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Kolmogorov's and Mourier's Strong Laws For Arrays with Independent and Identically Distributed Columns

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Abstract

Almost sure convergence of averages of the rows is proven for arrays of random variables and for random elements in Banach spaces, which can be extended to square arrays with independent and identically distributed columns. The result requires that each column converge to a limiting random variable. A counterexample is given to show that the result fails without the condition on the columns.

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Let $\{X_{ij}\}$ for $i \le j < n_i + 1$, $i = 1,2,\ldots$ be an indexed set of real-valued random variables on a probability space (Ω, B, P). We shall say that $\{X_{ij}\}$ is a square array if $n_i = \infty$ for all i, a triangular array if $n_i = i$, and an array if n_i is any sequence increasing to infinity. Denote by $X_{\cdot j}$ the column $(X_{ij}, X_{i+1,j}, X_{i+2,j}, \ldots)$ where $i = \inf\{\ell : j \le n_\ell\}$ is the first index for which X_{ij} is defined. The columns $X_{\cdot j}$ are random variables taking values in R^∞, the space of sequences of real numbers. Independence of the columns is defined as usual, but unlike the usual definition, $X_{\cdot j}$ and $X_{\cdot k}$ will be called identically distributed if for $i = \inf\{\ell : j \le n_\ell$ and $k \le n_\ell\}$, the random sequences $(X_{ij}, X_{i+1,j}, \ldots)$ and $(X_{ik}, X_{i+1,k}, \ldots)$ have the same distribution.

Suppose an array has independent and identically distributed columns. By analogy with Kolmogorov's law for i.i.d. sequences, one might suppose that if $EX_{ij} = 0$ for all i and j then $\lim_{i\to\infty} n_i^{-1} \sum_{j=1}^{n_i} X_{ij} = 0$ almost surely. However, this is not correct, as example 1 below demonstrates. In Theorem 1, we obtain a strong law with the additional condition that each column (equivalently, the first column) converges to a limiting random variable. This result is an extension of Kolmogorov's Law for i.i.d. sequences. The method yields the same result for X_{ij} random elements in a separable Banach space. This result is given in Theorem 2, which extends Mourier's (1953) law of large numbers to arrays.

Example 1: Let every element of a triangular array be i.i.d. obeying a symmetric stable law with characteristic function $e^{-|t|^\alpha}$, for some $1 < \alpha < 2$. Two properties of this distribution are that $\sum_{j=1}^{n} X_{nj}$ has the same distribution as $n^{1/\alpha} X_{11}$, and that $E|X_{11}|^\beta = \infty$ for all

$\beta \geq \alpha$. Letting $f(x)$ be the density of X_{11},

$$\sum_{n=1}^{\infty} P[\sum_{j=1}^{n} X_{nj} > n] = \sum_{n=1}^{\infty} P[X_{11} > n^{1-1/\alpha}]$$

$$\geq \int_1^\infty \int_{n^{1-1/\alpha}}^\infty$$

$$= \int_1^\infty x^{\alpha/(\alpha-1)} f(x)\, dx = \infty, \qquad (1)$$

the next to last inequality resulting from an application of Fubini's theorem, and the final equality follows from the non existence of the $\alpha/(\alpha-1)$ moment coupled with the fact that X_{11} has a symmetric distribution. The sequence $n^{-1} \sum_{j=1}^{n} X_{nj}$ is independent for different n, so the Borell-Cantelli theorem implies that $n^{-1} \sum_{j=1}^{n} X_{nj} > 1$ infinitely often, and hence it does not converge to zero almost surely.

Evidently additional conditions are necessary to obtain a strong law. Theorem 1 below shows that convergence of the first column to a limiting random variable suffices. We do not know if it is possible to further weaken this condition.

Theorem 1: Let $\{X_{ij}\}$ be an array of real variables with independent and identically distributed columns. Suppose that the first column converges almost surely, i.e. there exists a random variable X_1 such that $\lim_{i\to\infty} X_{i1}(\omega) = X_1(\omega)$ a.s. If $\lim_{i\to\infty} E(x_{i1}) = 0$ and $E(\sup_i |X_{i1} - X_1|) < \infty$, then for almost all $\omega \in \Omega$,

$$\lim_{n\to\infty} \frac{1}{n_i} \sum_{j=1}^{n_i} X_{nj} = 0$$

Proof: This is a special case of theorem 2, and does not appear to have a simpler proof.

For the case where the array consists of random elements in a separable Banach space, we have the following result, given in theorem 2.

Theorem 2: Let $\{X_{ij}\}$ be an array of random elements in a separable Banach space B. Suppose the array has independent and identically distributed columns, and there exists a sequence of random elements X_j such that $\lim_{i\to\infty} X_{ij}(\omega) = X_j(\omega)$ almost surely. Defining EX_{ij} via the Bochner intergral, if $\lim_{i\to\infty} EX_{i1} = 0$, and $E(\sup_i \|X_{i1} - X_1\|) < \infty$, then

$$\lim_{i\to\infty} \left\|\frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}\right\| = 0,$$

almost surely.

Proof: Let $\{X'_{ij}\}$ be a square array on a probability space $(\Omega', \mathcal{B}, P')$, such that the columns $X'_{\cdot j}$ are i.i.d. copies of $X_{\cdot 1}$. Define $C_0(B)$ to be the Banach of sequences converging to 0 in B, normed in the obvious way be $|||\cdot|||$, defined as

$$|||(b_1,b_2,\ldots)||| = \sup_i \|b_i\|,$$

where $\|\cdot\|$ is the norm on B. It is easily verified that $C_0(B)$ is separable. Define random elements Y_j in $C_0(B)$ by

$$Y_j = (X'_{1j}-X_j, X'_{2j}-X_j, \ldots).$$

It is easily checked that $\lim_{i\to\infty} X_{ij} = X_j$ implies $\lim_{i\to\infty} X'_{ij} = X_j$, and hence Y_j take values in $C_0(B)$, the space of sequences convergent to zero. Clearly Y_j are also i.i.d. Also, $E|||Y_j||| = E(\sup_i \| X_{ij} - X_i \|) < \infty$ by assumption. Thus Mourier's (1953) Law of Large Numbers applies, and implies that

$$\lim_{n\to\infty} |||\frac{1}{n} \sum_{j=1}^{n} Y_j - EY_1||| = 0,$$

almost surely. It is easily checked that this implies the result.

Some concluding comments are in order. Strong laws for arrays, unlike weak laws (see Petrov (1975), for example) or the central limit theorem, appear not to have been widely studied. Thus there do not exist any results for arrays directly comparable to ours. The closest one can come is the following result. Under the conventional assumption that rows consist of independent variables, and no restriction on the columns, the average of the rows centered to have zero means will converge almost surely to zero, provided that $\sup_{i,j} E|X_{ij}|^{2+\epsilon} < \infty$, for some $\epsilon > 0$. This is easily proven by an appeal to the Borel-Cantelli Lemma, and is also a special case a result due to Taylor (1979).

In terms of applications, the following corollary may be of interest.

<u>Corollary to Theorem 1</u>: Let $\{X_i\}$ be a sequence of i.i.d. real-valued random variables. Let ϕ_n: $R \to R$ be a sequence of measurable functions such that $\lim \phi_n(x) = \phi(x)$ for some limiting measurable function ϕ, for all $x \in R$. If $\lim_{n\to\infty} E\, \phi_n(X_1) = 0$, and $E \sup|\phi_n(X_1)| < \infty$, then $\frac{1}{n} \sum_{j=1}^{n} \phi_n(X_j)$ converges to zero almost surely.

Finally, we note that the problem of proving strong convergence of kernel density estimates can be posed as a question about convergence in norm of the averages of rows of a certain array of random variables. Taylor (1979) uses this approach supplemented with a strong law for arrays of random elements in a Banach space to obtain strong convergence of kernel density estimates. Although the array of random elements used in this approach has independent and indentically distributed columns, Taylor's strong law uses only the row-wise independence. It seems plausible that use of the i.i.d. structure of the array should permit us to obtain a stronger result. Unfortunately, our Theorem 2 cannot be applied since the columns fail to converge to a limit, in the case of kernel density estimation. Perhaps some variant of Theorem 2 can be obtained whcih will generate the desired result.

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