Properties of gravity waves at the interface between two unbounded fluids of different densities in relative motion are examined. The conditions which limit the existence of steady waves are determined.
1. **INTRODUCTION.**

We consider gravity waves at the interface between two uniform, unbounded fluids of different densities in the presence of a current or relative horizontal velocity $U$. The fluids are supposed to be immiscible, incompressible and inviscid, and the motion is assumed to be irrotational. We are concerned with the properties and existence of finite amplitude two-dimensional, periodic waves of permanent form which propagate steadily without change of shape. By two-dimensional, we mean that the flow field depends only on the horizontal direction of propagation, which will be the $x$-axis, and the vertical $y$-direction. In the field of surface gravity waves, which is the limit of the present study when the density of the upper fluid is zero, it has been found recently that three-dimensional waves of permanent form exist and are observed experimentally (see e.g. [5]). It is expected that such waves will also exist and be important for interfacial waves, but they will not be considered in the present work.

For the purpose of calculating steady waves, there is no loss of generality in taking the speed of propagation $c$ parallel to the current $U$, as an arbitrary constant transverse velocity may be linearly superposed on any...
two-dimensional steady wave without affecting its properties (the stability characteristics would, however, be affected). The wave can be reduced to rest by choosing a frame of reference moving with the wave. The problem is then to calculate steady irrotational solutions of the Euler equations which satisfy continuity of pressure across a common streamline. It follows from dimensional analysis that apart from scaling factors all flow variables will depend upon three dimensionless parameters:

\[
\frac{h}{L}, \frac{\rho_2}{\rho_1}, \frac{\rho_2 U^2}{\rho_1 g L},
\]

where \( h \) is the height of the wave defined as the vertical distance between crest and trough, \( L \) is the wavelength (the horizontal distance over which the flow field repeats itself which in the present work will be the distance between crests), \( \rho_2 \) and \( \rho_1 \) are the densities of the upper and lower fluid respectively, and \( g \) is the acceleration due to gravity. For example, the speed of the waves is given by

\[
c = (g L/2\pi)^{1/2} C \left( \frac{h}{L}, \frac{\rho_2}{\rho_1}, \frac{\rho_2 U^2}{\rho_1 g L} \right)
\]

where \( C \) is a dimensionless function of its arguments. For surface waves, where \( \rho_2 = 0 \) and there is dependence on only one parameter, namely \( h/L \), it is known that many interesting and unexpected phenomena exist, especially when the wave steepness becomes large. When there is dependence on three parameters, it is to be expected that many more phenomena are likely. However, in the absence of exact solutions for large \( h/L \), it is a highly non-trivial task to search a three-dimensional parameter space. The results to be presented below are limited to those phenomena which seem currently to be of the most interest.

In contrast to the voluminous work on surface waves, relatively little seems to have been done on interfacial waves of permanent form, and that work seems to have been confined to the case of zero current, i.e. \( U = 0 \). Tsuji
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and Nagata [7] calculated Stokes type expansions to order \((h/L)^5\), and Holyer [3] used the computer to compute the coefficients in such an expansion to order \((h/L)^3\). Nagata then used Padé approximants to estimate the behavior for large \(h/L\). We are not aware of any work for waves with current.

For the mathematical formulation, there is no loss of generality in taking \(g = 1\), \(L = 2\pi\), and \(\rho_1 = 1\). The mathematical problem is to determine the \(x\)-periodic velocity potentials and stream functions, for the lower and upper fluid respectively, which satisfy Laplace's equation and are harmonic conjugate pairs, so that at the unknown interface \(y = Y(x)\),

\[
\psi_1(x, Y(x)) = 0, \quad \psi_2(x, Y(x)) = 0, \tag{1.3}
\]

\[
\frac{1}{2} (\psi_1 Y(x)) + Y(x) + b = \frac{1}{2} \rho_2 (\psi_2 Y(x)) + \rho_B Y(x). \tag{1.4}
\]

In general, \(\rho_B = \rho_2\), but we allow for the possibility of Boussinesq waves (in which the inertia of the two fluids is the same and density differences only matter when multiplied by \(g\)) by setting \(\rho_2 = 1\) and \(\rho_B = 0\). Surface tension is neglected throughout. The quantity \(b\) is the Bernoulli constant, which by suitable choice of the origin of pressure may be set equal to zero in the lower fluid. Infinitely far from the interface, we have

\[
\psi_1 \sim -Cx, \quad \psi_2 \sim (U - c)x. \tag{1.5}
\]

The vertical origin is set by requiring that the mean elevation of the interface is zero and the horizontal origin can be fixed by placing the crest at \(x = 0\). This problem now appears to be free of arbitrary constants and the wave is determined by the crest to trough height \(h\). It is expected that isolated families of solutions exist in connected regions in \((h, \rho_2, U)\) space, although this does not yet appear to have been proved.

One question of considerable interest is the domain of parameter space in which solutions exist. Suppose that we consider a fixed value of \(\rho_2\) and vary \(h\) and \(U\). It is found that as \(U\) increases with \(h\) kept constant, the
system of equations describing steady solutions fails to have a solution, even though the 'limiting' wave profile is smooth and exhibits no singular properties. For $U > 0$ there are, when solutions exist, at least two physically distinct waves corresponding to the two wave speeds for propagation with and against the current. As $U$ increases, the wave propagating against the current is 'entrained' by the current and at a certain value of $U$, which depends on $h$ and $\rho_2'$, the two waves become identical and for larger $U$ there are no real solutions of the equations. Mathematically, this is like the disappearance of roots of a quadratic. We shall term this factor which limits existence a 'dynamical limit'.

The second factor is what we term a 'geometrical limit'. The mathematical formulation remains well-behaved but the solutions cease to make physical sense as the wave profiles cross themselves. This occurs for fixed $U$ and increasing $h$. Examples of this phenomenon are found in pure capillary and capillary-gravity waves [2,1] for which the wave profile crosses itself at a critical value of $h$. If $U \neq 0$, this limit is going to be different for the two solutions of waves moving with and against the current. In the case of surface waves, this limit corresponds to a 120° cusp. It is easy to see that except for two special cases (see §4), this cannot happen for interfacial waves. Holyer [3] identified the geometrical limit for $U = 0$ with the existence of a vertical tangent. We shall present evidence that waves can exist with a vertical tangent and significant overhang, and the evidence indicates that the geometrical limit is associated with the wave crossing itself when it is sufficiently high for $U > 0$.

2. WEAKLY NONLINEAR WAVES.

The properties of weakly nonlinear steady waves may be obtained by using the Stokes expansion in which all variables are expanded as power series in $h/L$. However, the algebra can be simplified somewhat by using Whitham's variational approach. Proceeding in the usual manner, one finds after some algebra that the average Lagrangian is
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\[
L = \frac{1}{4} (\rho_B - 1)(\frac{1}{4} h^2 + a_2^2)
\]
\[
+ \frac{h^2}{16k} \left[ \omega^2 + \rho_2(Uk - \omega)^2 \right] - \left( \frac{a_2^2}{2k} + \frac{kh^2}{256} \right) \left[ \omega^2 + \rho_2(Uk - \omega)^2 \right]
\]
\[
- \frac{h^2a_2}{16} \left[ \omega^2 - \rho_2(Uk - \omega)^2 \right] + O(h^6)
\]

(2.1)

for the wave with interface shape

\[
\gamma(x) = \frac{h}{2} \cos(kx - \omega t) + a_2 \cos 2(kx - \omega t)
\]

(2.2)

The value of \( a_2 \) is found from \( \partial L / \partial a_2 = 0 \) to be

\[
a_2 = \frac{h^2}{8(1 - \rho_B)} [C^2 - \rho_2(U - C)^2] + O(h^4)
\]

where \( C = \omega/k \) is the phase speed. The dispersion relation for the weakly nonlinear wave then follows from \( \partial L / \partial h = 0 \):

\[
C^2 + \rho_2(U - C)^2
\]

\[
= (1 - \rho_B) \left[ 1 + \frac{h^2}{8} \left( \frac{2C^2}{1 - \rho_B} - 1 \right)^2 + \frac{h^2}{8} \right] + O(h^4).
\]

(2.4)

For \( U = 0 \), the values of \( C \) agree with those in [7].

The values of the energy, momentum and action densities and fluxes follow from the expression (2.1) for \( L \) in the usual way. In particular, the total energy density \( E \) is given by

\[
E = kCL - L.
\]

(2.5)

It is to be noted that for \( U > 0 \), the energy is measured relative to the energy of the uniform state with a flat interface. Negative energies may therefore exist and mean that the energy of the state with waves is less than that of the undisturbed flow.

It follows from the dispersion relation (2.4) that for linear waves \( (h = 0) \) and given values of \( \rho_2 \) and \( U \), there are two solutions corresponding to the two roots of the quadratic equation for \( C \) in terms of \( \rho_2 \) and \( U \). We denote these two solutions by \( C_+ \) and \( C_- \), where
C_+ > C_- For the linear case, steady solutions cease to exist when U exceeds a critical value \( U_{c0} \) given by

\[
U_{c0} = \left[ (1 + \rho_2)(1 - \rho_B)/\rho_2 \right]^{1/2}
\]  

(2.6)

for which the two wave speeds are equal with the value

\[
C_+ = C_- = \rho_2 U_{c0}/(1 + \rho_2).
\]

The values of \( C_+ \) and \( C_- \) are

\[
\rho_2 U \pm \sqrt{\rho_2^2 U^2 - (1 + \rho_2)(\rho_2 U^2 - 1 + \rho_B)}^{1/2}.
\]  

(2.7)

For \( U = 0 \), the values are equal and opposite. As U increases, the speed of the wave propagating with the current originally increases but eventually decreases. The speed of the wave propagating against the current increases monotonically (in the algebraic sense), becomes zero when \( U = [(1 + \rho_B)/\rho_2]^{1/2} \) and then increases to equal \( C_+ \) when U is given by (2.6). According to the linear approximation, the energy density \( E \) equals \( \frac{1}{8} h^2 [C^2(1+r) - rCU] \), and it is interesting that the energy becomes negative when the direction of the \( C_- \) waves changes.

For finite amplitude waves, the two solutions corresponding to \( C_+ \) and \( C_- \) waves continue into two families of solutions marked by wave speeds \( C_+(h, \rho_2, U) \) and \( C_-(h, \rho_2, U) \). For any given value of \( h \) and \( \rho_2 \), there will again be a critical current \( U_c \) beyond which steady solutions no longer exist. For the weakly nonlinear approximation, this value is given by

\[
U_c = U_{c0}[1 + \frac{h^2}{4}(1 + \rho_2^2)]^{1/2}
\]  

(2.9)

It is noteworthy that increasing \( h \) increases \( U_c \).

3. **NUMERICAL METHODS.**

For values of \( h \) that are not small, it is necessary to employ numerical methods. Three different techniques were employed. The first was to compute in physical space, i.e. the interface, potentials and stream function were expanded as Fourier series in \( x \) with coefficients which
are exponential in $y$. The series were truncated to $N$ modes and the boundary conditions were then satisfied at $N + 1$ equally horizontally spaced points on the interface. This procedure gives $3N + 4$ equations for $3N + 4$ unknowns. These equations were solved by Newton's method, using continuation in either $U$ or $h$ to give the first guesses. Note that this formulation is essentially equivalent to calculating numerically the coefficients of the Stokes expansion as done in [3].

The second method used the potential and stream function as the independent variables and expands the physical coordinates as series in these. The boundary conditions are now satisfied at equally spaced values of the velocity potential and the resulting system of $3N + 3$ equations in $3N + 3$ variables, the expansions being truncated to $N$ modes, was also solved by Newton's method with continuation in $U$ and $h$ employed to give a first guess.

The third method used a vortex sheet representation in which the unknowns are the shape of the interface and the dipole strength of the equivalent double layer. This gives a nonlinear integrodifferential equation, which was solved by discretization and collocation, the resulting system of nonlinear equations again being solved by Newton's method with continuation.

For details, see [4,5]. All methods worked extremely well for small values of $h/L$, which generally meant $h < 0.6$, with some dependence on $\rho_2$ and $U$. (With our scaling, the surface wave of greatest height has $h = 0.89$). The first method was the first to fail as $h$ increased. It is of course clear that this approach of working in physical space must fail when the wave becomes very steep, but the failure, marked by the apparent failure of the Fourier series to converge, seemed to be due to other causes. What actually happened was that the singularities of the analytic continuation of the lower velocity potential, say, into the upper half plane moved down below the crest. In this case, the expansion of the velocity potential would have to diverge near the crest, even though the solution was
perfectly well behaved and physically meaningful. This difficulty would not affect the other two methods which were used for values of $h$ up to 1.2 for various values of $\rho_2$ and $U$. For large values of $h$, 100 modes were used in the second method and this seemed adequate except for the largest $h$. The vortex sheet method with 65 intervals was employed for this case. This method offers in principle the advantage of being able to concentrate points near regions of high curvature, although this was not done.

The accuracy of the calculations was checked by comparing the results of the somewhat different methods with each other in regions of apparent validity and by performing the usual tests of internal consistency by investigating the dependence on number of retained modes. The calculations were carried out on a PRIME 750 and the CRAY-1 at NCAR.

4. A SPECIAL CLASS OF SOLUTIONS.

It is interesting to note that a special class of solutions exist which are simple transformations of the well-known surface permanent wave solutions, which have been extensively studied both numerically and theoretically by many authors. For each value of $\rho_2$, these solutions describe the shape of the interface for the $C_+$ case when

$$C_+ = U = (1 - \rho_B)^{1/2}C_s(h) \quad (4.1)$$

where $C_s(h)$ is the wave speed of the surface wave of permanent form for the given wave height $h$. Since $C_+ = U$, the upper fluid is stagnant in the wave-fixed coordinates. The dynamic boundary condition for the motion in the lower fluid then becomes that for surface waves with a reduced gravity $g(1 - \rho_B)^{1/2}$. The velocities and wave speed are therefore those of the surface wave multiplied by the factor $(1 - \rho_B)^{1/2}$.

For the $C_-$ branch, special solutions exist with

$$C_- = 0, \quad U = [(1 - \rho_B)/\rho_2]^{1/2}C_s(h) \quad . \quad (4.2)$$

In this case, the lower fluid is stagnant and the dynamic boundary condition on the motion of the upper fluid is that with a reduced upside down gravity. The wave profiles are
inverted surface waves, with negative gravity multiplied by the factor \[ \frac{(1 - \rho_B)}{\rho_2} \] 

These special solutions have geometrical limits when \( h = 0.892 \), where the waves have a corner at the crest for the \( C_+ \) wave, and a corner at the trough for the \( C_- \) wave, with an interior angle of 120°. However, these special geometrical limits are only for the case when one of the fluids is moving with the wave. In general, it is expected (see below) that the geometrical limit is associated with the wave surface crossing itself.

5. RESULTS.

The equations have been solved numerically for various values of \( h, r = \frac{\rho_2}{\rho_1}, \) and \( U \). The existence of the dynamical limit was confirmed, and it was found that the weakly nonlinear approximation (2.9) is a good approximation for values of \( h \) up to 0.6. A typical set of results is shown in figure 1. These results are for \( r = 0.5 \) and \( h = 0.6 \), and show the wave speeds \( C \), total energies \( E \), and kinetic energies \( T \) for both the \( \pm \) waves as functions of the current velocity \( U \). The existence of the dynamical limit where \( C_+ = C_- \) is clearly demonstrated.

One feature of remarkable interest is the existence of a region of negative energy. This implies that there will be a range of parameters in which the energy of the state with finite amplitude waves is less than that with the same current and a flat surface. Spontaneous generation of such flows is then a definite possibility. The computed results and linear analysis suggest that negative energies appear when \( C_- \) is zero and continue for values of \( U \) up to that for which the dynamical limit is reached. This aspect of the solutions needs to be explored further in detail.

The geometrical limit or the shape of the wave of greatest height has been addressed for the case of zero current. Solutions were obtained for three values of \( r \) (1.0, 0.9, 0.1), using the vortex sheet method as this seemed to provide the best resolution when the waves are large. Values of the wave speed \( C \) for \( \pm \) waves are shown in figure 2. For the larger values of \( r \), it was possible to calculate solutions with vertical slope and the shapes
Figure 1. Properties of interfacial waves as functions of current velocity $U$ for a fixed wave height $h = 0.6$ and density ratio $r = 0.5$. Wave speed $C$, total energy $E$, and kinetic energy $T$ are shown for both branches.
Figure 2. Phase speed $C$ vs wave steepness $a/\pi = h/L$ for $r = 0.1, 0.9, 1.0$. $x$ denotes point of vertical tangency.
Figure 3. Profiles of steady interfacial waves for $r = 0.9$. The x-axis is the mean level.
Figure 4. Profiles of steady interfacial waves for $r = 1.0$. The x-axis is the mean level. These waves are symmetrical about $y = 0$. 
are shown in figures 3 and 4 for $r = 0.9$ and $r = 1.0$, respectively. These demonstrate clearly the existence of waves of permanent form with a substantial overhang region in which heavy fluid lies on top of light fluid. It is interesting to note that the fluid particles on the interface between the points of vertical tangency in the overhang region are moving faster than the wave. For the smallest value of the density ratio, we did not have sufficient resolution to distinguish the wave shape near the geometrical limit. This difficulty is to be expected, since the smaller the density ratio, the closer the geometrical limit will be to the 120° cusp and the smaller the size of the overhang region.

REFERENCES

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