STATE ESTIMATION AND CONTROL OF CONDITIONALLY LINEAR SYSTEMS*

Wojciech J. Kolodziej and Ronald R. Mohler
Department of Electrical and Computer Engineering
Oregon State University
Corvallis, Oregon 97331

Submitted for publication to:
SIAM J. Control and Optimization

Prepared for:
The Office of Naval Research
under
Contract No. N00014-81-K0814
R.R. Mohler, Principal Investigator

April 1984
STATE ESTIMATION AND CONTROL OF CONDITIONALLY LINEAR SYSTEMS*
WOJCIECH J. KOLODZIEJ¹ AND RONALD R. MÖHLER¹,²

Abstract. The filtering problem for a partially observable stochastic system, with linear in observable states dynamics and non-Gaussian initial conditions is studied here. It is shown that the conditional expected value of the unobservable states, given the past observations, can be expressed in terms of a finite dimensional set of statistics. This result, which generalizes the conditionally Gaussian filter is used to derive a separation principle for a linear-quadratic control problem.

Key Words. Optimal filtering, stochastic control, non-Gaussian stochastic systems.

*Sponsored by ONR Contract No. N00014-81-K0814
¹Department of Electrical and Computer Engineering, Oregon State University, Corvallis, OR 97331.
²During 1983-84, NAVALEX Professor, Department of Electrical and Computer Engineering, Naval Postgraduate School, Monterey, CA 93943
Introduction

Stochastic, partially observable systems, with linear-in-observable state dynamics are termed conditionally linear systems here. It is well known that the solution of a state estimation problem for a conditionally linear system with Gaussian distribution of the initial state is given in terms of two sets of sufficient statistics, satisfying stochastic differential equations [4].

Solved here is the state estimation problem which generalizes the above result for the case of an arbitrary a priori distribution. The method applied in this study is based on the derivation of an explicit formula for the conditional characteristic function of the state, given the past and present observations. This approach seems to impose less restrictive conditions on the system structure than the methods based on the derivation of the conditional distribution function. The latter can be found in [1] where the filter is derived for a linear system with a priori distribution having a well-defined density function.

It is shown here that the conditional characteristic function of the present and past states, given the present and past observations, is parametrically determined by a finite number of sufficient statistics. This result leads to the derivation of a filter, in the form of a finite set of stochastic differential equations which extends the result of [1] in a similar manner as a conditionally Gaussian filter generalizes a Kalman filter.

Also discussed here and illustrated by the examples, is the suitability of the filter structure for the study of stochastic control and parameter estimation.

1. Problem Formulation and the Main Result

Given the following system of stochastic differential equations
(1.1) \[ dx_t = (f_0(t,y) + f_1(t,y)x_t) \, dt + g_0(t,y)dw_t + q_0(t,y)dv_t, \]

(1.2) \[ dy_t = (h_0(t,y) + h_1(t,y)x_t) \, dt + dv_t, \quad 0 \leq t \leq T, \]

where \( f_0, f_1, g_0, q_0, h_0, h_1 \) are the nonanticipative functionals of \( y \) (i.e., \( \mathcal{Y}_t \) measurable with \( \mathcal{Y}_t = \sigma - \text{alg} \{ y_s \mid 0 \leq s \leq t \} \)), and \( w_t, v_t \) are independent Wiener processes.

The objective is to find \( \hat{x}_t = \mathbb{E}(x_t / \mathcal{Y}_t) \), assuming that \( x_t, y_t \) satisfy (1.1) and (1.2), and that the conditional distribution of the initial states \( F(x) = P(x_0 < a \mid y_0) \) is given.

The organization of this section starts with Lemma 1, whereby it is shown that the conditional characteristic function of \( (x_{t0}, x_{t1}, ..., x_{tn}) \mid \mathcal{Y}_t \), for an arbitrary decomposition \( 0 < t_0 < t_1 < ... < t_n < t \leq T \), of the interval \([0,T]\) is of a particular form. Results from the theory of conditionally Gaussian processes are used here.

Next, Lemma 2, the explicit formula for the characteristic function of \( x_t \mid \mathcal{Y}_t \) is derived, and finally, in Lemma 3, all the results are organized to yield the recursive, finite-dimensional set of filter equations.

The assumptions used in the proof of Lemma 1 and 2 are listed below:

Let \( C_T \) denote the space of continuous functions \( \eta = \{ \eta_t, 0 \leq t \leq T \} \). It is assumed that for each \( \eta \in C_T \),

(1.3) \[ \int_0^T \left( \sum_{k=0}^{\infty} \left( |f_k(t,\eta)| + |h_k(t,\eta)| + |g_0(t,\eta)|^2 + |q_0(t,\eta)|^2 \right) \right) \, dt < \infty. \]

The above assumption assures the existence of the (Ito) integrals in (1.1) and (1.2) [3]. In order to use the results for conditionally Gaussian processes it is also assumed that [4]:
(1.4) for all \( n \in \mathbb{C}_t \), \( t \in [0,T] \), \(|f_1(t,n)| + |h_1(t,n)| < \text{const} \), and

\[
\int_0^T \mathbb{E}(|f_0(t,y)|^4 + |g_0(t,y)|^4 + |q_0(t,y)|^4)dt < \infty, \quad \mathbb{E}(|x_0|^4) < \infty
\]

Lemma 1.

Let

\[
\phi_t = \exp \left( i \sum_{k=0}^n z_k x_{t_k} \right), \quad z = \begin{bmatrix} z_1 \\ \vdots \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{R}^n, \quad 0 < t_0 < t_1 < \ldots < t_n < t < T.
\]

Then the conditional characteristic function of \((x_{t0}, x_{t1}, \ldots, x_{tn}) \mid Y_t\) is given by

\[
e_t(z) = \mathbb{E}(\phi_t \mid Y_t) = \int_{-\infty}^{\infty} \exp \left( Q(t,a,z,y) \right) dF(a)
\]

where \( Q(t,a,z,y) \) is quadratic in the variables \( a \) and \( z \).

Proof of Lemma 1

First notice that (1.1) solves as

\[
x_t = \phi_t(x_0 + \int_0^t \phi_s^{-1}(f_0 - q_0 h_0)ds + \int_0^t \phi_s^{-1} g_0 dy_s + \int_0^t \phi_s^{-1} q_0 dw_s)
\]

where \( \phi_t = \exp \left( \int_0^t (f_1 - g_0 h_1)ds \right) \).

Rewrite (1.7) in the symbolic way as

\[
x_t = \psi_t(x_0, w, y).
\]
Now, the following version of the Bayes formula will be used [2, p. 8]:
Let $\phi_t(x_0,w,y)$ be a nonanticipative functional of its arguments with $E(|\phi_t|) < \infty$ for all $t \in [0,T]$. Then

\begin{equation}
E(\phi_t | Y_t) = \int_{C_T} \int_{-\infty}^{\infty} \phi_t(a,n,y) \rho_t(a,n,y) d\nu_w(n) dF(a)
\end{equation}

where $\nu_w$ is a Wiener measure in the measurable space of continuous functions $\eta$ on $[0,T],
\rho_t(a,n,y) = \exp\left(\int_0^t (\psi_s(a,n,y) - \hat{x}_s(y)) dv_s - \frac{1}{2} \int_0^t h_1^2(\psi_s(a,n,y) - \hat{x}_s(y))^2 ds\right)
\end{equation}

with $dv_s = dy_s - (h_0 + h_1 \hat{x}_s) ds$, and $\psi_s(a,n,y)$ defined by (1.8). The random process $v_t$ can be represented by

$$v_t = \int_0^t (dy_s - (h_0(s,y) + h_1(s,y) \hat{x}_s(y)) ds) = v_t + \int_0^t h_1(s,y)(x_s - \hat{x}_s(y)) ds .$$

Now using the Ito formula we have

$$e^{-izv_t} = e^{-izv_s} + iz \int_s^t h_1(\tau,y)e^{-izv_\tau}(x_\tau - \hat{x}_\tau(y)) d\tau$$

$$+ iz \int_s^t e^{-izv_\tau} dv_\tau - \frac{2}{2} \int_s^t e^{-izv_\tau} d\tau .$$
Multiplying both sides of the above equation by $e^{izv_s}$ and taking the conditional expectation $E(\cdot | Y_s)$ gives

$$E(e^{iz(v_t-v_s)}| Y_s) = 1 - \frac{z^2}{2} \int_s^t E(e^{iz(v_{\tau}-v_s)}| Y_s) d\tau.$$ 

Solving the last equation yields

$$(1.11) \quad E(e^{iz(v_{t}-v_s)}| Y_s) = e^{-\frac{z^2}{2}(t-s)},$$

which shows that $(v_t, Y_t)$ is a Wiener process.

Now rewrite $\rho_t(a, n, y)$ in a more convenient form. To this end introduce the following notation:

$$A_1(t,y) = h_1(\phi_t^{-1}(\xi_0 - q_0 h_0)ds + \int_0^t \phi_s^{-1} q_0 dy) - x_t,$$

$$A_2(t,y) = h_1 \phi_t,$$

$$A_3(t,y) = \phi_t^{-1} \xi_0,$$

$$C_1(t,y) = \int_0^t A_1(s,y)dv_s - \frac{1}{2} \int_0^t A_1^2(s,y)ds,$$

$$C_2(t,y) = \int_0^t A_2(s,y)dv_s - \int_0^t A_1(s,y)A_2(s,y)ds,$$

$$C_3(t,y) = (\int_0^t A_2^2(s,y)ds)^{1/2},$$

$$C_4(t,y,w) = \int_0^t A_2(s,y) \int_0^s A_3(\tau,y)dw_{\tau}dv_s - \int_0^t A_1(s,y)A_2(s,y) \int_0^s A_3(\tau,y)dw_{\tau}ds.$$
\[ C_5(t,y,w) = - \int_0^t A_2^2(s,y) \int_0^s A_3(s,y) dw \, ds. \]

Note also that \( C_4(t,y,w) \) and \( C_5(t,y,w) \) can be rewritten with the use of the Ito formula by:

\[
C_4(t,y,w) = \int_0^t A_4(t,s,y) dw \quad ,
\]
\[
C_5(t,y,w) = \int_0^t A_5(t,s,y) dw \quad ,
\]

where

\[
A_4(t,s,y) = \left( \int_0^t A_2(s,y) dv - \int_0^s A_2(\tau,y) d\tau \right) - \left( \int_0^t A_1(s,y) A_2(s,y) ds - \int_0^s A_1(\tau,y) A_2(\tau,y) d\tau \right) A_3(s,y)
\]
\[
A_5(t,s,y) = \left( \int_0^s A_2^2(\tau,y) d\tau \right) - \int_0^t A_2^2(s,y) ds \right) A_3(s,y) \quad .
\]

Now, using the above notation we have from (1.8) and (1.10)

\[
\rho_t(a,w,y) = \exp(C_1 + a(C_2 + C_5) + C_4 - \frac{a^2}{2} C_3^2 - \frac{1}{2} \int_0^t A_2^2(\int_0^s A_3 dw) dt ds)
\]
\[
= \exp(C_1 + aC_2 - \frac{a^2}{2} C_3^2 + \int_0^t (aA_5 + A_4) dw - \frac{1}{2} \int_0^t A_2^2(\int_0^s A_3 dw) dt ds) .
\]

(1.12)

The arguments in (1.12) were omitted for brevity.

From (1.8) it follows that
\[ x_t = \phi_t(x_0, w, y) = \phi_t(x_0 + A_5(t, y) + \int_0^t A_3(s, y)dw_s) \]

where

\[ A_5(t, y) = \int_0^t \phi_s^{-1} (f_0 - q_0 h_0) ds + \int_0^t \phi_2^{-1} q_0 dy_s. \]

Combining (1.12) and the above

\[ \exp(Q(t, a, z, y)) = \int_{C_T} \phi_t(a, n, y) \rho_t(a, n, y) du_w(n) \]

\[ = \exp(C_1 + aC_2 - \frac{a^2}{2} C_3^2 + a(\sum_{k=1}^{n} \phi_{tk}iz_k)) \]

\[ + \sum_{k=1}^{n} \phi_{tk}A_6(t_k, y)iz_k \int \exp(\int_0^t (aA_3 + A_4) d\eta_s) \]

\[ + \sum_{k=1}^{n} iz_k \phi_{tk} \int k A_3 d\eta_s - \frac{1}{2} \int A_2^2(\int A_3 d\eta_s)^2 ds du_w(n). \]

(1.13)

In order to evaluate the integral in (1.13) the following results will be used:

(i) Since the above integral represents a conditional expected value of its integrand, under the condition that \( y_s, s \in [0, t] \) and \( x_0 = a \) are given, the resulting distributions are of conditionally Gaussian type [4]. Note that this fact does not depend on the \( F(a) \).

(ii) With all the variables in (1.13) being conditionally Gaussian we can use a convenient theorem:
Theorem [4, pp. 12-13]

Let \( \omega_t, t \in [0,T] \) be a Wiener process and let \( R(t), G(t), \) and \( H(t) \geq 0 \) be such that

\[
\int_0^T (|R(t)| + G(t)^2 + H(t)) \, dt < \infty.
\]

Then for all \( t \in [0,T] \)

\[
(1.14) \quad \mathbb{E} \left( \exp \left( \int_0^t R(s)G(s) \, dw_s - \int_0^t H(s) \left( \int_0^s G(\tau) \, d\tau \right)^2 \, ds \right) \right)
= \exp \left( \frac{1}{2} \int_0^t D(t) \, dt + \frac{1}{2} \int_0^t G(s)^2 \Gamma(s) \, ds \right)
\]

where

\[
d\Gamma(s) = (2H(s) - \Gamma(s)^2 G(s)^2) \, ds, \quad \Gamma(t) = 0,
\]

and \( D(t) \) is the covariance of \( \int_0^t R(s) \, d\xi_s \), where

\[
d\xi_s = G(s)^2 \Gamma(s) \xi_s \, ds + G(s) \, dw_s, \quad \xi_0 = 0.
\]

Comparing the last integral in (1.13) with the equation given by (1.14), we note that the corresponding \( R(t) \) is a linear function of \( a \) and \( z \). Now (1.9), (1.13), (1.14), and the definition of \( D(t) \) conclude the proof of Lemma 1.

From Lemma 1 it follows in particular that for \( z \in \mathbb{R} \), the characteristic function of \( x_t \, | \, \gamma_t \) is given by
\[ e_{t}(z) = C(t,y) \int_{-\infty}^{\infty} \exp(a^{2}F_{1}(t,y) + aF_{2}(t,y) + izF_{3}(t,v) + izF_{4}(t,v) \]
\[ + z^{2}F_{5}(t,v))dF(a) , \]

where \( F_{1}, F_{2}, F_{3}, F_{4}, F_{5} \) do not depend on \( F(a) \). Normalizing \( e_{t}(z) \) (i.e., requiring that \( e_{t}(0) = 1 \)) yields

\[ e_{t}(z) = \exp(izF_{4} + z^{2}F_{5}) \frac{\int \exp(a^{2}F_{1} + aF_{2} + izF_{3})dF(a)}{\int \exp(a^{2}F_{1} + aF_{2})dF(a)} \]

Then from the general properties of the characteristic function, it follows that

\[ \left. \frac{1}{i} \frac{de_{t}(z)}{dz} \right|_{z=0} = \hat{\chi}_{t} , \]
\[ \left. (\frac{1}{i})^{2} \frac{d^{2}e_{t}(z)}{dz^{2}} \right|_{z=0} = \hat{\tau}_{t} + \hat{x}^{2}_{t} , \]

where \( \hat{\tau}_{t} = \mathbb{E}(x_{t} - \hat{x}^{2}_{t}|y_{t}) \) i.e., the conditional variance of \( x_{t}|y_{t} \).

From the above and (1.16)

\[ \hat{x}_{t} = F_{3}I_{t}(1) + F_{4} , \]
\[ \hat{\tau}_{t} = -2F_{5} + F_{3}^{2} (I_{t}(2) - I_{t}^{2}(1)) , \]

where

\[ I_{t}(n) = \frac{\int a^{n}\exp(a^{2}F_{1} + aF_{2})dF(a)}{\int \exp(a^{2}F_{1} + aF_{2})dF(a)} , \quad n = 1,2 . \]

The following Lemma defines \( F_{i} \) i = 1,2,3,4,5 in (1.16).
Lemma 2

The characteristic function of $x_t \mid \mathcal{F}_t$ is given by

$$e_t(z) = \exp \left( -\frac{1}{2} z^2 \bar{F}_t(0) \right) \frac{\int_{-\infty}^{\infty} \exp \left( a^2 F_1 + a F_2 + iz \bar{x}_t(a,0) \right) dF(a)}{\int_{-\infty}^{\infty} \exp \left( a^2 F_1 + a F_2 \right) dF(a)},$$

where $\bar{x}_t(a,0), \bar{F}_t(0)$ are given as the solutions to the following set of differential equations with $\sigma = 0$:

$$d\bar{x}_t(a,\sigma) = (\bar{F}_0 + \bar{F}_1 \bar{x}_t(a,\sigma)) dt + (q_0 + \bar{F}_t(0) h_1)(dy_t - (h_0 + h_1 \bar{x}_t(a,\sigma)) dt)$$

$$\bar{x}_0(a,\sigma) = a,$$

$$d\bar{F}_t(\sigma) = (2F_1 \bar{F}_t(\sigma) + g_0^2 - q_0^2 - (q_0 + \bar{F}_t(0) h_1)^2) dt, \quad \bar{F}_0(\sigma) = \sigma^2,$$

and

$$F_1 = -\frac{1}{2} \int_0^t h_1^2 s^2 ds,$$

$$F_2 = \int_0^t \phi_s h_1 (du_s + h_1 \phi_s I_1(1) ds),$$

$$\phi_t = \exp \left( \int_0^t (f_1 - h_1 (q_0 + \bar{F}_s(0) h_1)) ds \right).$$

Proof of Lemma 2

Since the $F_i$ do not depend on $F(a)$ (see Lemma 1) take

$$dF(a) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(a-m)^2}{2\sigma^2}) da, \ a, \sigma > 0.$$
In this case the resulting conditionally Gaussian distribution allows for explicit \( e_t(z) \) calculation [4]. Accordingly,

\[
(1.27) \quad e_t(z) = \exp(iz\bar{x}_t(m,\sigma) - \frac{1}{2}z^2\bar{P}_t(\sigma)),
\]

where \( \bar{x}_t(m,\sigma) \) and \( \bar{P}_t(\sigma) \) satisfy (1.21) and (1.22) respectively. With \( F(a) \) given by (1.26) it follows from (1.16) that

\[
(1.28) \quad e_t(z) = \exp(iz(F_4 + \hat{\sigma}^2(F_2 + \frac{m}{\sigma^2})F_3) + z^2(F_5 - \frac{1}{2}\hat{\sigma}^2 F_3)),
\]

where \( \hat{\sigma}^2 = \sigma^2 - 2F_1 \).

Comparing (1.27) and (1.28), we have

\[
(1.29) \quad \bar{x}_t(m,\sigma) = F_4 + \hat{\sigma}^2 (F_2 + \frac{m}{\sigma^2})F_3,
\]

and

\[
(1.30) \quad \bar{P}_t(\sigma) = \hat{\sigma}^2 F_3 \quad - 2F_5 .
\]

Letting now \( \sigma \rightarrow 0 \) in (1.29) and (1.30), it follows that

\[
(1.31) \quad F_4 + mF_3 = \bar{x}_t(m,0),
\]

and

\[
(1.32) \quad F_5 = -\frac{1}{2}\bar{P}_t(0).\]
The above allows \( e_t(z) \) to be of the form of (1.20) with \( F_1 \) and \( F_2 \) yet to be defined. Using now (1.17) and (1.18) and explicitly calculating \( I_t(n) \), \( n = 1, 2, \)

\[
\Delta_t (\sigma^{-2} - 2F_1) = F_3^2
\]

and

\[
\Delta_t (\frac{m}{\sigma^2} + F_2) = F_3 (\hat{x}_t - F_4),
\]

with \( \Delta_t = P_t - \overline{P}_t(0) \).

The formulae for \( F_1 \) and \( F_2 \) will be obtained by differentiating (1.33) and (1.34). However, before this is done recall from the theory of nonlinear filtering [3] that in general for \( x_t, y_t \) given as a solution to (1.1) and (1.2) \( \hat{x}_t, P_t \) satisfy

\[
\begin{align*}
\dot{\hat{x}}_t &= (f_0 + f_1 \hat{x}_t)dt + (q_0 + P_t h_1)dv_t, \quad \hat{x}_0 = \int_{-\infty}^{\infty} adF(a), \\
\dot{P}_t &= (2f_1 P_t + g_0^2 + q_0^2 - (q_0 + P_t h_1)^2)dt + h_1 R_t dv_t, \quad P_0 = \int_{-\infty}^{\infty} (a - \hat{x}_0)^2 dF(a), \\
\end{align*}
\]

(1.36)

where \( R_t = \mathbb{E}(\hat{x}_t - x_t)^3 | Y_t \).
Remark. Direct application of Eqs. (1.35) and (1.36) meets the difficulty of infinite coupling between the subsequent moments.

From Eqs. (1.22), (1.35), and (1.36), and the fact that for conditionally Gaussian processes $R_t = 0$,

\begin{equation}
(1.37) \quad \Delta_t = \Delta_t(2f_1 - h_1(2q_0 + h_1(P_t + \bar{P}_t(0))))dt, \quad \Delta_0 = \sigma^2
\end{equation}

Now from (1.33) and (1.34) (upon differentiation) and using (1.35), (1.37) it follows that

\begin{equation}
(1.38) \quad dF_1 = -\frac{1}{2} h_1^2 F_2^2 dt, \quad F_1(0) = 0,
\end{equation}

and

\begin{equation}
(1.39) \quad dF_2 = F_3 h_1(dv_t + h_1 F_2 I_t(1)dt), \quad F_2(0) = 0.
\end{equation}

To define $F_3$, notice that Eq. (1.21) solves as

\[ x_t(a, \sigma) = \Phi_t(a + \int_0^t \Phi_s^{-1}(f_0 - h_0(q_0 + \bar{P}_s(\sigma)h_1))ds + \int_0^t \Phi_s^{-1}(q_0 + \bar{P}_s(\sigma)h_1)dy_s), \]

\[ (1.40) \]

where

\begin{equation}
(1.41) \quad \Phi_t = \exp\left(\int_0^t (f_1 - h_1(q_0 + \bar{P}_s(\sigma)h_1))ds\right)
\end{equation}

Comparing the above with (1.31) shows that $F_3 = \Phi_t$ for $\sigma = 0$, which ends the proof of Lemma 2.
Lemma 3 below merely organizes all the results into the filter equations and the final form of the conditional characteristic function.

**Lemma 3**

Given the system (1.1) and (1.2) together with the a priori distribution $F(a) = P(x_0 < a \mid y_0)$. The following are the filter equations (i.e., formulae of the recursive type, which calculate $\hat{x}_t = E(x_t \mid y_t)$).

\[ \frac{dx_t}{dt} = (f_0 + f_1 \hat{x}_t)dt + (q_0 + P_t h_1)dv_t \, , \quad \hat{x}_0 = \int_{-\infty}^{\infty} \alpha \, dF(a) , \]

\[ dv_t = dy_t - (h_0 + h_1 \hat{x}_t)dt \, , \]

\[ P_t = \bar{P}_t + \phi_t^2 (I_t(2) - I_t(1)) \, , \]

\[ \frac{d\bar{P}_t}{dt} = (2\phi_t \bar{P}_t + s_0^2 + q_0^2 - (q_0 + \bar{P}_t h_1)^2)dt \, , \quad \bar{P}_0 = 0 \, , \]

\[ I_t(n) = \frac{\int_{-\infty}^{\infty} a^n \exp(a^2 F_1 + a F_2) \, dF(a)}{\int_{-\infty}^{\infty} \exp(a^2 F_1 + a F_2) \, dF(a)} \, , \quad n = 1, 2 \, , \]

\[ dF_1 = -\frac{1}{2} \phi_t^2 \hat{F}_t^2 dt \, , \quad F_1(0) = 0 \, , \]

\[ dF_2 = \phi_t h_1 (dv_t + \phi_t h_1 I_t(1) dt) \, , \quad F_2(0) = 0 \, , \]

\[ d\phi_t = (f_1 - h_1 (q_0 + \bar{P}_t h_1)) \phi_t dt \, , \quad \phi_0 = 1 \, . \]

The characteristic function of $x_t \mid y_t$ is given by:
\[ e_t(z) = \exp(iz(x_t - \frac{1}{2} z^2 \sum \phi_t)) - \frac{1}{2} z^2 \sum \phi_t - \int_{-\infty}^{\infty} \exp(a^2 F_1 + a F_2 + iz a) d\Phi(a) \]

(1.50)

2. Control and Special Cases

Two special cases of Eqs. (1.1) and (1.2) result in significant simplification of the filter equations. The first case occurs when \( g_0(t, y) = 0 \), \( 0 < t < T \). From (1.7) it follows then that \( x_t \) is of the form

\[ x_t = A_t(y)x_0 + B_t(y). \]

Using the above equation in (1.2) we have the following estimation problem:

Let \( x_0 \) be a random variable with distribution \( F(a) = P(x_0 < a | y_0) \). Assume that the observation process \( y_t, 0 < t < T \), admits a differential

\[ dy_t = (h_0(t, y) + h_1(t, y)x_0) dt + dv_t, \]

where the notation stays the same as in (1.2) and \( h_0, h_1 \) satisfy (1.3) and (1.4).

From Lemma 2 it follows now that the conditional characteristic function of \( x_0 \) given \( Y_t \) is of the form

\[ E_t(z) = \frac{\int_{-\infty}^{\infty} \exp(a^2 F_1 + a F_2 + iz a) d\Phi(a)}{\int_{-\infty}^{\infty} \exp(a^2 F_1 + a F_2) d\Phi(a)}. \]

(2.1)

The above results from the fact that \( dx_0 = 0 \) replaces Eq. (1.1) implying \( \Phi_t(0) = 0 \) and \( \Phi_t(a, 0) = a \), as defined by Eqs. (1.21) and (1.22). Now

\[ \left. \frac{dE_t(z)}{dz} \right|_{z=0} = i x_t, \]
where \( \hat{x}_t = \mathbb{E}(x_0 | Y_t) \), combined with the general filter equations (1.42) + (1.50) yields

\[
\hat{x}_t = -\frac{\int_{-\infty}^{\infty} a \exp\left(-\frac{1}{2} a^2 \int_0^t h_1^2 ds + a \int_0^t h_1(dy - h_0 ds)\right) dF(a)}{\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} a^2 \int_0^t h_1^2 ds + a \int_0^t h_1(dy - h_0 ds)\right) dF(a)}.
\]

(2.2)

In particular if \( dF(a) = \frac{1}{\sqrt{2\pi}\sigma_0} \exp(-\frac{1}{2}(a - \mu_0)^2) da \), (2.2) results in

\[
\hat{x}_t = \frac{\mu_0 + \sigma_0^2 \int_0^t h_1(dy - h_0 ds)}{1 + \sigma_0^2 \int_0^t h_1^2 ds}.
\]

(2.3)

The above agrees with the result presented in [4, pp. 22-24].

The second special case for which the filter takes a simple form follows if \( h_1(t, y) = 0 \) (i.e., the state is not observable directly), \( 0 < t < T \). Now the filter equations (1.42) + (1.50) reduce to

\[
\dot{x}_t = (f_0 + f_1 \hat{x}_t) dt + q_0(dy - h_0 dt),
\]

(2.4)

\[
\hat{x}_0 = \int_{-\infty}^{\infty} a dF(a),
\]

\[
dP_t = (2f_1 P_t + g_0^2) dt,
\]

\[
P_0 = \int_{-\infty}^{\infty} (a - \hat{x}_0)^2 dF(a).
\]
In order to discuss a control problem using the results obtained here, assume that all the coefficients in (1.1) and (1.2), except \( f_0(t, y) \) which is denoted here by \( u_t(y) \), are functions of time only. If \( u_t(y) \) satisfies the assumption (1.5) we say that \( u = \{u_t(y), 0 < t < T\} \) is an admissible control and write \( u \in U \).

Let \( x^u_t, x^v_t \) denote the solutions to (1.1) and (1.42) respectively, for some \( u \in U \), and let \( x^0_t, x^0_t \) correspond to \( u_t \equiv 0, 0 < t < T \).

Define \( e^u_t = x^u_t - x^0_t, e^0_t = x^0_t - x^0_t = e^0_t \), \( e^u_t = x^0_t - x^0_t \). Substracting Eq. (1.42) from (1.1) we have

\[
(2.5) \quad de^u_t = f_1e^u_t dt + g_0 dw_t + q_0 dv_t - \left( q_0 + P_t(e^u_t)h_1 \right) \left( h_1e^u_t + dv_t \right),
\]

where \( P_t = P_t(e^u_t) \) shows that \( P_t \) depends only on \( e^u_s, 0 < s < t \) which is seen from Eq. (1.48) rewritten as:

\[
(2.6) \quad dF_t = \phi_t h_1(e^u_t dt + dv_t + \phi_t h_1 I_t(1)) dt.
\]

From (2.5) it follows that with probability one the values of \( e^u_t \) and \( e^u_t \) coincide for all \( u \in U \). Now, since

\[
(2.7) \quad dv^0_t = dy^0_t - \left( h_0 + h_1 x^0_t \right) dt = h_1 e^0_t dt + dv_t = h_1 e^0_t dt + dv_t = dv^0_t,
\]

\( v^u_t \) and \( v^0_t \) coincide with probability one. From (2.7) it follows that Eq. (1.42) can be rewritten as

\[
(2.8) \quad d\tilde{x}^u_t = (f_1\tilde{x}^u_t + u_t) dt + (q_0 + h_1 P_t(e^0_t)) dv^0_t, \quad \tilde{x}^u_t = \tilde{x}_0 - \int_a^\infty \text{adF}(a)
\]

Now let \( \tilde{u}_t = F_t(\tilde{x}^u_t) \), where \( F_t \) is a nonanticipative functional of \( x^u_s, 0 < s < t \), and satisfies
\[ (2.9) \quad E\left( \int_0^T |F_t|^4 \, dt \right) < \infty. \]

From (2.8) it follows that \( \tilde{u}_t \) is \( \sigma \)-alg\( \{ \nu_t^0, 0 < s < t \} \) measurable. Now, let \( \bar{u}_t \) be any admissible control and let \( y_t^u \) be an observation process associated with \( \bar{u}_t \). From (2.7),

\[ y_t^u = \sigma \text{-alg}\{ y_s^u, 0 < s < t \} \supseteq \sigma \text{-alg}\{ y_s^0, 0 < s < t \} = \sigma \text{-alg}\{ \nu_s^0, 0 < s < t \}. \]

(2.10)

The above shows that \( \tilde{u}_t \) is \( \bar{y}_t^u \) measurable. This fact combined with (2.9) states that \( \tilde{u}_t \in U \), and that we can expect the separation of the stochastic control of \( \tilde{u}_t \) type and the filtering problem. As an illustration of the statement, consider the following control problem.

**Linear-Quadratic Control Problem with Non-Gaussian Initial Distributions**

The partially observable controlled process \( (x_t, y_t), 0 \leq t \leq T \), is given by the stochastic equations

\[ (2.11) \quad dx_t = (f_1(t)x_t + u_t) \, dt + g_0(t) \, dw_t, \]
\[ dy_t = h_1(t)x_t \, dt + dv_t, \quad y_0 = 0. \]

The independent Wiener processes \( w_t \) and \( v_t \) entering into (2.11) do not depend on the random variable \( x_0 \) (the initial state). \( x_0 \) is assumed to have distribution function \( F(a) = P(x_0 < a) \) with \( \int_{-\infty}^{\infty} a^4 \, dF(a) < \infty \) (finite fourth order moment).
The \( \mathcal{Y}_t = \sigma\text{-alg}(\mathcal{Y}_s, 0 < s < t) \) measurable, stochastic process \( u_t \) is called a control at time \( t \) and is assumed to satisfy

\[
E\left( \int_0^T |u_t|^4 dt \right) < \infty.
\]

For \( u = \{u_t, 0 < t < T\} \), satisfying the above we write \( u \in \mathcal{U} \), where \( \mathcal{U} \) is the class of admissible controls. It is also assumed that \( f_1, g_0, h_1 \) satisfy the deterministic version of (1.3) \( \pm \) (1.5).

Consider now the performance functional

\[
(2.12) \quad J(u) = E\left( x_T^2 h_T + \int_0^T (x_t^2 H(t) + u_t^2 R(t)) dt \right)
\]

where \( h_T > 0, H(t) > 0, 0 < R^{-1}(t) < \text{const.}, 0 < t < T \). The admissible control \( \hat{u} \in \mathcal{U} \) is called optimal if

\[
J(\hat{u}) = \inf_{u \in \mathcal{U}} J(u).
\]

**Lemma**

The optimal control for the process (2.11) and the performance index (2.12) exists and is defined by

\[
(2.13) \quad \hat{u}_t = -R^{-1}(t)Q(t)\hat{x}_t, \ 0 < t < T
\]

where \( Q(t) > 0 \) satisfies the Riccati equation

\[
(2.14) \quad \frac{dQ(t)}{dt} = 2f_1(t)Q(t) + H(t) - Q^2(t)R^{-1}(t), \ Q(T) = h_T,
\]

and \( \hat{x}_t \) is defined by
\begin{align}
\dot{x}_t &= (f_1(t) - R^{-1}(t)Q(t))x_t dt + P_t(v)h_1(t)dv_t \\
\dot{x}_0 &= \int_{-\infty}^{\infty} \text{adF(a)} \\
\dot{v}_t &= dy_t - h_1(t)x_t dt \\
F_t &= \bar{P}(t) + \phi(t)^2(I_t(2) - I_t(1)) \\
\frac{d\bar{P}(t)}{dt} &= 2f_1(t)\bar{P}(t) + g_0^2(t) - h_1^2(t)\bar{P}^2(t), \quad \bar{P}(0) = 0 \\
\phi(t) &= \exp\left(\int_0^t (f_1(s) - h_1^2(s)\bar{P}(s))ds\right) \\
F_1(t) &= -\frac{1}{2} \int_0^t h_1^2(s)\phi^2(s)ds \\
F_2(t, v) &= \int_0^t \phi(t)h_1(t)(dv_t + \phi(t)h_1(t)I_t(1)dt) \\
I_t(n) &= \frac{\int \alpha^n \exp(a^2F_1(t) + aF_2(t, v))dF(a)}{\int \exp(a^2F_1(t) + aF_2(t, v))dF(a)}, \quad n = 1, 2. 
\end{align}

Remark. The structure of the optimal control law is identical with the optimal controller for LQG problem.

**Proof of the Lemma**

First note that the assumptions made in the control problem statement assure well-defined filter for the system (2.11) and $u \in U$. Next rewrite the performance index as follows:
\begin{align}
J(u) &= \mathbb{E}(\mathbb{E}(x_T^2|Y_T)) + \int_0^T \mathbb{E}(\hat{x}_t^2H(t) + u_t^2R(t)|Y_t)dt \\
&= \mathbb{E}((\hat{x}_T^u)^2h_T) + \int_0^T (\hat{x}_t^u)^2H(t) + u_t^2R(t)dt \\
&+ \mathbb{E}(h_Tp_t^u + \int_0^T p_t^uH(t)dt)
\end{align}

From (2.6) we conclude that $p_t^u$ does not depend on the control $u$ and coincides with the function $p_t^0$ obtained from the filter equations for $u_t \equiv 0$, $0 < t < T$. The process $\hat{x}_t^u$ entering (2.15) satisfies equation (see 2.8)

\begin{align}
\dot{x}_t^u &= (\ell(t)x_t^u + u_t)dt + h_1(t)p_t^o dv_t^o,
\end{align}

where $v_t^o = v_t^u = \int_0^t (d\xi_t^u - h_1(s)x_s^u ds)$, according to (2.7), and $v_t^o$ is a Wiener process (see (1.11)). Introduce now the function

\begin{align}
V(t, \xi) &= \xi^2Q(t) + \int_t^T Q(t)h_1^2(\tau)(p_t^o)^2d\tau
\end{align}

where $0 < t < T$, $-\infty < \xi < \infty$, and $Q(t)$ satisfies (2.14). It is easy to verify that $V(t, \xi)$ satisfies the following Bellman equation:

\begin{align}
\xi^2H(t) + \xi\ell(t) \frac{\partial V(t, \xi)}{\partial \xi} + \frac{1}{2} (p_t^o)^2h_1^2(t) \frac{\partial^2 V(t, \xi)}{\partial \xi^2} + \frac{\partial V(t, \xi)}{\partial t} \\
+ \min_{n} (n^2R(t) + n \frac{\partial V(t, \xi)}{\partial \xi}) = 0,
\end{align}

and that $V(T, \xi) = \xi^2h_T$.

Note that $\hat{\eta}$ which minimizes the above for positive definite $R(t)$, is given by

\begin{align}
\hat{\eta} &= -R(t)^{-1}(t)Q(t)x.
\end{align}
Calculate now, with the use of the Ito formula,

\[ V(T, x_T^u) - V(0, x_0) = \int_0^T \left( \frac{\partial V(t, x_t)}{\partial t} \right)_{x_t = x_0^u} + \frac{1}{2} h_1^2(t) \left( p^0_t \right)^2 \left( \frac{\partial^2 V(t, x_t)}{\partial x_t^2} \right)_{x_t = x_0^u} dt \]

\[ + \left. \frac{\partial V(t, x_t)}{\partial x_t^u} \right|_{x_t = x_0^u} d\xi_t. \]

Taking into account (2.8) and (2.19) we obtain

\[ V(T, x_T^u) - V(0, x_0^u) > - \int_0^T \left( (x_T^u)^2 H(t) + u_t^2 R(t) \right) dt \]

\[ + 2 \int_0^T x_t^u Q(t) p^0_t h_1(t) d\nu_t^0. \]

After taking the expectation of both sides of the above inequality,

\[ V(0, x_0^u) < E((x_T^u)^2 h_T + \int_0^T ((x_T^u)^2 H(t) + u_t^2 R(t)) dt) \]

The equality in (2.21) holds, according to (2.20) only if

\[ \tilde{u}_t = -R^{-1}(t)Q(t) x_t^u. \]

Comparing (2.22) with (2.16),

\[ J(\tilde{u}) < J(u) \text{ for all } u \in U. \]

The admissibility of \( \tilde{u} \) defined by (2.22) follows from (2.10) and the fact that

\[ E(\sup_{0 < t < T} (x_t^u)^4) < \infty. \]

The above can be proven in the same way as in the derivation of a conditionally Gaussian filter [Lemma 12.1; 4, pp. 18-19]. This ends the proof of the Lemma.
It seems to be possible, (following e.g. [2]) to show that the separation principle holds also for nonquadratic performance functionals.
REFERENCES


