + A(u)u = f(u),$$

(3.27)

$$u(0) = \varphi,$$

generalizes (1) and is in turn generalized by

$$\frac{du}{dt} + A(t,u)u = f(t,u),$$

(3.28)

$$u(0) = \varphi.$$

The methods of this paper succeed, under appropriate assumptions, in the generality of (3.28). However, these methods do not entirely subsume the t-dependence in either A or f assumed in Kato's original works. Roughly, the conditons on the t-dependence are required to be more uniform in u than Kato needs (but are otherwise quite general). We will not discuss this point further here - see, e.g., [5] and [6]. Instead, let us indicate the situation with respect to (3.27). Kato used the following two conditions on f:

(f1) f maps W into a bounded subset of Y and there is a constant μ_f such that for every u, $\hat{u} \in W$ we have

$$\|f(u) - f(\hat{u})\|_X \leq \mu_f \|u - \hat{u}\|_X.$$

and

(f2) There is a constant μ_f such that for every u, $\hat{u} \in W$ we have

$$\|f(u) - f(\hat{u})\|_Y \leq \mu_f \|u - \hat{u}\|_Y.$$

The *following modification of* Theorem 1 is true and has essentially the same proof: If (X), (S), (A1), (A2), (A3) and (f1) hold, the difference scheme in (1.1) is replaced by

$$\frac{x_i - x_{i-1}}{\lambda} + A(x_{i-1})x_i = f(x_{i-1})$$

and (1.4) by the equation of (3.27), then the assertions preceeding (1.4) remain true. If also (A4) and (f2) hold, then the convergence holds in Y uniformly on [0,T].

Remark. Results completely analogous to the above can be proved for the fully implicit approximation

$$\frac{v_\lambda(t) - v_\lambda(t-\lambda)}{\lambda} + A(v_\lambda(t))v_\lambda(t) = 0 \text{ for } t > 0,$$

$$v_\lambda(t) = \varphi \text{ for } t \leq 0.$$

in place of the semi-implicit scheme (2).

Remark. One can play a bit with the assumptions on $P(u)$. For example, it is enough to require $u \to P(u)$ to be continuous into the strong operator topology from the X topology on W in order to assert the convergence in Y. (However, this does not seem a good assumption from point of view of applications.)

Section 4. Continuity with Respect to Data

In this section we state and prove a result of [8] concerning the continuous dependence of the solution of (1) as an element of $C[0,T:Y]$ on the data. Nothing new is proved in the process and we include this section primarily to indicate how one might prove such results in the current setting. To formulate the result, we consider a sequence of equations

$$(1)^n \qquad \begin{aligned} &\frac{du^n}{dt} + A^n(u^n)u^n = f^n(u^n), && n = 1,\ldots,\infty, \\ &u^n(0) = \varphi^n, && n = 1,\ldots,\infty, \end{aligned}$$

where $n = \infty$ is explicitly allowed. We assume

(4.1) A^n and f^n satisfy (A1) - (A4) and (f1) - (f2) with the same X, Y, S, W and constants independent of $n = 1,2,\ldots,\infty$.

The result is:

Theorem 3. Let (4.1) hold. Moreover, for each $w \in W$ let

$$(4.2) \qquad \begin{aligned} &\text{(i)} \quad A^n(w) \to A^\infty(w) \text{ strongly in } B(X,Y) \text{ as } n \to \infty. \\ &\text{(ii)} \quad P^n(w) \to P^\infty(w) \text{ strongly in } B(X) \text{ as } n \to \infty. \\ &\text{(ii)} \quad f^n(w) \to f^\infty(w) \text{ in } Y \text{ as } n \to \infty. \end{aligned}$$

If $\varphi^n \in W$ for $n = 1,\ldots,\infty$ and $\varphi^n \to \varphi^\infty$ as $n \to \infty$, then there is a $T > 0$ such that the solution of $(1)^n$ constructed in Section 3 satisfies $u^n \in C[0,T:Y] \cap C^1[0,T:X]$ (i.e., the interval of definition of u^n includes $[0,T]$) for $n = 1,2,\ldots,\infty$. Moreover,

$$u^n \to u^\infty \text{ in } Y \text{ uniformly on } [0,T].$$

Remark. Theorem 3 shows, in particular, that u depends continuously on φ in the Y norm.

Proof of Theorem 3.

For simplicity (and of necessity, since we did so before) we assume that $f^n = 0$. The existence of T and u^n as in the statement of the Theorem is an immediate consequence of Theorem 1, (3.4) and the assumption that $\varphi^n \to \varphi^\infty$.

We need the following lemma:

Lemma 5. Let $U^n(t,s)$ correspond to A^n and φ^n for $n = 1,\ldots,\infty$ as U corresponded to A and φ in (3.21). Let $x \in X$. Then

$$\lim_{n\to\infty} U^n(t,s)x = U^\infty(t,s)x \text{ in } X \text{ uniformly on } \Delta. \tag{4.3}$$

We continue with the proof of Theorem 3 and then we prove the lemma. In Step 4 of the demonstration of Theorem 1 we established that

$$Su^n(t) = U^n(t,0)S\varphi^n - \int_0^t U^n(t,s)P^n(u^n(s))Su^n(s)ds \text{ for } n = 1,\ldots,\infty.$$

If we subtract the n^{th} and ∞^{th} equations and use the triangle inequality several times we can obtain

$$\|u^n(t) - u^\infty(t)\|_Y \leq e^{2\theta T}\|\varphi^n - \varphi^\infty\|_Y + \|(U^n(t,0) - U^\infty(t,0))S\varphi^\infty\|_X +$$

$$+ \int_0^t \|U^n(t,s)P^\infty(u^\infty(s))Su^\infty(s) - U^\infty(t,s)P^\infty(u^\infty(s))Su^\infty(s)\|_X ds +$$

$$+ e^{2\theta T}\int_0^t \|P^n(u^\infty(s))Su^\infty(s) - P^\infty(u^\infty(s))Su^\infty(s)\|_X ds +$$

$$+ e^{2\theta T}(\gamma + C)\int_0^t \|u^n(s) - u^\infty(s)\|_Y ds$$

where θ is given by (3.5), γ is the bound on $\|P^n(w)\|_X$ and $C = \gamma(\text{bound on } \|Su^n(t)\|_X)$. Using Lemma 5, (4.2)(ii) and $\varphi^n \to \varphi^\infty$ in Y we see that all the terms on the right hand side above except the last one tend to zero as $n \to \infty$. Elementary estimates complete the proof.

Sketch of Proof of Lemma 5. This result is follows from those in [5], but we sketch the proof here in this context for completeness. Since Y is dense in X and (A1) - (A4) hold with constants uniform in n, U^n is bounded in $B(X)$ and it suffices to check (4.3) for $x \in Y$. To this end, we note (with the obvious notations) that

$$\lim_{\lambda\downarrow 0} u_\lambda^n(t) = u^n(t) \text{ in } X \text{ uniformly for } 0 \leq t \leq T \text{ and } n = 1,\ldots,\infty. \tag{4.4}$$

This is evident from Step 2 in the proof of Theorem 1. In addition, one can easily check that

$$\lim_{n\to\infty} u_\lambda^n(t) = u_\lambda^\infty(t) \text{ in } X \text{ uniformly for } 0 \le t \le T. \tag{4.5}$$

for small $\lambda > 0$. Using (4.4) and $x \in Y$, the proof of Theorem 1 adapts to show that

$$\lim_{\lambda\downarrow 0} V_\lambda^n(t,s)x = U^n(t,s)x \text{ in } X \text{ uniformly in } (t,s) \in \Delta \text{ and } n. \tag{4.6}$$

Finally, a straightforward estimate shows that

$$\begin{aligned} \|V_\lambda^n(t,s)x - V_\lambda^\infty(t,s)x\|_X &\le \text{Const.}\sup_{0\le\tau\le T} (\|u_\lambda^n(\tau) - u_\lambda^\infty(\tau)\|_X + \\ &+ \sup_{0\le\tau\le T} \|(A^n(u_\lambda^\infty(\tau)) - A^\infty(u^\infty(\tau)))V_\lambda^\infty(t,s)x\|_X) \end{aligned} \tag{4.7}$$

and the right hand side can be made small for fixed $\lambda > 0$ by choosing n large. (Recall that every function subscripted by λ has finitely many values, (4.5) and (4.2)(i).) But then (4.5), (4.6) and (4.7) together yield (4.3).

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A very successful theory of quasilinear evolution equations, which applies to many problems of mathematical physics, has been developed by T. Kato. The theory obtains solutions of quasilinear problems via contraction mappings which are defined by means of a theory of linear evolution equations also developed by Kato. In the current work we show how the existence and continuous dependence theorems obtained by Kato can be proved by discretization in time. As opposed to earlier work in this direction, the current results are much sharper concerning the continuity properties of the solutions of the discretized problem and the strength of the norms in which they converge.

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