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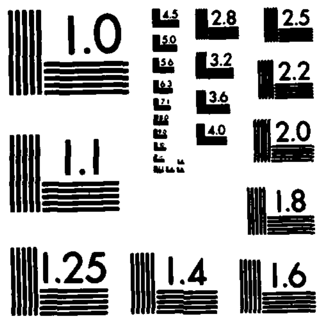
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Roger F. Harrington

Department of
Electrical and Computer Engineering
Syracuse University
Syracuse, New York 13210

Technical Report No. 25
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A numerical solution for the excess inductance of a microstrip discontinuity is obtained by the method of moments. This discontinuity is a via between two semi-infinite infinitesimally thin perfectly conducting strips. Above a ground plane, these strips are immersed in a medium that has constant permeability. Both strips run in the z-direction, but they are not necessarily collinear. The excess inductance is the inductance in excess of that which would result if

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20. ABSTRACT (continue)

the asymptotic inductance per unit length was assumed to exist up to z=0 on the left-hand strip and beginning with z=0 on the right-hand strip. The asymptotic inductance per unit length of one of the strips is that which would exist if the strip was infinitely long.

CONTENTS

# I. STATEMENT OF THE PROBLEM

Consider the microstrip discontinuity shown in Figs. 1 and 2. In region a ($-\infty < z < z_a$) of Fig. 1, an infinitesimally thin perfectly conducting strip of width $w_a$ and height $h_a$ above a perfectly conducting ground plane runs parallel to the z axis. In region c ($z_c < z < \infty$) of Fig. 1, an infinitesimally thin perfectly conducting strip of width $w_c$ and height $h_c$ above the ground plane runs parallel to the z axis. These two strips are called strips a and c. They communicate by means of what is called a via in region b ($z_a \leq z \leq z_c$) of Fig. 1. It is assumed that the extent ($z_c-z_a$) of the via in the z direction is greater than or equal to zero. The via is a perfectly conducting surface of arbitrary shape. It is assumed that the via establishes an electrical connection between the two strips.

Above the ground plane, the strips and via are immersed in a medium that has constant permeability $\mu$. Although both of the strips run in the z direction, they may be offset from each other in the transverse plane as shown in Fig. 2. In Fig. 2, the strip in region a begins at ($x=x_a$) while that in region c begins at ($x=x_c$).

The objective is to calculate the excess inductance of the structure shown in Fig. 1. This excess inductance is called $L_e$ and is defined by

$$L_e = \lim_{\substack{\ell_a \to \infty \\ \ell_c \to \infty}} \left( \frac{\psi - \ell_a \psi_a - \ell_c \psi_c}{I} \right) \tag{1}$$

In (1), the quantities $\psi$, $\psi_a$, and $\psi_c$ are defined in terms of the

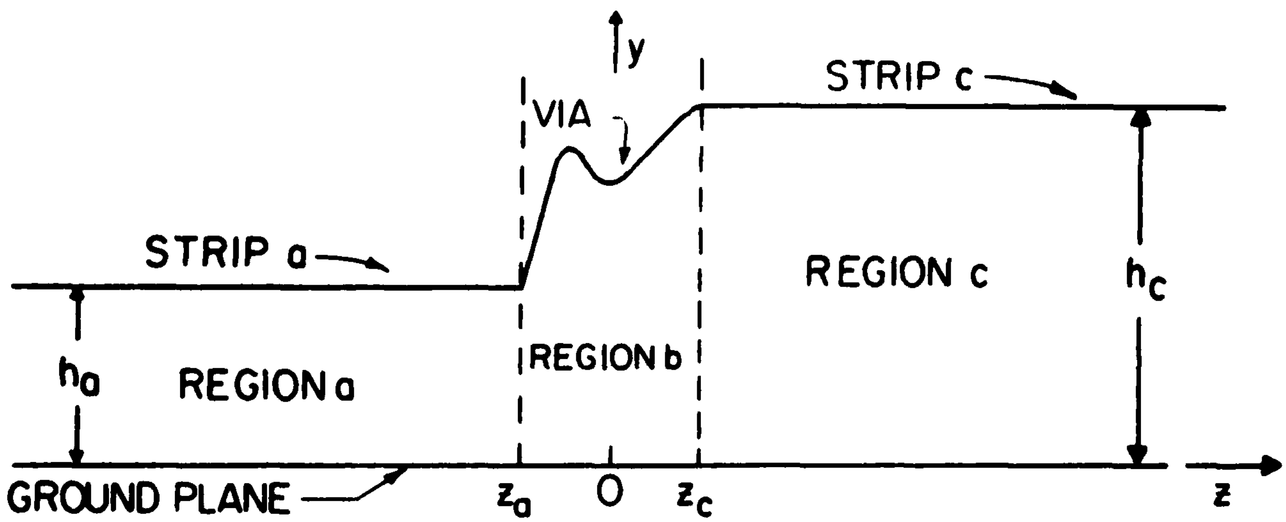


Fig. 1. Two perfectly conducting strips above a ground plane. The strip in region a communicates with that in region c by means of the via in region b.

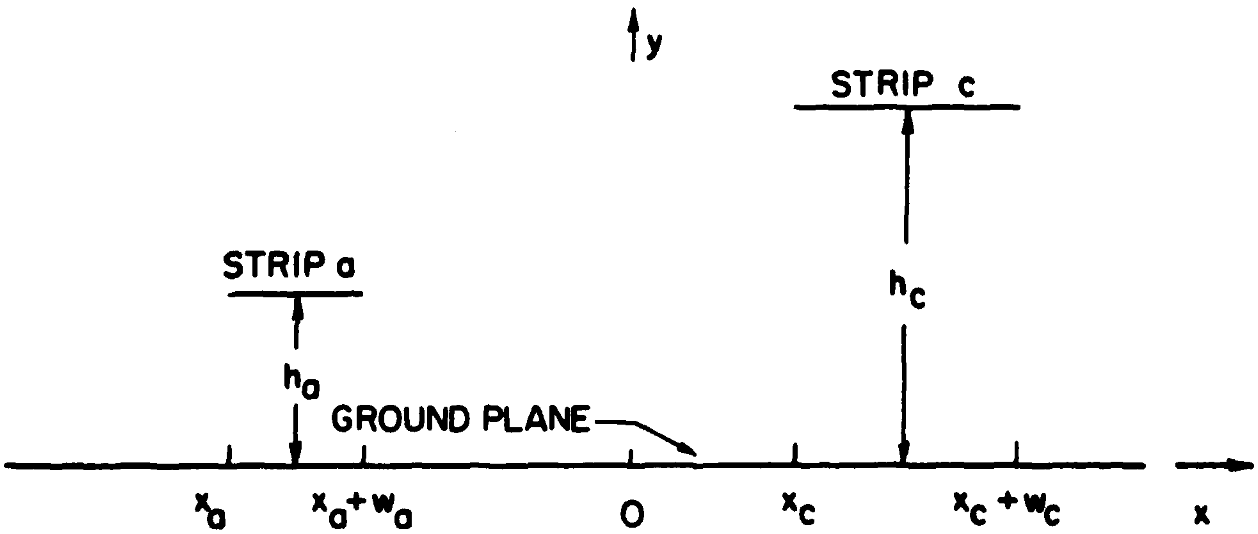


Fig. 2. Cross sectional view of the two strips and the ground plane. The via is not shown here.

magnetostatic problem in which the total current I flows from $(z = -\infty)$ on strip a through the via to $(z = \infty)$ on strip c.

The quantity $\psi$ is the magnetic flux which passes through a loop formed by the portion of the strips and via for which $(-\ell_a \le z \le \ell_c)$, the ground plane, and connecting lines at $(z = -\ell_a)$ and at $(z = \ell_c)$. The reference direction for $\psi$ is into the paper in Fig. 1.

The quantity $\psi_a$ is the magnetic flux per unit length of strip a far from the via. More precisely, if $z_o$ is such that the intersection of strip a with the $(z=z_o)$ plane is far from the via, then $\psi_a$ is the magnetic flux which passes through a loop formed by the portion of strip a for which $(z_o - 1m \le z \le z_o)$, the ground plane, and connecting lines at $(z = z_o - 1m)$ and at $(z = z_o)$. Here, "1m" denotes one meter and represents a unit length along the z axis. The reference direction for $\psi_a$ is into the paper in Fig. 1.

The quantity $\psi_c$ is the magnetic flux per unit length of strip c far from the via. More precisely, if $z_o$ is such that the intersection of strip c with the $(z=z_o)$ plane is far from the via, then $\psi_c$ is the magnetic flux which passes through a loop formed by the portion of strip c for which $(z_o \le z \le z_o + 1m)$, the ground plane, and connecting lines at $(z=z_o)$ and at $(z=z_o+1m)$. The reference direction for $\psi_c$ is into the paper in Fig. 1.

Because all three quantities $\psi$, $\psi_a$, and $\psi_c$ are proportional to I, expression (1) for $L_e$ does not depend on I. If $\ell_a$ is sufficiently large, the addition of any positive quantity $\Delta\ell_a$ to $\ell_a$ increases $\psi$ by $\Delta\ell_a\psi_a$ so that the quantity $(\psi - \ell_a\psi_a)$ on the right-hand side of (1) does not change. If $\ell_c$ is sufficiently large, the addition of any positive quantity $\Delta\ell_c$ to $\ell_c$ increases $\psi$ by $\Delta\ell_c\psi_c$ so that the quantity $(\psi - \ell_c\psi_c)$ on the right-hand side of (1) does not change. Therefore, the limit in (1) exists and is independent of I. Expression (1) for $L_e$ is similar to [1, Eq. (1)] for the excess inductance of a microstrip right-angle bend.

If the origin of the z coordinate were shifted a distance $\Delta z$ to the right and if the loop through which the flux $\psi$ threads were physically the same, then $\psi$ would not change, $\ell_a$ would increase by $\Delta z$, and $\ell_c$ would decrease by $\Delta z$. Therefore, the right-hand side of (1) would increase by the amount $\Delta z(\psi_c - \psi_a)/I$. Thus, as defined by (1), $L_e$ depends on the origin of the z coordinate.

## II. EXCESS INDUCTANCE IN TERMS OF MAGNETOSTATIC CURRENTS

In this section, the limit on the right-hand side of (1) is expressed in terms of magnetostatic currents. Consider the right-hand side of (1). Since all three quantities $\psi$, $\ell_a\psi_a$, and $\ell_c\psi_c$ approach infinity as $\ell_a$ and $\ell_c$ approach infinity, the two subtractions in (1) must be performed before passing to the limit. The flux $\psi$ is given by

$$\psi = \iint_{S_\ell} \underline{B}(\underline{J}) \cdot \underline{n}\, ds \tag{2}$$

where $\underline{J}$ is the magnetostatic current density on the entirety $(-\infty < z < \infty)$ of the strips and via. The amplitude of $\underline{J}$ is adjusted so that the total current flowing from left to right on the strips and via in Fig. 1 is I. In (2), $\underline{B}(\underline{J})$ is the magnetic field due to the existence of $\underline{J}$ in the presence of the ground plane, $S_\ell$ is a surface that caps the loop described in the paragraph that follows (1), $\underline{n}$ is a unit vector normal to $S_\ell$, and ds is the differential element of surface area. The vector $\underline{n}$ points into the paper in Fig. 1.

In the following manner, the integral over $S_\ell$ in (2) is transformed into an integral over C where C is a contour that goes from $(z = -\ell_a)$ on strip a through the via to $(z=\ell_c)$ on strip c. First, $\underline{B}(\underline{J})$ is expressed as

$$\underline{B}(\underline{J}) = \nabla \times \underline{A}(\underline{J}) \tag{3}$$

where $\underline{A}(\underline{J})$ is the magnetic vector potential due to the existence of $\underline{J}$ in the presence of the ground plane. Substitution of (3) into (2) yields

$$\psi = \iint_{S_\ell} (\nabla \times \underline{A}(\underline{J})) \cdot \underline{n}\, ds \tag{4}$$

Stokes' theorem [2, Eq. (42) on p. 489] reduces (4) to

$$\psi = \int_{C_\ell} \underline{A}(\underline{J}) \cdot d\underline{\ell} \tag{5}$$

where the contour $C_\ell$ is the loop described in the paragraph that follows (1). On the strips and via, $C_\ell$ proceeds from left to right in Fig. 1. In (5), $d\underline{\ell}$ is the differential of the radius vector on $C_\ell$. From the method of images, we have

$$\underline{A}_{tan}(\underline{J}) = 0 \quad \text{on the ground plane} \tag{6}$$

where the subscript tan denotes the component tangent to the ground plane. According to (6), the part of $C_\ell$ on the ground plane does not contribute to the integral in (5). Far from the via, $\underline{J}$ is entirely z directed and so is $\underline{A}$. Therefore, if $\ell_a$ and $\ell_c$ are sufficiently large, the parts of $C_\ell$ along the connecting lines at $(z = -\ell_a)$ and at $(z = \ell_c)$ do not contribute to the integral in (5). As a result, (5) reduces to

$$\psi = \int_C \underline{A}(\underline{J}) \cdot d\underline{\ell} \tag{7}$$

where C is defined in the first sentence of this paragraph.

The magnetostatic current density $\underline{J}$ exists on $S_\infty$ where $S_\infty$ is the entire surface $(-\infty < z < \infty)$ of the strips and via. The current $\underline{J}$ satisfies

$$\nabla_s \cdot \underline{J} = 0 \tag{8}$$

and

$$\underline{n} \cdot B(\underline{J}) = 0 \quad \text{on} \quad S_\infty \tag{9}$$

In (8), $\nabla_s \cdot \underline{J}$ is the surface divergence of $\underline{J}$. In (9), $\underline{n}$ is a unit vector normal to $S_\infty$, and $\underline{B}(\underline{J})$ is the magnetic field due to the existence of $\underline{J}$ in the presence of the ground plane. Equation (8) is written with the understanding that, at any edge of $S_\infty$ that is not at $(z = \pm\infty)$, $\underline{J}$ must be tangent to that edge. If $S_\infty$ is multiply connected because the via has holes in it, then the surface integral of (9) must be zero over each of the holes [3, Eq. (43)]. The total current associated with $\underline{J}$ is I. In Fig. 1, I flows from $(z = -\infty)$ on strip a through the via to $(z = \infty)$ on strip c. For this reason, $\underline{J}$ must have a z component at the edges of $S_\infty$ at $(z = \pm\infty)$. Therefore, $\underline{J}$ can not be tangent to the edges of $S_\infty$ at $(z = \pm\infty)$. Far from the via on strip a, $\underline{J}$ reduces to the current density which would exist on strip a if both strip c and the via were absent and if strip a extended from $(z = -\infty)$ to $(z = \infty)$. Far from the via on strip c, $\underline{J}$ reduces to the current density which would exist on strip c if both strip a and the via were absent and if strip c extended from $(z=-\infty)$ to $(z=\infty)$. If the value of I is specified, the conditions stated in this paragraph should suffice to determine $\underline{J}$.

In the following manner, the integral over C in (7) is transformed into an integral over S where S is the surface of the portion of the strips and via for which $(-\ell_a \leq z \leq \ell_c)$. Substitution of (3) for $\underline{B}(\underline{J})$ in (9) gives

$$\underline{n} \cdot \nabla \times \underline{A}(\underline{J}) = 0 \qquad \text{on } S_\infty \tag{10}$$

Equation (10) is also true on S because S is part of $S_\infty$. Since (10) holds on S, there is, according to (A-2), a scalar function $\Psi$ on S such that

$$\underline{A}_{\tan}(\underline{J}) = \nabla_s \Psi \qquad \text{on S} \tag{11}$$

where tan denotes the component tangent to S and $\nabla_s \Psi$ is the surface gradient of $\Psi$ on S. Because of (11), expression (7) for $\psi$ becomes

$$\psi = \int_C (\nabla_s \Psi) \cdot d\underline{\ell} \tag{12}$$

Equation (12) reduces to

$$\psi = \Psi_c - \Psi_a \tag{13}$$

where $\Psi_a$ is the value of $\Psi$ at the beginning of C at $(z=-\ell_a)$ on strip a, and $\Psi_c$ is the value of $\Psi$ at the end of C at $(z=\ell_c)$ on strip c.

If $\ell_a$ is sufficiently large, then $\underline{A}(\underline{J})$ is z directed on the $(z=-\ell_a)$ line on strip a, and $\nabla_s \Psi$ is also z directed there so that $\Psi$

is constant on the $(z=-\ell_a)$ line on strip a. If $\ell_c$ is sufficiently large, then $\underline{A}(\underline{J})$ is z directed on the $(z = \ell_c)$ line on strip c, and $\nabla_s \Psi$ is also z directed there so that $\Psi$ is constant on the $(z = \ell_c)$ line on strip c.

Let $\hat{\underline{J}}$ be an electric current density on S that satisfies

$$\nabla_s \cdot \hat{\underline{J}} = 0 \tag{14}$$

Equation (14) is written with the understanding that, at any edge of S that is neither at $(z=-\ell_a)$ nor at $(z = \ell_c)$, $\hat{\underline{J}}$ is tangent to that edge. It is assumed that $\hat{\underline{J}}$'s total current flowing from left to right in Fig. 1 is I. For this reason, $\hat{\underline{J}}$ must have a z component at the edges of S at $(z=-\ell_a)$ and at $(z = \ell_c)$. Therefore, $\hat{\underline{J}}$ cannot be tangent to the edges of S at $(z=-\ell_a)$ and at $(z = \ell_c)$. If $\ell_a$ and $\ell_c$ are sufficiently large so that $\Psi$ is constant on the $(z=-\ell_a)$ line on strip a and on the $(z = \ell_c)$ line on strip c, then [2, Eq. (42) on p. 503]

$$(\Psi_c - \Psi_a)I = \iint_S \nabla_s \cdot (\Psi\hat{\underline{J}})ds \tag{15}$$

where $\Psi_a$ is the value of $\Psi$ on the $(z=-\ell_a)$ line on strip a, and $\Psi_c$ is the value of $\Psi$ on the $(z = \ell_c)$ line on strip c. Substituting $(\Psi_c - \Psi_a)$ of (15) into (13), we obtain

$$\psi = \frac{1}{I} \iint_S \nabla_s \cdot (\Psi\hat{\underline{J}})ds \tag{16}$$

which becomes [2, Eq. (29) on p. 502]

$$\psi = \frac{1}{I} \iint_S (\nabla_s\Psi) \cdot \hat{\underline{J}}\, ds + \frac{1}{I} \iint_S \Psi(\nabla_s \cdot \hat{\underline{J}})ds \tag{17}$$

Equations (11) and (14) reduce (17) to

$$\psi = \frac{1}{I} \iint_S \underline{A}(\underline{J}) \cdot \hat{\underline{J}} \, ds \tag{18}$$

In contrast with (18), the expression for $\psi$ in terms of stored energy is [1, Eq. (4)]

$$\psi = \frac{1}{I} \iint_S \underline{A}(\underline{J}) \cdot \underline{J} \, ds \tag{19}$$

The current $\underline{J}$ is expressed as

$$\underline{J} = \underline{J}_{abc} + \underline{J}_e \tag{20}$$

where

$$\underline{J}_{abc} = \begin{cases} \underline{J}_a \text{ , on strip a,} & -\infty < z < z_a \\ \underline{J}_b \text{ , on the via,} & z_a \le z \le z_c \\ \underline{J}_c \text{ , on strip c,} & z_c < z < \infty \end{cases} \tag{21}$$

In (21), $\underline{J}_a$ is the current density that $\underline{J}$ tends toward far from the via on strip a, and $\underline{J}_c$ is the current density that $\underline{J}$ tends toward far from the via on strip c. Since the total current of $\underline{J}$ is I, the total current of $\underline{J}_a$ is I, and the total current of $\underline{J}_c$ is I. In Fig. 1, I flows from left to right. In (21), $\underline{J}_b$ is any solenoidal via current density that is tangent to the edges of S and that provides for a continuous flow of current from $\underline{J}_a$ on line a through the via to $\underline{J}_c$ on line c. Line a is the line at $(z=z_a)$ where the via connects with strip a. Line c is the line at $(z=z_c)$ where the via connects with strip c. Since the current flows continuously across line a, no electric charge can accumulate on line a so that, on line a, the component of $\underline{J}_b$ normal to line a entering the via is equal to the z component of $\underline{J}_a$. Similarly, the component of $\underline{J}_b$ normal to line c leaving the via is equal to the z component of $\underline{J}_c$. With $\underline{J}_a$, $\underline{J}_b$, and $\underline{J}_c$ so defined, $\underline{J}_{abc}$ is solenoidal and is tangent to any edge of $S_\infty$ not at $(z = \pm\infty)$. The total current of $\underline{J}_{abc}$ is I. This current flows continuously from $(z = -\infty)$ on strip a through the via to $(z = \infty)$ on strip c. Since both $\underline{J}$ and $\underline{J}_{abc}$ are solenoidal, are tangent to any edge of $S_\infty$ not at $(z = \pm\infty)$, and have a total current of I, the excess current $\underline{J}_e$ in (20) is solenoidal, is tangent to any edge of $S_\infty$ not at $(z = \pm\infty)$, and has no total current associated with it. Moreover, $\underline{J}_e$ tends toward zero far from the via on strip a and far from the via on strip c.

Substituting $\underline{J}_{abc}$ for $\hat{\underline{J}}$ and (20) for $\underline{J}$ in (18), we obtain

$$\psi = \frac{1}{I} \iint_S \underline{A}(\underline{J}_{abc}) \cdot \underline{J}_{abc} ds + \frac{1}{I} \iint_S \underline{A}(\underline{J}_e) \cdot \underline{J}_{abc} ds \tag{22}$$

In the following manner, an alternative to (21) is obtained for substitution into the first integral in (22). In (21), $\underline{J}_a$ is the current density that would exist on strip a if both strip c and the via were absent and if strip a extended from $(z = -\infty)$ to $(z = \infty)$. In (21), $\underline{J}_c$ is the current density that would exist on strip c if both strip a and the via were absent and if strip c extended from $(z = -\infty)$ to $(z = \infty)$. Taking $\underline{J}_a$ and $\underline{J}_c$ to extend from $(z = -\infty)$ to $(z = \infty)$, we have

$$\underline{J}_a = \underline{J}_a^- + \underline{J}_a^+ \tag{23}$$

$$\underline{J}_c = \underline{J}_c^- + \underline{J}_c^+ \tag{24}$$

where

$$\underline{J}_a^- = \underline{J}_a \quad , \quad -\infty < z < z_a \tag{25a}$$

$$\underline{J}_a^+ = \underline{J}_a \quad , \quad z_a \leq z < \infty \tag{25b}$$

$$\underline{J}_c^- = \underline{J}_c \quad , \quad -\infty < z \leq z_c \tag{25c}$$

$$\underline{J}_c^+ = \underline{J}_c \quad , \quad z_c < z < \infty \tag{25d}$$

Using (25a) and (25d) and assuming that $\underline{J}_b$ exists only on the via, we can recast (21) as

$$\underline{J}_{abc} = \underline{J}_a^- + \underline{J}_b + \underline{J}_c^+ \tag{26}$$

Substitution of (26) for $\underline{J}_{abc}$ in the first integral in (22) yields

$$\psi = \frac{1}{I} \iint_S (\underline{A}(\underline{J}_b + \underline{J}_c^+) \cdot \underline{J}_a^- + \underline{A}(\underline{J}_a^- + \underline{J}_b + \underline{J}_c^+) \cdot \underline{J}_b + \underline{A}(\underline{J}_a^- + \underline{J}_b) \cdot \underline{J}_c^+) ds$$

$$+ T_a + T_c + \frac{1}{I} \iint_S \underline{A}(\underline{J}_e) \cdot \underline{J}_{abc} \, ds \tag{27}$$

where

$$T_a = \frac{1}{I} \iint_S \underline{A}(\underline{J}_a^-) \cdot \underline{J}_a^- \, ds \tag{28}$$

$$T_c = \frac{1}{I} \iint_S \underline{A}(\underline{J}_c^+) \cdot \underline{J}_c^+ \, ds \tag{29}$$

Substituting the solution of (23) for $\underline{J}_a^-$ in the argument of $\underline{A}$ in (28), we obtain

$$T_a = \frac{1}{I} \iint_S \underline{A}(\underline{J}_a) \cdot \underline{J}_a^- ds - \frac{1}{I} \iint_S \underline{A}(\underline{J}_a^+) \cdot \underline{J}_a^- \, ds \tag{30}$$

In the first integral in (30), the domain of $\underline{J}_a^-$ is extended to include $(z = z_a)$, and $\underline{J}_a^-$ is taken to be equal to $\underline{J}_a$ at $(z = z_a)$. Now, the first integral in (30) is viewed as being over the extended domain of $\underline{J}_a^-$ rather than over all of S. From this viewpoint, the first term in (30) is expression (18) with S replaced by the portion of strip a for which $(-\ell_a \leq z \leq z_a)$, $\underline{J}$ replaced by $\underline{J}_a$, and $\hat{\underline{J}}$ replaced by $\underline{J}_a^-$. With these replacements, (18) is valid representation for the flux $(\ell_a + z_a)\ \psi_a$ where $\psi_a$, the magnetic flux per unit length of strip a far from the via, appears in (1). With its first term so disposed, (30) reduces to

$$T_a = (\ell_a + z_a)\psi_a - \frac{1}{I}\iint_S \underline{A}(\underline{J}_a^+) \cdot \underline{J}_a^- \, ds \tag{31}$$

Substituting the solution of (24) for $\underline{J}_c^+$ in the argument of $\underline{A}$ in (29), we obtain

$$T_c = \frac{1}{I}\iint_S \underline{A}(\underline{J}_c) \cdot \underline{J}_c^+ \, ds - \frac{1}{I}\iint_S \underline{A}(\underline{J}_c^-) \cdot \underline{J}_c^+ \, ds \tag{32}$$

In the first integral in (32), the domain of $\underline{J}_c^+$ is extended to include $(z = z_c)$, and $\underline{J}_c^+$ is taken to be equal to $\underline{J}_c$ at $(z = z_c)$. Now, the first integral in (32) is viewed as being over the extended domain of $\underline{J}_c^+$ rather than over all of S. From this viewpoint, the first term in (32) is expression (18) with S replaced by the portion of strip c for which $(z_c \leq z \leq \ell_c)$, $\underline{J}$ replaced by $\underline{J}_c$, and $\hat{\underline{J}}$ replaced by $\underline{J}_c^+$. With these replacements, (18) is a valid representation for the flux $(\ell_c - z_c)\psi_c$ where $\psi_c$, the magnetic flux per unit length of strip c far from the via, appears in (1). With its first term so disposed, (32) reduces to

$$T_c = (\ell_c - z_c)\psi_c - \frac{1}{I}\iint_S \underline{A}(\underline{J}_c^-) \cdot \underline{J}_c^+ \, ds \tag{33}$$

Substituting (31) and (33) into (27) and then using (26), we obtain

$$\psi = \frac{1}{I}\iint_S [\underline{A}(\underline{J}_c^+ - \underline{J}_a^+ + \underline{J}_e + \underline{J}_b) \cdot \underline{J}_a^- + \underline{A}(\underline{J}_a^- - \underline{J}_c^- + \underline{J}_e + \underline{J}_b) \cdot \underline{J}_c^+ + \underline{A}(\underline{J}_a^- + \underline{J}_c^+ + \underline{J}_e + \underline{J}_b) \cdot \underline{J}_b] ds$$

$$+ (\ell_a + z_a)\psi_a + (\ell_c - z_c)\psi_c \tag{34}$$

When (34) is substituted into (1), the flux $(\ell_a\psi_a + \ell_c\psi_c)$ cancels and we are left with

$$L_e = \lim_{\substack{\ell_a \to \infty \\ \ell_c \to \infty}} \left[ \frac{1}{I^2} \iint_S (\underline{A}(\underline{J}_c^+ - \underline{J}_a^+ + \underline{J}_e + \underline{J}_b) \cdot \underline{J}_a^- + \underline{A}(\underline{J}_a^- - \underline{J}_c^- + \underline{J}_e + \underline{J}_b) \cdot \underline{J}_c^+ + \underline{A}(\underline{J}_a^- + \underline{J}_c^+ + \underline{J}_e + \underline{J}_b) \cdot \underline{J}_b) ds \right.$$

$$\left. + \frac{z_a\psi_a - z_c\psi_c}{I} \right] \tag{35}$$

where S is the surface of the portion of the strips and via for which $(-\ell_a \le z \le \ell_c)$.

If $\ell_a$ increases by the amount $\Delta\ell_a$, then the quantity in brackets on the right-hand side of (35) increases by

$$\frac{1}{I^2} \iint_{S_{\Delta\ell a}} \underline{A}(\underline{J}_c^+ - \underline{J}_a^+ + \underline{J}_e + \underline{J}_b) \cdot \underline{J}_a^- \, ds \tag{36}$$

where $S_{\Delta\ell a}$ is the portion of strip a for which $(-\ell_a - \Delta\ell_a \le z \le -\ell_a)$. If $\ell_a$ is large and if $\Delta\ell_a$ is positive, then the vector potential $\underline{A}$

in (36) is small on $S_{\Delta\ell a}$ because its current source is distant from $S_{\Delta\ell a}$. Moreover, it is conjectured that, if $\ell_a$ is sufficiently large, then expression (36) is negligibly small for $(0 < \Delta\ell_a < \infty)$.

If $\ell_c$ increases by the amount $\Delta\ell_c$, then the quantity in brackets on the right-hand side of (35) increases by

$$\frac{1}{I^2} \iint\limits_{S_{\Delta\ell c}} \underline{A}(\underline{J}_a^- - \underline{J}_c^- + \underline{J}_e + \underline{J}_b) \cdot \underline{J}_c^+ ds \tag{37}$$

where $S_{\Delta\ell c}$ is the portion of strip c for which $(\ell_c \leq z \leq \ell_c + \Delta\ell_c)$. If $\ell_c$ is large and if $\Delta\ell_c$ is positive, then the vector potential $\underline{A}$ in (37) is small on $S_{\Delta\ell c}$ because its current source is distant from $S_{\Delta\ell c}$. Moreover, it is conjectured that, if $\ell_c$ is sufficiently large, then expression (37) is negligibly small for $(0 < \Delta\ell_c < \infty)$.

If the conjectures in the previous two paragraphs are true, then the limit in (35) exists. In this case, a good approximation to $L_e$ is given by

$$L_e = \frac{1}{I^2} \iint\limits_{S} (\underline{A}(\underline{J}_c^+ - \underline{J}_a^+ + \underline{J}_e + \underline{J}_b) \cdot \underline{J}_a^- + \underline{A}(\underline{J}_a^- - \underline{J}_c^- + \underline{J}_e + \underline{J}_b) \cdot \underline{J}_c^+ + \underline{A}(\underline{J}_a^- + \underline{J}_c^+ + \underline{J}_e + \underline{J}_b) \cdot \underline{J}_b) ds$$

$$+ \frac{z_a \psi_a - z_c \psi_c}{I} \tag{38}$$

where S is the surface of the portion of the strips and via for which $(-\ell_a \leq z \leq \ell_c)$ where $\ell_a$ and $\ell_c$ are sufficiently large. In (38), $\underline{J}_a^{\pm}$ and $\underline{J}_c^{\pm}$ are given by (25). Furthermore, $\underline{J}_a$, $\underline{J}_c$, $\underline{J}_e$, and $\underline{J}_b$ are described in the paragraph that contains (21). In (38), $\underline{A}(\underline{J}_c^+ - \underline{J}_a^+ + \underline{J}_e + \underline{J}_b)$ is the magnetic vector potential due to the existence of $(\underline{J}_c^+ - \underline{J}_a^+ + \underline{J}_e + \underline{J}_b)$ in the presence of the ground plane. The remaining $\underline{A}$'s in (38) are similarly defined. Moreover, $\psi_a$ is the magnetic flux per unit length of strip a far from the via, and $\psi_c$ is the magnetic flux per unit length of strip c far from the via. Otherwise stated, $\psi_a/I$ is the inductance per unit length of strip a far from the via, and $\psi_c/I$ is the inductance per unit length of strip c far from the via. In Section III, the fluxes $\psi_a$ and $\psi_c$ will emerge as by-products of the solutions for $\underline{J}_a$ and $\underline{J}_c$. The right-hand side of (38) does not depend on I because the $\underline{A}$'s, the $\underline{J}$'s, and the $\psi$'s in (38) are proportional to I.

## III. INTEGRAL EQUATION FORMULATION

The objective of Section III is to derive integral equations for $\underline{J}_a$, $\underline{J}_c$, and $\underline{J}_e$ and to obtain expressions for $\psi_a$ and $\psi_c$.

By definition, $\underline{J}_a$ is the current density that $\underline{J}$ tends toward far from the via on strip a. Equivalently, $\underline{J}_a$ is what $\underline{J}$ would be if both strip c and the via were absent and if strip a extended from $(z = -\infty)$ to $(z = \infty)$. Hence, the problem of determining $\underline{J}_a$ is a two-dimensional one. The situation is shown in Fig. 3. Since $\underline{J}$'s total current flowing from left to right in Fig. 1 is I, $\underline{J}_a$'s total z-directed current is I.

Because $\underline{J}$ satisfies (8) and (9), $\underline{J}_a$ must satisfy

$$\nabla_s \cdot \underline{J}_a = 0 \tag{39}$$

and

$$\underline{u}_y \cdot \underline{B}(\underline{J}_a) = 0 \quad \text{on strip a} \tag{40}$$

In (40), $\underline{u}_y$ is the unit vector in the y direction, and $\underline{B}(\underline{J}_a)$ is the

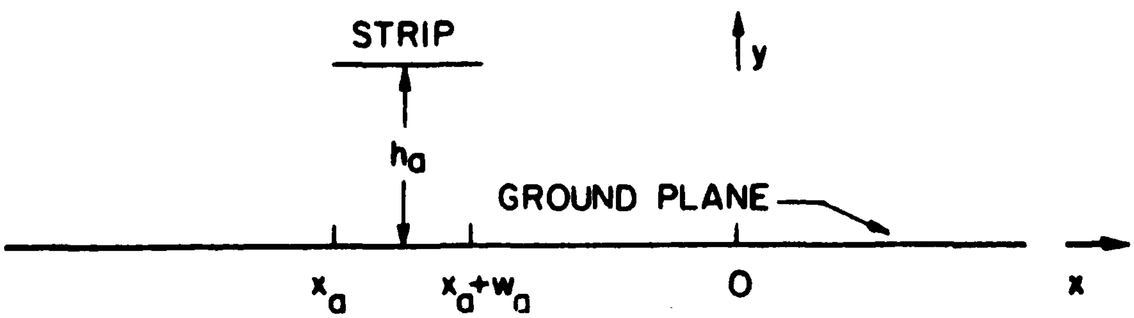


Fig. 3. An infinitely long perfectly conducting strip of width $w_a$ at height $h_a$ above a perfectly conducting ground plane. On the strip, there is a magnetostatic current density whose total z-directed current is 1.

magnetic field due to the existence of $\underline{J}_a$ in the presence of the ground plane. Equation (39) is written with the understanding that, at any one of the two lateral edges of strip a, $\underline{J}_a$ must be tangent to that edge.

The magnetic field $\underline{B}(\underline{J}_a)$ that appears in (40) is expressed as

$$\underline{B}(\underline{J}_a) = \nabla \times \underline{A}(\underline{J}_a) \tag{41}$$

where $\underline{A}(\underline{J}_a)$ is the magnetic vector potential due to the existence of $\underline{J}_a$ in the presence of the ground plane. If $\underline{J}_a$ was an arbitrarily directed current above the ground plane, then

$$\underline{A}(\underline{J}_a) = \frac{\mu}{4\pi} \iint_{\text{strip a}} \left[\frac{\underline{J}_a(\underline{r}')}{|\underline{r} - \underline{r}'|} + \frac{\underline{u}_y J_{ay}(\underline{r}') - \underline{u}_x J_{ax}(\underline{r}') - \underline{u}_z J_{az}(\underline{r}')}{|\underline{r} - \underline{r}''|}\right] ds' \tag{42}$$

where $(\underline{u}_x, \underline{u}_y, \underline{u}_z)$ are the unit vectors in the (x,y,z) directions, and $(J_{ax}, J_{ay}, J_{az})$ are the (x,y,z) components of $\underline{J}_a$. In (42), ds' is the differential element of area at $\underline{r}'$ on strip a, $\underline{r}''$ is the image of $\underline{r}'$ about the ground plane, and $\underline{r}$ is the point at which $\underline{A}(\underline{J}_a)$ is evaluated.

Because the problem of determining $\underline{J}_a$ is two-dimensional, $\underline{J}_a$ is independent of z so that

$$\underline{J}_a = J_{ax}(x)\underline{u}_x + J_{az}(x)\underline{u}_z \tag{43}$$

Now, (39) expands to [2, Eq. (18) on p. 501]

$$\frac{\partial J_{ax}(x)}{\partial x} + \frac{\partial J_{az}(x)}{\partial z} = 0 \tag{44}$$

Since $J_{az}$ does not depend on z, (44) implies that $J_{ax}$ does not depend on x. Moreover, $J_{ax}$ is zero at $(x = x_a)$ and at $(x = x_a + w_a)$ because $\underline{J}_a$ was constrained to be tangent to the lateral edges of strip a. Therefore, $J_{ax}$ is zero everywhere on strip a. Hence, (43) simplifies to

$$\underline{J}_a = J_{az}(x)\underline{u}_z \tag{45}$$

Equation (45) reduces (42) to

$$\underline{A}(\underline{J}_a) = A_z(\underline{J}_a)\underline{u}_z \tag{46}$$

where

$$A_z(\underline{J}_a) = \frac{\mu}{4\pi}\int_{x_a}^{x_a+w_a} dx' J_{az}(x') \int_{-\infty}^{\infty} dz' \left(\frac{1}{\sqrt{(x-x')^2+(y-h_a)^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y+h_a)^2+(z-z')^2}}\right) \tag{47}$$

In (47), (x,y,z) are the rectangular coordinates of the point at which $A_z(\underline{J}_a)$ is evaluated.

If the integral with respect to z' in (47) is called $I_{z1}$, then [4, Formula 200.01]

$$I_{z1} = \lim_{\substack{\alpha\to\infty \\ \beta\to\infty}} \left[\ln\left(\frac{\gamma + \sqrt{\gamma^2 + (x-x')^2 + (y-h_a)^2}}{\gamma + \sqrt{\gamma^2 + (x-x')^2 + (y+h_a)^2}}\right)\right]_{\gamma=-\alpha}^{\gamma=\beta} \tag{48}$$

The logarithm in (48) vanishes at its upper limit so that (48) reduces to

$$I_{z1} = \lim_{\alpha\to\infty} \ln\left(\frac{-\alpha + \sqrt{\alpha^2 + (x-x')^2 + (y+h_a)^2}}{-\alpha + \sqrt{\alpha^2 + (x-x')^2 + (y-h_a)^2}}\right) \tag{49}$$

Expression (49) is rewritten as

$$I_{z1} = \lim_{\alpha\to\infty} \ln \left[ \left(\frac{(x-x')^2 + (y+h_a)^2}{(x-x')^2 + (y-h_a)^2}\right) \left(\frac{\alpha + \sqrt{\alpha^2 + (x-x')^2 + (y-h_a)^2}}{\alpha + \sqrt{\alpha^2 + (x-x')^2 + (y+h_a)^2}}\right) \right] \quad (50)$$

The function of $\alpha$ in parentheses in (50) approaches unity so that (50) becomes

$$I_{z1} = \ln \left(\frac{(x-x')^2 + (y+h_a)^2}{(x-x')^2 + (y-h_a)^2}\right) \quad (51)$$

Substituting expression (51) for the integral with respect to z' in (47), we obtain

$$A_z(\underline{J}_a) = \frac{\mu}{4\pi} \int_{x_a}^{x_a+w_a} J_{az}(x') \ln \left(\frac{(x-x')^2 + (y+h_a)^2}{(x-x')^2 + (y-h_a)^2}\right) dx' \quad (52)$$

In view of (46), substitution of (41) into (40) gives

$$\underline{u}_y \cdot \nabla \times (A_z(\underline{J}_a)\underline{u}_z) = 0 \quad \textit{on strip a} \quad (53)$$

Equation (53) reduces to [2, Eq. (51) on p. 490]

$$\frac{\partial A_z(\underline{J}_a)}{\partial x} = 0 \quad \text{on strip a} \quad (54)$$

It is evident from (52) and (54) that $A_z(\underline{J}_a)$ is constant on strip a. Hence, using (52), we have

$$\int_{x_a}^{x_a+w_a} J_{az}(x') \ln \left(\frac{(x-x')^2 + (2h_a)^2}{(x-x')^2}\right) dx' = \alpha, \quad x_a \le x \le x_a+w_a \quad (55)$$

where $\alpha$ is a constant that is determined by requiring $J_{az}$'s total z-directed current to be I.

$$\int_{x_a}^{x_a+w_a} J_{az}(x')dx' = I \tag{56}$$

Equation (55) is rewritten as

$$\int_{x_a}^{x_a+w_a} \hat{J}_{az}(x') \ln\left(\frac{(x-x')^2 + (2h_a)^2}{(x-x')^2}\right)dx' = 1, \quad x_a \le x \le x_a + w_a \tag{57}$$

where

$$\hat{J}_{az}(x') = \frac{1}{\alpha} J_{az}(x') \tag{58}$$

From (58), we obtain

$$J_{az}(x') = \alpha \hat{J}_{az}(x') \tag{59}$$

Substituting the right-hand side of (59) for $J_{az}(x')$ in (56), we find that

$$\alpha = \frac{I}{\int_{x_a}^{x_a+w_a} \hat{J}_{az}(x')\, dx'} \tag{60}$$

Substitution of (60) into (59) with x' replaced by x gives

$$J_{az}(x) = \frac{I\, \hat{J}_{az}(x)}{\int_{x_a}^{x_a+w_a} \hat{J}_{az}(x')dx'} \tag{61}$$

where $\hat{J}_{az}(x')$ satisfies (57).

The flux $\psi_a$ is given, in analogy with (7), by

$$\psi_a = \int \underline{A}(\underline{J}_a) \cdot d\underline{\ell} \tag{62}$$

where the line integral in (62) is over a unit length of strip a in the z direction. Substituting (46) into (62), we obtain

$$\psi_a = A_z(\underline{J}_a) \tag{63}$$

where $A_z(\underline{J}_a)$ is evaluated on strip a. Expression (52) gives

$$A_z(\underline{J}_a) = \frac{\mu}{4\pi} \int_{x_a}^{x_a+w_a} J_{az}(x') \ln \left(\frac{(x-x')^2 + (2h_a)^2}{(x-x')^2}\right) dx' \quad \text{on strip a} \tag{64}$$

Substituting (61) into (64) and using (57) and (63), we obtain

$$\psi_a = \frac{\mu I}{4\pi \int_{x_a}^{x_a+w_a} \hat{J}_{az}(x') dx'} \tag{65}$$

where $\hat{J}_{az}(x')$ satisfies (57).

The inductance per unit length of strip a is called $L_a$ and is defined by

$$L_a = \frac{\psi_a}{I} \tag{66}$$

Substituting (65) into (66), we obtain

$$L_a = \frac{\mu}{4\pi \int_{x_a}^{x_a+w_a} \hat{J}_{az}(x')\, dx'} \tag{67}$$

where $\hat{J}_{az}(x')$ satisfies (57)

The capacitance per unit length of strip a in [5] is called $C_a$ and is defined by

$$C_a = \frac{Q_a}{V} \tag{68}$$

where

$$Q_a = \int_{x_a}^{x_a+w_a} q_a(x')dx' \tag{69}$$

In (69), $q_a(x')$ satisfies [5, Eq. (9)]. Substitution of (69) into (68) gives

$$C_a = \frac{1}{V}\int_{x_a}^{x_a+w_a} q_a(x')dx' \tag{70}$$

where $q_a(x')$ satisfies [5, Eq. (9)]. If

$$V = \frac{1}{4\pi\varepsilon} \tag{71}$$

where $\varepsilon$ is the permittivity above the ground plane in [5], then comparison of [5, Eq. (9)] with (57) gives

$$q_a(x') = \hat{J}_{az}(x') \tag{72}$$

Because of (71) and (72), the product of the right-hand sides of (67) and (70) is $\mu\varepsilon$ so that

$$L_a C_a = \mu\varepsilon \tag{73}$$

Relationship (73) is well-known for a two conductor lossless transmission line [6, Eq. (10b) on p. 123].

By definition, $\underline{J}_c$ is the current density that $\underline{J}$ tends toward far from the via on strip c. Equivalently, $\underline{J}_c$ is what $\underline{J}$ would be if both strip a and the via were absent and if strip c extended from $(z=-\infty)$ to $(z=\infty)$. Hence, the problem of determining $\underline{J}_c$ is a two-dimensional one. The result of a development similar to (39)-(61) is that

$$\underline{J}_c = J_{cz}(x)\underline{u}_z \tag{74}$$

where

$$J_{cz}(x) = \frac{I\,\hat{J}_{cz}(x)}{\int_{x_c}^{x_c+w_c} \hat{J}_{cz}(x')dx'} \tag{75}$$

where $\hat{J}_{cz}(x')$ satisfies

$$\int_{x_c}^{x_c+w_c} \hat{J}_{cz}(x') \ln \left(\frac{(x-x')^2 + (2h_c)^2}{(x-x')^2}\right) dx' = 1, \quad x_c \le x \le x_c+w_c \tag{76}$$

The analogue of (65) is

$$\psi_c = \frac{\mu I}{4\pi \int_{x_c}^{x_c+w_c} \hat{J}_{cz}(x')dx'} \tag{77}$$

where $\hat{J}_{cz}(x')$ satisfies (76).

Now, the objective is to derive an integral equation for $\underline{J}_e$. In (20), $\underline{J}$ is solenoidal, is tangent to all the edges of $S_\infty$ except those at $(z=\pm\infty)$, and its total current flowing from left to right in Fig. 1 is I. Moreover, $\underline{J}_{abc}$ is solenoidal, is tangent to all the edges of S except those at $(z=\pm\infty)$, and its total current is I. Therefore, $\underline{J}_e$ is solenoidal, is tangent to all the edges of $S_\infty$ with the possible exception of those at ($z=\pm\infty$), and has no total current associated with it. Since $\underline{J}_e$ tends toward zero far from the via on strip a and far from the via on strip c, $\underline{J}_e$ is zero on the edges of $S_\infty$ that occur at ($z=\pm\infty$). Hence, $\underline{J}_e$ is tangent to all the edges of $S_\infty$. The solenoidal property of $\underline{J}_e$ is expressed as

$$\nabla_s \cdot \underline{J}_e = 0 \tag{78}$$

Substitution of (20) into (9) gives

$$\underline{n} \cdot \underline{B}(\underline{J}_e) = -\,\underline{n} \cdot \underline{B}(\underline{J}_{abc}) \quad \text{on} \quad S_\infty \tag{79}$$

where $\underline{B}(\underline{J}_e)$ is the magnetic field due to the existence of $\underline{J}_e$ in the presence of the ground plane, and $\underline{B}(\underline{J}_{abc})$ is the magnetic field due to the existence of $\underline{J}_{abc}$ in the presence of the ground plane.

Because of (78), there exists a scalar function $u(\underline{r})$ such that [3, Eq. (B-1)]

$$\underline{J}_e(\underline{r}) = \underline{n} \times \nabla_s\, u(\underline{r}) \tag{80}$$

where $\nabla_s$ is the surface gradient on $S_\infty$. Since $\underline{J}_e$ is tangent to each edge of $S_\infty$, (80) implies that u is constant on each edge of $S_\infty$. If the via is a simple strip, then $S_\infty$ has, in the domain of finite z, only two lateral edges. In this case (80) predicts that the value of u will be the same on both of these edges because $\underline{J}_e$ has no total current associated with it. In general, however, the value of u on one edge of $S_\infty$ is not necessarily the same as that on any other edge of $S_\infty$. Only derivatives of u appear in (80). Hence, if (80) is substituted into (79), then (79) can only determine u to within an arbitrary additive constant. To make u unique, we specify (u = 0) at an arbitrary point on S . If the via is a simple strip, it is natural to specify (u = 0) at a point on one of the two lateral edges of $S_\infty$ because this specification will render (u = 0) on the entirety of both lateral edges of $S_\infty$. In summary, the boundary conditions on u in (80) are that u is constant on each edge of $S_\infty$ and that (u = 0) at some point on $S_\infty$.

In (79), $\underline{B}(\underline{J}_e)$ is expressed as

$$\underline{B}(\underline{J}_e) = \nabla \times \underline{A}(\underline{J}_e) \tag{81}$$

where

$$\underline{A}(\underline{J}_e) = \frac{\mu}{4\pi} \iint_{S_\infty} \left[\frac{\underline{J}_e(\underline{r}')}{|\underline{r}-\underline{r}'|} + \frac{\underline{u}_y J_{ey}(\underline{r}') - \underline{u}_x J_{ex}(\underline{r}') - \underline{u}_z J_{ez}(\underline{r}')}{|\underline{r}-\underline{r}''|}\right] ds' \tag{82}$$

where $(J_{ex}, J_{ey}, J_{ez})$ are the (x,y,z) components of $\underline{J}_e$. In (82), $\underline{r}'$ is the point at which the differential element of area ds' is located, $\underline{r}''$ is the image of $\underline{r}'$ about the ground plane, and $\underline{r}$ is the point at which $\underline{A}(\underline{J}_e)$ is evaluated. Equations (81) and (82) are also valid with $\underline{J}_e$ replaced by $\underline{J}_{abc}$. With $\underline{B}$ given by (81), (79) becomes

$$\underline{n} \cdot \nabla \times \underline{A}(\underline{J}_e) = -\,\underline{n} \cdot \nabla \times \underline{A}(\underline{J}_{abc}) \quad \text{on } S_\infty \tag{83}$$

where $\underline{A}$ is given by (82), and $\underline{J}_e$ is given by (80) in which u is an unknown scalar that is constant on each edge of $S_\infty$ and whose value is zero at some point on $S_\infty$. If $S_\infty$ is multiply connected because the via has holes in it, then (83) must be accompanied by the auxiliary condition that the surface integral of the left-hand side of (83) over any one of the holes must equal the surface integral of the right-hand side of (83) over that hole.

Equation (83) represents a surface integral equation for u. If (83) can be solved for u, then $\underline{J}_e$ will be given by (80). Rigorously, the domains of both u and the point $\underline{r}$ at which both sides of (83) are observed extend to $(z=-\infty)$ in negative z direction and to $(z = \infty)$ in the positive z direction. However, as $|z|$ grows, $\nabla_s u$ approaches zero, and both sides of (83) approach zero. Since $\nabla_s u$ approaches zero as $|z|$ grows, the point at which $(u = 0)$ can be chosen so that u approaches zero as $|z|$ grows. In Section IV, a numerical solution for u is obtained by truncating both the unknown u and its integral equation at a finite negative value of z and at a finite positive value of z.

## IV. SOLUTION BY THE METHOD OF MOMENTS

In Section IV, the method of moments is used to numerically solve the integral equations (57), (76), and (83) for $J_{az}$, $J_{cz}$, and u, respectively. After these equations have been solved, $\underline{J}_a$ can be obtained by substituting (61) into (45), $\underline{J}_c$ can be obtained by substituting (75) into (74), and $\underline{J}_e$ can be obtained by simply substituting u into (80). The quantities $\underline{J}_a^-$, $\underline{J}_a^+$, $\underline{J}_c^-$, $\underline{J}_c^+$, $\psi_a$, and $\psi_c$ in expression (38) for the excess inductance $L_e$ will then be given by (25a), (25b), (25c), (25d), (65), and (77), respectively.

The moment solution for $J_{az}$ is constructed by expanding $J_{az}$ as a linear combination of pulse functions and by point matching (57) at the centers of the domains of the pulses. The moment solution for $\hat{J}_{cz}$ is constructed in the same manner. The moment solution for u is constructed by truncating $S_\infty$ at $(z = -\ell_a)$ on the negative z axis and at $(z = \ell_c)$ on the positive z axis, calling the truncated surface S, modeling S by triangular patches [7], calling the resulting triangulated surface $S_T$, expanding u as a linear combination of pyramid functions on $S_T$, and testing (83) with the pyramid functions. Such testing consists of multiplying (83) by each of the pyramid functions successively and integrating over $S_T$. In the resulting simultaneous equations, the derivatives on $\underline{A}$ are transferred to the pyramid testing functions, and the integrals over the pyramid testing functions are approximated by sampling $\underline{A}$ at the centroids of the triangular patches.

To construct the moment solution of (57) for $\hat{J}_{az}$, we approximate $\hat{J}_{az}$ by

$$\hat{J}_{az}(x') = \sum_{j=1}^{N_a} I_{aj} P_a(x' - x_{aj}) \tag{84}$$

where

$$x_{aj} = x_a + (j-1)\Delta_a, \quad j=1,2,\ldots,N_a+1 \tag{85}$$

$$\Delta_a = w_a/N_a \tag{86}$$

In (84), $P_a(x)$ is the pulse function defined by

$$P_a(x) = \begin{cases} 1, & 0 \leq x < \Delta_a \\ 0, & \text{otherwise} \end{cases} \tag{87}$$

Not used in (84), the quantity $x_{a,N_a+1}$ of (85) is the x coordinate of the farther edge of strip a. This quantity is used later.

Next, $N_a$ equations involving the unknown constants $\{I_{aj}\}$ are obtained by substituting (84) into (57) and enforcing (57) at the centers of the domains of the pulses in (84). These equations are written in matrix notation as

$$P_a \vec{I}_a = \vec{V}_a \tag{88}$$

where $\vec{I}_a$ and $\vec{V}_a$ are column vectors. The jth element of $\vec{I}_a$ is $I_{aj}$, and each element of $\vec{V}_a$ is unity. In (88), $P_a$ is a square matrix. Its ijth element is given by

$$P_{aij} = \int_{x_{aj}}^{x_{a,j+1}} \ln \left(\frac{(x^+_{ai} - x')^2 + (2h_a)^2}{(x^+_{ai} - x')^2}\right) dx' \quad , \quad \begin{cases} i=1,2,\ldots,N_a \\ j=1,2,\ldots,N_a \end{cases} \tag{89}$$

where

$$x^+_{ai} = x_a + (i - \tfrac{1}{2})\Delta_a \tag{90}$$

Expression (89) for $P_{aij}$ becomes [5, Eq. (B-8)]

$$P_{aij} = \left[ x \ln \left(1 + \left(\frac{2h_a}{x}\right)^2\right) + 4h_a \tan^{-1} \left(\frac{x}{2h_a}\right) \right]_{(|j-i| - \frac{1}{2})\Delta_a}^{(|j-i|+\frac{1}{2})\Delta_a} \tag{91}$$

The moment solution of (57) for $\hat{J}_{az}$ is completed by solving (88) for $\vec{I}_a$ and substituting the elements of $\vec{I}_a$ into expression (84) for $\hat{J}_{az}$.

Similar to the moment solution of (57) for $\hat{J}_{az}$, the moment solution of (76) for $\hat{J}_{cz}$ is given by

$$\hat{J}_{cz}(x') = \sum_{j=1}^{N_c} I_{cj} P_c (x' - x_{cj}) \tag{92}$$

where

$$x_{cj} = x_c + (j-1)\Delta_c, \quad j=1,2,\ldots,N_c+1 \tag{93}$$

$$\Delta_c = w_c/N_c \tag{94}$$

$$P_c(x) = \begin{cases} 1, & 0 \leq x < \Delta_c \\ 0, & \text{otherwise} \end{cases} \tag{95}$$

Not used in (92), the quantity $x_{c,N_c+1}$ of (93) is the x coordinate of the farther edge of strip c. In (92), $I_{cj}$ is the jth element of the column vector $\vec{I}_c$ that satisfies

$$P_c\vec{I}_c = \vec{V}_c \tag{96}$$

Here, $\vec{V}_c$ is a column vector of $N_c$ elements. Each element of $\vec{V}_c$ is unity. In (96), $P_c$ is a square matrix of order $N_c$. In analogy with (91), the ijth element of $P_c$ is given by

$$P_{cij} = \left[ x \ln\left(1 + \left(\frac{2h_c}{x}\right)^2\right) + 4h_c \tan^{-1}\left(\frac{x}{2hc}\right) \right]_{(|j-i| - \frac{1}{2})\Delta_c}^{(|j-i| + \frac{1}{2})\Delta_c} \tag{97}$$

Before constructing the moment solution of (83) for u, we will show that the solution u to (83) is proportional to I. Certainly, u will be proportional to I if $\underline{J}_{abc}$ on the right-hand side of (83) is proportional to I. According to (21), $\underline{J}_{abc}$ consists of the part of $\underline{J}_a$ for which $(-\infty < z < z_a)$, $\underline{J}_b$ on the via, and the part of $\underline{J}_c$ for which $(z_c < z < \infty)$. Because of (45) and (61), $\underline{J}_a$ is proportional to I. Because of (74) and (75), $\underline{J}_c$ is proportional to I. Since $\underline{J}_b$ is any solenoidal via current density that is tangent to the edges of S and that provides for a continuous flow of current from $\underline{J}_a$ on strip a to $\underline{J}_c$ on strip c, $\underline{J}_b$ can be chosen to be proportional to I. With $\underline{J}_b$ so chosen, each of the currents $\underline{J}_a$, $\underline{J}_b$, and $\underline{J}_c$ is proportional to I. Therefore, $\underline{J}_{abc}$ is proportional to I. Hence, u is proportional to I.

Construction of the moment solution of (83) for u is begun by expanding u as

$$u(\underline{r}) = I \sum_{j=1}^{N} I_j u_j(\underline{r}) \tag{98}$$

where $\{u_j\}$ are expansion functions and $\{I_j\}$ are unknown constants to be determined. Because $\underline{u}$ was shown to be proportional to I in the previous paragraph, none of the unknowns $\{I_j\}$ will depend on I. Before defining the expansion functions $\{u_j\}$, we truncate strip a at $(z = -\ell_a)$ far from the via and truncate strip c at $(z = \ell_c)$ far from the via. The surface of the via and the truncated strips is called S and is modeled by triangular patches [7]. The resulting triangular patch surface is called $S_T$. The vertices of the triangles are called the nodes of $S_T$. The triangles are chosen so that none of them straddles strip a and the via, and none of them straddles strip c and the via. Each triangle is either entirely on strip a, or entirely on the via, or entirely on strip c.

If the via is a simple strip, then there is a one-to-one correspondence between the expansion functions and the interior nodes of $S_T$. An interior node of $S_T$ is a node of $S_T$ not on any edge of $S_T$. The expansion function $u_j$ is associated with the jth interior node of $S_T$. Consider each triangle of which one vertex is the jth interior node of $S_T$. Called ($T_{jn}$, n=1,2,...,$N_j$), these triangles encircle the jth interior node of $S_T$. The set of these triangles is the domain of $u_j$. On $T_{jn}$, $u_j$ decreases linearly from unity at the jth interior node of $S_T$ to zero at the side of $T_{jn}$ opposite the jth interior node of $S_T$. More precisely,

$$u_j(\underline{r}) = \xi_{jn}(\underline{r}) \qquad \text{on } T_{jn}, \quad n=1,2,\ldots,N_j \tag{99}$$

where $\xi_{jn}$ is the area coordinate associated with the vertex of $T_{jn}$ that is the jth interior node of $S_T$. At the point $\underline{r}$ on $T_{jn}$, $\xi_{jn}$ is defined by

$$\xi_{jn}(\underline{r}) = \frac{A^r_{jn}}{A_{jn}} \tag{100}$$

where $A_{jn}$ is the area of $T_{jn}$, and $A^r_{jn}$ is the area $T_{jn}$ would have if its vertex which is the jth interior node of $S_T$ were replaced by the point $\underline{r}$. As defined by (99), the expansion function $u_j$ is unity at the jth interior node of $S_T$, is zero on the edges of $S_T$, and is continuous everywhere. Furthermore, $\nabla_s u_j$ is continuous everywhere except on the sides of the triangles that encircle the jth interior node of $S_T$.

If $\underline{J}_j$ is the expansion function for $\underline{J}_e$ associated with $u_j$, then according to (80),

$$\underline{J}_j(\underline{r}) = \underline{n} \times \nabla_s u_j(\underline{r}) \tag{101}$$

Substituting (99) into (101), we obtain

$$\underline{J}_j = \frac{\underline{\ell}_{jn}}{2A_{jn}} \quad \text{on} \quad T_{jn}, \quad n=1,2,\ldots,N_j \tag{102}$$

where $\underline{\ell}_{jn}$ is a vector whose length is the length of the side of $T_{jn}$

opposite the jth interior node of $S_T$ and whose direction is parallel to this side. The sense of $\underline{\ell}_{jn}$ is such that $\underline{J}_j$ encircles $\underline{n}$ in the left-hand sense. A left-handed screw whose axis is parallel to $\underline{n}$ would advance in the direction of $\underline{n}$ when rotated in the direction of $\underline{\ell}_{jn}$. The current $\underline{J}_j$ flows continuously around the jth interior node of $S_T$ in the sense that, on each triangle side that fans out from this node, the component of current normal to this side is continuous. Here, the word continuous is used loosely because, if $S_T$ bends sharply at the side of a triangle, the component of current normal to this side must change direction suddenly in order to remain on $S_L$. Obviously, $\underline{J}_j$ is tangent to all triangle sides that are edges of $S_T$. As defined by (102), $\underline{J}_j$ is $\underline{J}_L/2$ where $\underline{J}_L$ is shown in [8, Fig. 2.2].

If the via is not a simple strip, $S_T$ may contain junctions of surfaces. Each junction of surfaces is a chain of straight line segments drawn between nodes. This chain of straight line segments is called a junction line, and these nodes are called junction nodes.

If some electrical contacts are ignored along junction lines, then the junctions will disappear, and $S_T$ will separate into several isolated surfaces. Let $S_k$ be a typical one of these surfaces. If $S_k$ is closed, all the nodes of $S_k$ are interior nodes, and an expansion function is associated with each of these nodes except one. If $S_k$ is open, an expansion function is associated with each interior node of $S_k$. Just after (83), it was stated that the value of u is zero at some point on the surface. With the expansion functions chosen previously in this paragraph, u is zero at the edges of $S_k$ if $S_k$ is open. If $S_k$ is closed, u is zero at the interior node which does not have an expansion function associated with it. If $S_k$ is open and multiply connected because it has holes in it, then $S_K$ has more than one edge. In this case, additional expansion functions are needed because, although it is proper to have (u=0) on one edge, it is too restrictive to have (u=0) on all edges. Currents can circulate about the holes. None of these circulating currents can be expressed as a linear combination of the expansion functions associated with the interior nodes of $S_k$.

An additional expansion function must be associated with each hole of $S_k$. A typical hole of $S_k$ has a closed contour which is a chain of straight line segments drawn between nodes. This chain of straight line segments is called the contour $C_{hole}$. The expansion function associated with the hole whose contour is $C_{hole}$ is called $u_{hole}$. The domain of $u_{hole}$ consists of all the triangles that are attached to $C_{hole}$. A triangle is attached to $C_{hole}$ if at least one of its vertices is a node of $C_{hole}$. By definition, $u_{hole}$ is unity on $C_{hole}$. The nodes of $C_{hole}$ are called $C_{hole}$ nodes. On each triangle of which exactly one vertex is a $C_{hole}$ node, $u_{hole}$ decreases linearly from unity at the $C_{hole}$ node to zero at the side opposite this node. On each triangle of which exactly two vertices are $C_{hole}$ nodes, $u_{hole}$ decreases linearly from unity on the line segment which connects these nodes to zero at the remaining vertex. Finally, $u_{hole}$ is unity everywhere on each triangle of which all three vertices are $C_{hole}$ nodes. The expansion function for $\underline{J}_e$ associated with $u_{hole}$ is called $\underline{J}_{hole}$ and is given, according to (80), by

$$\underline{J}_{hole}(\underline{r}) = \underline{n} \times \nabla_s \, u_{hole}(\underline{r}) \tag{103}$$

If the electrical contacts that were previously ignored along junction lines are restored, loops about which current can circulate are formed. Around each loop, a closed contour is chosen. For each of these contours, an additional expansion function is needed. Let $C_{loop}$ be a typical one of these contours and let $u_{loop}$ be the expansion function associated with it. Similar to the expansion function $u_{hole}$ defined in the previous paragraph, $u_{loop}$ is unity on $C_{loop}$, and $u_{loop}$ decreases linearly to zero at the string of adjacent nodes that run along one side of $C_{loop}$. The expansion function $u_{hole}$ was defined only to one side of $C_{hole}$ because $C_{hole}$ was a boundary of $S_k$. However, $C_{loop}$ is not always a boundary of $S_T$ so that $u_{loop}$ must be defined to both sides of $C_{loop}$. To one side of $C_{loop}$, $u_{loop}$ was required to decrease linearly to zero at the adjacent nodes. To the other side of $C_{loop}$, we want $u_{loop}$ to suddenly drop to zero. Unfortunately, the ensuing discontinuity in $u_{loop}$ will give rise to an impulse in $\nabla_s u_{loop}$. The expansion function for $\underline{J}_e$ associated with $u_{loop}$ is called $\underline{J}_{loop}$. According to (80), $\underline{J}_{loop}$ is given by

$$\underline{J}_{loop}(\underline{r}) = \underline{n} \times \nabla_s \, u_{loop}(\underline{r}) \tag{104}$$

However, not wanting any impulse in $\underline{J}_{loop}$, we alter $\underline{J}_{loop}$ so that $\underline{J}_{loop}$ is given, not strictly by (104), but by (104) with $\nabla_s \, u_{loop}(\underline{r})$ stripped of its impulse.

When the previously ignored electrical contacts are restored along junction lines, it may be necessary to associate additional expansion functions with some of the junction nodes. A typical expansion function associated with a junction node is called $u_{junc}$. Now, $u_{junc}$ is either an expansion function that was associated with the junction node before the electrical contacts were restored or an expansion function that must be appended when the electrical contacts are restored. The domain of $u_{junc}$ completely surrounds the junction node, part of the domain being on one branch of surface and the rest of it on another branch of surface. These two branches of surface must have consistent unit normal vectors $\underline{n}$. If one branch with its vector $\underline{n}$ rigidly attached to it was rotated about the junction line until it coincided with the other branch, the vectors $\underline{n}$ of the two branches should point in opposite directions. The expansion function $u_{junc}$ is unity at the junction node and exists on each triangle that is simultaneously on one of the two branches of surface in the domain of $u_{junc}$ and attached to the junction node. On any one of these triangles, $u_{junc}$ decreases linearly from unity at the vertex which is the junction node to zero at the side opposite this vertex. The expansion function for $\underline{J}_e$ associated with $u_{junc}$ is called $\underline{J}_{junc}$. According to (80), $\underline{J}_{junc}$ is given by

$$\underline{J}_{junc}(\underline{r}) = \underline{n} \times \nabla_s u_{junc}(\underline{r}) \tag{105}$$

As many such linearly independent expansion functions as possible are associated with each junction node.

Previously, $S_T$ separated into several isolated surfaces when some electrical contacts were ignored along junction lines. In effect, the function u in (80) was specified to be zero at several points, one on each isolated surface. When the electrical contacts are restored, two of these isolated surfaces may combine to form a composite surface. As a result, the u's on both of these formerly isolated surfaces will combine to form a composite u. The composite u is not necessarily continuous on the junction line along which the two formerly isolated surfaces are connected to each other. Presumably, any discontinuity in u gives rise to an impulse in $\nabla_s u$. This impulse is not wanted because, according to (80), it would appear in the electric current. Fortunately, both the discontinuity in u and the accompanying impulse in $\nabla_s u$ will be suppressed automatically if u is viewed as a linear combination of expansion functions, each of which exists in the absence of all the others.

We now generalize (98) to include all necessary expansion functions, the $\{u_j\}$ of (99), the $\{u_{hole}\}$ in (103), the $\{u_{loop}\}$ in (104), and the $\{u_{junc}\}$ in (105). It is more convenient to test (83) before inserting the generalized expansion (98) for the unknown u. Following Galerkin's method in which the set of testing functions is the same as the set of expansion functions, we choose the testing functions to be the collection of expansion functions in the generalized expansion (98). This collection of functions is called $\{u_i,\ i=1,2,\ldots,N\}$.

The integral over $S_T$ of the product of (83) with the testing function $u_i$ is

$$\iint_{S_T} u_i \underline{n} \cdot \nabla \times \underline{A}(\underline{J}_e) ds = - \iint_{S_T} u_i \underline{n} \cdot \nabla \times \underline{A}(\underline{J}_{abc}) ds \tag{106}$$

We want $u_i$ to be continuous in (106) so that Stokes' theorem can be applied to both integrals in (106). Now, $u_i$ may be $u_j$ of (99), $u_i$ may be $u_{hole}$ in (103), $u_i$ may be $u_{loop}$ in (104), or $u_i$ may be $u_{junc}$ in (105). If $u_i$ is $u_j$ of (99) or $u_{hole}$ in (103), then $u_i$ is continuous on $S_T$, the range of integration in (106).

If $u_i$ is $u_{loop}$ in (104), then $u_i$ drops suddenly from unity on $C_{loop}$ to zero immediately to one side of $C_{loop}$. In this case, we restrict the range of integration in (106) to the part of $S_T$ on $C_{loop}$ and to the side of $C_{loop}$ where $u_i$ decreases linearly from unity to zero. This means that the part of $S_T$ to the other side of $C_{loop}$ where $u_i$ drops suddenly to zero is suppressed from the range of integration in (106). Not affecting the values of the integrals in (106), this suppression of the part of $S_T$ to one side of $C_{loop}$ makes $C_{loop}$ a boundary of the range of integration so that $u_i$ is continuous on the range of integration and equal to unity on the boundary $C_{loop}$.

If $u_i$ is $u_{junc}$ in (105), then $u_i$ exists on two branches of surface that meet on a junction line. Attaining unity on the junction line, $u_i$ is continuous on these two branches. However, an observer who approaches the junction line from another branch of surface will see $u_i$ jump suddenly from zero on that branch to unity on the junction line. Hence, $u_i$ is discontinuous on the junction line. In this case, we restrict the range of integration in (106) to the two branches of surface on which $u_i$ exists. Not affecting the values of the integrals in (106), this restriction of the range of integration renders $u_i$ continuous in (106). Of course, this $u_i$ is zero on the boundary of the range of integration.

With the range of integration restricted as described in the two previous paragraphs, (106) becomes

$$\iint_{S'} u_i \underline{n} \cdot \nabla \times \underline{A}(\underline{J}_e)ds = -\iint_{S'} u_i \underline{n} \cdot \nabla \times \underline{A}(\underline{J}_{abc})ds \tag{107}$$

where S' is the restricted range of integration. Stokes' theorem converts (107) to [3, Eq. (C-1)]

$$\iint_{S'}(\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_e)ds - \int_{C'} u_i \underline{A}(\underline{J}_e) \cdot d\underline{\ell} = -\iint_{S'} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_{abc})ds$$

$$+ \int_{C'} u_i \underline{A}(\underline{J}_{abc}) \cdot d\underline{\ell} \tag{108}$$

where C' is the contour that bounds S'. The direction of C' is such that a right-handed screw would advance in the direction of $\underline{n}$ when rotated in the direction of C'. Equation (108) is rewritten as

$$\iint_{S'}(\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_e)ds = -\iint_{S'}(\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_{abc})ds + \int_{C'} u_i \underline{A}(\underline{J}) \cdot d\underline{\ell} \tag{109}$$

where $\underline{J}$ is given by (20).

If $u_i$ is $u_j$ of (99) or $u_{junc}$ in (105), then $u_i$ is zero on C' so that (109) reduces to

$$\iint_{S'} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_e)ds = -\iint_{S'} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_{abc})ds \tag{110}$$

If $u_i$ is $u_{hole}$ in (103), then the only part of C' on which $u_i$ is not zero is $C_{hole}$. On $C_{hole}$, $u_i$ is unity so that (109) reduces to

$$\iint_{S'} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_e)ds = -\iint_{S'} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_{abc})ds + \int_{C_{hole}} \underline{A}(\underline{J}) \cdot d\underline{\ell} \tag{111}$$

If $u_i$ is $u_{loop}$ in (104), then the only part of C' on which $u_i$ is not zero is $C_{loop}$. On $C_{loop}$, $u_i$ is unity so that (109) reduces to

$$\iint_{S'} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_e)ds = -\iint_{S'} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_{abc})ds + \iint_{C_{loop}} \underline{A}(\underline{J}) \cdot d\underline{\ell} \tag{112}$$

Both $C_{hole}$ and $C_{loop}$ are closed contours. In view of (3), Stokes' theorem [2, Eq. (51) on p. 503] gives

$$\int_{C_{hole}} \underline{A}(\underline{J}) \cdot d\underline{\ell} = \int_{S_{hole}} \underline{B}(\underline{J}) \cdot \underline{n}\, ds \qquad (113)$$

and

$$\int_{C_{loop}} \underline{A}(\underline{J}) \cdot d\underline{\ell} \quad \int_{S_{loop}} \underline{B}(\underline{J}) \cdot \underline{n}\, ds \qquad (114)$$

where $S_{hole}$ is a surface that caps $C_{hole}$, and $S_{loop}$ is a surface that caps $C_{loop}$. Because of [3, Eq. (43)], the right-hand sides of both (113) and (114) vanish. As a result, both (111) and (112) reduce to (110). Therefore, (109) always reduces to (110), regardless of whether $u_i$ is $u_j$ of (99), $u_{hole}$ in (103), $u_{loop}$ in (104), or $u_{junc}$ in (105).

If all the impulses in $\nabla_s u_i$ are suppressed, (110) can be written as

$$\iint_{S_T} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_e) ds = - \iint_{S_T} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_{abc}) ds \qquad (115)$$

On the right-hand side of (115), the integration over the parts of $S_T$ on strips a and c is difficult to perform because strips a and c on which $\underline{A}(\underline{J}_{abc})$ must be evaluated are covered with the source current $\underline{J}_{abc}$. Furthermore, as the domain of $u_i$ moves on either strip father and farther from the via, the right-hand side of (115) approaches zero in a manner that is not obvious.

Now, the objective is to replace the right-hand side of (115) by a form that is more suitable for calculation. The right-hand side of (115) is called R and is recast as

$$R = R_a + R_b + R_c \tag{116}$$

where

$$R_a = -\iint_{S_a} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_{abc})\, ds \tag{117}$$

$$R_b = -\iint_{S_b} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_{abc})\, ds \tag{118}$$

$$R_c = -\iint_{S_c} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_{abc})\, ds \tag{119}$$

In (117), $S_a$ is the part of $S_T$ for which $(-\ell_a \leq z < z_a)$. In (118), $S_b$ is the part of $S_T$ for which $(z_a \leq z \leq z_c)$. In (119), $S_c$ is the part of $S_T$ for which $(z_c < z \leq \ell_c)$. Otherwise stated, $S_a$ is the part of $S_T$ on strip a, $S_b$ is the part of $S_T$ on the via, and $S_c$ is the part of $S_T$ on strip c. As in (115), all impulses are suppressed from $\nabla_s u_i$ in (117)-(119).

Consider $R_a$ of (117). According to (21), $\underline{J}_{abc}$ is equal to $\underline{J}_a$ on $S_a$. The discontinuities in $\underline{J}_a$ due to the expansion (84) could seriously affect the accuracy of $\underline{A}(\underline{J}_{abc})$ on $S_a$. Solving (23) for $\underline{J}_a^-$ and substituting this $\underline{J}_a^-$ into (26), we obtain

$$\underline{J}_{abc} = \underline{J}_b + \underline{J}_c^+ - \underline{J}_a^+ + \underline{J}_a \tag{120}$$

Because the operator $\underline{A}$ is linear, (120) allows us to write

$$\underline{A}(\underline{J}_{abc}) = \underline{A}(\underline{J}_b + \underline{J}_c^+ - \underline{J}_a^+) + \underline{A}(\underline{J}_a) \tag{121}$$

Substitution of (121) into (117) gives

$$R_a = -\iint_{S_a} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_b + \underline{J}_c^+ - \underline{J}_a^+) ds - \iint_{S_a} (\underline{n} \times \nabla_s u_i) \cdot (\underline{A}(\underline{J}_a) ds \quad (122)$$

Concerning the second integral in (122), Stokes' theorem yields [3, Eq. (C-1)]

$$\iint_{S_a} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_a) ds = -\iint_{S_a} u_i \underline{n} \cdot \nabla \times \underline{A}(\underline{J}_a) ds + \int_{C_a} u_i \, \underline{A}(\underline{J}_a) \cdot d\underline{\ell} \quad (123)$$

where $C_a$ is the contour that bounds $S_a$. For convenience, $S_a$ is extended to include the line at $(z=z_a)$. Now, $C_a$ consists of the lateral edges of strip a for $(-\ell_a \le z \le z_a)$, the edge at $(z = -\ell_a)$, and the edge at $(z = z_a)$. Because $u_i$ vanishes on the lateral edges of strip a and because $\underline{A}(\underline{J}_a)$ is z directed on the edge at $(z = -\ell_a)$ and the edge at $(z = z_a)$, the line integral over $C_a$ on the right-hand side of (123) vanishes. Moreover, thanks to (40) and (41), the surface integral over $S_a$ on the right-hand side of (123) vanishes. Hence, (123) reduces to

$$\iint_{S_a} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_a) \, ds = 0 \quad (124)$$

so that (122) becomes

$$R_a = -\iint_{S_a} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_b + \underline{J}_c^+ - \underline{J}_a^+) \, ds \quad (125)$$

Consider $R_c$ of (119). According to (21), $\underline{J}_{abc}$ is equal to $\underline{J}_c$ on $S_c$. The discontinuities in $\underline{J}_c$ due to the expansion (92) could seriously affect the accuracy of $\underline{A}(\underline{J}_{abc})$ on $S_c$. Solving (24) for $\underline{J}_c^+$ and substituting this $\underline{J}_c^+$ into (26), we obtain

$$\underline{J}_{abc} = \underline{J}_b + \underline{J}_a^- - \underline{J}_c^- + \underline{J}_c \tag{126}$$

Because the operator $\underline{A}$ is linear, (126) allows us to write

$$\underline{A}(\underline{J}_{abc}) = \underline{A}(\underline{J}_b + \underline{J}_a^- - \underline{J}_c^-) + \underline{A}(\underline{J}_c) \tag{127}$$

Substitution of (127) into (119) gives

$$R_c = -\iint_{S_c} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_b + \underline{J}_a^- - \underline{J}_c^-) ds - \iint_{S_c} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_c) ds \tag{128}$$

Concerning the second integral in (128), Stokes' theorem yields [3, Eq. (C-1)]

$$\iint_{S_c} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_c) ds = -\iint_{S_c} u_i \underline{n} \cdot \nabla \times \underline{A}(\underline{J}_c) ds + \int_{C_c} u_i \underline{A}(\underline{J}_c) \cdot d\underline{\ell} \tag{129}$$

where $C_c$ is the contour that bounds $S_c$. The right-hand side of (129) vanishes in the same way that the right-hand side of (123) did so that (128) reduces to

$$R_c = -\iint_{S_c} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_b + \underline{J}_a^- - \underline{J}_c^-) ds \tag{130}$$

Substituting (125), (118), and (130) into (116), we obtain

$$R = -\iint_{S_a} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_b + \underline{J}_c^+ - \underline{J}_a^+) ds - \iint_{S_b} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_{abc}) ds$$

$$- \iint_{S_c} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_b + \underline{J}_a^- - \underline{J}_c^-) ds \tag{131}$$

Hence, (115) becomes

$$\iint_{S_T} (\underline{n} \times \nabla_s u_i) \cdot \underline{A}(\underline{J}_e)ds = - \iint_{S_T} (\underline{n} \times \nabla_s u_i) \cdot \hat{\underline{A}}(\underline{J}_{abc})ds \tag{132}$$

Where $\underline{A}(\underline{J}_e)$ is given by (82), and $\hat{\underline{A}}(\underline{J}_{abc})$ is given by

$$\hat{\underline{A}}(\underline{J}_{abc}) = \begin{cases} \underline{A}(\underline{J}_b + \underline{J}_c^+ - \underline{J}_a^+) & \text{on } S_a \\ \underline{A}(\underline{J}_{abc}) & \text{on } S_b \\ \underline{A}(\underline{J}_b + \underline{J}_a^- - \underline{J}_c^-) & \text{on } S_c \end{cases} \tag{133}$$

From (101), (103), (104), and (105), we obtain

$$\underline{J}_i = \underline{n} \times \nabla_s u_i \tag{134}$$

where all the impulses in $\nabla_s u_i$ are suppressed. Equation (134) reduces (132) to

$$\iint_{S_T} \underline{J}_i \cdot \underline{A}(\underline{J}_e)ds = - \iint_{S_T} \underline{J}_i \cdot \hat{\underline{A}}(\underline{J}_{abc})ds \tag{135}$$

In (135), the part of $\hat{\underline{A}}(\underline{J}_{abc})$ of (133) on $S_a$ is easy to evaluate because the points of evaluation are free of source current. Similarly, the part of $\hat{\underline{A}}(\underline{J}_{abc})$ of (133) on $S_c$ is easy to evaluate because the points of evaluation are free of source current. Furthermore, the right-hand side of (135) obviously approaches zero as the domain of $\underline{J}_i$ moves farther and farther from the via because $\hat{\underline{A}}$ is being evaluated farther and farther from any source current. On $S_b$, however, $\hat{\underline{A}}$ must be evaluated on the source current $\underline{J}_{abc}$. According to (21), $\underline{J}_{abc}$ is equal to $\underline{J}_b$ on $S_b$. We choose $\underline{J}_b$ to be a constant vector on each triangular patch and approximate the integration over $S_b$ on the right-hand side of (135) by sampling $\hat{\underline{A}}$ at the centroids of the triangles. Since the centroids of the triangles are relatively far from the discontinuities of $\underline{J}_b$ on the sides of the triangles, good accuracy should be obtained.

In view of (101), (103), (104), and (105), the generalized expansion (98) for u is accompanied by

$$\underline{J}_e(\underline{r}) = I \sum_{j=1}^{N} I_j \underline{J}_j(\underline{r}) \tag{136}$$

If j is such that $u_j$ is one of the $u_j$'s in (98) proper, then $\underline{J}_j$ is given by (102). If j is such that $u_j$ is $u_{hole}$ in (103), $u_{loop}$ in (104), or $u_{junc}$ in (105), then (102) still holds provided that $\{T_{jn}, n=1,2,\ldots,N_j\}$ are the triangles on which $u_j$ exists, $A_{jn}$ is the area of $T_{jn}$, and $\underline{\ell}_{jn}$ is an appropriate vector side of $T_{jn}$. As defined by this generalization of (102), the generalized $\underline{J}_j$ flows continuously across all sides of triangles in the sense that, on any one of these sides, the component of current normal to this side is continuous. Obviously, the generalized $\underline{J}_j$ is tangent to all triangle sides that are edges of $S_T$.

If one side of a triangle is an edge of $S_T$, then, on this triangle, $\underline{J}_e$ of (136) is a constant vector parallel to this side. However, if none of the sides of a triangle is an edge of $S_T$, then, on this triangle, $\underline{J}_e$ of (136) is an arbitrarily directed constant vector. Therefore, $\underline{J}_e$ of (136) can, if necessary, annihilate $\underline{J}_b$ on any triangle on the via.

Substituting the generalized expansion (136) for $\underline{J}_e$ in (135), dividing (135) by I, and letting (i=1,2,...,N), we obtain the matrix equation

$$P\vec{I} = \vec{V} \tag{137}$$

where $\vec{I}$ and $\vec{V}$ are $N \times 1$ column vectors, and P is a square matrix of order N. The jth element of $\vec{I}$ is $I_j$ of (136). The ith element of $\vec{V}$ is called $V_i$ and is given by

$$V_i = -\frac{1}{I} \iint_{S_T} \underline{J}_i \cdot \hat{\underline{A}}(\underline{J}_{abc}) ds , \qquad i=1,2,\ldots,N \tag{138}$$

The ijth element of P is called $P_{ij}$ and is given by

$$P_{ij} = \iint_{S_T} \underline{J}_i \cdot \underline{A}(\underline{J}_j) ds , \qquad \begin{cases} i=1,2,\ldots,N \\ j=1,2,\ldots,N \end{cases} \tag{139}$$

Replacing jn by im in the generalization of (102), we obtain

$$\underline{J}_i = \frac{\underline{\ell}_{im}}{2A_{im}} \quad \text{on} \quad T_{im} , \qquad m=1,2,\ldots,N_i \tag{140}$$

Substitution of (140) for $\underline{J}_i$ in (138) and (139) yields

$$V_i = -\sum_{m=1}^{N_i} \frac{\underline{\ell}_{im}}{2IA_{im}} \iint_{T_{im}} \hat{\underline{A}}(\underline{J}_{abc}) ds, \qquad i=1,2,\ldots,N \tag{141}$$

and

$$P_{ij} = \sum_{m=1}^{N_i} \frac{\underline{\ell}_{im}}{2A_{im}} \iint_{T_{im}} \underline{A}(\underline{J}_j) ds , \qquad \begin{cases} i=1,2,\ldots,N \\ j=1,2,\ldots,N \end{cases} \tag{142}$$

If, in the integrands of (141) and (142), $\hat{\underline{A}}(\underline{J}_{abc})$ and $\underline{A}(\underline{J}_j)$ are approximated by their values at the centroid of $T_{im}$, then (141) and (142) reduce to

$$V_i = -\frac{1}{2I} \sum_{m=1}^{N_i} \underline{\ell}_{im} \cdot [\hat{\underline{A}}(\underline{J}_{abc})]^{cim} , \quad i=1,2,\ldots,N \tag{143}$$

and

$$P_{ij} = \frac{1}{2} \sum_{m=1}^{N_i} \underline{\ell}_{im} \cdot [\underline{A}(\underline{J}_j)]^{cim} , \quad \begin{cases} i=1,2,\ldots,N \\ j=1,2,\ldots,N \end{cases} \tag{144}$$

where the superscript cim denotes evaluation at the centroid of $T_{im}$.

Consider the $\underline{A}$ that appears in (144). Since $\underline{A}$ is given by (82) and $\underline{J}_j$ is given by the generalization of (102),

$$[\underline{A}(\underline{J}_j)]^{cim} = \frac{\mu}{8\pi} \sum_{n=1}^{N_j} \frac{1}{A_{jn}} \left[ \underline{\ell}_{jn} \iint_{T_{jn}} \frac{ds'}{|\underline{r}^{cim} - \underline{r}'|} + (\underline{u}_y \ell_{jny} - \underline{u}_x \ell_{jnx} - \underline{u}_z \ell_{jnz}) \iint_{T_{jn}} \frac{ds'}{|\underline{r}^{cim} - \underline{r}''|} \right] \tag{145}$$

where $(\ell_{jnx}, \ell_{jny}, \ell_{jnz})$ are the (x,y,z) components of $\underline{\ell}_{jn}$, $\underline{r}'$ is the position vector of the differential element of area ds', $\underline{r}''$ is the image of $\underline{r}'$ about the ground plane, and $\underline{r}^{cim}$ is the radius vector to the centroid of $T_{im}$. Each integral in (145) is the integral over a triangle of the reciprocal of the distance from a fixed point. The value of this integral is given by [5, Eq. (46)], an expression which was adapted from [9, Eq. (5)]. If $T_{jn}$ is on either $S_a$ or $S_c$, then the term in square brackets on the right-hand side of (145) has no y component. If $T_{im}$ is on either $S_a$ or $S_c$, then $\underline{\ell}_{im}$ in (144) has no y component so that the y component of (145) does not come into play.

Consider $[\hat{\underline{A}}(\underline{J}_{abc})]^{cim}/I$ of (143). Substituting (26) into (133) and using the fact that the operator $\underline{A}$ is linear, we obtain

$$[\hat{\underline{A}}(\underline{J}_{abc})]^{cim}/I = \begin{cases} [\underline{A}(\underline{J}_b)]^{cim}/I + [\underline{A}(\underline{J}_c^+)]^{cim}/I - [\underline{A}(\underline{J}_a^+)]^{cim}/I, & -\ell_a \le z < z_a \\ [\underline{A}(\underline{J}_b)]^{cim}/I + [\underline{A}(\underline{J}_a^-)]^{cim}/I + [\underline{A}(\underline{J}_c^+)]^{cim}/I, & z_a \le z \le z_c \\ [\underline{A}(\underline{J}_b)]^{cim}/I + [\underline{A}(\underline{J}_a^-)]^{cim}/I - [\underline{A}(\underline{J}_c^-)]^{cim}/I, & z_c < z \le \ell_c \end{cases} \tag{146}$$

where the superscript cim denotes evaluation at the centroid of $T_{im}$. Now, $\underline{J}_b$ is expressed as

$$\underline{J}_b = \frac{I\underline{\ell}_{bn}}{2A_{bn}} \quad \text{on} \quad T_{bn} \,, \quad n=1,2,\ldots,N_{N+1} \tag{147}$$

where ($T_{bn}$, $n=1,2,\ldots,N_{N+1}$) are the triangles on which $\underline{J}_b$ exists, $A_{bn}$ is the area of $T_{bn}$, and $\underline{\ell}_{bn}$ is a vector that is independent of position on $T_{bn}$. Since $\underline{J}_b$ was chosen to be proportional to I in the paragraph preceding (98), $\underline{\ell}_{bn}$ can not depend on I. Hence, $\underline{\ell}_{bn}$ is truly a constant, constant with respect to spatial coordinates and with respect to I. Moreover, $\{\underline{\ell}_{bn}\}$ must be such that $\underline{J}_b$ exists only on the via, is solenoidal, is tangent to the edges of $S_T$, and provides for a continuous flow of current from $\underline{J}_a$ *on line a through the via to* $\underline{J}_c$ on line c. Lines a and c are defined in the paragraph that contains (21). Construction of $\underline{J}_b$ is described in Appendix B. Replacing $\underline{J}_e$ in (82) by $\underline{J}_b$ of (147) and replacing $\underline{r}$ in (82) by $\underline{r}^{cim}$, we obtain

$$[\underline{A}(\underline{J}_b)]^{cim}/I = \frac{\mu}{4\pi} \sum_{n=1}^{N_{N+1}} \frac{1}{2A_{bn}} \left[ \underline{\ell}_{bn} \iint_{T_{bn}} \frac{ds'}{|\underline{r}^{cim}-\underline{r}'|} + (\underline{u}_y \ell_{bny} - \underline{u}_x \ell_{bnx} - \underline{u}_z \ell_{bnz}) \iint_{T_{bn}} \frac{ds'}{|\underline{r}^{cim}-\underline{r}''|} \right] \tag{148}$$

where ($\ell_{bnx}$, $\ell_{bny}$, $\ell_{bnz}$) are the (x,y,z) components of $\underline{\ell}_{bn}$. The integrals in (148) are evaluated in the same manner as those in (145).

Next, we consider $[\underline{A}(\underline{J}_a^-)]^{cim}/I$ in (146). Since $\underline{J}_a^-$ is given by (25a), $[\underline{A}(\underline{J}_a^-)]^{cim}$ has only a z component which is given by the right-hand side of (47) with the upper limit on z' replaced by $z_a$. Hence,

$$[\underline{A}(\underline{J}_a^-)]^{cim}/I = \underline{u}_z \frac{\mu}{4\pi I} \int_{x_a}^{x_a+w_a} dx' J_{az}(x') \int_{-\infty}^{z_a} dz' \left( \frac{1}{\sqrt{(z^{cim}-z')^2+\rho_1^2}} - \frac{1}{\sqrt{(z^{cim}-z')^2+\rho_2^2}} \right) \tag{149}$$

where

$$\rho_1 = \sqrt{(x^{cim}-x')^2 + (y^{cim}-h_a)^2} \tag{150}$$

and

$$\rho_2 = \sqrt{(x^{cim}-x')^2 + (y^{cim}+h_a)^2} \tag{151}$$

Here, $(x^{cim}, y^{cim}, z^{cim})$ are the $(x,y,z)$ coordinates of $\underline{r}^{cim}$. If the integral with respect to $z'$ in (149) is called $I_{z2}$, then [4, Formula 200.01]

$$I_{z2} = \lim_{\alpha\to\infty} \left[ \ln\left( \frac{\gamma + \sqrt{\gamma^2 + \rho_1^2}}{\gamma + \sqrt{\gamma^2 + \rho_2^2}} \right) \right]_{\gamma = -\alpha - z^{cim}}^{\gamma = z_a - z^{cim}} \tag{152}$$

Expression (152) is recast as

$$I_{z2} = \lim_{\alpha\to\infty} \left[ \ln\left( \frac{-\gamma + \sqrt{\gamma^2 + \rho_2^2}}{-\gamma + \sqrt{\gamma^2 + \rho_1^2}} \right) + 2 \ln \left( \frac{\rho_1}{\rho_2} \right) \right]_{\gamma=-\alpha-z^{cim}}^{\gamma=z_a-z^{cim}} \tag{153}$$

which reduces to

$$I_{z2} = \ln \left( \frac{z^{cim}-z_a + \sqrt{(z^{cim}-z_a)^2 + \rho_2^2}}{z^{cim}-z_a + \sqrt{(z^{cim}-z_a)^2 + \rho_1^2}} \right) \tag{154}$$

Substitution of (154) into (149) gives

$$[\underline{A}(\underline{J}_a^-)]^{cim}/I = \underline{u}_z \frac{\mu}{4\pi I} \int_{x_a}^{x_a+w_a} J_{az}(x') \ln \left( \frac{z^{cim}-z_a + \sqrt{(z^{cim}-z_a)^2 + \rho_2^2}}{z^{cim}-z_a + \sqrt{(z^{cim}-z_a)^2 + \rho_1^2}} \right) dx' \tag{155}$$

If $(z^{cim}-z_a)$ in (155) is viewed as the axial distance from the end of the strip of current $\underline{J}_a^-$, then (155) generalizes to

$$[\underline{A}(\underline{J}_a^{\pm})]^{cim}/I = \underline{u}_z \frac{\mu}{4\pi I} \int_{x_a}^{x_a+w_a} J_{az}(x') \ln \left( \frac{\mp(z^{cim}-z_a) + \sqrt{(z^{cim}-z_a)^2 + \rho_2^2}}{\mp(z^{cim}-z_a) + \sqrt{(z^{cim}-z_a)^2 + \rho_1^2}} \right) dx' \tag{156}$$

The current $J_{az}$ that appears in (156) is given by (61) which is recast as

$$J_{az}(x') = \frac{4\pi}{\mu} L_a I \hat{J}_{az}(x') \tag{157}$$

where $L_a$, the inductance per unit length of strip a, is given by (67) which is

$$L_a = \frac{\mu}{4\pi \int_{x_a}^{x_a+w_a} \hat{J}_{az}(x')dx'} \tag{158}$$

Since $\hat{J}_{az}(x')$ is given by (84), (158) becomes

$$L_a = \frac{\mu}{4\pi\Delta_a \sum_{j=1}^{N_a} I_{aj}} \tag{159}$$

where $\Delta_a$ is given by (86), and $I_{aj}$ is the jth element of the column vector $\vec{I}_a$ that satisfies (88). Substitution of (84) into (157) yields

$$J_{az}(x') = \frac{4\pi L_a I}{\mu} \sum_{j=1}^{N_a} I_{aj} P_a(x'-x_{aj}) \tag{160}$$

Substituting (160) for $J_{az}$, (150) for $\rho_1$, and (151) for $\rho_2$ in (156) and then changing the variable of integration in (156) to eliminate the offset $x^{cim}$, we obtain

$$[\underline{A}(\underline{J}_a^{\pm})]^{cim}/I = \underline{u}_z L_a \sum_{j=1}^{N_a} I_{aj} \int_{x_{aj}-x^{cim}}^{x_{a,j+1}-x^{cim}} \ln\left(\frac{\mp(z^{cim}-z_a)+\sqrt{x^2+(y^{cim}+h_a)^2+(z^{cim}-z_a)^2}}{\mp(z^{cim}-z_a)+\sqrt{x^2+(y^{cim}-h_a)^2+(z^{cim}-z_a)^2}}\right)dx \tag{161}$$

The integral in (161) is of the same form as the integral in [5, Eq. (C-4)]. The value of the integral in [5, Eq. (C-4)] is given by [5, Eqs. (C-23) and (C-24)].

If the inductance per unit length of strip c is called $L_c$, then, in analogy with (159),

$$L_c = \frac{\mu}{4\pi\Delta_c \sum_{j=1}^{N_c} I_{cj}} \tag{162}$$

where $\Delta_c$ is given by (94), and $I_{cj}$ is the jth element of the column vector $\vec{I}_c$ that satisfies (96). Replacing $\underline{J}_a^{\pm}$ and its associated quantities in (161) by $\underline{J}_c^{\pm}$ and its corresponding associated quantities, we obtain

$$[\underline{A}(\underline{J}_c^{\pm})]^{cim}/I = \underline{u}_z L_c \sum_{j=1}^{N_c} I_{cj} \int_{x_{cj}-x^{cim}}^{x_{c,j+1}-x^{cim}} \ln\left(\frac{\mp(z^{cim}-z_c)+\sqrt{x^2+(y^{cim}+h_c)^2+(z^{cim}-z_c)^2}}{\mp(z^{cim}-z_c)+\sqrt{x^2+(y^{cim}-h_c)^2+(z^{cim}-z_c)^2}}\right)dx \tag{163}$$

The integral in (163) is evaluated in the same manner as that in (161).

Equations (148), (161), and (163) are substituted into (146), and then (146) is substituted into (143) to obtain $V_i$. If $T_{im}$ is on either $S_a$ or $S_c$, then $\underline{\ell}_{im}$ in (143) has no y component so that the y component of (148) does not come into play. The resulting $V_i$ is, as expected, independent of I. Substitution of (145) into (144) gives $P_{ij}$. After $V_i$ and $P_{ij}$ have been calculated, the moment solution of (83) for u is completed by solving (137) for $\vec{I}$ and substituting the elements of $\vec{I}$ into expression (98) for u.

## V. CONCLUSION

To conclude, we calculate the excess inductance $L_e$ by substituting the magnetostatic currents of Section IV in (38) and by numerically performing the surface integration explicit in (38). Calculated values of the excess current density $\underline{J}_e$ on the strips, the total current density $(\underline{J}_e + \underline{J}_b)$ on the via, and the excess inductance $L_e$ are given for a specific example.

Replacing S by $S_T$ in (38) and using the fact that the operator $\underline{A}$ is linear, we obtain

$$L_e = I_a + I_c + I_b + z_a L_a - z_c L_c \tag{164}$$

where $L_a$ is given by (66), and $L_c$ is given by

$$L_c = \frac{\psi_c}{I} \tag{165}$$

The I's in (164) are given by

$$I_a = \frac{1}{I^2} \iint_{S_a} [\hat{\underline{A}}(\underline{J}_{abc}) + \underline{A}(\underline{J}_e)] \cdot \underline{J}_a^- \, ds \tag{166}$$

$$I_c = \frac{1}{I^2} \iint_{S_c} [\hat{\underline{A}}(\underline{J}_{abc}) + \underline{A}(\underline{J}_e)] \cdot \underline{J}_c^+ \, ds \tag{167}$$

$$I_b = \frac{1}{I^2} \iint_{S_b} [\hat{\underline{A}}(\underline{J}_{abc}) + \underline{A}(\underline{J}_e)] \cdot \underline{J}_b \, ds \tag{168}$$

where $S_a$, $S_b$, and $S_c$ are defined immediately after (119). Furthermore, $\underline{\hat{A}}(\underline{J}_{abc})$ is given by (133) in which $\underline{J}_{abc}$ is given by (26).

To numerically perform the integrations in (166) and (167), we enumerate the triangles on $S_a$ by $\{T_i, i=1,2,\ldots,N_{ta}\}$ and those on $S_c$ by $\{T_i, i=N_{tb}+1, N_{tb}+2,\ldots,N_{tc}\}$. The intervening triangles $\{T_i, i=N_{ta}+1, N_{ta}+2,\ldots,N_{tb}\}$ are on $S_b$ and will come into play later when $\underline{J}_e$ is calculated. Using (25a) to replace $\underline{J}_a^-$ by $\underline{J}_a$ in (166) and then approximating $(\underline{\hat{A}}(\underline{J}_{abc}) + \underline{A}(\underline{J}_e))$ on each triangle of $S_a$ in (166) by its value at the centroid of the triangle, we obtain

$$I_a = \frac{1}{I^2} \sum_{i=1}^{N_{ta}} [\underline{\hat{A}}(\underline{J}_{abc}) + \underline{A}(\underline{J}_e)]^{ci} \cdot \iint_{T_i} \underline{J}_a ds \tag{169}$$

where the superscript ci denotes evaluation at the centroid of $T_i$. Using (25d) to replace $\underline{J}_c^+$ by $\underline{J}_c$ in (167) and then approximating $(\underline{\hat{A}}(\underline{J}_{abc}) + \underline{A}(\underline{J}_e))$ on each triangle of $S_c$ by its value at the centroid of the triangle, we obtain

$$I_c = \frac{1}{I^2} \sum_{i=N_{tb}+1}^{N_{tc}} [\underline{\hat{A}}(\underline{J}_{abc}) + \underline{A}(\underline{J}_e)]^{ci} \cdot \iint_{T_i} \underline{J}_c ds \tag{170}$$

Replacement of n by m in (147) and subsequent substitution of (147) into (168) produce

$$I_b = \frac{1}{I} \sum_{m=1}^{N_{N+1}} \frac{\underline{\ell}_{bm}}{2A_{bm}} \cdot \iint_{T_{bm}} [\underline{\hat{A}}(\underline{J}_{abc}) + \underline{A}(\underline{J}_e)] ds \tag{171}$$

Approximating the integral in (171) by sampling the integrand at the centroid of $T_{bm}$, we obtain

$$I_b = \frac{1}{2I} \sum_{m=1}^{N_{N+1}} \underline{\ell}_{bm} \cdot [\underline{\hat{A}}(\underline{J}_{abc}) + \underline{A}(\underline{J}_e)]^{cbm} \tag{172}$$

where the superscript cbm denotes evaluation at the centroid of $T_{bm}$.

As for $\underline{J}_a$ in (169), combining (45) and (160) with x' replaced by x, we obtain

$$\underline{J}_a(x) = \underline{u}_z \frac{4\pi L_a I}{\mu} \sum_{j=1}^{N_a} I_{aj} P_a(x-x_{aj}) \tag{173}$$

where $L_a$ is given by (159), and $I_{aj}$ is the jth element of the column vector $\vec{I}_a$ that satisfies (88). On $T_i$, $\underline{J}_a(x)$ of (173) is approximated by

$$\underline{J}_a(x) = \underline{u}_z \frac{4\pi L_a I}{\mu} I_a^{ci} \tag{174}$$

where $I_a^{ci}$ is the average value over x on $T_i$ of the sum in (173).

$$I_a^{ci} = \frac{1}{x_{max}^i - x_{min}^i} \int_{x_{min}^i}^{x_{max}^i} [\sum_{j=1}^{N_a} I_{aj} P_a(x-x_{aj})] dx \tag{175}$$

In (175), $x_{min}^i$ is the minimum value of x on $T_i$, and $x_{max}^i$ is the maximum value of x on $T_i$. Substitution of (174) for $\underline{J}_a(x)$ in (169) gives

$$I_a = \frac{4\pi L_a}{\mu I} \sum_{i=1}^{N_{ta}} A_i I_a^{ci} [\hat{A}_z(\underline{J}_{abc}) + A_z(\underline{J}_e)]^{ci} \tag{176}$$

where $A_i$ is the area of $T_i$, and $\hat{A}_z$ and $A_z$ are the z components of $\underline{\hat{A}}$ and $\underline{A}$, respectively.

As for $\underline{J}_c$ in (170), we have, in analogy with (173),

$$\underline{J}_c(x) = \underline{u}_z \frac{4\pi L_c I}{\mu} \sum_{j=1}^{N_c} I_{cj} P_c(x-x_{cj}) \tag{177}$$

where $L_c$ is given by (162), and $I_{cj}$ is the jth element of the column vector $I_c$ that satisfies (96). On $T_i$, $\underline{J}_c(x)$ of (177) is approximated by

$$\underline{J}_c(x) = \underline{u}_z \frac{4\pi L_c I}{\mu} I_c^{ci} \tag{178}$$

where $I_c^{ci}$ is the average value over x on $T_i$ of the sum in (177).

$$I_c^{ci} = \frac{1}{x_{max}^i - x_{min}^i} \int_{x_{min}^i}^{x_{max}^i} [\sum_{j=1}^{N_c} I_{cj} P_c(x - x_{cj})] dx \tag{179}$$

In (179), $x_{min}^i$ is the minimum value of x on $T_i$, and $x_{max}^i$ is the maximum value of x on $T_i$. Substitution of (178) for $\underline{J}_c(x)$ in (170) gives

$$I_c = \frac{4\pi L_c}{\mu I} \sum_{i=N_{tb}+1}^{N_{tc}} A_i I_c^{ci} [\hat{A}_z(\underline{J}_{abc}) + A_z(\underline{J}_e)]^{ci} \tag{180}$$

where $A_i$ is the area of $T_i$, and $\hat{A}_z$ and $A_z$ are the z components of $\underline{\hat{A}}$ and $\underline{A}$, respectively.

Expression (176) is recast as

$$I_a = \frac{4\pi L_a}{\mu} \sum_{i=1}^{N_{ta}} A_i I_a^{ci} \{[\hat{A}_z(\underline{J}_{abc})]^{ci}/I + [A_z(\underline{J}_e)]^{ci}/I\} \tag{181}$$

and (180) is recast as

$$I_c = \frac{4\pi L_c}{\mu} \sum_{i=N_{tb}+1}^{N_{tc}} A_i I_c^{ci} \{[\hat{A}_z(\underline{J}_{abc})]^{ci}/I + [A_z(\underline{J}_e)]^{ci}/I\} \tag{182}$$

Similarly, (172) is recast as

$$I_b = \frac{1}{2} \sum_{m=1}^{N_{N+1}} \underline{\ell}_{bm} \cdot \{[\underline{\hat{A}}(\underline{J}_{abc})]^{cbm}/I + [\underline{A}(\underline{J}_e)]^{cbm}/I\} \tag{183}$$

The next task is to evaluate the $\hat{A}_z$ and $\underline{\hat{A}}$ terms in (181)-(183) and the $A_z$ and $\underline{A}$ terms in (181)-(183). The $\hat{A}_z$ term in (181) and (182) is given by the z component of (146) with the superscript cim replaced by ci. The $\underline{\hat{A}}$ term in (183) is given by (146) with the superscript cim replaced by cbm. As indicated in the final paragraph of Section IV, the terms on the right-hand side of (146) are given by (148), (161), and (163).

The $A_z$ term in (181) and (182) and the $\underline{A}$ term in (183) are due to the current $\underline{J}_e$. These terms are evaluated by first obtaining a suitable expression for $\underline{J}_e$. Since $\underline{J}_e$ is given by (136) in which $\underline{J}_j$ is given by (102), we have

$$\underline{J}_e = I \sum_{j=1}^{N} I_j \sum_{n=1}^{N_j} \frac{\underline{\ell}_{jn}}{2A_{jn}} \tag{184}$$

where $I_j$ is the jth element of the column vector $\vec{I}$ that satisfies (137), $A_{jn}$ is the area of $T_{jn}$, and $\underline{\ell}_{jn}$ is an appropriate vector side of $T_{jn}$. In (184), $\underline{\ell}_{jn}$ is viewed as existing only on $T_{jn}$. To eliminate the overlapping that may occur in (184) in the sense that several different $T_{jn}$'s, different in that the jn's are different, may actually be the same physical triangle, we rearrange the terms in (184) into a sum of vectors each of which exists on one of the triangles $\{T_i\}$ enumerated in the paragraph containing (169). Thus, (184) becomes

$$\underline{J}_e = I \sum_{k=1}^{N_{tc}} \underline{\ell}_k \tag{185}$$

where $\underline{\ell}_k$ is a constant vector on $T_k$. Elsewhere, $\underline{\ell}_k$ is zero. On $T_k$, $\underline{\ell}_k$ is given by

$$\underline{\ell}_k = \Big(\sum_{j,n} \frac{I_j\, \underline{\ell}_{jn}}{2\, A_{jn}}\Big)_{T_{jn} = T_k} \tag{186}$$

where the subscript ($T_{jn} = T_k$) denotes that $\sum\limits_{j,n}$ is over all combinations of j and n in (186) such that ($T_{jn} = T_k$). If all three vertices of $T_k$ are on an edge of $S_T$, there may be no combination of j and n such that ($T_{jn} = T_k$) in which case $\sum\limits_{j,n}$ is zero.

Substituting $\underline{J}_e$ of (185) into (82), attaching the superscript ci to $\underline{r}$ and $\underline{A}$, taking the z component of (82), and dividing by I, we obtain

$$[A_z(\underline{J}_e)]^{ci}/I = \frac{\mu}{4\pi} \sum_{k=1}^{N_{tc}} \ell_{kz} \Big( \iint_{T_k} \frac{ds'}{|\underline{r}^{ci} - \underline{r}'|} - \iint_{T_k} \frac{ds'}{|\underline{r}^{ci} - \underline{r}''|} \Big) \tag{187}$$

where $\underline{r}'$ is the radius vector to the differential element of surface ds', $\underline{r}''$ is the image of $\underline{r}'$ about the ground plane, and $\ell_{kz}$ is the z component of $\underline{\ell}_k$. As in (181) and (182), the superscript ci in (187) denotes evaluation at the centroid of $T_i$ so that, in (187), $\underline{r}^{ci}$ is the radius vector to the centroid of $T_i$. Since the operator $\underline{A}$ is given by (82) and the current $\underline{J}_e$ is given by (185), it is evident that

$$[\underline{A}(\underline{J}_e)]^{cbm}/I = \frac{\mu}{4\pi} \sum_{k=1}^{N_{tc}} \Big( \underline{\ell}_k \iint_{T_k} \frac{ds'}{|\underline{r}^{cbm} - \underline{r}'|} + (\underline{u}_y \ell_{ky} - \underline{u}_x \ell_{kx} - \underline{u}_z \ell_{kz}) \iint_{T_k} \frac{ds'}{|\underline{r}^{cbm} - \underline{r}''|} \Big) \tag{188}$$

where $\underline{r}'$, ds', and $\underline{r}''$ have the same meanings as in (187), and ($\ell_{kx}, \ell_{ky}, \ell_{kz}$) are the (x,y,z) components of $\underline{\ell}_k$. As in (183), the superscript cbm in (188) denotes evaluation at the centroid of $T_{bm}$. Thus, in (188), $\underline{r}^{cbm}$ is the radius vector to the centroid of $T_{bm}$. The integrals in (187) and (188) are evaluated in the same manner as those in (145).

The excess inductance $L_e$ can now be calculated from (164) with $I_a$ given by (181), $I_c$ by (182), $I_b$ by (183), $L_a$ by (159), and $L_c$ by (162). The evaluation of the $\hat{A}_z$ and $\underline{\hat{A}}$ terms in (181)-(183) is described in the paragraph that follows (183). The $A_z$ term in (181) and (182) is given by (187). The $\underline{A}$ term in (183) is given by (188).

A computer program was written to calculate the excess inductance $L_e$. Intermediate output consists of the excess current density $\underline{J}_e$ on the strips and the total current density $(\underline{J}_e + \underline{J}_b)$ on the via. This computer program will be described and listed in a forthcoming report. Sample input and output data are provided to verify that the program is running properly. For the sample input data, $N_a = 8$, $N_c = 10$, and the triangulated surface of Fig. 4 is used. In Fig. 4, the triangular patches are labeled $\{T_j, j=1,2,\ldots,32\}$. The vertices of the triangles are called nodes and are numbered from 1 to 27. All dimensions in Fig. 4 are in meters. The rise from $(h_a = 3)$ on strip a to $(h_c = 5)$ on strip c is linear on the via. When these sample input data were entered, the computer program calculated an excess inductance $L_e$ of 0.704 micro-henrys and the current density shown in Tables 1 and 2. The permeability $\mu$ was not entered as data but was indirectly set equal to the permeability of free space by substituting $0.5 \times 10^{-7}$ for $\mu/(8\pi)$ near the end of the main program.

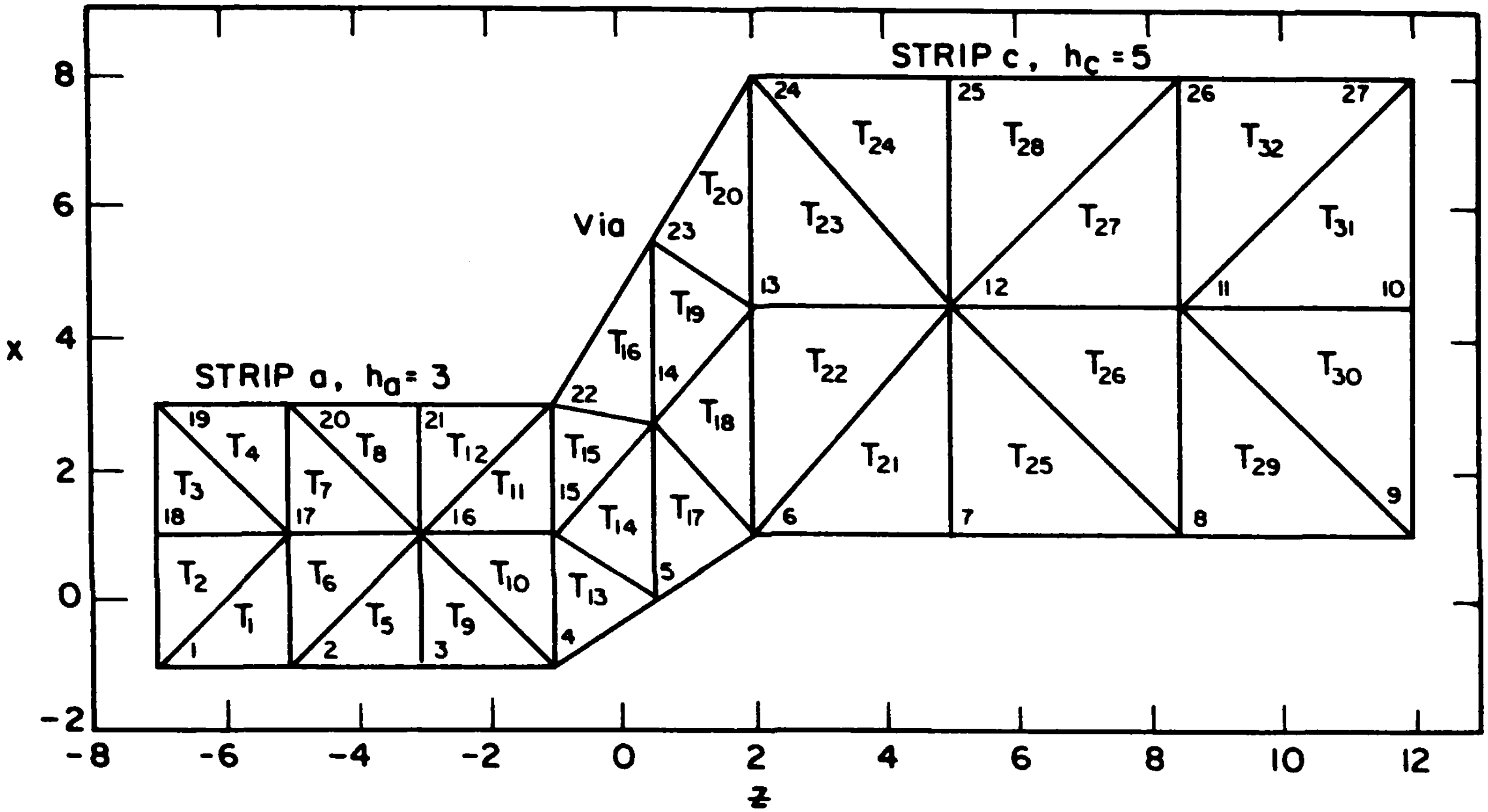


Fig. 4. Triangulation of the strips and via for the sample input data for the computer program.

Table 1. The x component of current density $\hat{\underline{J}}_e/I$ on the triangular patches $\{T_{i+4(j-1)},\ i=1,2,3,4,\ j=1,2,\ldots,8\}$ of the surface in Fig. 4. On the strips, $\hat{\underline{J}}_e$ is the excess current density $\underline{J}_e$ of (185). On the via, $\hat{\underline{J}}_e$ is the total current density $\underline{J}_e + \underline{J}_b$ where $\underline{J}_e$ is given by (185) and $\underline{J}_b$ is given by (B-1). Table 1 gives the x component of $\hat{\underline{J}}_e/I$ in units of 0.1 amperes per meter.

| i ↑ \ j → | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 4 | 0.00 | 0.00 | 0.00 | 2.29 | 1.40 | 0.00 | 0.00 | 0.00 |
| 3 | 0.08 | 0.18 | 0.23 | 2.11 | 1.04 | 0.26 | 0.14 | 0.05 |
| 2 | 0.08 | 0.18 | 0.23 | 1.13 | 1.23 | 0.26 | 0.14 | 0.05 |
| 1 | 0.00 | 0.00 | 0.00 | 1.12 | 1.10 | 0.00 | 0.00 | 0.00 |

Table 2. The orthogonal component of the current density $\hat{\underline{J}}_e/I$ on the triangular patches $\{T_{i+4(j-1)},\ i=1,2,3,4,\ j=1,2,\ldots,8\}$ of the surface in Fig. 4 where $\hat{\underline{J}}_e$ is defined in the caption for Table 1. The orthogonal component of $\hat{\underline{J}}_e/I$ is the component of $\hat{\underline{J}}_e/I$ perpendicular to the x direction. This component is in the $\underline{u}_z$ direction on the strips and in the $(2\underline{u}_y + 3\underline{u}_z)$ direction on the via. Table 2 gives the orthogonal component of $\hat{\underline{J}}_e/I$ in units of 0.1 amperes per meter.

| i ↑ \ j → | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 4 | 0.08 | 0.26 | 0.26 | 1.65 | 1.01 | -0.19 | -0.19 | -0.05 |
| 3 | 0.00 | 0.08 | 0.49 | 2.99 | 1.65 | -0.42 | -0.05 | 0.00 |
| 2 | 0.00 | -0.08 | -0.49 | 1.98 | 1.85 | 0.42 | 0.05 | 0.00 |
| 1 | -0.08 | -0.26 | -0.26 | 2.01 | 1.98 | 0.19 | 0.19 | 0.05 |

## APPENDIX A

In Appendix A, it is shown that if $\underline{A}(\underline{r})$ is a vector function whose component tangent to a surface S is differentiable and if

$$\underline{n} \cdot \nabla \times \underline{A}(\underline{r}) = 0 , \quad \underline{r} \text{ on S} \tag{A-1}$$

where $\underline{n}$ is a unit vector normal to S, then $\underline{A}_{tan}(\underline{r})$ can be written as

$$\underline{A}_{tan}(\underline{r}) = \nabla_s \Psi(\underline{r}) , \quad \underline{r} \text{ on S} \tag{A-2}$$

where

$$\Psi(\underline{r}) = \Psi(\underline{r}_o) + \int_{C_r} \underline{A}(\underline{r}') \cdot d\underline{r}' \tag{A-3}$$

In (A-2), the subscript tan denotes the component tangent to S, and $\nabla_s \Psi(\underline{r})$ is the surface gradient of $\Psi(\underline{r})$. In (A-3), $\underline{r}_o$ is the position vector of an arbitrary point on S, and $C_r$ is any contour on S from $\underline{r}_o$ to $\underline{r}$.

The following reasoning is used to show that (A-2) is true. According to [2, Eq. 166 on p. 497], we have

$$\underline{n} \cdot \nabla \times \underline{A} = \frac{1}{h_1 h_2} \left[\frac{\partial}{\partial v_1} (h_2 A_2) - \frac{\partial}{\partial v_2} (h_1 A_1)\right] \tag{A-4}$$

where $(v_1, v_2)$ are orthogonal curvilinear coordinates on S, and $(h_1, h_2)$ are the corresponding metrical coefficients. Moreover, $A_1$ is the component of $\underline{A}$ in the direction of increasing $v_1$, and $A_2$ is the component of $\underline{A}$ in the direction of increasing $v_2$. Equations (A-4) and (A-1) imply that

$$\frac{\partial}{\partial v_1}(h_2A_2) = \frac{\partial}{\partial v_2}(h_1A_1) \qquad \text{(A-5)}$$

In view of (A-5), the differential form

$$h_1A_1dv_1 + h_2A_2dv_2 \qquad \text{(A-6)}$$

is exact. Therefore, there is a scalar function $\Psi$ such that

$$h_1A_1 = \frac{\partial\Psi}{\partial v_1} \qquad \text{(A-7)}$$

$$h_2A_2 = \frac{\partial\Psi}{\partial v_2} \qquad \text{(A-8)}$$

From (A-7) and (A-8), we obtain the desired result (A-2) in which

$$\nabla_s\Psi(\underline{r}) = \frac{1}{h_1}\frac{\partial\Psi}{\partial v_1}\underline{u}_1 + \frac{1}{h_2}\frac{\partial\Psi}{\partial v_2}\underline{u}_2 \qquad \text{(A-9)}$$

In (A-9), $\underline{u}_1$ and $\underline{u}_2$ are, respectively, the unit vectors in the directions of increasing $v_1$ and $v_2$.

If $\underline{r}$ changes by the infinitesimal amount $d\underline{r}$, then $\Psi(\underline{r})$ changes by the infinitesimal amount $(\nabla_s\Psi(\underline{r}))\cdot d\underline{r}$ so that

$$\Psi(\underline{r}) = \Psi(\underline{r}_o) + \iint_{C_r} (\nabla_s'\Psi(\underline{r}'))\cdot d\underline{r}' \qquad \text{(A-10)}$$

where $\underline{r}_o$ and $C_r$ are the same as in (A-3). In (A-10), $\nabla_s'\Psi(\underline{r}')$ is the surface gradient of $\Psi$ with respect to the coordinates of $\underline{r}'$. The result (A-3) follows from (A-10) and (A-2).

# APPENDIX B

In Appendix B, the current $\underline{J}_b$ is constructed from the specifications in the paragraph containing (147).

In agreement with (147), we expand $\underline{J}_b$ as

$$\underline{J}_b = I \sum_{j=1}^{N_b} I_{bj} \underline{J}_{bj} \tag{B-1}$$

where $\{\underline{J}_b\}$ are vector functions each of which is constant on each triangle on the via, and $\{I_{bj}\}$ are unknown constants to be determined. Since $\underline{J}_b$ is solenoidal and tangent to the edges of $S_T$, it is natural to choose $\underline{J}_{bj}$ to be solenoidal and tangent to the edges of $S_T$. In order for $\underline{J}_{bj}$ to be solenoidal, the component of $\underline{J}_{bj}$ normal to any common side of two triangles on the via must be continuous across this side. Here, the word continuous is used loosely because, if $S_T$ bends sharply at the side of a triangle, the component of $\underline{J}_{bj}$ normal to this side must change direction suddenly in order to remain on $S_T$. Guided by these conditions, we choose some of the functions $\{\underline{J}_{bj}\}$ to be similar to $\underline{J}_{loop}$ of (104). As opposed to $\underline{J}_j$ of (102) and $\underline{J}_{loop}$ of (104), the functions $\{\underline{J}_{bj}\}$ must terminate abruptly on line a and on line c. Line a is the line at $(z=z_a)$ where the via connects with strip a. Line c is the line at $(z=z_c)$ where the via connects with strip c.

Now, $\underline{J}_b$ must provide for a continuous flow of current from $\underline{J}_a$ on line a through the via to $\underline{J}_c$ on line c. Therefore, on line a, we must have

$$\underline{J}_b \cdot \underline{n}_a = J_{az}(x) \tag{B-2}$$

where $J_{az}$ appears in (45), and $\underline{n}_a$ is the unit vector that is tangent to $S_T$ on the via side of line a and perpendicular to line a. The vector $\underline{n}_a$ points from line a toward the via. On line c, we must have

$$\underline{J}_b \cdot \underline{n}_c = J_{cz}(x) \tag{B-3}$$

where $J_{cz}$ appears in (74), and $\underline{n}_c$ is the unit vector that is tangent to $S_T$ on the via side of line c and perpendicular to line c. The vector $\underline{n}_c$ points from the via toward line c.

If $S_T$ has $N_{\ell a}$ nodes on line a, then line a consists of $(N_{\ell a}-1)$ sides of triangles. On any one of these sides, $J_{az}(x)$ varies continuously while $\underline{J}_b$ of (B-1) remains constant so that (B-2) can not be satisfied everywhere on this side. *Setting* $\underline{J}_b \cdot \underline{n}_a$ *equal to the average* value of $J_{az}(x)$ over the ith triangle side on line a and letting i run from one to $(N_{\ell a}-1)$, we obtain

$$I \sum_{j=1}^{N_b} I_{bj}(\underline{J}_{bj} \cdot \underline{n}_a)_{bai} = \frac{1}{x_{bai}^{+} - x_{bai}^{-}} \int_{x_{bai}^{-}}^{x_{bai}^{+}} J_{az}(x)dx, \quad i=1,2,\ldots,N_{\ell a}-1 \tag{B-4}$$

where $x_{bai}^{-}$ is the x coordinate of the beginning of the ith triangle side on line a, and $x_{bai}^{+}$ is the x coordinate of the end of the ith triangle side on line a. The subscript bai on the left-hand side of (B-4) denotes evaluation on the ith triangle side on line a.

If $S_T$ has $N_{\ell c}$ nodes on line c, then line c consists of $(N_{\ell c}-1)$ sides of triangles. On any one of these sides, $J_{cz}(x)$ varies continuously while $\underline{J}_b$ of (B-1) remains constant so that (B-3) can not be satisfied everywhere on this side. Setting $\underline{J}_b \cdot \underline{n}_c$ equal to the average value of $J_{cz}(x)$ over the kth triangle side on line c and letting k run from one to $(N_{\ell c}-1)$, we obtain

$$I \sum_{j=1}^{N_b} I_{bj}(\underline{J}_{bj} \cdot \underline{n}_c)_{bck} = \frac{1}{x^+_{bck} - x^-_{bck}} \int_{x^-_{bck}}^{x^+_{bck}} J_{cz}(x)dx, \quad k=1,2,\ldots,N_{\ell c}-1 \tag{B-5}$$

where $x^-_{bck}$ is the x coordinate of the beginning of the kth triangle side on line c, and $x^+_{bck}$ is the x coordinate of the end of the kth triangle side on line c. The subscript bck on the left-hand side of (B-5) denotes evaluation on the kth triangle side on line c.

Together, (B-4) and (B-5) give $(N_{\ell a} + N_{\ell c} - 2)$ equations. However, one of these equations is redundant because, due to the nature of $\{\underline{J}_{bj}\}$, the flux of $\underline{J}_b$ entering the via at line a is equal to the flux of $\underline{J}_b$ leaving the via at line c, and, accordingly, the flux of $\underline{u}_z J_{az}(x)$ entering the via at line a is equal to the flux of $\underline{u}_z J_{cz}(x)$ leaving the via at line c. Here, $\underline{u}_z$ is the unit vector in the z direction. Now, $(N_{\ell a} + N_{\ell c} - 3)$ equations remain to be satisfied.

Taking the number of functions $\{\underline{J}_{bj}\}$ in (B-1) equal to the number of equations to be satisfied, we have

$$N_b = N_{\ell a} + N_{\ell c} - 3 \tag{B-6}$$

Each of the first $(N_{\ell a} - 1)$ of the functions $\{\underline{J}_{bj}\}$ is chosen to be similar to $\underline{J}_{loop}$ of (104). There is a one-to-one correspondence between these $(N_{\ell a} - 1)$ functions and the $(N_{\ell a} - 1)$ triangle sides on line a. The function associated with a particular triangle side on line a is not zero on that triangle side and is zero on all other triangle sides on line a. This function is a current that flows from its associated triangle side on line a through the via to line c. Each of the remaining $(N_{\ell c} - 2)$ of the functions $\{\underline{J}_{bj}\}$ is chosen to be similar to $\underline{J}_j$ of (102) proper. There is a one-to-one correspondence between these $(N_{\ell c} - 2)$ functions and the $(N_{\ell c} - 2)$ interior nodes of $S_L$ on line c. The function associated with a particular interior node of $S_L$ on line c is a current that circulates halfway around the node. Being restricted to the via, this current can not completely encircle the node as $\underline{J}_j$ of (102) does.

Dividing both sides of (B-4) by I, taking all $(N_{\ell a} - 1)$ equations in (B-4), dividing both sides of (B-5) by I, and taking only the first $(N_{\ell c} - 2)$ equations in (B-5), we obtain the matrix equation

$$P_b \vec{I}_b = \vec{V}_b \tag{B-7}$$

where $\vec{V}_b$ and $\vec{I}_b$ are column vectors, and $P_b$ is a square matrix. The jth element of $\vec{I}_b$ is the unknown coefficient $I_{bj}$ in (B-1). The ith element of $\vec{V}_b$ is called $V_{bi}$ and is given by

$$V_{bi} = \frac{1}{(x^+_{bai} - x^-_{bai})I} \int_{x^-_{bai}}^{x^+_{bai}} J_{az}(x)dx, \qquad i=1,2,\ldots,N_{\ell a}-1 \tag{B-8}$$

and

$$V_{bi} = \frac{1}{(x^+_{bck} - x^-_{bck})I} \int_{x^-_{bck}}^{x^+_{bck}} J_{cz}(x)dx, \qquad i = N_{\ell a}, N_{\ell a}+1,\ldots,N_b \tag{B-9}$$

where $N_b$ is given by (B-6), and k is given by

$$k = i - N_{\ell a} + 1 \tag{B-10}$$

The ijth element of $P_b$ is called $P_{bij}$ and is given by

$$P_{bij} = (\underline{J}_{bj} \cdot \underline{n}_a)_{bai} \;, \quad \begin{cases} i=1,2,\ldots,N_{\ell a}-1 \\ j=1,2,\ldots,N_b \end{cases} \tag{B-11}$$

and

$$P_{bij} = (\underline{J}_{bj} \cdot \underline{n}_c)_{bck} \;, \quad \begin{cases} i=N_{\ell a},\ N_{\ell a}+1,\ldots,N_b \\ j=1,2,\ldots,N_b \end{cases} \tag{B-12}$$

where $N_b$ is given by (B-6), and k is given by (B-10). In (B-11), $\underline{n}_a$ is the unit vector that is tangent to $S_T$ on the via side of line a and perpendicular to line a. The vector $\underline{n}_a$ points from line a toward the via. The subscript bai in (B-11) denotes evaluation on the ith triangle side on line a. In (B-12), $\underline{n}_c$ is the unit vector that is tangent to $S_T$ on the via side of line c and perpendicular to line c. The vector $\underline{n}_c$ points from the via toward line c. The subscript bck in (B-12) denotes evaluation on the kth triangle side on line c.

Substitution of (61) for $J_{az}(x)$ in (B-8) gives

$$V_{bi} = \frac{1}{(x^+_{bai} - x^-_{bai}) \displaystyle\int_{x_a}^{x_a+w_a} \hat{J}_{az}(x')dx'} \int_{x^-_{bai}}^{x^+_{bai}} \hat{J}_{az}(x)dx, \quad i=1,2,\ldots,N_{\ell a}-1 \tag{B-13}$$

where $\hat{J}_{az}(x)$ satisfies (57), and $x^-_{bai}$ and $x^+_{bai}$ are, respectively, the x coordinates of the beginning and end of the ith triangle side on line a. Substitution of (75) for $J_{cz}(x)$ in (B-9) gives

$$V_{bi} = \frac{1}{(x^+_{bck} - x^-_{bck}) \int_{x_c}^{x_c+w_c} \hat{J}_{cz}(x')dx'} \int_{x^-_{bck}}^{x^+_{bck}} \hat{J}_{cz}(x)dx, \quad i = N_{\ell a}, N_{\ell a}+1, \ldots, N_b \tag{B-14}$$

where $\hat{J}_{cz}(x)$ satisfies (76), $N_b$ is given by (B-6), k is given by (B-10), and $x^-_{bck}$ and $x^+_{bck}$ are, respectively, the x coordinates of the beginning and end of the kth triangle side on line c.

After the elements of $P_b$ have been calculated from (B-11) and (B-12) and those of $\vec{V}_b$ from (B-13) and (B-14), the matrix equation (B-7) is solved for $\vec{I}_b$. Then, $\underline{J}_b$ is given by (B-1) in which $I_{bj}$ is the jth element of $\vec{I}_b$.

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