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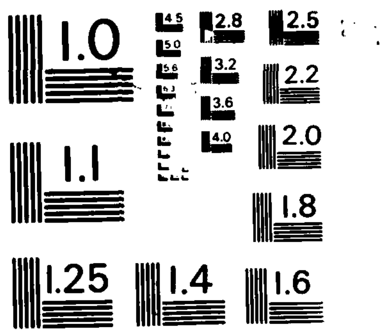

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ANALYSIS AND CONTROL OF A CLASS OF STIFF LINEAR DISTRIBUTED SYSTEMS

HASSEN SALHI

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This thesis examines a class of systems whose models are described by linear partial differential equations that depend on a small parameter ε. First, the spectral decomposition of the so-called "stiff" operators (using the terminology of [24]) is investigated, including the convergence of their eigenvalue-eigenvector pairs as $\varepsilon \to 0$, with the objective of clarifying their singular behavior. Second, asymptotic approximation of the solution boundary value problems involving stiff operators are constructed, using the weak limits of

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20. ABSTRACT (continued)

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By

Hassen Salhi

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ANALYSIS AND CONTROL OF A CLASS OF STIFF LINEAR DISTRIBUTED SYSTEMS

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B.S., University of Illinois, 1977
M.S., University of Illinois, 1979

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Electrical Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 1983

Thesis Advisor: Professor Douglas P. Looze

Urbana, Illinois

ANALYSIS AND CONTROL OF A CLASS OF STIFF LINEAR DISTRIBUTED SYSTEMS

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This thesis examines a class of systems whose models are described by linear partial differential equations that depend on a small parameter ε. First, the spectral decomposition of the so-called "stiff" operators (using the terminology of [24]) is investigated, including the convergence of their eigenvalue-eigenvector pairs as $\varepsilon \to 0$, with the objective of clarifying their singular behavior. Second, asymptotic approximation of the solution boundary value problems involving stiff operators are constructed, using the weak limits of their eigenvectors. This approach leads to a decomposition into "regular" approximation and "internal layer" approximation, which are found separately and then combined to provide an approximation to the original problem. This methodology is not complicated. Moreover, it alleviates the inherent stiffness when numerical algorithms are employed. Third, the same approach is applied to some control problems. In this case, similar results are obtained, provided additional requirements are satisfied, due to the type of control, which may drastically alter the system behavior.

TO MY FATHER, AND IN MEMORY OF MY DEAR MOTHER

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CHAPTER 1

INTRODUCTION

1.1. Motivation

Many physical systems are modeled adequately by a system of ordinary and/or difference equations. However, the need to consider models with partial differential equations arises in many areas of the physical sciences. Examples include mass or heat transfer [10,14,38], el cicity [10,33,34,37,38,40], electromagnetic wave propagation [10,37,38 uclear reactor theory [9,34], fluid flow [19], and stochastic processes [.

Due to physical considerations and the desire to obtain simpler models, the engineer or the applied mathematician often redefines the variables of the model at hand, so that small parameters appear explicitly. In systems described by partial differential equations, the small parameter may represent a small diffusivity or a small convection coefficient in heat transfer, the thinness of a vibrating membrane in elasticity, or "cheapness" of control in optimal control problems. However, the introduction of small parameters may be purely artificial. Such is the case in regularized and penalized problems [23]. In this situation, the interest usually lies in the properties of the solution of the limiting problem as $\varepsilon \to 0$ and not the problem itself.

The dependence of these models on ε is singular, i.e., the formal limit of these operators as $\varepsilon \to 0$, may or may not exist. In the case it exists, ellipticity of the original operator is often lost (order reduction). Hence the solution of a boundary value problem involving the perturbed operator converges to the corresponding boundary value problem involving the formal operator limit in a larger Hilbert space. Thus, the need to introduce "correctors" concentrated in the vicinity of the boundary of the set over which the Hilbert space of functions is defined, is inevitable [1,5,11-14], [16,24,25,27,29,31,41]. In some instances, the formal operator limit may not be well-defined (e.g., not elliptic). Depending upon the type of boundary value problems at hand, their solutions may be expanded in Laurent series expansion of ε or may be approximated by "regular" expansions with the addition of correctors [24,27]. In this class of problems, if ellipticity of the formal operator limit is not uniformly lost (e.g., in some stiff operators), one would expect some "separation" in the spectrum of the original operator. This last intuitive observation is the driving force behind the present investigation of the spectral decomposition of stiff operators.

1.2. Literature Survey

Asymptotic expansions of linear and nonlinear differential operators depending on a small parameter ε have been studied by several scientists over the past several decades. The underlying theory (such as order and validity of approximation, asymptotic error estimates, etc.) is discussed in detail in [11]. Eigenvalue problems of some of these operators are considered in [5,12,13,15,16,31,32] and the references therein. Most of these references assume that the formal limits as $\varepsilon \to 0$ of such operators are uniformly elliptic. In this instance, the eigenvalues are uniformly bounded away from zero as $\varepsilon \to 0$, or may become dense in a subset of the real line [13].

In stiff operators, there are several conditions that cause stiffness, some of which are explored in this thesis. In general, the eigenvalues (and the corresponding eigenvectors) of stiff operators can be decomposed into groups depending upon their convergence as $\varepsilon \to 0$.

Boundary value problems for some classes of operators that depend upon a small parameter ε (including several control problems) are studied in [1,11,14,24,27,29], just to name a few. Formal asymptotic expansions of the solutions of stiff elliptic boundary value problems are considered in [24,27,29] without any reference to their spectral decomposition. Parabolic and hyperbolic problems involving stiff operators seem not to have been previously investigated.

The main contributions of this thesis are:

1) The spectral analysis of some stiff operators, including the convergence of their eigenvalue-eigenvector pairs as $\varepsilon \to 0$. An appropriate terminology such as flattening, attenuation and oscillation, is introduced to describe the deformations of the eigenvectors as $\varepsilon \to 0$.

2) The approximation of the solutions of boundary value (including some control) problems involving stiff operators, using the weak limits of the eigenvectors of the aforementioned operators. The advantage of this method is its simplicity. Moreover, it alleviates stiffness when numerical algorithms are employed.

1.3. Thesis Overview

In Chapter 2, the eigenvalue problem of stiff operators that have coefficients $O(1), O(\varepsilon), \ldots, O(\varepsilon^p)$ in the different interfaced subsets $\Omega_0, \Omega_1, \ldots, \Omega_p$ (whose union constitutes the open connected set Ω ($\subset \mathbb{R}^n$)), are analyzed. The interfaces are the counterpart of interconnections between "areas" in large scale lumped systems. One way to understand the singular behavior of stiff systems is to analyze their spectral decomposition. Indeed, for small values of ε, the eigenvalues of stiff operators can be separated, depending upon their convergence as $\varepsilon \to 0$. Their corresponding eigenvectors are also classified accordingly.

In Chapter 3, using the convergence results of the eigenvalue-eigenvectors of stiff operators as $\varepsilon \to 0$, approximations to some classical boundary value problems (namely elliptic, parabolic, and hyperbolic) are constructed and asymptotic error estimates are derived. Most of the ideas are specialized to second order operators for simplicity. However, the approach is general enough and hence may be applied in many other similar problems.

In Chapter 4, two control problems are considered using the approach developed in Chapter 3. For optimization problems (including control problems), some caution is advised in applying this approach, because of the inherent dependence of the optimality systems on the type of observation and control [23,29]. There are several control and observation mechanisms, i.e., distributed, boundary, pointwise, etc. The control action transforms the characteristics of the system. Consequently, it may not be possible to use the eigenvectors of the uncontrolled system to solve the controlled one.

Chapter 5 gives the numerical results concerning boundary value and control problems. The approximations derived using the approach of Chapter 3 are compared with those of the direct approach using a finite element method.

The last chapter contains some concluding remarks and some possible extensions of the results presented in this thesis.

CHAPTER 2

SPECTRAL ANALYSIS OF STIFF OPERATORS

2.1. Introduction

This chapter considers the eigenvalue problem of the following formal selfadjoint operator, written in matrix form as:

$$A_\varepsilon = \begin{bmatrix} A_0 & 0 \\ 0 & \varepsilon A_1 \end{bmatrix}$$

where A_i, $i = 0,1$ are unbounded operators.

Many physical problems can be described by models containing the operators A_ε. Examples of such problems in distributed parameter systems are numerous. Without being exhaustive, examples include the following:

1) Nuclear reactor operations [9]
2) Heat or mass transfer in interfaced media having different diffusivities [10,38]
3) Electromagnetic wave propagation in waveguides made of materials having different permittivities [10]
4) Small vibrations of elastic interfaced through membranes with different material densities [40]
5) Continuous stochastic processes when the noise intensity level is different from one part to another of a medium [4].

There are several motivations for investigating the present eigenvalue problem. First, the operator A_ε may be used in many instances in models of interfaced media of mathematical physics, as previously indicated. Second, the operator A_ε, at first sight, seems to conceal some "singular" behavior, using the terminology of singular perturbation of lumped systems [18].

Third, the introduction of the small parameter ε in models of interfaced media may be purely artificial, in order to obtain an approximation of the problem at hand. Fourth, a formal Laurent series expansion in powers of ε is derived for some elliptic boundary value problems in [24,29]. It is not clear what the relationship is between the terms of this expansion and the eigenvalue-eigenvector pairs of the operator A_ε. In the sequel, these observations will be fully investigated.

The eigenvalue problem involving several perturbed operators has been studied in the literature [12,13,16,17,31], and the references therein. However, the spectral analysis of stiff operators (using the terminology of [24]) has not been investigated previously. This chapter presents a general formulation using bilinear forms to avoid possible cumbersome and complex boundary and interface conditions. The chapter is organized as follows. The eigenvalue problem formulation of a class of stiff operators involving two bilinear forms is presented in Section 2.2. In Section 2.3, the convergence of the eigenvalues and the corresponding eigenvectors as $\varepsilon \to 0$ is investigated. Several examples are given to illustrate the results obtained in this section. In Section 2.4, a generalization of the analysis of Section 2.3 to $p + 1$ $(p > 1)$ bilinear forms, is undertaken. In Section 2.5, it is shown, with the aid of three examples, that some of the results derived in Section 2.3 are applicable to a larger class of eigenvalue problems. This is accomplished by relaxing some of the assumptions made in Section 2.2. In Section 2.6, formal asymptotic expansion in powers of ε of the eigenvalues and eigenvectors is discussed. In Section 2.7, two numerical examples are solved. The first example illustrates the properties of stiff operators of Section 2.3. However, the second example elucidates the properties of stiff operators of Section 2.5.3. Finally, in the last section, some concluding remarks as well as some extensions of the forthcoming analysis are given.

2.2. Eigenvalue Problem Formulation

In this section, the eigenvalue problems of a class of stiff operators is formulated. Let V, H be two given real Hilbert spaces such that V is dense in H and

A1) the injection of V into H is compact.

Let V^* denote the dual space of V. After identifying H with H^*, one has

A2) $V \subset H \subset V^*$.

Let $a_i(\varphi,\psi)$, i = 0,1 be two forms on V such that the following assumptions hold:

A3) $a_i(\varphi,\psi)$ is bilinear, symmetric on V

A4) $a_i(\varphi,\psi)$ is continuous on V, i.e., there exists β_i such that

$$a_i(\varphi,\psi) \le \beta_i \|\varphi\|_V \|\psi\|_V, \ \forall\varphi\in V, \ \forall\psi\in V$$

A5) $a_i(\varphi,\varphi) \ge \alpha_i p_i(\varphi)^2$ where $\alpha_i > 0$ and $p_i(\cdot)$ is continuous semi-norm on V

A6) $p_0(\varphi) + p_1(\varphi)$ is a norm equivalent to $\|\varphi\|_V$

A7) $a_i(\varphi,\varphi) = 0$ on $V_i \subset V$, where V_i is an infinite-dimensional subspace of V, i=0,1.

A8) If $\psi \mapsto L_0(\psi)$ is a continuous linear form on V, null on V_0, there exists $\varphi\in V$ (modulo V_0) such that

$$a_0(\varphi,\psi) = L_0(\psi), \ \forall\varphi\in V$$

Let $a_\varepsilon(\varphi,\psi)$, $a(\varphi,\psi)$ be defined as

$$a_\varepsilon(\varphi,\psi) = a_0(\varphi,\psi) + \varepsilon a_1(\varphi,\psi) \tag{2.1}$$

$$a(\varphi,\psi) = a_0(\varphi,\psi) + a_1(\varphi,\psi) \quad . \tag{2.2}$$

Now some important remarks clarifying the above assumptions and definitions are in order:

Remark 2.1:

It can be easily seen from (A3) that $a_\varepsilon(\varphi,\psi)$, $a(\varphi,\psi)$ are bilinear, symmetric forms on V.

Remark 2.2:

From (A4-A6), one concludes that $a_\varepsilon(\varphi,\varphi)$, $a(\varphi,\varphi)$ are coercive and bounded on V. In particular, for sufficiently small ε, they satisfy

$$\begin{aligned} \alpha\varepsilon\|\varphi\|_V^2 \leq a_\varepsilon(\varphi,\varphi) \leq \nu\|\varphi\|_V^2 \quad &, \quad \forall\varphi\in V \\ \alpha\|\varphi\|_V^2 \leq a(\varphi,\varphi) \leq \nu\|\varphi\|_V^2 \quad &, \quad \forall\varphi\in V \end{aligned} \tag{2.3}$$

where α (resp. ν) is independent of ε and depends solely on α_0,α_1 (resp. ν_0,ν_1) and the semi-norms $p_i(\cdot)$, $i = 0,1$.

Remark 2.3:

The bilinear forms $a_\varepsilon(\varphi,\psi)$, $a(\varphi,\psi)$ define selfadjoint operators [2]

$$A_\varepsilon,\ A \subset \mathcal{L}(V;V^*),$$

i.e.,

$$a_\varepsilon(\varphi,\psi) = \langle A_\varepsilon\varphi,\psi\rangle \quad ,\forall\ \varphi,\psi \in V$$

$$a(\varphi,\psi) = \langle A\varphi,\psi\rangle \quad , \ \forall\ \varphi,\psi \in V$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V and its dual V^*.

From the preceding remarks, one concludes that the spectra of A_ε, A are subsets of $\mathbb{R}^+$, consisting only of the point spectrum [2,15,42].

The eigenvalue problem for A_ε is, then, to seek $\{\gamma_\varepsilon^k, \chi_\varepsilon^k\} \in \mathbb{R}^+ \times V$ such that

$$A_\varepsilon \chi_\varepsilon^k = \gamma_\varepsilon^k \chi_\varepsilon^k \quad . \tag{2.4}$$

The equivalent variational formulation is

$$a_\varepsilon(\chi_\varepsilon^k, \varphi) = \gamma_\varepsilon^k (\chi_\varepsilon^k, \varphi) \ , \qquad \forall \varphi \in V \tag{2.5}$$

Now, some well-known facts are summarized in:

Proposition 2.1:

If (A1-A6) hold, then there exist unique sequences $\{\gamma_\varepsilon^k\}_{k=1}^\infty \in \mathbb{R}^+$, $\{\chi_\varepsilon^k\}_{k=1}^\infty \in V$ such that (2.4)(or, equivalently, (2.5)) is satisfied. Furthermore,

1) $0 < \gamma_\varepsilon^1 \le \gamma_\varepsilon^2 \le \ldots, \ \lim_{k \to +\infty} \gamma_\varepsilon^k = +\infty$

2) $\{\chi_\varepsilon^k\}_{k=1}^\infty$ is a complete orthonormal set in H, i.e., in particular,

$$(\chi_\varepsilon^k, \chi_\varepsilon^\ell) = \delta^{k\ell} \qquad \text{(Kronecker delta)}$$

3) The multiplicity of each eigenvalue is finite

4) The eigenvalues satisfy the following minimax formula:

$$\gamma_\varepsilon^k = \min_{W \subset V, \ \dim W = k} \ \max_{\chi \in V, \|\chi\|_H = 1} a_\varepsilon(\chi, \chi) \tag{2.6}$$

Proof: See [2,15,42]

Remark 2.4:

It is noteworthy to mention that Prop. 2.1 is valid for any positive value of ε, e.g., $\varepsilon = 1$. In this case, the eigenvalue-eigenvector pair of A is obtained.

2.3. Analysis of the Spectrum of A_ε

This section starts with a series of lemmas which characterize the various properties of the spectrum of A_ε. Then the convergence of the eigenvalues and their corresponding eigenvectors as $\varepsilon \to 0$ is stated and proved in Theorem 2.1. Some typical examples are given at the end of this section, to illustrate the ideas advanced in the course of the present analysis.

One way to gather information about the behavior of a single eigenvalue as $\varepsilon \to 0$, is to bound it from below and from above by known functions of ε. This task is accomplished in:

Lemma 2.1:

For sufficiently small positive ε, the following estimate holds:

$$\varepsilon\gamma^k \leq \gamma_\varepsilon^k \leq \gamma^k \tag{2.7}$$

for $k = 1,2,\ldots$, where $\{\gamma^k\}_{k=1}^\infty$ are the eigenvalues of the operator A, i.e., they satisfy

$$a_0(\rho^k,\varphi) + a_1(\rho^k,\varphi) = \gamma^k(\rho^k,\varphi), \forall \varphi \in V$$

Proof: For sufficiently small ε, one has

$$\varepsilon a(\varphi,\varphi) \leq a_\varepsilon(\varphi,\varphi) \leq a(\varphi,\varphi) \quad , \quad \forall \varphi \in V \quad . \tag{2.8}$$

Using the minimax characterization of eigenvalues (2.6), one readily deduces (2.7) from (2.8).

Now an upper bound for the eigenvector norm in V is derived:

Lemma 2.2:

If χ_ε^k is any normalized eigenvector of A_ε, corresponding to γ_ε^k, i.e., $\|\chi_\varepsilon^k\|_H = 1$, then

$$\alpha\varepsilon\|\chi_\varepsilon^k\|_V^2 \leq \gamma_\varepsilon^k \leq \gamma^k \tag{2.9}$$

for $k = 1,2,\ldots$.

Proof: For each k, the sequence γ_ε^k is bounded by Lemma 2.1. Let $\varphi = \chi_\varepsilon^k$ in (2.5) to get

$$a_\varepsilon(\chi_\varepsilon^k,\chi_\varepsilon^k) = \gamma_\varepsilon^k(\chi_\varepsilon^k,\chi_\varepsilon^k)$$

$$= \gamma_\varepsilon^k$$

$$\leq \gamma^k$$

Now one easily gets (2.9) by using (2.3). At this point, the tools necessary for finding the limits of the eigenvalues are available.

Lemma 2.3:

The sequence $\{\gamma_\varepsilon^k\}_{k=1}^\infty$ is decomposable into two subsequences $\{\lambda_\varepsilon^k\}_{k=1}^\infty$, $\{\mu_\varepsilon^k\}_{k=1}^\infty$ such that, for each k,

$$\lim_{\varepsilon \to 0} \lambda_\varepsilon^k = 0 \tag{2.10}$$

$$\lim_{\varepsilon \to 0} \mu_\varepsilon^k = \mu_0^k > 0 \tag{2.11}$$

$k = 1,2,\ldots$.

Proof: The following three-step contradiction argument is used to ascertain the above lemma.

1) Suppose $\lim_{\varepsilon \to 0} \gamma_\varepsilon^k = 0$, $k = 1,2,\ldots$. Let $V = V_0 \oplus V_0^\perp$, where the orthogonality is that of V. Take $v \in V_0^\perp$ and write it as

$$v = \sum_{k=1}^\infty (v,\chi_\varepsilon^k)\chi_\varepsilon^k \tag{2.12}$$

With this choice of v, the following inequality holds:

$$a_\varepsilon(v,v) = a_0(v,v) + \varepsilon a_1(v,v) \geq C \tag{2.13}$$

for some strictly positive constant C, which is independent of ε.

Using (2.12), $a_\varepsilon(v,v) = \sum_{k=1}^{\infty} \gamma_\varepsilon^k (v,\chi_\varepsilon^k)^2$, which converges to zero as $\varepsilon \to 0$, contradicting (2.13).

2) Suppose $\lim_{\varepsilon \to 0} \gamma_\varepsilon^k = \gamma_0^k > 0$, $k = 1,2,\ldots$. Select $v \in V_0$ and write it as in (2.12). It is clear that the following inequality holds:

$$a_\varepsilon(v,v) = \sum_{k=1}^{\infty} \gamma_\varepsilon^k (v,\chi_\varepsilon^k)^2 > C \tag{2.14}$$

for some strictly positive constant C, which is independent of ε. However $a_\varepsilon(v,v) = \varepsilon a_1(v,v)$, which converges to zero as $\varepsilon \to 0$, contradicting (2.14).

3) Suppose $\lim_{\varepsilon \to 0} \gamma_\varepsilon^k = 0$, for $k = 1,2,\ldots,\ell$, without loss of generality. Let

$$V_\ell = \text{span}\{\chi_\varepsilon^1, \chi_\varepsilon^2, \ldots, \chi_\varepsilon^\ell\} .$$

Select $v \in V_0^\perp \cap V_\ell^\perp$ and go to step 2, to conclude that V_ℓ is infinite dimensional, unless V_0 is degenerate (i.e., finite-dimensional). An identical argument can be advanced to contradict the possibility that (2.11) is true for $k=1,2,\ldots,\ell$ (ℓ finite).

Now decompose $\{\gamma_\varepsilon^k, \chi_\varepsilon^k\}_{k=1}^\infty$ into $\{\lambda_\varepsilon^k, \varphi_\varepsilon^k\}_{k=1}^\infty$ if $\lim \gamma_\varepsilon^k = 0$ and into $\{\mu_\varepsilon^k, \psi_\varepsilon^k\}_{k=1}^\infty$, otherwise.

Some attention must be focused on how λ_ε^k converges to zero, $k = 1,2,\ldots$.

<u>Lemma 2.4</u>:

The sequence λ_ε^k converges to zero with a constant rate λ_1^k, i.e.,

$$\lambda_\varepsilon^k = \lambda_1^k \varepsilon + o(\varepsilon) \quad , \; k = 1,2,\ldots \quad .$$

<u>Proof</u>: By Lemma 2.1, one may assume, without loss of generality, that

$$\lambda_\varepsilon^k = \lambda_1^k \varepsilon^\nu + o(\varepsilon) \tag{2.15}$$

with $\nu \in (0,1]$. In order to complete the proof, it suffices to show that $\nu = 1$.

If φ_ε^k is a normalized eigenvector, i.e., $(\varphi_\varepsilon^k, \varphi_\varepsilon^k)_H = 1$, corresponding to λ_ε^k, then

$$\lambda_\varepsilon^k = a_0(\varphi_\varepsilon^k, \varphi_\varepsilon^k) + \varepsilon a_1(\varphi_\varepsilon^k, \varphi_\varepsilon^k) \tag{2.16}$$

from which one observes that

$$a_0(\varphi_\varepsilon^k, \varphi_\varepsilon^k) \leq O(\varepsilon^\nu) \quad . \tag{2.17}$$

Therefore, $a_0(\varphi_\varepsilon^k, \varphi_\varepsilon^k) \to 0$ as $\varepsilon \to 0$, which implies

$$p_0(\varphi_\varepsilon^k) \to 0, \text{ as } \varepsilon \to 0$$

by (A5). Using (A6), one has

$$p_1(\varphi_\varepsilon^k) > C \tag{2.18}$$

for some strictly positive constant C, which is independent of ε.

Suppose that $\nu < 1$. Then, from (2.15-2.18) one concludes that

$$\begin{aligned} \lambda_\varepsilon^k &= \lambda_1^k \varepsilon^\nu + o(\varepsilon) \\ &= a_0(\varphi_\varepsilon^k, \varphi_\varepsilon^k) + o(\varepsilon). \end{aligned} \tag{2.19}$$

From this, there exists an element of V, $\bar{\varphi}_\varepsilon^k = \dfrac{\varphi_\varepsilon^k}{\varepsilon^{\nu/2}}$ such that

$$\lambda_1^k = a_0(\bar{\varphi}_\varepsilon^k, \bar{\varphi}_\varepsilon^k) + o(\varepsilon^{1-\nu}) \quad .$$

However, such a claim is false because

$$(\bar{\varphi}_\varepsilon^k, \bar{\varphi}_\varepsilon^k)_H = \frac{1}{\varepsilon^\nu} \to +\infty \text{ as } \varepsilon \to 0 .$$

Since the injection of V into H is continuous, $\|\bar{\varphi}_\varepsilon^k\|_V \to +\infty$ as $\varepsilon \to 0$. In conclusion, there is no element of V such that (2.19) is satisfied.

Remark 2.5:

For $\nu < 1$, the major contribution to λ_ε^k is supplied from $V_0^\perp$ (Cf. (2.19)), but the norm of the contributor is concentrated on V_0 (Cf. (2.18)), which is the paradox.

Hereafter, the focus will be on the asymptotic behavior of the eigenvectors. The following lemma summarizes the norm bounds of the eigenvectors:

Lemma 2.5:

Let $\{\lambda_\varepsilon^k, \varphi_\varepsilon^k\}_{k=1}^\infty$, $\{\mu_\varepsilon^k, \psi_\varepsilon^k\}_{k=1}^\infty$ be as in the proof of Lemma 2.3, with the eigenvectors normalized in H. Then, for sufficiently small ε, the following estimates hold:

$$1) \quad \|\varphi_\varepsilon^k\|_V \leq C_1 \tag{2.20}$$

$$2) \quad \sqrt{\varepsilon}\, \|\psi_\varepsilon^k\|_V \leq C_2 \tag{2.21}$$

$$k = 1,2,\ldots$$

where C_1, C_2 denote constants independent of ε.

Proof: Use Lemmas 2.2-2.4.

The forthcoming theorem is the main result of this section. It states the convergence of the eigenvalues and their corresponding eigenvectors as $\varepsilon \to 0$.

Theorem 2.1

Let $\{\gamma_\varepsilon^k\}_{k=1}^\infty$ be the eigenvalues of A_ε and $\{\chi_\varepsilon^k\}_{k=1}^\infty$ the corresponding normalized system of eigenvectors. Then, given a sequence of ε converging to zero, $\{\gamma_\varepsilon^k,\chi_\varepsilon^k\}_{k=1}^\infty$ can be decomposed into two subsequences $\{\lambda_\varepsilon^k,\varphi_\varepsilon^k\}_{k=1}^\infty$ and $\{\mu_\varepsilon^k,\psi_\varepsilon^k\}_{k=1}^\infty$ which have the following asymptotic properties, for each k,

1) $\lambda_\varepsilon^k \to 0$ linearly in ε, $\varphi_\varepsilon^k \to \varphi^k$ strongly in V

2) $\mu_\varepsilon^k \to \mu_0^k > 0$, $\psi_\varepsilon^k \to \psi^k$ weakly in H

where $\{\varphi^k\}_{k=1}^\infty$ and $\{\psi^k\}_{k=1}^\infty$ satisfy

$$a_1(\varphi^k,\chi) = \lambda_1^k(\varphi^k,\chi),\ \varphi^k \in V_0 \subset V,\quad \forall\chi \in V_0 \tag{2.22}$$

$$a_0(\psi^k,\chi) = \mu_0^k(\psi^k,\chi),\ \psi^k \in H_1 \subset H,\quad \forall\chi \in V\ . \tag{2.23}$$

Proof: Using the fact that $\|\varphi_\varepsilon^k\|_H = 1$, the estimates (2.17), (2.20), one concludes that, given a sequence of ε converging to zero, $\varphi_\varepsilon^k \to \varphi^k$ weakly in V (hence strongly in H by compactness). From (2.17), it results that $\varphi^k \in V_0$. By Lemma 2.4, λ_ε^k is asymptotically equal to $\lambda_1^k\varepsilon$. Hence (2.5) degenerates into (2.22) in the limit.

Now let $w_\varepsilon^k = \varphi_\varepsilon^k - \varphi^k$ which satisfies

$$\frac{1}{\varepsilon}a_0(w_\varepsilon^k,w_\varepsilon^k) + a_1(w_\varepsilon^k,w_\varepsilon^k) = \frac{\lambda_\varepsilon^k - \lambda_1^k\varepsilon}{\varepsilon}(\varphi_\varepsilon^k,w_\varepsilon^k) + \lambda_1^k(w_\varepsilon^k,w_\varepsilon^k)$$

$$\geq \alpha\,\|w_\varepsilon^k\|_V^2$$

The right-hand side of the above equation converges to zero as $\varepsilon \to 0$. Hence $\|w_\varepsilon^k\|_V \to 0$, indicating that $\varphi_\varepsilon^k \to \varphi^k$ strongly in V. It is clear that

$$\mu_\varepsilon^k = a_0(\psi_\varepsilon^k,\psi_\varepsilon^k) + \varepsilon a_1(\psi_\varepsilon^k,\psi_\varepsilon^k) \tag{2.24}$$

for ψ_ε^k normalized to 1 in H. Since μ_ε^k is $O(1)$, using (2.24) and the minimax characterization of eigenvalues, i.e., (2.6), it results that $a_0(\psi_\varepsilon^k,\psi_\varepsilon^k)$ is $O(1)$ and $a_1(\psi_\varepsilon^k,\psi_\varepsilon^k)$ is $O(\frac{1}{\varepsilon})$ or equivalently

$$p_0(\psi_\varepsilon^k) = O(1) \tag{2.25}$$

$$p_1(\psi_\varepsilon^k) = O(\frac{1}{\sqrt{\varepsilon}}) \quad . \tag{2.26}$$

Hence, the estimate (2.21) is tight. Therefore

$$\|\psi_\varepsilon^k\|_V \to +\infty \text{ as } \varepsilon \to 0 \quad . \tag{2.27}$$

Note that ψ_ε^k also satisfies

$$a_0(\psi_\varepsilon^k,\chi) + \varepsilon a_1(\psi_\varepsilon^k,\chi) = \mu_\varepsilon^k(\psi_\varepsilon^k,\chi), \quad \forall\, \chi \in V \ . \tag{2.28}$$

From (2.25-2.26), one deduces that $a_0(\psi_\varepsilon^k,\chi)$ is bounded as $\varepsilon \to 0$ and $a_1(\psi_\varepsilon^k,\chi)$ is $O(\frac{1}{\sqrt{\varepsilon}})$. Since $\|\psi_\varepsilon^k\|_H = 1$, taking formally the limit as $\varepsilon \to 0$ in (2.28) yields:

$$a_0(\psi^k,\chi) = \mu_0^k(\psi^k,\chi), \quad \forall\, \chi \in V \tag{2.29}$$

where $\psi^k \in H_1$ (a subspace of H).

Now consider the following boundary value problem

$$a_0(v_\varepsilon^k,\chi) + \varepsilon a_1(v_\varepsilon^k,\chi) = \mu_0^k(\psi^k,\chi), \qquad \forall\, \chi \in V$$

which admits a unique solution $v_\varepsilon^k \in V$ for positive values of ε. As $\varepsilon \to 0$, $v_\varepsilon^k \to \psi^k$ strongly in H.

Let $w_\varepsilon^k = \psi_\varepsilon^k - v_\varepsilon^k$ which satisfies

$$a_0(w_\varepsilon^k,\chi) + \varepsilon a_1(w_\varepsilon^k,\chi) = \mu_\varepsilon^k(\psi_\varepsilon^k - \psi^k,\chi)$$

$$+ (\mu_\varepsilon^k - \mu_0^k)(\psi^k,\chi), \qquad \forall\, \chi \in V .$$

The left-hand side of this equation goes to zero as $\varepsilon \to 0$, implying

$$(\psi_\varepsilon^k - \psi^k,\chi) \to 0 \text{ as } \varepsilon \to 0.$$

Therefore

$$\psi_\varepsilon^k \to \psi^k \text{ weakly in H.}$$

Remark 2.6:

The weak convergence in Theorem 2.1 cannot be improved in general. This will be illustrated by Example 2.4.

Remark 2.7:

A careful examination of the steps of the analysis undertaken in the present section yields the following observation: the weak limits $\{\varphi^k\}_{k=1}^\infty$, $\{\psi^k\}_{k=1}^\infty$ form an orthonormal system in H. This remark is of paramount importance in approximating the solution of boundary value problems involving the operator A_ε.

Remark 2.8:

In the sequel, the fact that $\|\psi_\varepsilon^k\|_V \to +\infty$ as $\varepsilon \to 0$ is referred to as the resonant behavior of ψ_ε^k .

Now some examples are given as concrete illustrations of the above abstract results. Only operators of order less than or equal to four are considered, due to their frequent usage in modeling of physical processes.

Let $\Omega = \Omega_0 \cup \Omega_1 \cup S$ be a bounded set in $\mathbb{R}^n$ with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. The manifold S denotes the interface between Ω_0 and Ω_1, as indicated in Figure 2.1:

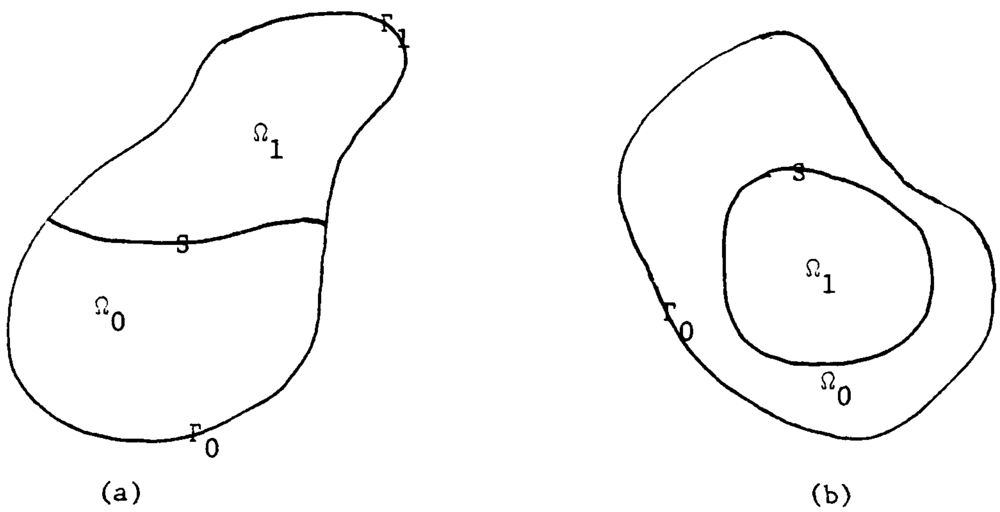

Figure 2.1 a-b. Examples of interfaced sets.

<u>Example 2.1: A second order operator</u>

Let $H = L^2(\Omega)$, $V = H_0^1(\Omega)$

$$a_i(\varphi,\psi) = \sum_{j=1}^{n} \int_{\Omega_i} \frac{\partial \varphi}{\partial x_j} \frac{\partial \psi}{\partial x_j}\, dx \ , \quad i = 0,1$$

then (2.4) becomes:

$$\left.\begin{array}{l} \left.\begin{array}{l} -\Delta \chi_{\varepsilon 0}^{k} = \gamma_{\varepsilon}^{k} \chi_{\varepsilon 0}^{k} \text{ on } \Omega_0 \\ -\varepsilon\Delta \chi_{\varepsilon 1}^{k} = \gamma_{\varepsilon}^{k} \chi_{\varepsilon 1}^{k} \text{ on } \Omega_1 \end{array}\right\} \\ \chi_{\varepsilon 0}^{k}\big|_{\Gamma_0} = 0, \quad \chi_{\varepsilon 1}^{k}\big|_{\Gamma_1} = 0 \\ \left.\begin{array}{l} \chi_{\varepsilon 0}^{k} = \chi_{\varepsilon 1}^{k} \\ \dfrac{\partial \chi_{\varepsilon 0}^{k}}{\partial \nu} = \varepsilon \dfrac{\partial \chi_{\varepsilon 1}^{k}}{\partial \nu} \end{array}\right\} \text{ on } S \end{array}\right\} \tag{2.30}$$

where Δ stands for the Laplacian in $\mathbb{R}^n$

ν is the unit normal on Γ or S, outward relative to Ω_0.

In this example, (2.22) becomes

$$\left.\begin{array}{l} \left.\begin{array}{l} \varphi_0^k = 0 \text{ on } \Omega_0 \\ -\Delta\varphi_1^k = \lambda_1^k \varphi_1^k \text{ on } \Omega_1 \end{array}\right\} \\ \varphi_1^k\big|_{\Gamma_1} = 0, \quad \varphi_1^k\big|_S = 0 \end{array}\right\} \tag{2.31}$$

which is a Dirichlet eigenvalue problem for the Laplacian operator in Ω_1. The subspace V_0 of V is:

$$V_0 = \{ \chi \in V : \chi_0 = 0, \quad \chi_1 \in H_0^1(\Omega_1) \}$$

The conclusions of Prop. 2.1 are applicable in this case. Hence $\{\varphi_1^k\}_{k=1}^{\infty}$ is a complete orthonormal system in $L^2(\Omega_1)$.

Equation (2.23) becomes

$$\left.\begin{array}{l}\left.\begin{array}{r}-\Delta\,\psi_1^k = \mu_0^k\,\psi_0^k \quad \text{on} \quad \Omega_0 \\ \psi_1^k = 0 \qquad \text{on} \quad \Omega_1\end{array}\right\} \\ \psi_1^k\big|_{\Gamma_0} = 0\ ,\ \dfrac{\partial\psi_0^k}{\partial\nu}\Big|_S = 0\end{array}\right\} \tag{2.32}$$

which is an eigenvalue problem with mixed boundary conditions for the Laplacian operator in Ω_0. Again, the conclusions of Prop. 2.1 are applicable in this instance, provided the interface S is sufficiently smooth. Therefore, $\{\psi_0^k\}_{k=1}^{\infty}$ form an orthonormal system in $L^2(\Omega_0)$. It is noteworthy to observe that $\psi_1^k = 0$ because ψ^k must be orthogonal (in $L^2(\Omega)$) to φ^ℓ, $\ell=1,2,\ldots$.

The subspace H_1 of H is then

$$H_1 = \{\chi \in H : \chi_0 \in H^1(\Omega_0;\Gamma_0),\ \frac{\partial\chi_0}{\partial\nu} = 0 \text{ on } S,\ \chi_1 = 0\}\ .$$

Remark 2.9:

In Example 2.1, one can consider

$$a_i(\varphi,\psi) = \sum_{j=1}^{n}\sum_{k=1}^{n}\int_{\Omega_i} a_{jk}^i \frac{\partial\varphi}{\partial x_j}\frac{\partial\psi}{\partial x_k}\,dx,\ i = 0,1$$

where a_{jk}^i satisfies

1) $a_{jk}^i \in C^1(\bar{\Omega}_i)$

2) $a_{jk}^i = a_{kj}^i$

3) $\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}^i\,\xi_j\xi_k \geq \alpha_i \sum_{k=1}^{n}\xi_k^2,\ \alpha_i > 0\ , \qquad \forall\,\xi \in \mathbb{R}^n,\ \xi \neq 0.$

The discussion therein remains unchanged.

Example 2.2: A fourth order operator

Let $H = L^2(\Omega)$, $V = H_0^2(\Omega)$

$$a_i(\varphi,\psi) = \int_{\Omega_i} \Delta\varphi\Delta\psi \, dx \ , \ i = 0,1$$

then (2.4) becomes:

$$\left.\begin{array}{r}
\left.\begin{array}{rl}
\Delta^2\chi^k_{\varepsilon 0} = \gamma^k_\varepsilon \chi^k_{\varepsilon 0} & \text{on } \Omega_0 \\
\varepsilon\Delta^2\chi^k_{\varepsilon 1} = \gamma^k_\varepsilon \chi^k_{\varepsilon 1} & \text{on } \Omega_1
\end{array}\right\} \\
\chi^k_{\varepsilon 0} = \dfrac{\partial\chi^k_{\varepsilon 0}}{\partial\nu} = 0 \text{ on } \Gamma_0 \\
\chi^k_{\varepsilon 1} = \dfrac{\partial\chi^k_{\varepsilon 1}}{\partial\nu} = 0 \text{ on } \Gamma_1 \\
\left.\begin{array}{r}
\chi^k_{\varepsilon 0} = \chi^k_{\varepsilon 1}, \ \dfrac{\partial\chi^k_{\varepsilon 0}}{\partial\nu} = \dfrac{\partial\chi^k_{\varepsilon 1}}{\partial\nu} \\
\Delta\chi^k_{\varepsilon 0} = \varepsilon\, \Delta\chi^k_{\varepsilon 1}, \ \dfrac{\partial\Delta\chi^k_{\varepsilon 0}}{\partial\nu} = \varepsilon\, \dfrac{\partial\Delta\chi^k_{\varepsilon 1}}{\partial\nu}
\end{array}\right\} \text{on } S
\end{array}\right\} \qquad (2.33)$$

Equation (2.22) becomes

$$\left.\begin{array}{r}
\left.\begin{array}{rl}
\varphi^k_0 = 0 & \text{on } \Omega \\
\Delta^2\varphi^k_1 = \lambda^k_1\varphi^k_1 & \text{on } \Omega_1
\end{array}\right\} \\
\varphi^k_1 = \dfrac{\partial\varphi^k_1}{\partial\nu} = 0 \text{ on } \Gamma_1 \\
\varphi^k_1 = \dfrac{\partial\varphi^k_1}{\partial\nu} = 0 \text{ on } S
\end{array}\right\} \qquad (2.34)$$

This is a Dirichlet eigenvalue problem for the biharmonic operator in Ω_1.

Equation (2.23) becomes

$$\left.\begin{array}{l}\left.\begin{array}{l}\Delta^2\psi_0^k = \mu_0^k\psi_0^k \text{ on } \Omega_0 \\ \psi_1^k = 0 \quad \text{ on } \Omega_1\end{array}\right\} \\ \psi_0^k = \dfrac{\partial\psi_0}{\partial\nu} = 0 \text{ on } \Gamma_0 \\ \Delta\psi_0^k = \dfrac{\partial\Delta\psi_0^k}{\partial\nu} = 0 \text{ on } S\end{array}\right\} \tag{2.35}$$

Identical comments to those of Example 2.1 can be made here, provided some of the function spaces are changed, to reflect the increase in the operator order from two to four, as seen in (2.33).

For simplicity considerations, a one-dimensional version of Example 2.1 is studied in the sequel. This example will be useful for illustrating later developments. It will clarify many aspects of the eigenvalue problem at hand, such as nonanalyticity and oscillatory behavior of $\{\psi_\varepsilon^k\}_{k=1}^\infty$, "flattening" (and sometimes "attenuation") of $\{\varphi_\varepsilon^k\}_{k=1}^\infty$.

<u>Example 2.3</u>: (Cf. Example 2.1)

Let $\Omega_0 = (a,b) \cup (c,d)$, $\Omega_1 = (b,c)$

$$\Gamma_0 = \{a,d\}, \Gamma_1 = \{d\}, S = \{b,c\}$$

with $a < b < c \leq d$ as in Figure 2.2.

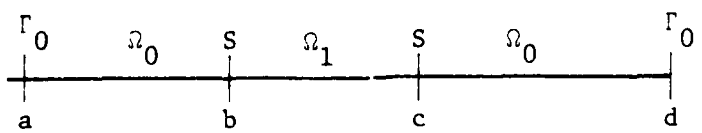

Figure 2.2. Structure of the set $\Omega \subset \mathbb{R}$.

In this instance, (2.31) can be readily solved, to get

$$\left.\begin{aligned}
&\lambda_1^k = \left(\frac{k\pi}{c-b}\right)^2 \\
&\left.\begin{aligned}
&\varphi_0^k = 0 \\
&\varphi_1^k = \frac{2}{c-b} \sin k\pi \frac{x-b}{c-b}
\end{aligned}\right\}
\end{aligned}\right\} \tag{2.36}$$

The solution to (2.32) is

$$\left.\begin{aligned}
&\mu_{ab}^k = \left(\frac{(2k-1)\,\frac{\pi}{2}}{b-a}\right)^2 \\
&\left.\begin{aligned}
&\psi_{0_{ab}}^k = \begin{cases} \sin(2k-1)\,\frac{\pi}{2}\,\frac{x-a}{b-a} & \text{on } (a,b) \\ 0 & \text{on } (c,d) \end{cases} \\
&\psi_{1_{ab}}^k = 0
\end{aligned}\right\} \\
&\mu_{cd}^k = \left(\frac{((2k-1)\,\frac{\pi}{2})}{d-c}\right)^2 \\
&\left.\begin{aligned}
&\psi_{0_{cd}}^k = \begin{cases} 0 & \text{on } (a,b) \\ \sin(2k-1)\,\frac{\pi}{2}\,\frac{x-c}{d-c} & \text{on } (c,d) \end{cases} \\
&\psi_{1_{cd}}^k = 0 \qquad \text{on } (b,c)
\end{aligned}\right\}
\end{aligned}\right\} \tag{2.37}$$

Remark 2.10:

Note that in the case when Ω_0 is not a connected set, each subset of it is associated with a subgroup of eigenvalue-eigenvector pairs as indicated in (2.37).

Using Example 2.3, some exact eigenvalue-eigenvector pairs are constructed for some specific values of ε to shed more light on the asymptotic behavior of the spectrum of the operator A_ε. For simplicity, consider the following example:

Example 2.4: (Cf. Example 2.3)

Let a = -1, b = 0, c = d = 1 so that (2.36-2.37) become:

$$\left.\begin{array}{l} \lambda_1^k = (k\pi)^2 \\[2ex] \left.\begin{array}{l} \varphi_0^k = 0 \\[2ex] \varphi_1^k = \sqrt{2}\,\sin k\pi x \end{array}\right\} \end{array}\right\}$$

$$\left.\begin{array}{l} \mu_0^k = \left((2k-1)\,\frac{\pi}{2}\right)^2 \\[2ex] \left.\begin{array}{l} \psi_0^k = \cos(2k-1)\,\frac{\pi}{2}\,x \\[2ex] \psi_1^k = 0 \end{array}\right\} \end{array}\right\}$$

Direct computation of the eigenvalues yields that γ_ε^k must satisfy:

$$\cos\sqrt{\gamma_\varepsilon^k}\,\sin\sqrt{\frac{\gamma_\varepsilon^k}{\varepsilon}} + \sqrt{\varepsilon}\,\sin\sqrt{\gamma_\varepsilon^k}\,\cos\sqrt{\frac{\gamma_\varepsilon^k}{\varepsilon}} = 0\ . \tag{2.38}$$

Despite the transcendental nature of (2.38), it is possible to solve for γ_ε^k for some sequence of ε. For example, for $\varepsilon_\ell = \frac{1}{(4\ell+1)^2}$, $\ell \in \mathbb{N}$, $\{\mu_{\varepsilon_\ell}^1, \psi_{\varepsilon_\ell}^1\}$ given below is an exact eigenvalue-eigenvector pair

$$\left.\begin{aligned} \mu_{\varepsilon_\ell}^1 &= \left(\frac{\pi}{2}\right)^2 \\ \psi_{\varepsilon_\ell 0}^1 &= \cos \frac{\pi}{2} x \\ \psi_{\varepsilon_\ell 1}^1 &= \cos \frac{\pi}{2\sqrt{\varepsilon_\ell}} x \end{aligned}\right\} \tag{2.39}$$

Remark 2.11:

It is instructional to observe the oscillatory behavior of $\psi_{\varepsilon_\ell 1}^1$ as $\ell \to +\infty$. This is consequential to the weak convergence of $\psi_{\varepsilon_\ell 1}^1$ in $L^2(\Omega_1)$ to zero, i.e., $(\psi_{\varepsilon_\ell 1}^1, \chi)_{L^2(\Omega_1)} \to 0$ as $\ell \to +\infty$, $\forall\, \chi \in L^2(\Omega_1)$, which can be easily verified. Furthermore, it can be easily seen from (2.39) that $\{\psi_\varepsilon^k\}_{k=1}^\infty$ are not analytic functions of ε, in the vicinity of $\varepsilon = 0$. Therefore, the task of finding these eigenvectors (in order to solve boundary value problems involving the operator A_ε) is nearly impossible. The alternative is to use the weak limits of $\{\chi_\varepsilon^k\}_{k=1}^\infty$, i.e., $\{\psi^k\}_{k=1}^\infty$ and $\{\varphi^k\}_{k=1}^\infty$, since they possess some desirable properties (Cf. Remark 2.7).

Remark 2.12:

The computation of $\mu_{\varepsilon_\ell}^1$, using the Rayleigh quotient of $\psi_{\varepsilon_\ell}^1$, yields

$$\mu^1_{\varepsilon_\ell} = \frac{a_{\varepsilon_\ell}(\psi^1_{\varepsilon_\ell}, \psi^1_{\varepsilon_\ell})}{(\psi^1_{\varepsilon_\ell}, \psi^1_{\varepsilon_\ell})}$$

$$= \frac{\int_{-1}^{0} (-\frac{\pi}{2} \sin \frac{\pi}{2} x)^2 dx + \varepsilon_\ell \int_0^1 (-\frac{\pi}{2\sqrt{\varepsilon_\ell}} \sin \frac{\pi}{2\sqrt{\varepsilon_\ell}} x)^2 dx}{\frac{1}{2} + \frac{1}{2}}$$

$$= \frac{1}{2} (\frac{\pi}{2})^2 + \frac{1}{2} \varepsilon_\ell (\frac{\pi}{2\sqrt{\varepsilon_\ell}})^2$$

$$= (\frac{\pi}{2})^2$$

which shows clearly that the oscillatory behavior effect is to cancel the effect of ε_ℓ in the bilinear form in order to contribute an amount of O(1) to the value of $\mu^1_{\varepsilon_\ell}$ (Cf. Theorem 2.1).

Similarly, for $\varepsilon_k = \frac{1}{k^2}$, $k = 1,2,\ldots$, $\{\lambda^k_{\varepsilon_k}, \varphi^k_{\varepsilon_k}\}$ given below is an exact eigenvalue-eigenvector pair

$$\left.\begin{aligned} \lambda^k_{\varepsilon_k} &= (k\pi)^2 \\ \left.\begin{aligned} \varphi^k_{\varepsilon_k 0} &= \sqrt{\varepsilon_k} \sin \sqrt{\varepsilon_k} k\pi x \\ \varphi^k_{\varepsilon_k 1} &= \sin k\pi x \end{aligned}\right\} \end{aligned}\right\} \tag{2.40}$$

<u>Remark 2.13</u>:

Note that in (2.40), ε_k depends on the index of the eigenvalue. Hence, one cannot let ε_k go to zero and observe what happens to this eigenvalue-eigenvector pair.

Remark 2.14:

The results of Theorem 2.1 show that $\varphi^k_{\varepsilon 0} \to 0$ in the present example. This is certainly reflected in (2.40) by the presence of the factor $\sqrt{\varepsilon_k}$ multiplying $\varphi^k_{\varepsilon_k 0}$. This behavior of φ^k_ε is referred to as the "attenuation" of φ^k_ε. It depends generally upon the choice of the space V. See Example 2.5.

Remark 2.15:

The computation of $\lambda^k_{\varepsilon_k}$ using the Rayleigh quotient of $\varphi^k_{\varepsilon_k}$ yields

$$\lambda^k_{\varepsilon_k} = \frac{\frac{1}{2}(k\pi)^2 \ \varepsilon_k + \frac{1}{2}\ \varepsilon_k (k\pi)^2}{\frac{1}{2}\ \varepsilon_k + \frac{1}{2}} \ .$$

Observe that the relative contribution of $a_0(\varphi^k_{\varepsilon_k}, \varphi^k_{\varepsilon_k})$, i.e.,

$$\frac{a_0(\varphi^k_{\varepsilon_k}, \varphi^k_{\varepsilon_k})}{(\varphi^k_{\varepsilon_k}, \varphi^k_{\varepsilon_k})_{L^2(\Omega_0)}}$$

, is $O(\varepsilon_k)$. This property seems to be of a general nature.

See Example 2.5.

Remark 2.16:

In the proof of Lemma 2.4, it is mentioned that $a_0(\varphi^k_\varepsilon, \varphi^k_\varepsilon) \leq O(\varepsilon)$, which is certainly illustrated by the present example.

Remark 2.17

The presence of $\sqrt{\varepsilon_k}$ in the argument of the sine function in (2.40) indicates that "flattening" occurs in φ^k_ε as $\varepsilon \to 0$ which is predicted by the fact that $a_0(\varphi^k_\varepsilon, \varphi^k_\varepsilon) \leq O(\varepsilon)$.

The next example shows the presence of flattening without attenuation.

Example 2.5:

Let $H = L^2(\Omega)$, $V = H^1(\Omega;\Gamma_1)$ with $\Omega_0 = (-1,0)$, $\Omega_1 = (0,1)$, $\Gamma_0 = \{-1\}$, $\Gamma_1 = \{1\}$, $S = \{0\}$ and identical bilinear forms to those of Example 2.1. From (2.22), the limit of $\{\varphi_\varepsilon^k\}_{k=1}^\infty$ as $\varepsilon \to 0$ satisfies

$$\left.\begin{aligned} &\varphi_0^k = \varphi_1^k(0) \quad \text{on } \Omega_0 \\ &-\frac{d^2}{dx^2}\varphi_1^k = \lambda_1^k \varphi_1^k \quad \text{on } \Omega_1 \\ &\frac{d\varphi_1^k}{dx}(0) = 0,\ \varphi_1^k(1) = 0 \end{aligned}\right\} \tag{2.41}$$

whose solution is

$$\left.\begin{aligned} &\lambda_1^k = ((2k-1)\tfrac{\pi}{2})^2 \\ &\left.\begin{aligned} &\varphi_0^k = 1 \\ &\varphi_1^k = \cos(2k-1)\tfrac{\pi}{2}\, x \end{aligned}\right\} \end{aligned}\right\}$$

For $\varepsilon_k = \dfrac{1}{(2k-1)^2}$, $k = 1,2,\dots$ is an eigenvalue-eigenvector pair is

$$\left.\begin{aligned} &\lambda_{\varepsilon_k}^k = ((2k-1)\tfrac{\pi}{2})^2\ \varepsilon_k \\ &\left.\begin{aligned} &\varphi_{\varepsilon_k}^k = \cos\ \sqrt{\varepsilon_k}(2k-1)\tfrac{\pi}{2}\ x \\ &\varphi_{\varepsilon_k}^k = \cos\ (2k-1)\tfrac{\pi}{2}\ x \end{aligned}\right\} \end{aligned}\right\} \tag{2.42}$$

It is noteworthy to observe that $\varphi^k_{\varepsilon_k 0}$ in (2.42) is not multiplied by $\sqrt{\varepsilon_k}$, which indicates no attenuation. However, the argument of the cosine function does contain $\sqrt{\varepsilon_k}$, signaling flattening.

Remark 2.18

The computation of $\lambda^k_{\varepsilon_k}$ using the Rayleigh quotient of $\varphi^k_{\varepsilon_k}$ shows that

$$\lambda^k_{\varepsilon_k} = \frac{\frac{1}{2}((2k-1)\ \frac{\pi}{2})^2 \varepsilon_k + \frac{1}{2}((2k-1)\ \frac{\pi}{2})^2 \varepsilon_k}{\frac{1}{2} + \frac{1}{2}} \quad .$$

Note that $a_0(\varphi^k_{\varepsilon_k}, \varphi^k_{\varepsilon_k}) = O(\varepsilon_k)$ and $(\varphi^k_{\varepsilon_k 0}, \varphi^k_{\varepsilon_k 0})_{L^2(\Omega_0)} = \frac{1}{2}$, in contrast to that of Example 2.4. However, the relative contribution to the value of the eigenvalue is of the same order of magnitude.

In summary, flattening of φ^k_ε is inherent to the problem, but attenuation depends upon the order of magnitude of $a_0(\varphi^k_\varepsilon, \varphi^k_\varepsilon)$ which, in turn, depends upon the choice of the space V. If $a_0(\varphi^k_\varepsilon, \varphi^k_\varepsilon) < O(\varepsilon)$, then attenuation is present.

2.4. General Results

In this section, the results of Section 2.3 are generalized to p+1 ($p > 1$) bilinear forms. The generalization of Theorem 2.1 is stated, but its proof is omitted, because it follows similar steps to that of Theorem 2.1. A corollary is then presented, which considers positive as well as negative powers of ε.

Suppose one is given two Hilbert spaces V, H as in Section 2.2. Let $a_i(\varphi, \psi)$, $i = 0,1,2,\ldots,p$, be p+1 ($p > 1$) forms on V, with each of them satisfying (A3-A5).

It is assumed that:

(A6') $\sum_{i=0}^{p} p_i(\varphi)$ is a norm equivalent to $\|\varphi\|_V$

(A7') $\begin{cases} a_0(\varphi,\varphi) \text{ is null on } V_0 \subset V,\ V_0 \neq \{0\} \\ a_1(\varphi,\varphi) \text{ restricted to } V_0, \text{ is null on } V_1, \\ \ldots \\ a_p(\varphi,\varphi) \text{ restricted to } V_{p-1}, \text{ is null on } V_p \neq \{0\} \end{cases}$

(A8') If $\psi \mapsto L_j(\psi)$ is a continuous linear form on V, null on V_j, there exists $\varphi \in V_{j-1}$ (modulo V_j) such that

$$a_j(\varphi,\psi) = L_j(\psi),\ \forall\psi\in V_{j-1},\ j=0,1,2,\ldots,p \quad , \quad \text{with } V_{-1} = V.$$

Let $a_\varepsilon(\varphi,\psi)$ be defined as

$$a_\varepsilon(\varphi,\psi) = \sum_{i=0}^{p} \varepsilon^i a_i(\varphi,\psi) .$$

Remark 2.19:

Remarks 2.1-2.3 are applicable in this case.

The variational formulation of the eigenvalue problem is to seek $\{\gamma_\varepsilon^k, \chi_\varepsilon^k\} \in \mathbb{R}^+ \times V$ such that

$$a_\varepsilon(\chi_\varepsilon^k,\varphi) = \gamma_\varepsilon^k(\chi_\varepsilon^k,\varphi) \ , \ \forall\ \varphi \in V \ . \tag{2.43}$$

In this case, the conclusions of Proposition 2.1 are valid. Moreover, some additional properties are summarized in the following theorem, which is the counterpart of Theorem 2.1:

Theorem 2.2:

Let $\{\gamma_\varepsilon^k\}_{k=1}^\infty$ be the eigenvalues and $\{\chi_\varepsilon^k\}_{k=1}^\infty$ the corresponding normalized eigenvectors, as derived from (2.43). Then given a sequence of ε converging to zero, $\{\gamma_\varepsilon^k, \chi_\varepsilon^k\}_{k=1}^\infty$ can be decomposed into p+1 groups

$\{\mu^k_{\varepsilon,\ell}, \psi^k_{\varepsilon,r}\}^\infty_{k=1}$, $r=0,1,\ldots,p-1$ and $\{\lambda^k_\varepsilon, \varphi^k_\varepsilon\}^\infty_{k=1}$, with the following asymptotic properties:

1) $\lambda^k_\varepsilon = \lambda^k_p \varepsilon^p + o(\varepsilon^p)$, $\varphi^k_\varepsilon \to \varphi^k$ strongly in V

2) $\mu^k_{\varepsilon,r} = \mu^k_r \varepsilon^r + o(\varepsilon^r)$, $\psi^k_{\varepsilon,r} \to \psi^k_r$ weakly in H,
   $r=0,1,\ldots,p-1$

where $\{\varphi^k\}^\infty_{k=1}$ and $\{\psi^k_r\}^\infty_{k=1}$, $r=0,1,\ldots,p-1$ satisfy:

$$a_p(\varphi^k,\chi) = \lambda^k_p(\varphi^k,\chi), \quad \varphi^k \in V_p\ , \quad \forall \chi \in V_p \tag{2.44}$$

$$a_r(\psi^k_r,\chi) = \mu^k_r(\psi^k_r,\chi), \quad \psi^k_r \in H_{p-r} \subset H,\ \forall\ \chi \in V_{r-1}$$
$$r=0,1,\ldots,p-1. \qquad \square \tag{2.45}$$

Let

$$b_\varepsilon(\varphi,\psi) = \sum_{i=-m}^{n} \varepsilon^i b_i(\varphi,\psi), \quad n,m \in \mathbb{N} \quad (m+n > 1) \text{ where } b_i(\varphi,\psi),$$

$i = -m,-m+1,\ldots,n$ are $m+n+1$ forms on V, satisfying (A3-A5), (A6'-A7') then one has:

<u>Corollary 2.1</u>

Let $\{\nu^k_\varepsilon\}^\infty_{k=1}$ be the eigenvalues and $\{\chi^k_\varepsilon\}^\infty_{k=1}$ the corresponding normalized eigenvectors, derived from

$$b_\varepsilon(\chi^k_\varepsilon,\varphi) = \nu^k_\varepsilon(\chi^k_\varepsilon,\varphi)\ , \quad \forall\ \varphi \in V \tag{2.46}$$

Then, given a sequence of ε converging to zero, $\{\nu^k_\varepsilon,\chi^k_\varepsilon\}^\infty_{k=1}$ can be decomposed into $\{\lambda^k_\varepsilon,\varphi^k_\varepsilon\}^\infty_{k=1}$ and $\{\mu^k_{\varepsilon,\ell},\psi^k_{\varepsilon,\ell}\}^\infty_{k=1}$, $\ell = -m,-m+1,\ldots,n-1$, with the following properties:

1) $\lambda_\varepsilon^k = \lambda_n^k \varepsilon^n + o(\varepsilon^n)$, $\varphi_\varepsilon^k \to \varphi^k$ strongly in V

2) $\mu_{\varepsilon,r}^k = \mu_r^k \varepsilon^r + o(\varepsilon^r)$, $\psi_{\varepsilon,r}^k \to \psi_r^k$ weakly in H,

$r = -m,-m+1,\ldots,n-1$

where $\{\varphi^k\}_{k=1}^\infty$ and $\{\psi_r^k\}_{k=1}^\infty$, $r = -m,-m+1,\ldots,n-1$ satisfy:

$$b_n(\varphi^k,\chi) = \lambda_n^k(\varphi^k,\chi), \quad \varphi^k \in V_{m+n}, \quad \forall\chi \in V_{m+n} \tag{2.47}$$

$$b_r(\psi_r^k,\chi) = \mu_r^k(\psi_r^k,\chi), \quad \psi_r^k \in H_{m+n-r} \subset H, \ \forall\chi \in V_{r-1}$$

$$r = -m,-m+1,\ldots,n-1. \tag{2.48}$$

Proof:

Let $m+n = p$, $a_i(\varphi,\psi) = \varepsilon^m b_{i-m}(\varphi,\psi)$, $i = 0,1,\ldots,p$ and $\gamma_\varepsilon^k = \varepsilon^m \nu_\varepsilon^k$ in *Theorem 2.2, to get the desired results.*

2.5. Additional Examples of Stiff Operators

Several operators depending upon one or more small parameters, possess some of the properties discussed in the previous sections, although they do not fit in the axiomatization of Section 2.2.

In this section, three such examples are investigated. The differences and similarities with the operators of the preceding section are highlighted. In Section 2.3, the bilinear form of (φ,ψ) is assumed to be coercive on V, i.e., $a_\varepsilon(\varphi,\varphi) \geq \alpha\varepsilon \|\varphi\|_V^2$, $\forall\, \varphi \in V$. In the first example of this section, the bilinear form $a_\varepsilon(\varphi,\psi)$ satisfies

$$a_\varepsilon(\varphi,\varphi) + \lambda\|\varphi\|_H^2 \geq \alpha\varepsilon \|\varphi\|_V^2, \quad \forall \ \varphi \in V \tag{2.49}$$

for some positive constant λ.

2.5.1. Neumann eigenvalue problems

Let $H = L^2(\Omega)$, $V = H^1(\Omega)$ where $\Omega = \Omega_0 \cup \Omega_1 \cup S \subset \mathbb{R}^n$, with sufficiently regular boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ and interface S. Let $a_i(\varphi,\psi) = \sum_{j=1}^{n} \int_{\Omega_i} \frac{\partial\varphi}{\partial x_j} \frac{\partial\psi}{\partial x_j}\, dx$, $i = 0,1$. Then the eigenvalue problem is expressed as

$$\left.\begin{aligned}
&\left.\begin{aligned}
-\Delta\chi_{\varepsilon 0}^k &= \gamma_\varepsilon^k \chi_{\varepsilon 0}^k \quad \text{on } \Omega_0 \\
-\varepsilon\Delta\chi_{\varepsilon 1}^k &= \gamma_\varepsilon^k \chi_{\varepsilon 1}^k \quad \text{on } \Omega_1
\end{aligned}\right\} \\
&\left.\frac{\partial\chi_{\varepsilon 0}^k}{\partial\nu}\right|_{\Gamma_0} = 0, \quad \left.\frac{\partial\chi_{\varepsilon 1}^k}{\partial\nu}\right|_{\Gamma_1} = 0 \\
&\left.\begin{aligned}
\chi_{\varepsilon 0}^k &= \chi_{\varepsilon 1}^k \\
\frac{\partial\chi_{\varepsilon 0}^k}{\partial\nu} &= \varepsilon\,\frac{\partial\chi_{\varepsilon 1}^k}{\partial\nu}
\end{aligned}\right\} \text{on } S
\end{aligned}\right\} \tag{2.50}$$

It can readily be verified that this problem admits $\lambda^0 = 0$, $\varphi^0 =$ constant as an eigenvalue-eigenvector pair, i.e., $\mathcal{N}(A_\varepsilon) \neq \phi$ (null space of A_ε). If the operator A_ε is restricted to $H^1(\Omega)/\mathcal{N}(A_\varepsilon)$, then its spectrum can be decomposed as in Section 2.3, i.e., there are two groups of eigenvalue-eigenvectors $\{\lambda_\varepsilon^k, \varphi_\varepsilon^k\}_{k=1}^\infty$, $\{\mu_\varepsilon^k, \psi_\varepsilon^k\}_{k=1}^\infty$ having the properties of Theorem 2.1. The limits of the eigenvectors as $\varepsilon \to 0$ satisfy, respectively,

$$\left.\begin{aligned}
&\left.\begin{aligned}
\varphi_0^k &= \text{constant on } \Omega_0 \\
-\Delta\varphi_1^k &= \lambda_1^k \varphi_1^k \quad \text{on } \Omega_1
\end{aligned}\right\} \\
&\left.\frac{\partial\varphi_1^k}{\partial\nu}\right|_{\Gamma_1} = \left.\frac{\partial\varphi_1^k}{\partial\nu}\right|_S = 0
\end{aligned}\right\} \tag{2.51}$$

$$\left.\begin{array}{ll}\left.\begin{array}{ll} -\Delta\psi_0^k = \mu_0^k\psi_0^k & \text{on } \Omega_0 \\ \psi_1^k = 0 & \text{on } \Omega_1 \end{array}\right\} \\ \left.\dfrac{\partial\psi_0^k}{\partial\nu}\right|_{\Gamma_0} = \left.\dfrac{\partial\psi_1^k}{\partial\nu}\right|_S = 0 \end{array}\right\} \tag{2.52}$$

where the constant in (2.51) is chosen so that $\varphi^k \in H^1(\Omega)$.

Remark 2.20:

Note that $C_0^\infty(\Omega)$ is not dense in $H^1(\Omega)$. Therefore, the dual of $H^1(\Omega)$ is not a space of distributions. Hence (A2) is satisfied in a specific sense [30]. The same interpretation applies to Example 2.5.

The next example is quite different from the preceding examples. Mathematically, it does not satisfy (A5)-(A6). Physically, it arises in the field of heat conduction when boundary convection is present.

2.5.2. Robin eigenvalue problems

Let $H = L^2(\Omega)$, $V = H^1(\Omega)$ where Ω is a bounded set in $\mathbf{R}^n$ with sufficiently regular boundary Γ. Consider the following eigenvalue problem

$$\left.\begin{array}{ll} -\Delta\chi_\varepsilon^k = \gamma_\varepsilon^k\chi_\varepsilon^k & \text{on } \Omega \\ \dfrac{\partial\chi_\varepsilon^k}{\partial\nu} + \varepsilon\chi_\varepsilon^k = 0 & \text{on } \Gamma \end{array}\right\} \tag{2.53}$$

which can be also expressed as

$$a_0(\chi_\varepsilon^k,\varphi) + \varepsilon a_1(\chi_\varepsilon^k,\varphi) = \gamma_\varepsilon^k(\chi_\varepsilon^k,\varphi), \qquad \forall\ \varphi \in H^1(\Omega) \tag{2.54}$$

where

$$a_0(\varphi,\psi) = \sum_{j=1}^{n} \int_\Omega \frac{\partial\varphi}{\partial x_j} \frac{\partial\psi}{\partial x_j} \, dx$$

$$a_1(\varphi,\psi) = \int_\Gamma \varphi \, \psi \, d\Gamma \, .$$

Problem 2.53 has no zero eigenvalue. The limiting eigenvalue problem is

$$a_0(\chi^k,\varphi) = \gamma^k(\chi^k,\varphi), \qquad \forall \, \varphi \in H^1(\Omega) \tag{2.55}$$

which has a single zero eigenvalue $\gamma^0 = 0$ with corresponding eigenvector φ^0 = constant. Therefore, there is only one eigenvalue γ_ε^0 of (2.53) which converges to γ^0.

Now the above discussion and the convergence of the eigenvectors are formalized in:

Theorem 2.3:

Let $\{\gamma_\varepsilon^k\}_{k=1}^\infty$ be the eigenvalues of (2.53) and $\{\chi_\varepsilon^k\}_{k=1}^\infty$ the corresponding normalized eigenvectors. Then, given a sequence of ε converging to zero, $\{\gamma_\varepsilon^k,\chi_\varepsilon^k\}_{k=1}^\infty$ can be decomposed into $\{\gamma_\varepsilon^0,\chi_\varepsilon^0\}$ and $\{\gamma_\varepsilon^k,\chi_\varepsilon^k\}_{k=1}^\infty$ such that

1) $\gamma_\varepsilon^0 \to 0$, $\chi_\varepsilon^0 \to \chi^0$ strongly in V

2) $\gamma_\varepsilon^k \to \gamma^k > 0$, $\chi_\varepsilon^k \to \chi^k$ strongly in V, $k = 1,2,\ldots$

where $\{\gamma_\varepsilon^k,\chi_\varepsilon^k\}_{k=1}^\infty$ satisfy

$$a_0(\chi^k,\varphi) = \gamma^k(\chi^k,\varphi), \qquad \forall \, \varphi \in V \tag{2.56}$$

$k=0,1,2,\ldots$.

Proof:

First observe that

$$a_0(\varphi,\varphi) + \lambda \|\varphi\|_H^2 \geq \alpha_0 \|\varphi\|_V^2, \ \alpha_0 > 0, \quad \forall \ \varphi \in V \tag{2.57}$$

for some constant $\lambda > 0$. For $\varepsilon << 1$, the following inequality holds

$$a_0(\varphi,\varphi) + a_1(\varphi,\varphi) + \lambda \|\varphi\|_H^2 > a_0(\varphi,\varphi) + \varepsilon a_1(\varphi,\varphi) + \lambda \|\varphi\|_H^2, \quad \forall \ \varphi \in V \tag{2.58}$$

Let $\{\nu^k,\rho^k\}_{k=0}^\infty$ satisfy

$$a_0(\rho^k,\varphi) + a_1(\rho^k,\varphi) = \nu^k(\rho^k,\varphi) \qquad \forall \ \varphi \in V$$

with $\{\rho^k\}_{k=1}^\infty$ normalized in H. Then using the minimax characterization of eigenvalues, one concludes

$$\begin{aligned} \nu^k + \lambda &> a_0(\chi_\varepsilon^k,\chi_\varepsilon^k) + \varepsilon a_1(\chi_\varepsilon^k,\chi_\varepsilon^k) + \lambda \\ &\geq \alpha_0 \|\chi_\varepsilon^k\|_V^2 \quad \text{by (2.57)} \end{aligned} \tag{2.59}$$

Consequently, given a sequence of ε converging to zero, $\chi_\varepsilon^k \to \chi^k$ weakly in V (hence strongly in H), where χ^k satisfies (2.56).

Now let $w_\varepsilon^k = \chi_\varepsilon^k - \chi^k$

$$a_0(w_\varepsilon^k,\varphi) + \varepsilon a_1(w_\varepsilon^k,\varphi) = \gamma_\varepsilon^k(w_\varepsilon^k,\varphi) + (\gamma_\varepsilon^k - \gamma^k)(\chi^k,\varphi) + \varepsilon a_1(\chi^k,\varphi), \ \forall \ \varphi \in V. \tag{2.60}$$

Let $\varphi = w_\varepsilon^k$ in (2.60) and use (2.57) to get

$$a_0(w_\varepsilon^k,w_\varepsilon^k) + \varepsilon a_1(w_\varepsilon^k,w_\varepsilon^k) + \lambda \|w_\varepsilon^k\|_H^2 \geq \alpha_0 \|w_\varepsilon^k\|_V^2$$

from which it results, using (2.60), that

$$\|w_\varepsilon^k\|_V \to 0 \text{ as } \varepsilon \to 0.$$

Therefore,

$$\chi_\varepsilon^k \to \chi^k \text{ strongly in } V.$$

Now, a simpler version of this example is considered to illustrate the flattening of χ_ε^0 as $\varepsilon \to 0$.

Let $\Omega = (-1,1)$, $\Gamma = \{-1,1\}$.

In this instance, direct computation of $\{\gamma_\varepsilon^k, \chi_\varepsilon^k\}_{k=1}^\infty$ yields

$$\left.\begin{aligned} \gamma_\varepsilon^{2\ell} &= (\ell\pi)^2 \\ \chi_\varepsilon^{2\ell} &= \cos \ell\pi x + \frac{\varepsilon}{\ell\pi} \sin \ell\pi x \end{aligned}\right\}$$

$$\left.\begin{aligned} \gamma_\varepsilon^{2\ell-1} &= ((2\ell-1)\tfrac{\pi}{2})^2 \\ \chi_\varepsilon^{2\ell-1} &= \frac{-\varepsilon}{(2\ell-1)\frac{\pi}{2})} \cos(2\ell-1)\frac{\pi}{2}\, x + \sin(2\ell-1)\frac{\pi}{2}\, x \end{aligned}\right\}$$

$$\ell = 1,2,\ldots \quad .$$

Computation of the pair $\{\gamma_\varepsilon^0, \chi_\varepsilon^0\}$ is not trivial. However, the following approximation can readily be found:

$$\left.\begin{aligned} \gamma_\varepsilon^0 &= \varepsilon^2 + o(\varepsilon^2) \\ \chi_\varepsilon^0 &= \cos \varepsilon x + \sin \varepsilon x + O(\varepsilon) \ . \end{aligned}\right\} \qquad (2.61)$$

It is worthy of observation that χ_ε^0 flattens as $\varepsilon \to 0$, to become $\chi^0 = 1$ on Ω in the limit.

<u>Remark 2.21</u>:

The above analysis is unchanged if

$$a_0(\varphi,\psi) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \frac{\partial\varphi}{\partial x_i} \frac{\partial\psi}{\partial x_j}\, dx$$

$$a_1(\varphi,\psi) = \int_\Gamma \varphi\, \psi\, d\Gamma$$

where

$$\frac{\partial}{\partial\nu} = \sum_{i,j=1}^{n} a_{ij}(x)\cos(\nu,x_i) \frac{\partial}{\partial x_j}$$

ν = unit normal, outward relative to Ω

$a_{ij}(x) \in C^1(\bar{\Omega})$

$a_{ij}(x) = a_{ji}(x)$

$i,j = 1,2,\ldots,n$

2.5.3. Other examples of stiff operators

The third class of eigenvalue problems to be considered arises in heat transfer, when internal heat exchange with the surrounding by free convection is taken into account. It will be discovered that a different order of magnitude of the convection in interfaced media also produces stiffness. First, a general formulation of these problems is presented. Then the convergence of the eigenvalue-eigenvector pairs as $\varepsilon \to 0$ is derived. The results are then specialized to a second order operator. A detailed presentation is not pursued because most of the ideas are adapted from Sections 2.2-2.3.

Suppose two Hilbert spaces are given as in Section 2.2 and four forms $a_i(\varphi,\psi)$ on V, $b_i(\varphi,\psi)$ on H, i = 0,1. The forms $a_i(\varphi,\psi)$, i = 0,1 satisfy (A3-A7). The forms $b_i(\varphi,\psi)$, i = 0,1 satisfy

(B3) $b_i(\varphi,\psi)$ bilinear, symmetric on H

(B4) $b_i(\varphi,\psi)$ is continuous on H

(B5) $b_i(\varphi,\varphi) \geq \beta_i q_i(\varphi)^2$, where $\beta_i > 0$ and $q_i(\cdot)$ is continuous semi-norm on H

(B6) $q_i(\varphi) \leq \xi_i p_i(\varphi)$, $\forall \varphi \in V$, $\xi_i > 0$

(B7) $q_0(\varphi) + q_1(\varphi)$ is a norm equivalent to $\|\varphi\|_H$.

In the sequel, the behavior as $\varepsilon \to 0$ of the eigenvalue-eigenvector pairs of the following eigenvalue problem is investigated:

$$\varepsilon a_0(\chi_\varepsilon^k,\varphi) + \varepsilon a_1(\chi_\varepsilon^k,\varphi) + b_0(\chi_\varepsilon^k,\varphi) + \varepsilon b_1(\chi_\varepsilon^k,\varphi) = \gamma_\varepsilon^k(\chi_\varepsilon^k,\varphi),$$

$$k = 1,2,\ldots, \quad \forall\ \varphi \in V \tag{2.62}$$

<u>Theorem 2.4</u>:

Let $\{\gamma_\varepsilon^k\}_{k=1}^\infty$ be the eigenvalues of (2.62) and $\{\chi_\varepsilon^k\}_{k=1}^\infty$ the corresponding normalized eigenvectors. Then, given a sequence of ε converging to zero, $\{\gamma_\varepsilon^k,\chi_\varepsilon^k\}_{k=1}^\infty$ is decomposed into $\{\lambda_\varepsilon^k,\varphi_\varepsilon^k\}_{k=1}^\infty$ and $\{\mu_\varepsilon^k,\psi_\varepsilon^k\}_{k=1}^\infty$ having the following properties, for each k:

1) $\lambda_\varepsilon^k \to 0$ linearly in ε, $\varphi_\varepsilon^k \to \varphi^k$ strongly in V (2.63)

2) $\mu_\varepsilon^k \to \mu_0^k > 0$ affinely in ε, $\psi_\varepsilon^k \to \psi^k$ weakly in V (2.64)

where $\{\varphi^k\}_{k=1}^\infty$ and $\{\psi^k\}_{k=1}^\infty$ satisfy

$$a_1(\varphi^k,\chi) + b_1(\varphi^k,\chi) = \lambda_1^k(\varphi^k,\chi)\ , \quad \varphi^k \in V_0, \quad \forall\ \chi \in V_0 \tag{2.65}$$

$$\left.\begin{array}{l} a_0(\psi^k,\chi) = \mu_1^k(\psi^k,\chi), \quad \psi^k \in V_1, \quad \forall\ \chi \in V_1 \\ \mu_0^k = \dfrac{(\psi^k,\psi^k)}{b_0(\psi^k,\psi^k)} \\ k = 1,2,\ldots \end{array}\right\} \tag{2.66}$$

Proof:

Let

$$(\chi_\varepsilon^k,\chi_\varepsilon^k) = 1 \quad . \tag{2.67}$$

Using the simple arguments of Lemma 2.1 and Lemma 2.3 adapted to the present eigenvalue, one concludes that there are two groups of eigenvalue-eigenvector pairs $\{\lambda_\varepsilon^k,\varphi_\varepsilon^k\}_{k=1}^\infty$ and $\{\mu_\varepsilon^k,\psi_\varepsilon^k\}_{k=1}^\infty$ such that $\lambda_\varepsilon^k \to 0$ linearly in ε and $\mu_\varepsilon^k \to \mu_0^k > 0$. Adapt Lemma 2.2 to the problem at hand to get

$$\alpha\varepsilon \, \|\varphi_\varepsilon^k\|_V^2 + \beta\varepsilon \, \|\varphi_\varepsilon^k\|_H^2 \le \lambda_\varepsilon^k \tag{2.68}$$

since λ_ε^k converges to zero linearly, it results that

$$\|\varphi_\varepsilon^k\|_V \le C \tag{2.69}$$

for some strictly positive constant C, which is independent of ε, and, hence given a sequence of ε converging to zero

$$\varphi_\varepsilon^k \to \varphi^k \quad \text{weakly in V.}$$

From (2.62), one deduces that

$$b_0(\varphi_\varepsilon^k,\varphi_\varepsilon^k) \le O(\varepsilon^2)$$

since $\lambda_\varepsilon^k = O(\varepsilon)$. Let $\varphi = \chi \in V_0$ in (2.62) and take the limit as $\varepsilon \to 0$ (after dividing by ε) to get (2.65). Now let $w_\varepsilon^k = \varphi_\varepsilon^k - \varphi^k$ which satisfies

$$a_0(w_\varepsilon^k,w_\varepsilon^k) + a_1(w_\varepsilon^k,w_\varepsilon^k) + b_1(w_\varepsilon^k,w_\varepsilon^k) + O(\varepsilon) = \frac{(\lambda_\varepsilon^k-\lambda_1^k\varepsilon)}{\varepsilon}(\varphi_\varepsilon^k,w_\varepsilon^k) + \lambda_1^k(w_\varepsilon^k,w_\varepsilon^k)$$

$$\ge \alpha\|w_\varepsilon^k\|_V^2 + \beta_1 q_1^2(w_\varepsilon^k) + O(\varepsilon)$$

from which one concludes that

$$\varphi_\varepsilon^k \to \varphi^k \text{ strongly in } V.$$

For the remaining eigenvalues, i.e., $\{\mu_\varepsilon^k\}_{k=1}^\infty$, one observes that μ_ε^k is a uniformly bounded sequence due to (B4) and (2.67). Hence using Proposition 2.1 one deduces that

$$a_i(\psi_\varepsilon^k,\psi_\varepsilon^k) \le O(1), \quad i = 0,1 \tag{2.70}$$

which implies by (A5-A6) that

$$\|\psi_\varepsilon^k\|_V \le C \tag{2.71}$$

for some strictly positive constant C, which is independent of ε.

Due to (2.67), the following inequalities hold

$$b_i(\psi_\varepsilon^k,\psi_\varepsilon^k) \le O(1), \quad i = 0,1 \ . \tag{2.72}$$

In fact, a better estimate can be obtained, by a contradiction argument, i.e.,

$$b_1(\psi_\varepsilon^k,\psi_\varepsilon^k) \le O(\varepsilon) \ . \tag{2.73}$$

Suppose

$$b_1(\psi_\varepsilon^k,\psi_\varepsilon^k) = O(1) \tag{2.74}$$

for all bilinear forms $a_i(\varphi,\psi)$, $b_i(\varphi,\psi)$, $i = 0,1$ satisfying the assumptions of the present section. Then one may select bilinear forms (as in Example 2.6) such that

$$b_0(\chi,\chi) + b_1(\chi,\chi) = (\chi,\chi) \quad , \quad \forall \chi \in V \ .$$

Using (2.62) and the minimax characterization of eigenvalues to obtain

$$b_0(\psi_\varepsilon^k,\psi_\varepsilon^k) \to (\psi^k,\psi^k) = 1, \quad \text{as} \quad \varepsilon \to 0$$

and hence $b_1(\psi_\varepsilon^k,\psi_\varepsilon^k) \to 0$ as $\varepsilon \to 0$, contradicting (2.74). Therefore, (2.74) is valid. Hence (2.64) is deduced from (2.71) using the same argument used in the proof of Theorem 2.1. Now expand (2.62) formally in powers of ε (while recalling (2.73)). The zeroth and first order term yield (2.66) for $\chi \in V_1$.

Example 2.6:

Let $H = L^2(\Omega)$, $V = H_0^1(\Omega)$ where $\Omega = \Omega_0 \cup \Omega_1 \cup S \subset \mathbb{R}^n$, with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ and interface S, as in Figure 2.1.

Let

$$a_i(\varphi,\psi) = \sum_{j=1}^{n} \int_{\Omega_i} \frac{\partial\varphi}{\partial x_j} \frac{\partial\psi}{\partial x_j}\, dx \quad , \quad i = 0,1$$

$$b_i(\varphi,\psi) = \int_{\Omega_i} \varphi\, \psi\, dx \ , \ i = 0,1$$

then (2.62) is equivalent to

$$\left.\begin{array}{l} \left.\begin{array}{ll} -\varepsilon\Delta\chi_{\varepsilon 0}^k + \chi_{\varepsilon 0}^k = \gamma_\varepsilon^k \chi_{\varepsilon 0}^k & \text{on } \Omega_0 \\ -\varepsilon\Delta\chi_{\varepsilon 1}^k + \varepsilon\chi_{\varepsilon 1}^k = \gamma_\varepsilon^k \chi_{\varepsilon 1}^k & \text{on } \Omega_1 \end{array}\right\} \\ \chi_{\varepsilon 0}^k\big|_{\Gamma_0} = 0, \ \chi_{\varepsilon 1}^k\big|_{\Gamma_1} = 0 \\ \left.\begin{array}{l} \chi_{\varepsilon 0}^k = \chi_{\varepsilon 1}^k \\ \dfrac{\partial\chi_{\varepsilon 0}^k}{\partial\nu} = \dfrac{\partial\chi_{\varepsilon 1}^k}{\partial\nu} \end{array}\right\} \text{ on } S \end{array}\right\}$$

Equations (2.65-2.66) give respectively

$$\left.\begin{array}{l}\left.\begin{array}{ll}\varphi_0^k = 0 & \text{on } \Omega_0 \\ -\Delta\varphi_1^k + \varphi_1^k = \lambda_1^k\varphi_1^k & \text{on } \Omega_1\end{array}\right\} \\ \varphi_1^k\big|_{\Gamma_1} = 0, \quad \varphi_1^k\big|_S = 0\end{array}\right\}$$

$$\left.\begin{array}{l}\left.\begin{array}{ll}-\Delta\psi_0^k = \mu_1^k\psi_0^k & \text{on } \Omega_0 \\ \psi_1^k = 0 & \text{on } \Omega_1\end{array}\right\} \\ \psi_0^k\big|_{\Gamma_0} = 0, \quad \psi_1^k\big|_S = 0\end{array}\right\}$$

$$\mu_0^k = 1$$

$$k = 1,2,\ldots$$

Remark 2.22:

Once more, observe the flattening of $\{\varphi_\varepsilon^k\}_{k=1}^\infty$ on Ω_0. However, $\{\psi_\varepsilon^k\}_{k=1}^\infty$ oscillate and attenuate concurrently on Ω_1, which ascertain their ameliorated convergence, compared to those of Section 2.3.

Remark 2.23:

Remark 2.7 is also applicable in the present eigenvalue problem.

2.6. Formal Asymptotic Expansions of the Eigenvalues and Eigenvectors of A_ε

It was shown in Section 2.3 that $\{\psi_\varepsilon^k\}_{k=1}^\infty$ are not analytic functions of ε. Therefore, they do not have an analytic expansion in powers of ε. Their mode of convergence (weak in H) has the most profound impact in solving boundary value problems. This shall be clearly demonstrated in the next chapter.

It was also demonstrated in Section 2.3 that $\{\lambda_\varepsilon^k\}_{k=1}^\infty$ are analytic functions of ε. Thus, it may be possible to expand $\{\lambda_\varepsilon^k\}_{k=1}^\infty$ as well as their corresponding eigenvectors $\{\varphi_\varepsilon^k\}_{k=1}^\infty$ in powers of ε.

In the sequel, a formal asymptotic expansion of λ_ε^k and φ_ε^k are derived. Then the formal calculations are implicitly given for Example 2.1. Using Examples 2.4-2.5, some of the difficulties associated with such an expansion are mentioned.

For notational simplicity, let the index k of the eigenvalues be dropped and let

$$\lambda_\varepsilon = \lambda^1\varepsilon + \lambda^2\varepsilon^2 + \ldots \tag{2.75}$$

$$\varphi_\varepsilon = \varphi^0 + \varepsilon\varphi^1 + \ldots \quad . \tag{2.76}$$

Substitute (2.75)-(2.76) into (2.5) and identify formally equal powers of ε to get:

<u>Lemma 2.6</u>:

The sequences $\{\lambda^\ell\}_{\ell=1}^\infty$ and $\{\varphi^\ell\}_{\ell=0}^\infty$ formally satisfy:

$$\left.\begin{aligned} a_0(\varphi^0,\chi) &= 0, \qquad \forall\ \chi \in V \\ a_1(\varphi^0,\chi) &= \lambda^1(\varphi^0,\chi), \forall\ \chi \in V_0 \end{aligned}\right\} \tag{2.77}$$

$$\left.\begin{aligned} a_0(\varphi^\ell,\chi) &= \sum_{i=0}^{\ell-1} \lambda^{i+1}(\varphi^{\ell-i-1},\chi) - a_1(\varphi^{\ell-1},\chi), \quad \forall\ \chi \in V \\ a_1(\varphi^\ell,\chi) &= \sum_{i=0}^{\ell} \lambda^{i+1}(\varphi^{\ell-1},\chi), \quad \forall\ \chi \in V_0 \end{aligned}\right\} \tag{2.78}$$

$\ell = 1,2,\ldots$

Proof:

The derivation of (2.77-2.78) is straightforward.

Now their solvability is considered for Example 2.1:

Example 2.7:

In this case, (2.77-2.78) become respectively

$$\left.\begin{array}{l} \left.\begin{array}{ll} \varphi_0^0 = 0 & \text{on } \Omega_0 \\ -\Delta\varphi_1^0 = \lambda^1\varphi_1^0 & \text{on } \Omega_1 \end{array}\right\} \\ \varphi_1^0\big|_{\Gamma_1} = 0,\ \ \varphi_1^0\big|_S = 0 \end{array}\right\} \tag{2.79}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} -\Delta\varphi_0^\ell = \sum_{i=0}^{\ell-1} \lambda^{i+1}\varphi_0^{\ell-i-1} \quad \text{on } \Omega_0 \\ \varphi_0^\ell\big|_{\Gamma_0} = 0,\ \ \dfrac{\partial\varphi_0^\ell}{\partial\nu}\Big|_S = \dfrac{\partial\varphi_1^{\ell-1}}{\partial\nu}\Big|_S \end{array}\right\} \\ \left.\begin{array}{l} -\Delta\varphi_1^\ell = \sum_{i=0}^{\ell} \lambda^{i+1}\varphi_1^{\ell-i} \quad \text{on } \Omega_1 \\ \varphi_1^\ell\big|_{\Gamma_1} = 0,\ \ \varphi_1^\ell\big|_S = \varphi_0^\ell\big|_S \end{array}\right\} \end{array}\right\} \tag{2.80}$$

$\ell=1,2,\ldots$

First, (2.79) is identical to (2.31) with the obvious notational change. Hence one can solve uniquely for λ^1 and φ^0. It will be shown that (2.80) is solvable recursively. Let $\ell = 1$ in (2.80) to get:

$$\left.\begin{aligned} -\Delta\varphi_0^1 &= \lambda^1\varphi_0^0 \quad \text{on } \Omega_0 \\ \varphi_0^1\big|_{\Gamma_0} &= 0, \quad \frac{\partial\varphi_0^1}{\partial\nu}\Big|_S = \frac{\partial\varphi_1^0}{\partial\nu}\Big|_S \end{aligned}\right\} \tag{2.81}$$

$$\left.\begin{aligned} -\Delta\varphi_1^1 &= \lambda^1\varphi_1^1 + \lambda^2\varphi_1^0 \quad \text{on } \Omega_1 \\ \varphi_1^1\big|_{\Gamma_1} &= 0, \quad \varphi_1^1\big|_S = \varphi_0^1\big|_S \end{aligned}\right\} \tag{2.82}$$

The pair $\{\lambda^1, \varphi_1^0\}$ is computed from (2.79). Therefore, $\frac{\partial\varphi_1^0}{\partial\nu}\Big|_S \in H^{-1/2}(S)$. Hence (2.81) is a nonhomogeneous boundary value problem in $H^1(\Omega_0;\Gamma_0)$ from which one computes φ_0^1. In (2.82), λ^1, φ_1^0, φ_0^1 are known. Let $z_1^1 \in H^1(\Omega_1)$ such that

$$z_1^1\big|_{\Gamma_1} = 0, \quad z_1^1\big|_S = \varphi_0^1\big|_S$$

and let $\bar{\varphi}_1^1 = \varphi_1^1 - z_1^1$. Clearly $\bar{\varphi}_1^1$ satisfies

$$\left.\begin{aligned} (-\Delta - \lambda^1)\bar{\varphi}_1^1 &= \Delta z_1^1 + \lambda^1 z_1^1 + \lambda^2\varphi_1^0 \\ \bar{\varphi}_1^1\big|_{\Gamma_1} &= 0, \quad \bar{\varphi}_1^1\big|_S = 0 \end{aligned}\right\} \tag{2.83}$$

By the Fredholm Alternative [42], (2.83) is uniquely solvable for $\bar{\varphi}_1^1$ in $H_0^1(\Omega_1)$ provided its right-hand side is orthogonal to φ_1^0 in $L^2(\Omega_1)$, i.e.,

$$(\Delta z_1^1 + \lambda^1 z_1^1 + \lambda^2\varphi_1^0,\ \varphi_1^0)_{L^2(\Omega_1)} = 0$$

and consequently,

$$\lambda^2 = \frac{(-\Delta z_1^1 - \lambda^1 z_1^1, \varphi_1^0)_{L^2(\Omega_1)}}{(\varphi_1^0, \varphi_1^0)_{L^2(\Omega_1)}}$$

Using Green's formula, one obtains

$$\lambda^2 = \frac{-\int_S \varphi_0^1 \frac{\partial \varphi_1^0}{\partial \nu}\, dS}{(\varphi_1^0, \varphi_1^0)_{L^2(\Omega_1)}} \tag{2.84}$$

Using a similar argument and recursively on ℓ, one solves (2.80) and hence obtain all of the terms in (2.75-2.76).

Now the yield of the above calculations for Example 2.5 is presented.

Example 2.8: (Cf. Example 2.4)

In this case, one obtains

$$\left.\begin{aligned} &\lambda_\varepsilon^k = (k\pi)^2[\varepsilon + 2\varepsilon^2 + (1 - \tfrac{2}{3}(k\pi)^2)\varepsilon^3 + \ldots] \\ &\left.\begin{aligned} &\varphi_{\varepsilon 0}^k = k\pi(1+x)[\varepsilon + (\frac{(k\pi)^2}{6}(1+x)^2 + (\frac{(k\pi)^2}{2} - 1))\varepsilon^2 + \ldots] \\ &\varphi_{\varepsilon 1}^k = \sin k\pi x + k\pi(1-x)\cos k\pi x\, \varepsilon + \ldots \end{aligned}\right\} \end{aligned}\right\} \tag{2.85}$$

$k = 1,2,\ldots$

Now, some remarks about this iterative process are in order.

Remark 2.24:

This process yields a nonunique expansion. For example, the following expansion is also given by the same iterative process:

$$\left.\begin{aligned} \lambda_\varepsilon^k &= (k\pi)^2[\varepsilon - 2\varepsilon^2 - \tfrac{1}{3}(k\pi)^2\varepsilon^3 + \dots] \\ \left.\begin{aligned} \varphi_{\varepsilon 0}^k &= k\pi(1+x)[\varepsilon + \frac{(k\pi)^2}{2}(-\frac{(x-1)^2}{3} + 1)\varepsilon^2 + \dots] \\ \varphi_{\varepsilon 1}^k &= \sin k\pi x + (1-x)(k\pi\cos k\pi x + \sin k\pi x)\varepsilon + \dots \end{aligned}\right\} \end{aligned}\right\} \qquad (2.86)$$

$$k = 1,2,\dots$$

Remark 2.25:

It is not easy to compute the terms beyond φ_0^1 in the general case. See [5, 13, 16, 17, 31, 32] for related topics.

Remark 2.26:

The exact eigenvalue-eigenvector pairs given in Example 2.4 for particular values of ε suggest the following conjecture: the eigenvectors $\{\varphi_\varepsilon^k\}_{k=1}^\infty$ are not analytic functions of ε and therefore cannot be expanded as in (2.76).

Remark 2.27:

In this example as well as in Example 2.7, λ^2 and φ_1^1 do not depend upon the choice of z_1^1.

In the next example, this observation is not true.

Example 2.9: (Cf. Example 2.5)

In an iterative process similar to that of Example 2.8, one obtains

$$\lambda^2 = \frac{\int_S \frac{\partial z_1^1}{\partial \nu} \varphi_1^0 \, dS}{(\varphi_1^0, \varphi_1^0)_{L^2(\Omega_1)}} \tag{2.87}$$

i.e., φ^2 and φ_1^1 depend upon z_1^1 .

2.7 Numerical Results

In this section, Example 2.4 and a one-dimensional version of Example 2.6 are analyzed numerically, using the Finite Element Method, to supplement the analysis undertaken in the preceding sections. The set $\Omega = (-1,1)$ is divided into N equal intervals of length $h = \frac{2}{N}$. The roof functions $\{\varphi_h^i\}_{i=1}^{N-1}$ [39,43] are selected as a basis for the finite dimensional approximation of $H_0^1(\Omega)$. The finite dimensional approximation of the operator A_ε can be written in matrix form as

$$A_\varepsilon = (M^h)^{-1} K_\varepsilon^h \tag{2.88}$$

where the entries of M^h are

$$(M^h)_{i,j} = \int_\Omega \varphi_h^i \varphi_h^j \, dx$$
$$i,j = 1,2,\ldots,N-1.$$

In the forthcoming examples, the matrix M^h can be written explicitly, i.e.,

$$M^h = \frac{h}{6} \begin{bmatrix} 4 & 1 & & & 0 \\ 1 & 4 & \ddots & & \\ & \ddots & \ddots & & \\ & & 1 & 4 & 1 \\ 0 & & & 1 & 4 \end{bmatrix}$$

For all computer runs, N is selected so that an accurate plot is obtained (N is indicated under each plot). The subroutine RSG from EISPACK [36] with single precision is used to compute the eigenvalues and eigenvectors of A_ε^h.

Example 2.10: (Cf. Example 2.4)

In this example, the entries of K_ε^h are

$$(K_\varepsilon^h)_{i,j} = \int_\Omega a(x) \frac{d\varphi_h^i}{dx} \frac{d\varphi_h^j}{dx} dx \tag{2.89}$$

$$i,j = 1,2,\ldots,N-1$$

where

$$a(x) = \begin{cases} 1 & \text{if} \quad x \in (-1,0) \\ \varepsilon & \text{if} \quad x \in (0,1) \end{cases}$$

For N even, K_ε^h can be written as

$$K_\varepsilon^h = \frac{1}{h} \begin{bmatrix} 2 & -1 & & & & & & 0 \\ -1 & 2 & \ddots & & & & & \\ & \ddots & \ddots & -1 & & & & \\ & & -1 & 1+\varepsilon & -\varepsilon & & & \\ & & & -\varepsilon & 2\varepsilon & \ddots & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & & 2\varepsilon & -\varepsilon \\ 0 & & & & & & -\varepsilon & 2\varepsilon \end{bmatrix} \rightarrow \frac{N}{2} + 1 \tag{2.90}$$

The eigenvalues λ_ε^1, λ_ε^2, μ_ε^1 are tabulated in Table 2.1 for $\varepsilon = 0.1$, 0.04, 0.01, 0.001. The corresponding eigenvectors are plotted in Figures (2.3a-2.3d), (2.4a-2.4d), (2.5a-2.5d).

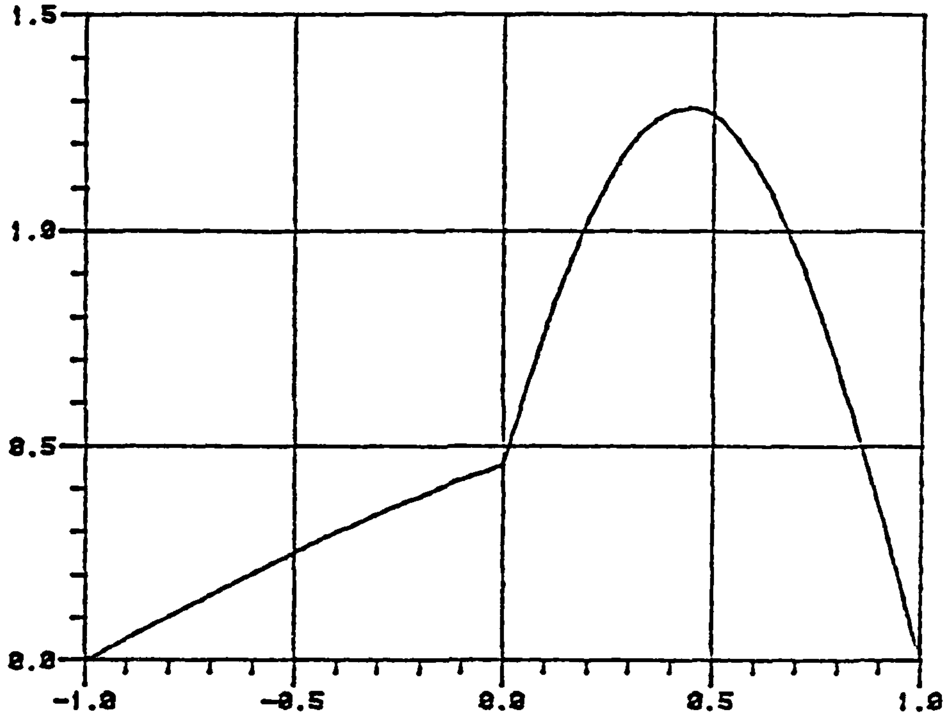

Figure 2.3a. $\varphi_{\varepsilon}^{1}$ for $\varepsilon = 0.1$, $N = 50$.

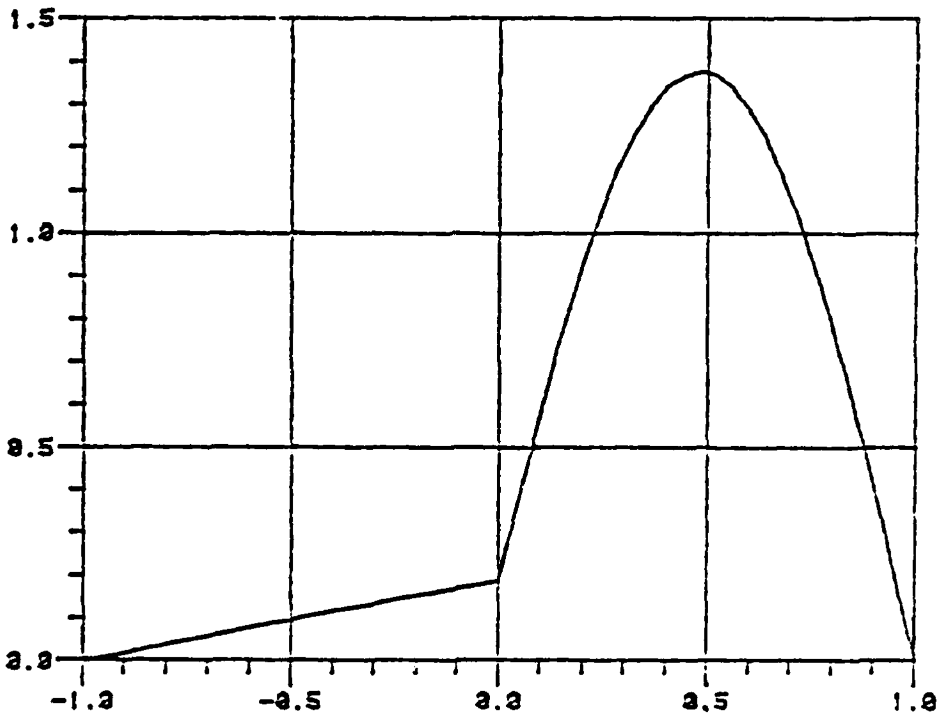

Figure 2.3b. $\varphi_{\varepsilon}^{1}$ for $\varepsilon = 0.04$, $N = 50$.

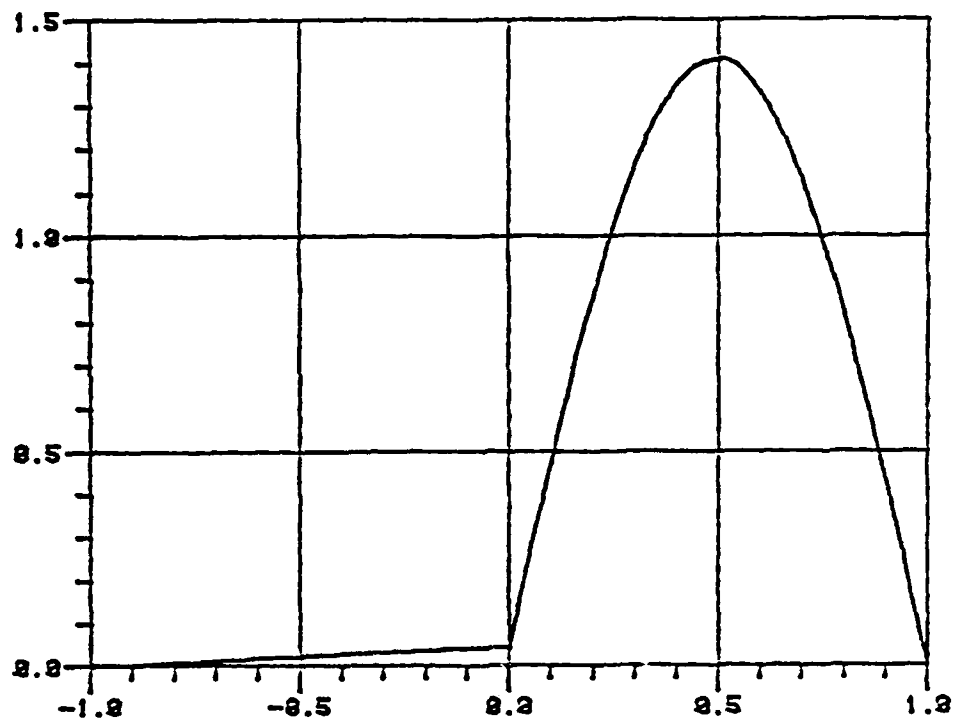

Figure 2.3c. $\varphi_{\varepsilon}^{1}$ for $\varepsilon = 0.01$, $N = 100$.

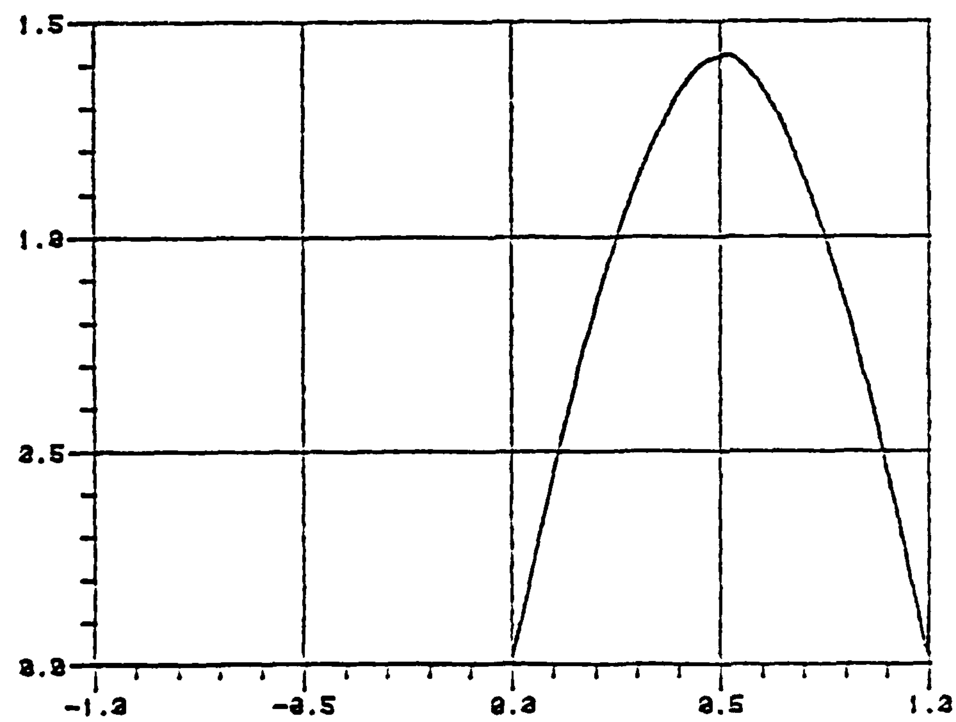

Figure 2.3d. $\varphi_{\varepsilon}^{1}$ for $\varepsilon = 0.001$, $N = 150$.

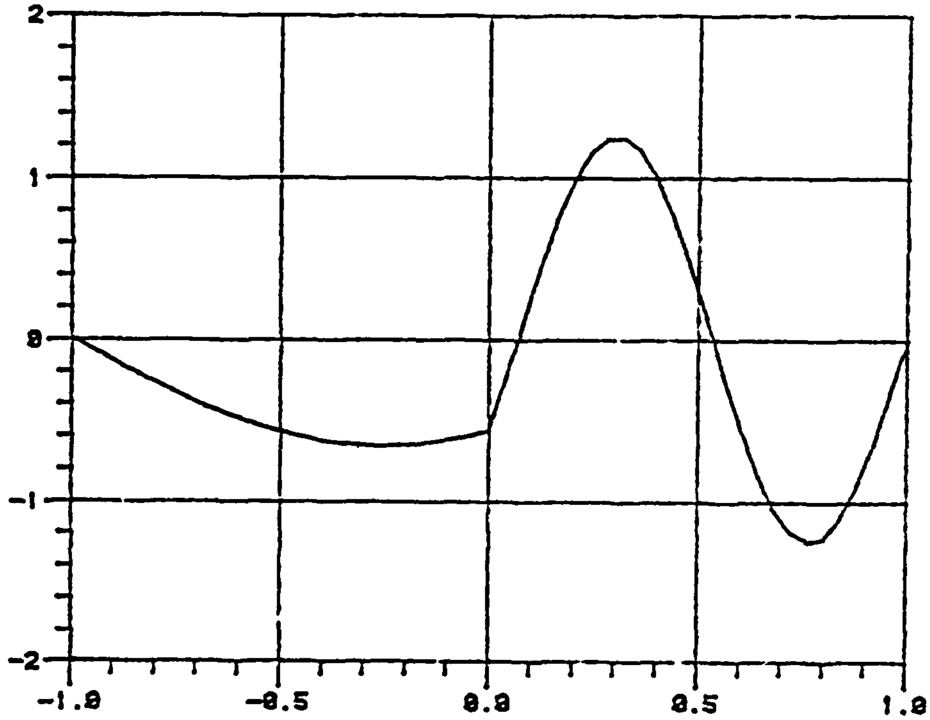

Figure 2.4a. $\varphi_{\varepsilon}^{2}$ for $\varepsilon = 0.1$, $N = 50$.

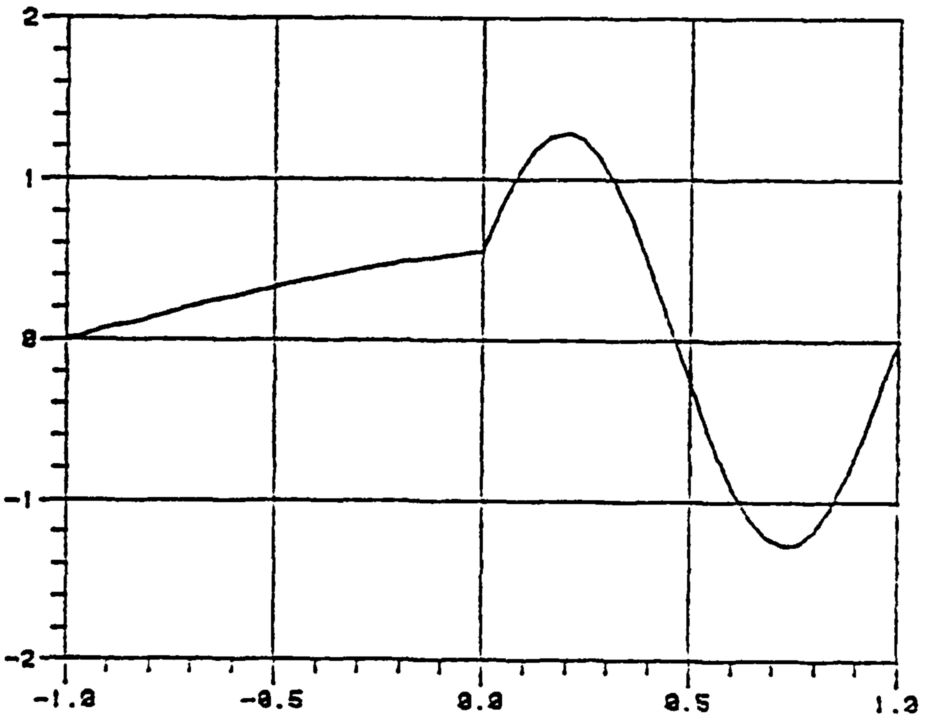

Figure 2.4b. $\varphi_{\varepsilon}^{2}$ for $\varepsilon = 0.04$, $N = 50$.

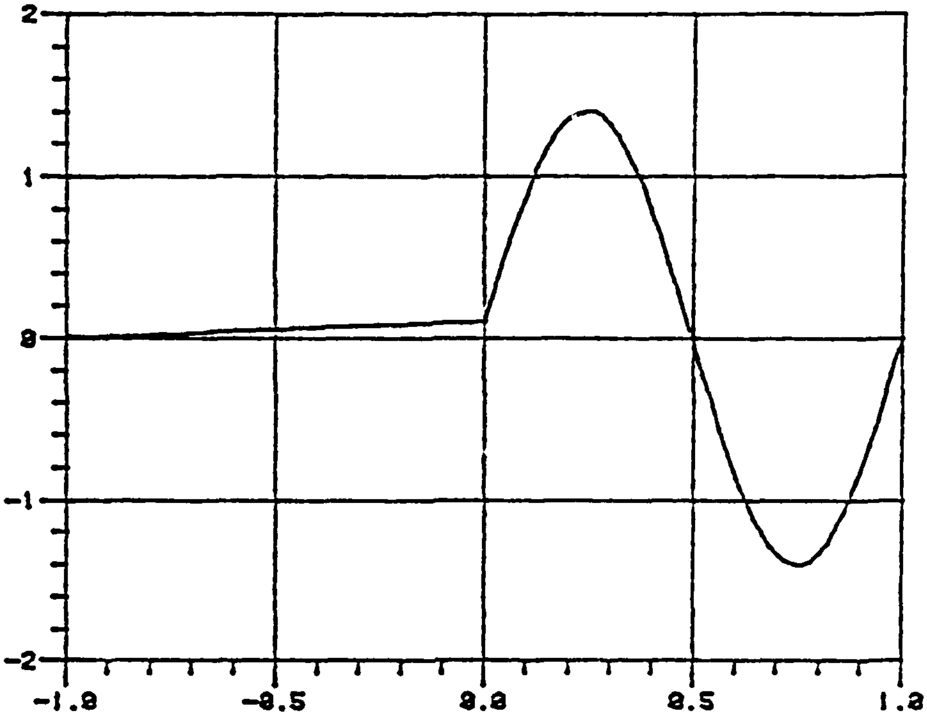

Figure 2.4c. φ_ε^2 for $\varepsilon = 0.01$, $N = 100$.

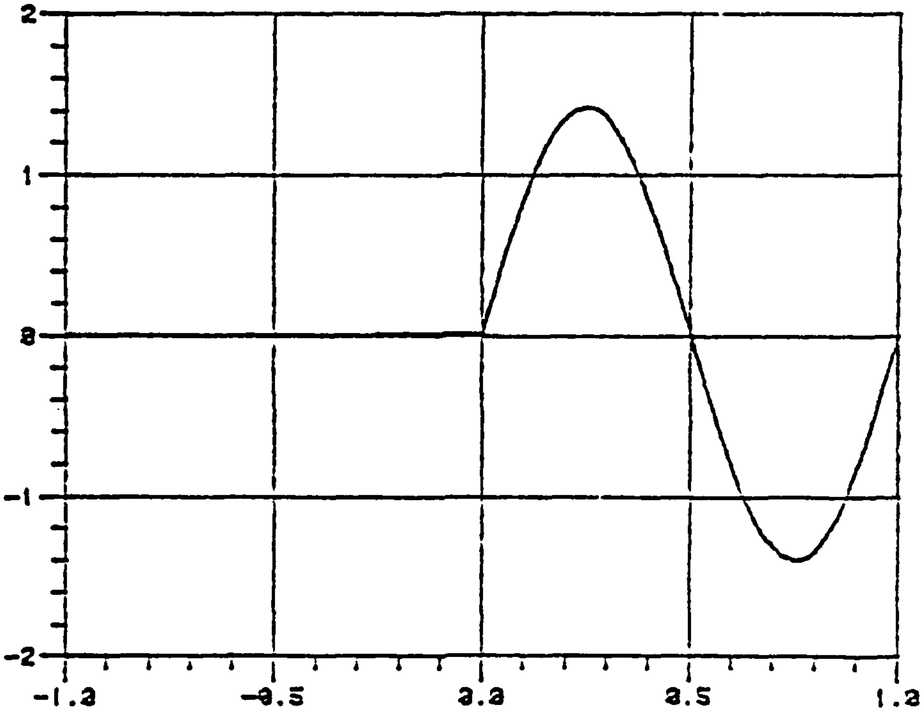

Figure 2.4d. φ_ε^2 for $\varepsilon = 0.001$, $N = 150$.

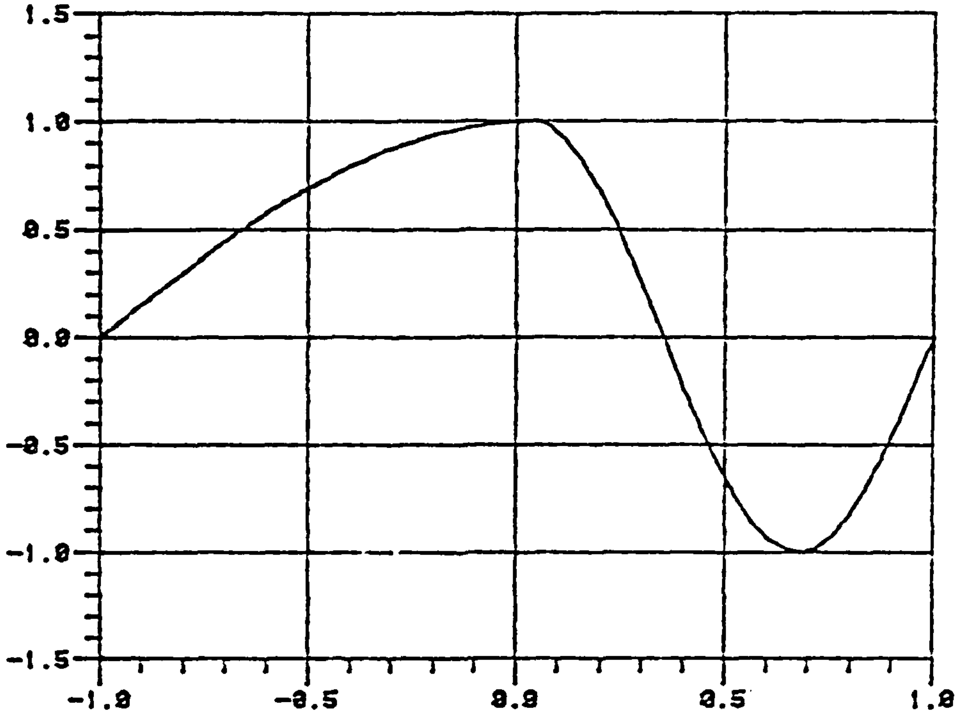

Figure 2.5a. ψ_{ε}^{1} for $\varepsilon = 0.1$, $N = 50$.

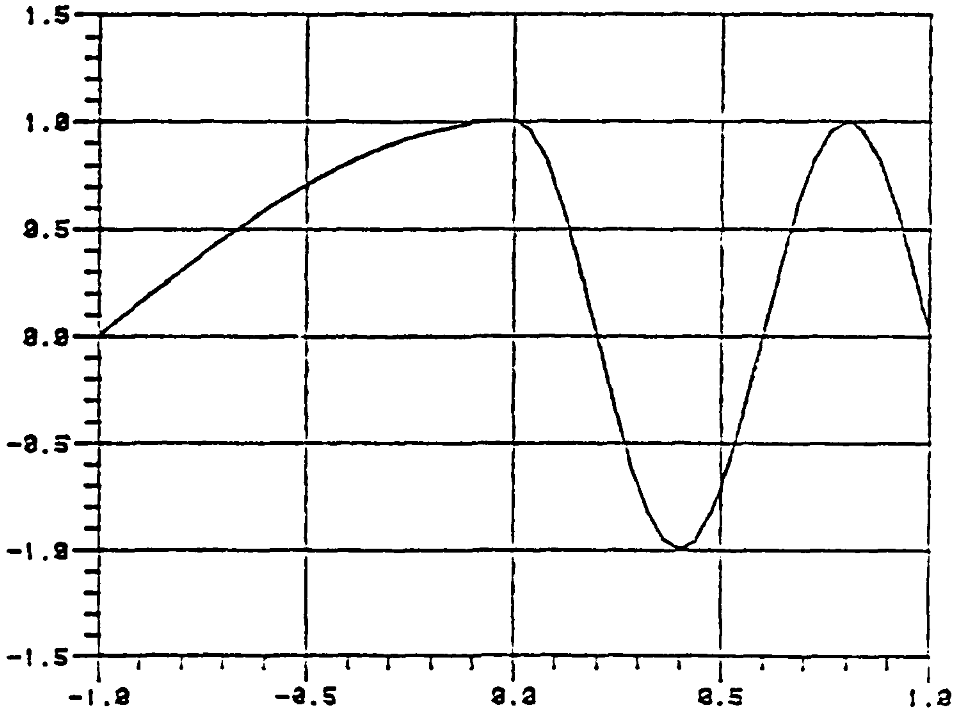

Figure 2.5b. ψ_{ε}^{1} for $\varepsilon = 0.04$, $N = 50$.

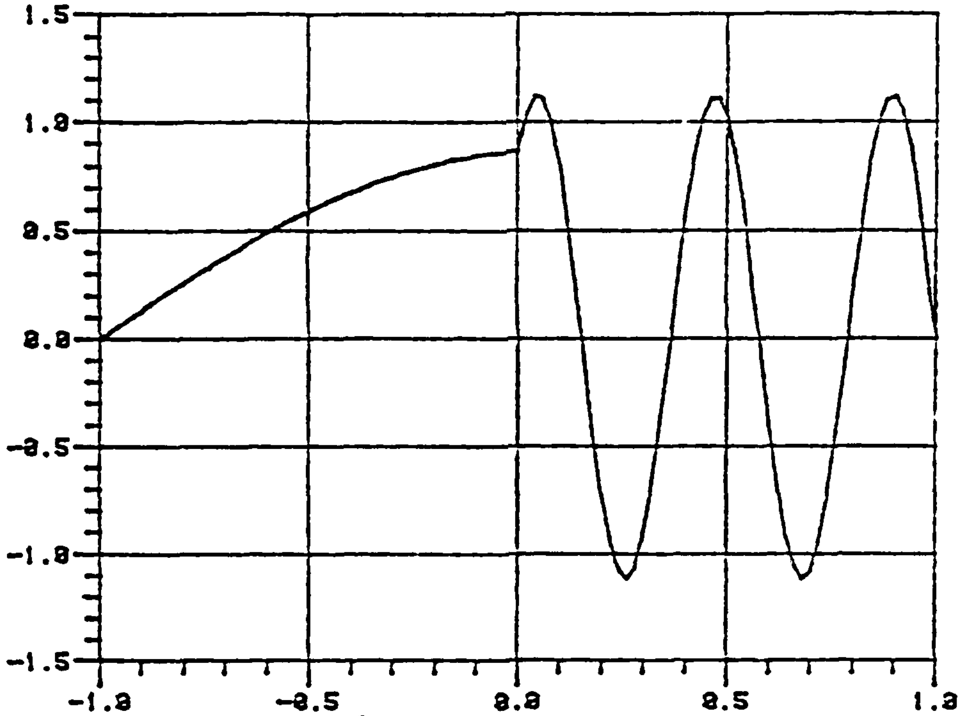

Figure 2.5c. ψ_ε^1 for $\varepsilon = 0.01$, $N = 100$.

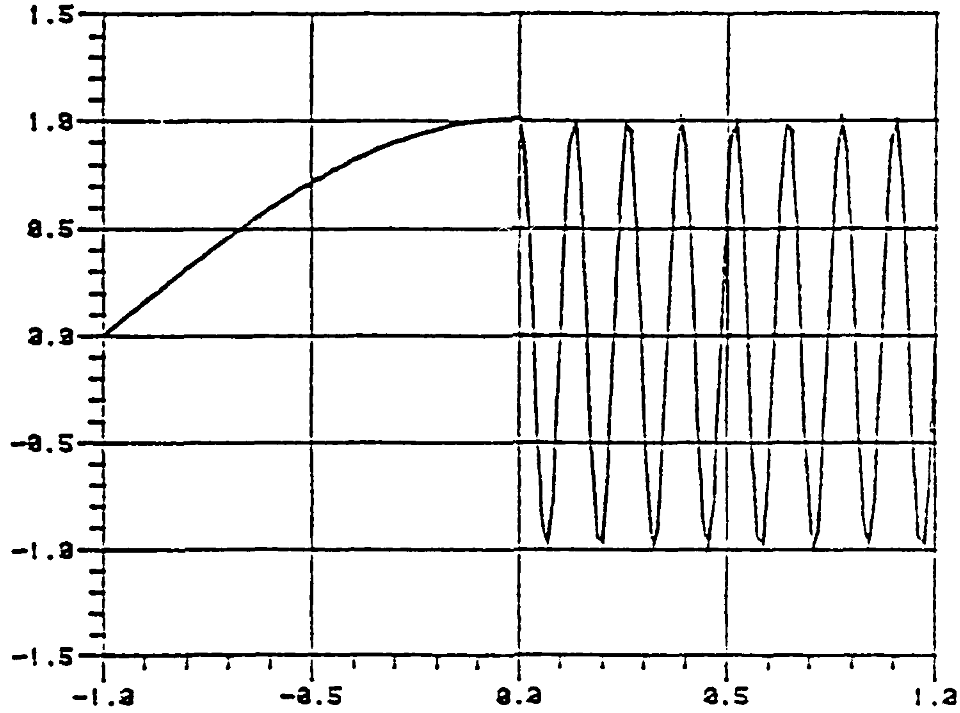

Figure 2.5d. ψ_ε^1 for $\varepsilon = 0.001$, $N = 150$.

TABLE 2.1. $\lambda_\varepsilon^1, \lambda_\varepsilon^2, \mu_\varepsilon^1$ for ε = 0.1,0.04,0.01,0.001

ε	λ_ε^1	λ_ε^2	μ_ε^1
0.1	0.77230	4.5742	2.3462
0.04	0.36160	1.3715	2.4782
0.01	0.09650	0.38677	2.2153
0.001	0.01069	0.03997	2.2538

It is clear that most of the features described in Section 2.3. are exhibited in these plots. First, observe the attenuation of φ_ε^k, k = 1,2 as $\varepsilon \to 0$ on Ω_0. Second, note the oscillatory behavior of ψ_ε^1 as $\varepsilon \to 0$ on Ω_1. In Figure 2.5d, the corners do not belong to φ_ε^k. They are inherent in the Finite Element Method due to the type of functions selected, i.e., $\{\varphi_h^i\}_{i=1}^{N-1}$. Furthermore, one may add that the first eigenvalue-eigenvector pairs are computed more accurately than their last conterparts [17,39]. Consequently, if the eigenvalues of A_ε^h are ordered ascendingly, $\{\mu_\varepsilon^1, \psi_\varepsilon^1\}$ is pushed higher and higher as $\varepsilon \to 0$ and hence computed less and less accurately. Moreover, since h is fixed, the oscillatory behavior of ψ_ε^1 would not be captured by this approximation, unless h is made smaller and hence increasing the order of the matrix A_ε^h .

Example 2.11: (Cf. Example 2.6)

The following eigenvalue problem is analyzed numerically

$$\left.\begin{aligned}
&\left.\begin{aligned}
&-\varepsilon \frac{d^2\chi^k_{\varepsilon 0}}{dx^2} + \chi^k_{\varepsilon 0} = \gamma^k_\varepsilon \chi^k_{\varepsilon 0} \quad \text{on } (-1,0) \\
&-\varepsilon \frac{d^2\chi^k_{\varepsilon 1}}{dx^2} + \varepsilon\chi^k_{\varepsilon 1} = \gamma^k_\varepsilon \chi^k_{\varepsilon 1} \quad \text{on } (0,1)
\end{aligned}\right\} \\
&\chi^k_{\varepsilon 0}(-1) = 0, \quad \chi^k_{\varepsilon 1}(1) = 0 \\
&\left.\begin{aligned}
&\chi^k_{\varepsilon 0}(0) = \chi^k_{\varepsilon 1} \\
&\frac{d\chi^k_{\varepsilon 0}}{dx}(0) = \frac{d\chi^k_{\varepsilon 1}}{dx}(0)
\end{aligned}\right\}
\end{aligned}\right\} \tag{2.91}$$

The finite dimensional approximation of the operator A_ε can be written as in (2.88) where M^h is given by (2.89). The entries of K^h_ε are:

$$(K^h_\varepsilon)_{i,j} = \varepsilon \int_\Omega \frac{d\varphi^i_h}{dx} \frac{d\varphi^j_h}{dx}\, dx + \int_{\Omega_0} \varphi^i_h \varphi^j_h \, dx + \varepsilon \int_{\Omega_1} \varphi^i_h \varphi^j_h \, dx$$

$$i,j = 1,2,\ldots ,$$

which can be explicitly written as

$$K^h_\varepsilon = \frac{\varepsilon}{h} \begin{bmatrix} 2 & -1 & & & & 0 \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & 2 & -1 \\ 0 & & & & -1 & 2 \end{bmatrix} + \frac{h}{6} \begin{bmatrix} 4 & 1 & & & & 0 \\ 1 & 4 & 1 & & & \\ & \ddots & \ddots & 1 & & \\ & & 1 & 2+2\varepsilon & \varepsilon & \\ & & & \varepsilon & 4\varepsilon & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix} \leftarrow \frac{N}{2}+1 \tag{2.92}$$

The major difference between the present example and the preceding one is that $\mu_\varepsilon^k \to \mu_0^k = 1$, and the corresponding eigenvector $\psi_\varepsilon^k \to \psi^k$ weakly in $H_0^1(-1,1)$, $k = 1,2,\ldots$. Consequently, their oscillatory behavior attenuate as $\varepsilon \to 0$.

The eigenvalues λ_ε^1, μ_ε^1, μ_ε^2 are tabulated in Table 2.2 for $\varepsilon = 0.1, 0.01, 0.001, 0.0001$. The corresponding eigenvectors are plotted in Figures (2.6a-2.6d), (2.7a-2.7d), (2.8a-2.8d).

TABLE 2.2. γ_ε^1, μ_ε^1, μ_ε^2 for $\varepsilon = 0.1, 0.01, 0.001, 0.0001$

ε	λ_ε^1	μ_ε^1	μ_ε^1
0.1	0.6279	1.6548	4.555
0.01	0.09148	1.0806	1.4521
0.001	0.01028	1.0100	1.0389
0.0001	0.001070	1.0010	1.0040

From the theoretical results of Section 2.5.3, it is shown that

$$\varphi_\varepsilon^1 \to \varphi^1 = (0, \sin\pi x) \text{ strongly in } H_0^1(-1,1)$$

$$\psi_\varepsilon^1 \to \psi^1 = (\sin\pi x, 0) \text{ weakly in } H_0^1(-1,1)$$

$$\psi_\varepsilon^2 \to \psi^2 = (\sin 2\pi x, 0) \text{ weakly in } H_0^1(-1,1)$$

and this is clearly depicted in the figures below.

The flattening of φ_ε^1 on Ω_0 is also clearly visible in Figures 2.6a-2.6d. The attenuating oscillatory behavior of ψ_ε^1, ψ_ε^2 on Ω_1 is unquestionably documented in Figures 2.7a-2.8d.

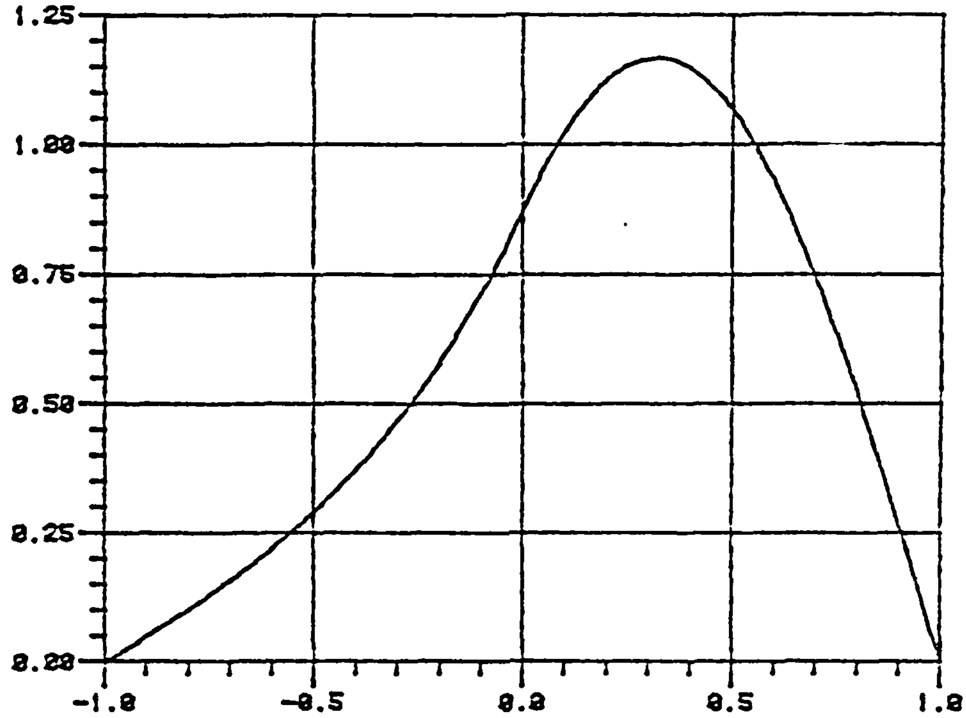

Figure 2.6a. φ_ε^1 for $\varepsilon = 0.1$, $N = 50$.

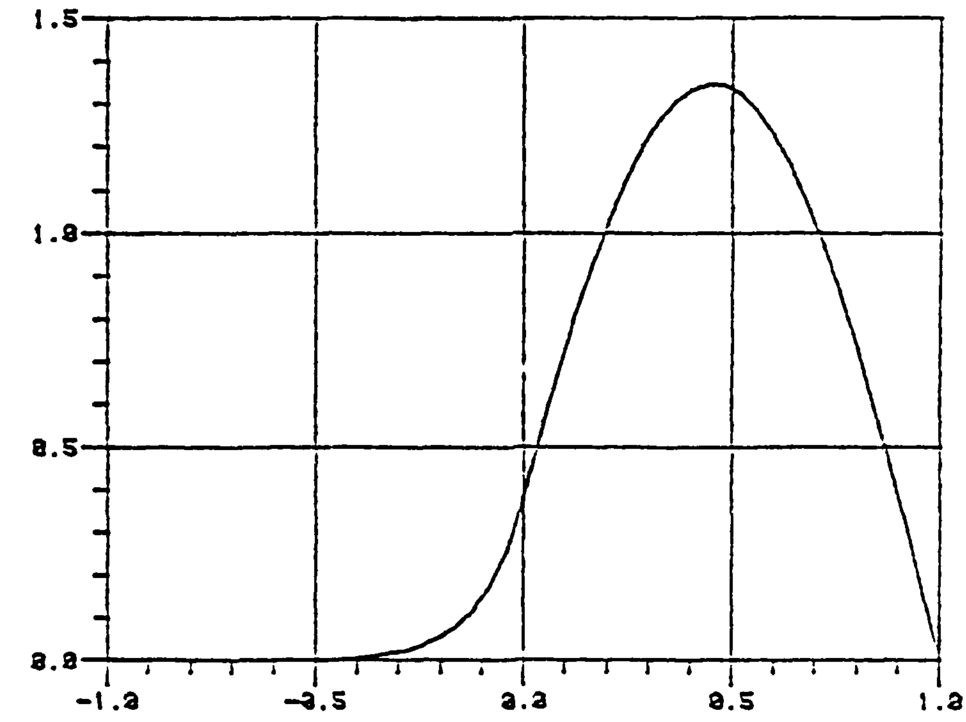

Figure 2.6b. φ_ε^1 for $\varepsilon = 0.01$, $N = 50$.

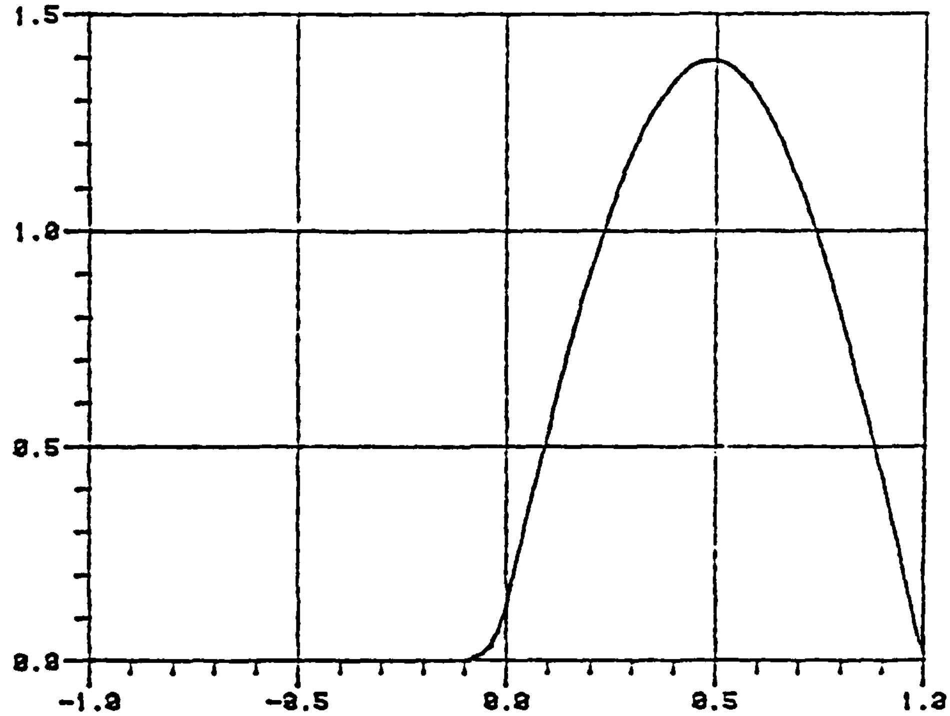

Figure 2.6c. $\varphi_{\varepsilon}^{1}$ for $\varepsilon = 0.001$, $N = 100$.

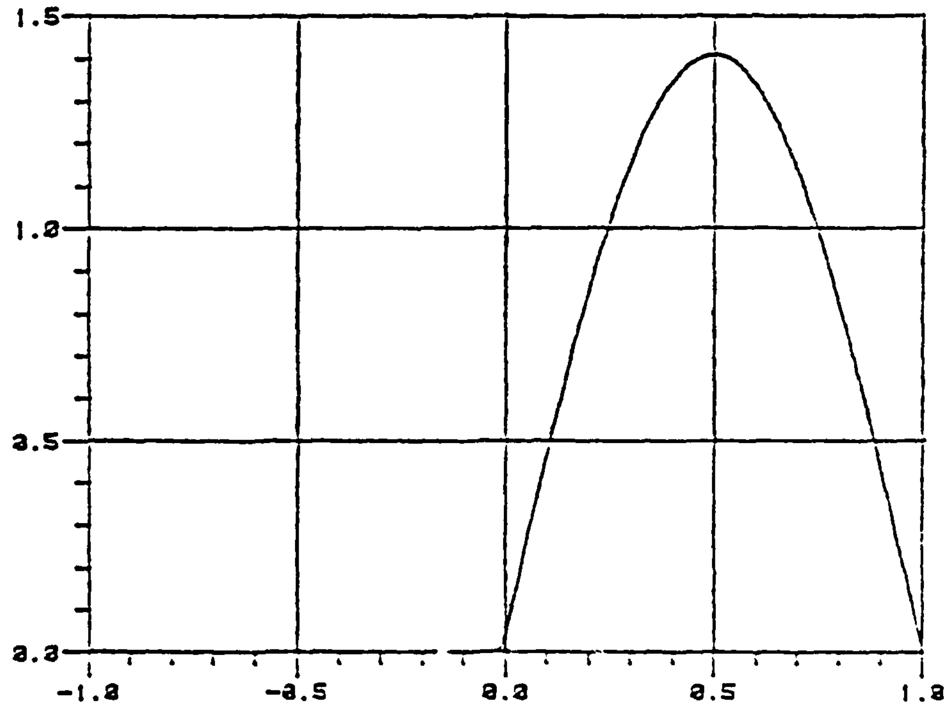

Figure 2.6d. $\varphi_{\varepsilon}^{1}$ for $\varepsilon = 0.0001$, $N = 150$.

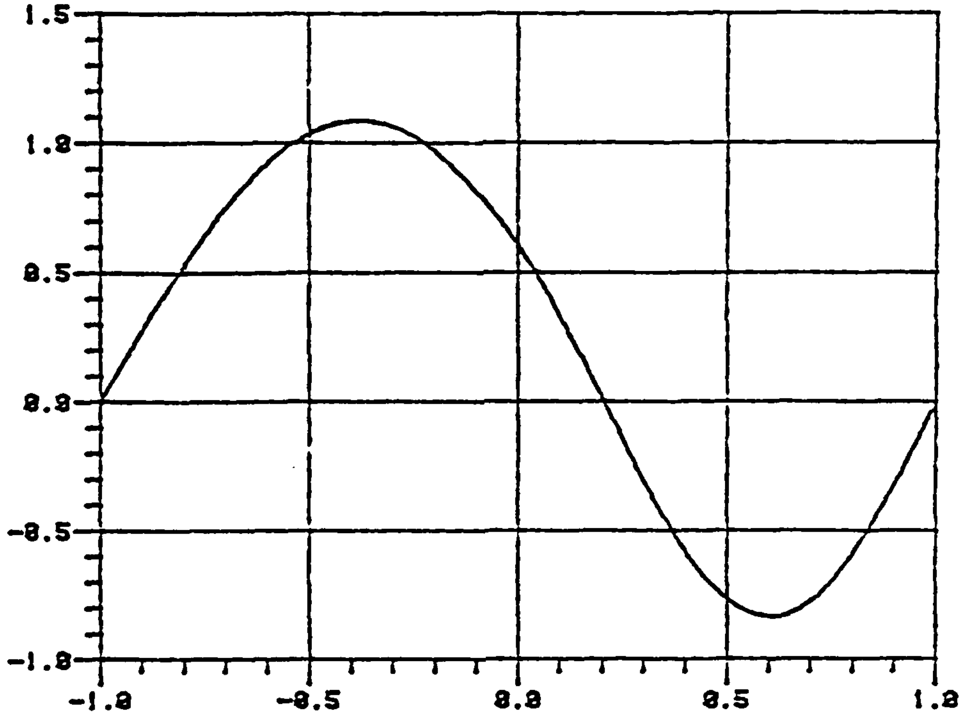

Figure 2.7a. ψ_{ε}^{1} for $\varepsilon = 0.1$, $N = 50$.

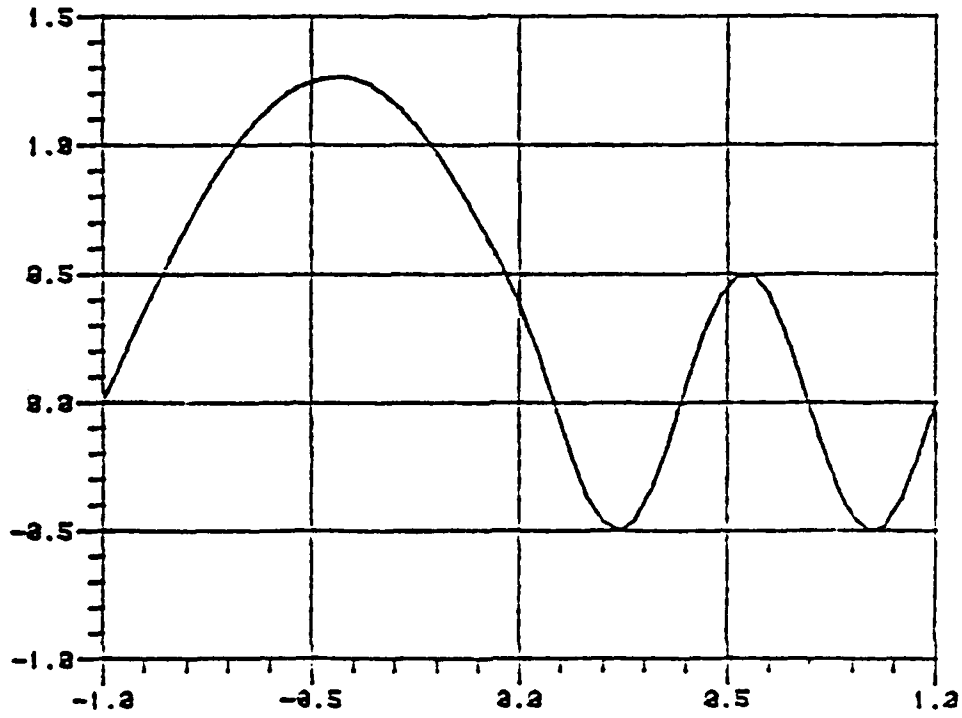

Figure 2.7b. ψ_{ε}^{1} for $\varepsilon = 0.01$, $N = 50$.

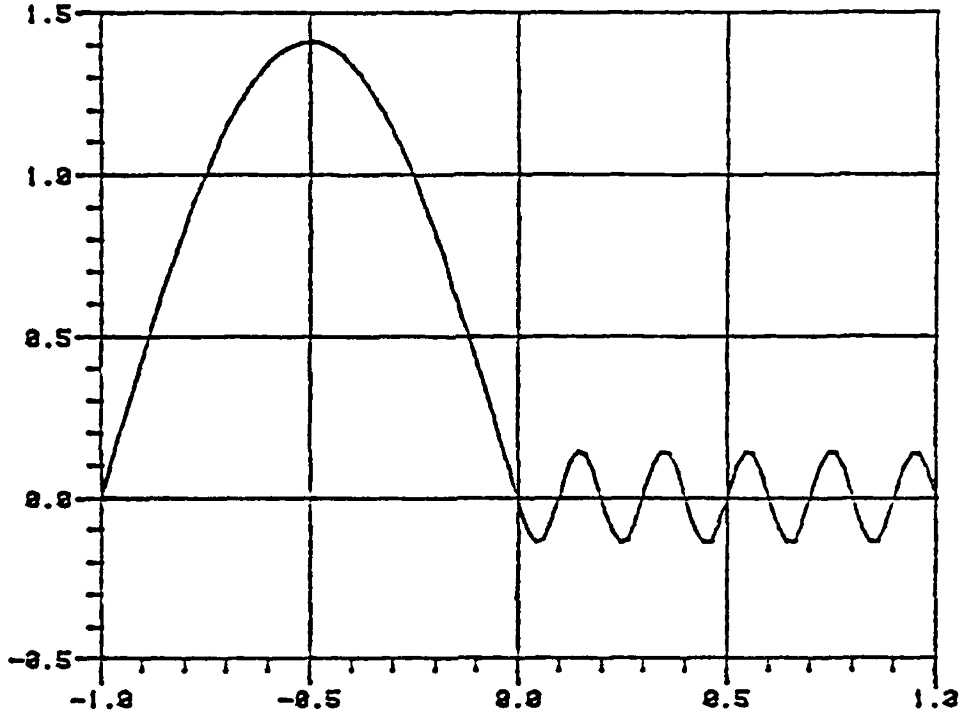

Figure 2.7c. ψ_ε^1 for $\varepsilon = 0.001$, $N = 100$.

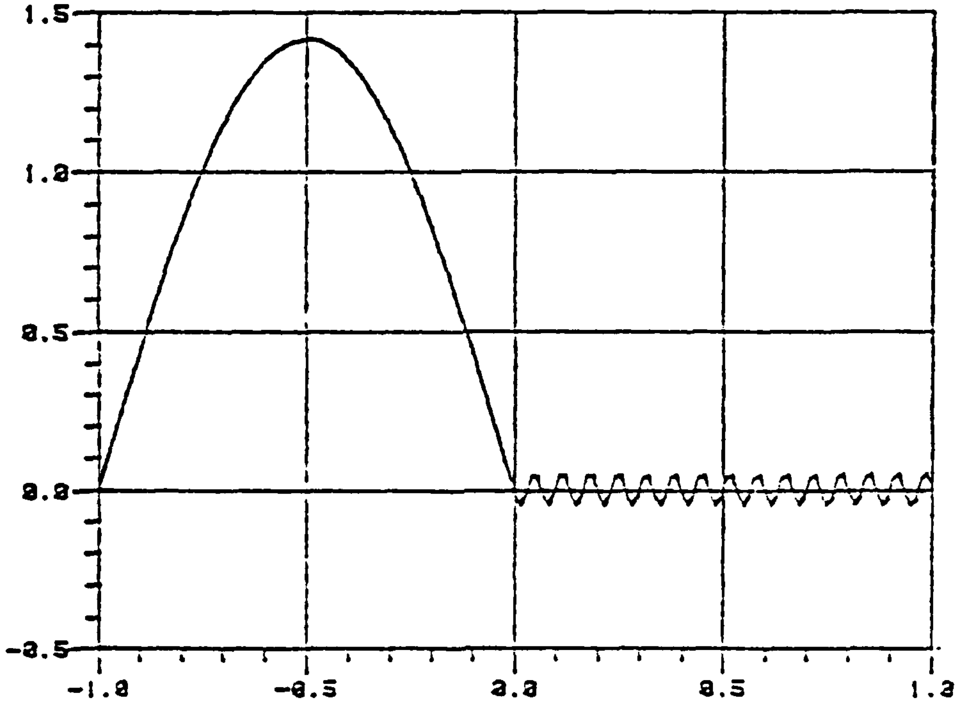

Figure 2.7d. ψ_ε^1 for $\varepsilon = 0.0001$, $N = 150$.

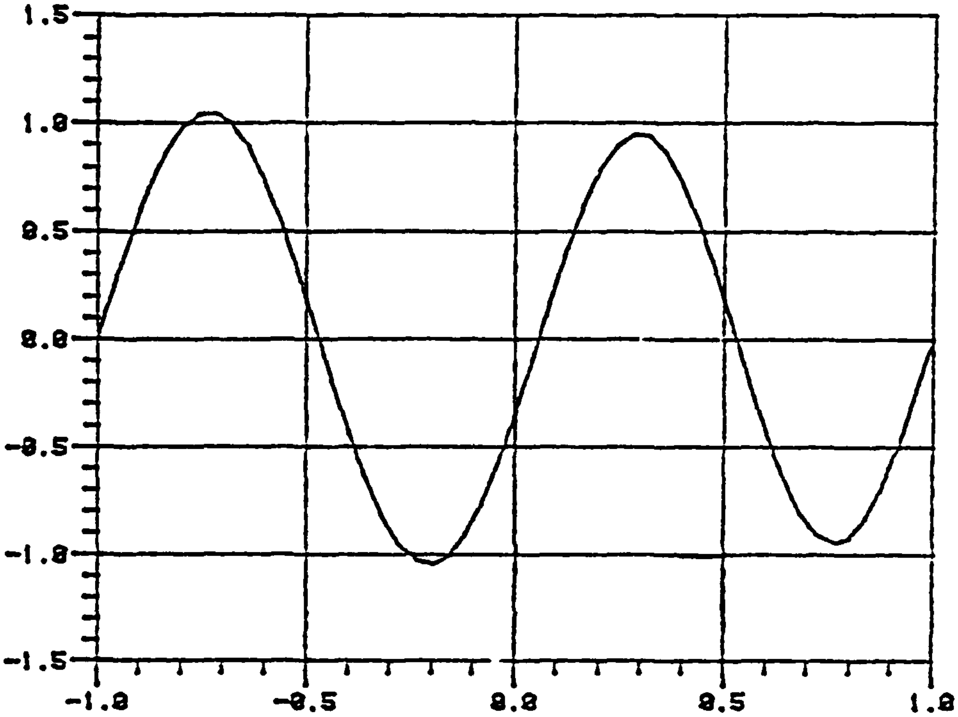

Figure 2.8a. ψ_ε^2 for $\varepsilon = 0.1$, $N = 50$.

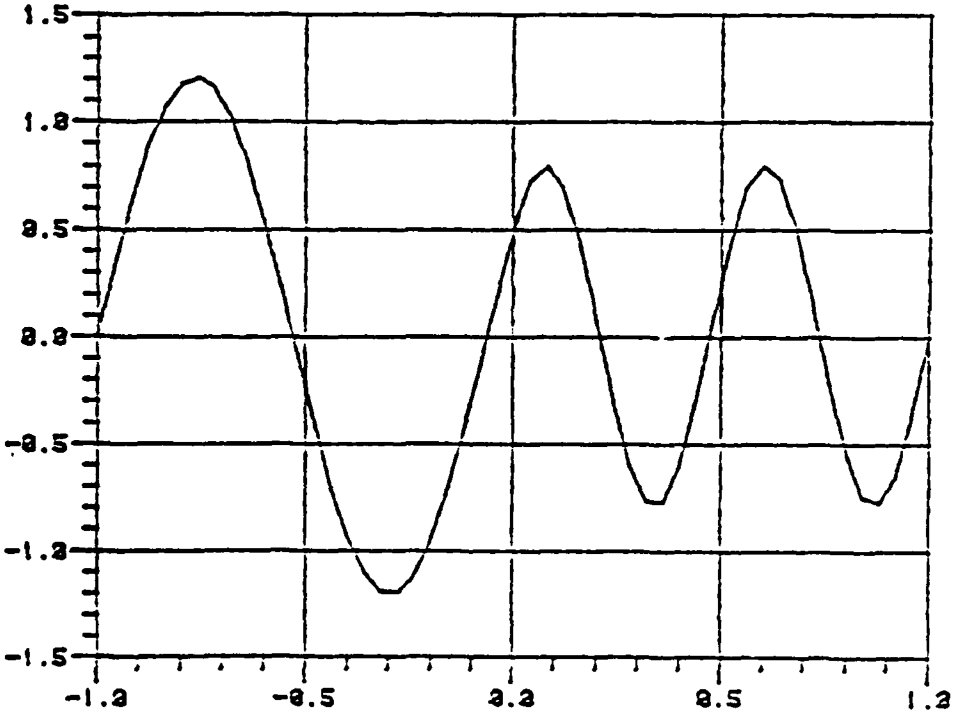

Figure 2.8b. ψ_ε^2 for $\varepsilon = 0.01$, $N = 50$.

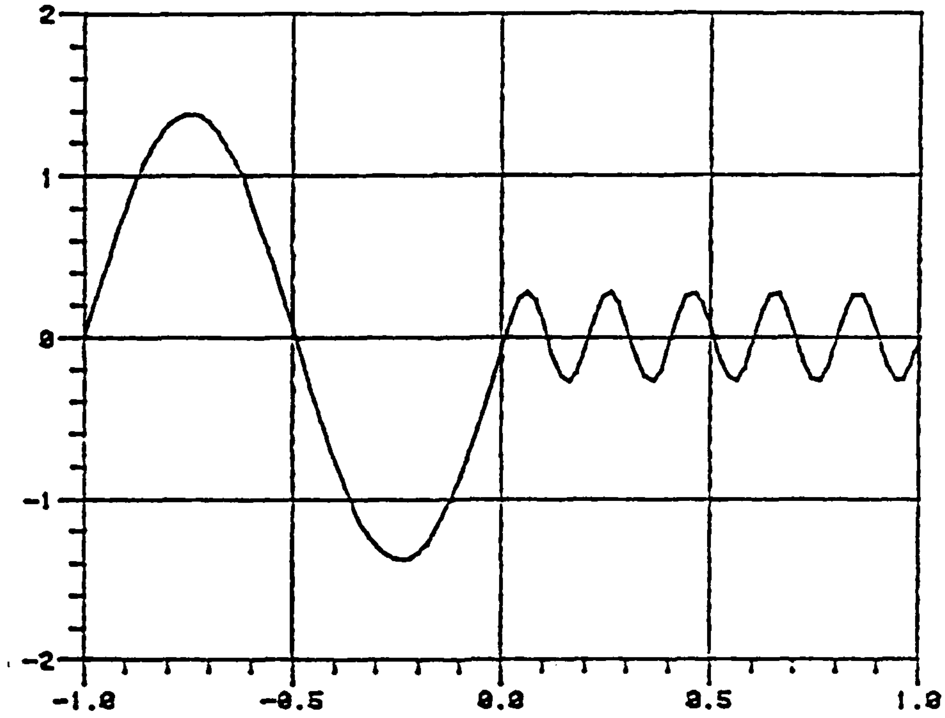

Figure 2.8c. ψ_{ε}^{2} for $\varepsilon = 0.001$, $N = 100$.

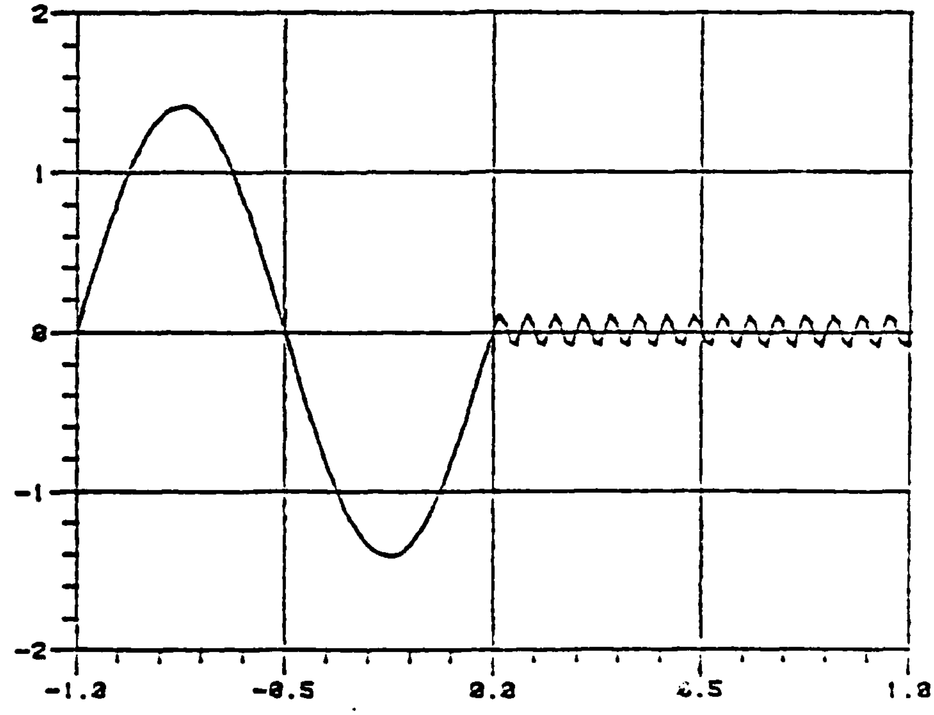

Figure 2.8d. ψ_{ε}^{2} for $\varepsilon = 0.0001$, $N = 150$.

Many of the observations advanced in the preceding example are valid in this example as well.

Finally, one can easily estimate the orders of magnitude of the attenuation and the oscillation frequency of ψ_ε^k on Ω_1. They are $O(\sqrt{\varepsilon})$ and $O(\frac{1}{\sqrt{\varepsilon}})$, respectively.

2.8. Conclusion

2.8.1. Concluding remarks

The eigenvalue problem of stiff operators has been analyzed in this chapter, via a general formulation using bilinear forms, to avoid the complexity of explicitly keeping track of the various boundary and interface conditions. First, the intuitive idea that the eigenvalues of stiff operators are of different order of magnitude as functions of the parameter ε, is rigorously verified. Second, many "concealed" features about the behavior of the eigenvectors have been exposed, such as flattening, attenuation, and oscillation. Third, the convergence of the eigenvectors of stiff operators as $\varepsilon \to 0$ has been investigated. This analysis is of paramount importance, because it will yield insight into how to approximate boundary value problems involving stiff operators. Table 2.3 summarizes the properties of some stiff operators.

TABLE 2.3. SUMMARY OF PROPERTIES OF SOME STIFF OPERATORS.

Operator	Convergence of $\{\lambda^k_\varepsilon, \varphi^k_\varepsilon\}^\infty_{k=1}$ as $\varepsilon \to 0$	Convergence of $\{\mu^k_\varepsilon, \psi^k_\varepsilon\}^\infty_{k=1}$ as $\varepsilon \to 0$	Section
$\left.\begin{array}{l} -\Delta\chi^k_{\varepsilon 0} = \gamma^k_\varepsilon \chi^k_{\varepsilon 0} \text{ on } \Omega_0 \\ -\varepsilon\Delta\chi^k_{\varepsilon 1} = \gamma^k_\varepsilon \chi^k_{\varepsilon 1} \text{ on } \Omega_1 \end{array}\right\}$ $\chi^k_{\varepsilon 0}\big\vert_{\Gamma_0} = 0,\ \chi^k_{\varepsilon 1}\big\vert_{\Gamma_1} = 0$ $\left.\begin{array}{l} \chi^k_{\varepsilon 0} = \chi^k_{\varepsilon 1} \\ \dfrac{\partial\chi^k_{\varepsilon 0}}{\partial\nu} = \varepsilon \dfrac{\partial\chi^k_{\varepsilon 1}}{\partial\nu} \end{array}\right\}$ on S	• $\lambda^k_\varepsilon \to 0$ linearly in ε • $\varphi^k_\varepsilon \to \varphi^k$ strongly in $H^1_0(\Omega)$ • Attenuation on Ω_0 • Flattening on Ω_0 • No oscillation $k=1,2,\ldots$	• $\mu_\varepsilon \to \mu^k_0 > 0$ • $\psi^k_\varepsilon \to \psi^k$ weakly in $L^2(\Omega)$ • No attenuation • No flattening • Oscillation on Ω_1 $k=1,2,\ldots$	2.3
$-\Delta\chi^k_\varepsilon = \gamma^k_\varepsilon \chi^k_\varepsilon$ on Ω $\dfrac{\partial\chi^k_\varepsilon}{\partial\nu} + \varepsilon\chi^k_\varepsilon = 0$ on Γ	• $\lambda^k_\varepsilon \to 0$ quadratically in ε • $\varphi^k_\varepsilon \to \varphi^k$ strongly in $H^2(\Omega)$ • No attenuation • Flattening on Ω • No oscillation $k = 0$	• $\mu^k_\varepsilon \to \mu^k_0 > 0$ • $\psi^k_\varepsilon \to \psi^k$ strongly in $L^2(\Omega)$ • No attenuation • No flattening • No oscillation $k=1,2,\ldots$	2.5.2
$\left.\begin{array}{l} -\varepsilon\Delta\chi^k_{\varepsilon 0} + \chi^k_{\varepsilon 0} = \gamma^k_\varepsilon \chi^k_{\varepsilon 0} \text{ on } \Omega_0 \\ -\varepsilon\Delta\chi^k_{\varepsilon 1} + \varepsilon\chi^k_{\varepsilon 1} = \gamma^k_\varepsilon \chi^k_{\varepsilon 1} \text{ on } \Omega_1 \end{array}\right\}$ $\chi^k_{\varepsilon 0}\big\vert_{\Gamma_0} = 0,\ \chi^k_{\varepsilon 1}\big\vert_{\Gamma_1} = 0$ $\left.\begin{array}{l} \chi^k_{\varepsilon 0} = \chi^k_{\varepsilon 1} \\ \dfrac{\partial\chi^k_{\varepsilon 0}}{\partial\nu} = \varepsilon \dfrac{\partial\chi^k_{\varepsilon 1}}{\partial\nu} \end{array}\right\}$ on S	• $\lambda^k_\varepsilon \to 0$ linearly in ε • $\varphi^k_\varepsilon \to \varphi^k$ strongly in $H^1_0(\Omega)$ • Attenuation on Ω_0 • Flattening on Ω_0 • No oscillation $k=1,2,\ldots$	• $\mu^k_\varepsilon \to \mu^k_0 > 0$ affinely in ε • $\psi^k_\varepsilon \to \psi^k$ weakly in $H^1_0(\Omega)$ • Attenuation on Ω_1 • No flattening • Oscillation on Ω_1 $k=1,2,\ldots$	2.5.3

2.8.2. Extensions

The concepts used in this chapter are very general. Consequently, several operators can be constructed from the ones examined herein. They can be analyzed using the techniques and the concepts developed in this chapter. For example, one may combine operators of Sections 2.3-2.4 with those of Section 2.5 to obtain multiparameter eigenvalue problems. One may also consider nonselfadjoint operators such as

$$A_\varepsilon = \begin{bmatrix} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}^0(x) \frac{\partial}{\partial x_j} & 0 \\ 0 & -\varepsilon \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}^1(x) \frac{\partial}{\partial x_j} \end{bmatrix} \tag{2.93}$$

where $a_{ij}^k(x)$ is as in Remark 2.9 but

$$a_{ij}^k(x) \neq a_{ji}^k(x), \qquad k = 0,1.$$

The same analysis holds for the selfadjoint part of A_ε, i.e., A_ε^0. The operator A_ε can be written as

$$A_\varepsilon = A_\varepsilon^0 + A_\varepsilon^1 \tag{2.94}$$

where

$$A_\varepsilon^0 \in \mathcal{L}(H_0^1(\Omega); H^{-1}(\Omega))$$

$$A_\varepsilon^1 \in \mathcal{L}(H_0^1(\Omega); L^2(\Omega))$$

given by

$$A_\varepsilon^0 = \begin{bmatrix} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{a_{ij}^0(x) + a_{ji}^0(x)}{2}\frac{\partial}{\partial x_j}\right) & 0 \\ 0 & -\varepsilon \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{a_{ij}^1(x)+a_{ji}^1(x)}{2}\frac{\partial}{\partial x_j}\right) \end{bmatrix}$$

$$A_\varepsilon^1 = \begin{bmatrix} \frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{\partial a_{ji}^0(x)}{\partial x_i} - \frac{\partial a_{ij}^0(x)}{\partial x_i}\right)\frac{\partial}{\partial x_j} & 0 \\ 0 & \frac{\varepsilon}{2}\sum_{i,j=1}^{n}\left(\frac{\partial a_{ji}^1(x)}{\partial x_i} - \frac{\partial a_{ij}^1(x)}{\partial x_i}\right)\frac{\partial}{\partial x_j} \end{bmatrix} .$$

CHAPTER 3

APPROXIMATION OF FORCED STIFF SYSTEMS

3.1. Introduction

In the preceding chapter, the spectral decomposition of stiff operators was investigated and the properties of their eigenvalues and eigenvectors as functions of the small parameter ε analyzed. One major property which is summarized below, is the convergence of these eigenvalues and their corresponding eigenvectors as $\varepsilon \to 0$. It was shown that the spectrum of A_ε is decomposable into two groups of eigenvalue-eigenvector pairs $\{\lambda_\varepsilon^k, \varphi_\varepsilon^k\}_{k=1}^\infty$, $\{\mu_\varepsilon^k, \psi_\varepsilon^k\}_{k=1}^\infty$ with the following convergence mode:

1) $\lambda_\varepsilon^k \to 0$ linearly in ε, $\varphi_\varepsilon^k \to \varphi^k$ strongly in V
2) $\mu_\varepsilon^k \to \mu_0^k > 0$, $\psi_\varepsilon^k \to \psi^k$ weakly in H

where V, H are two given Hilbert spaces, with V being a dense subspace of H, having a stronger topology than that of H. Moreover, some of the aforementioned eigenvectors are nonanalytic functions of ε. Consequently, they cannot be expanded in powers of ε. From these established facts, one concludes it may not be possible to obtain "strong" approximations of solutions of boundary value problems involving the operator A_ε.

In this chapter, the above results, as well as those discussed in Chapter 2, are employed to investigate the behavior of the solutions y_ε of the following three abstract equations:

$$A_\varepsilon y_\varepsilon = f \tag{3.1}$$

$$\frac{\partial y_\varepsilon}{\partial t} + A_\varepsilon y_\varepsilon = f \quad , \quad y_\varepsilon(0) = h \tag{3.2}$$

$$\frac{\partial^2 y_\varepsilon}{\partial t^2} + A_\varepsilon y_\varepsilon = f \ , \ y_\varepsilon(0) = h \ , \ \frac{\partial y_\varepsilon}{\partial t}(0) = g \qquad (3.3)$$

The occurrence of (3.1)-(3.3) is very frequent in mathematical models of distributed physical processes such as nuclear reactors, heat exchangers, chemical reactors, fluid systems, vibration systems, steel and glass processes, etc. Thus, it is important to focus on them. A logical question to ask is the following: since the eigenvectors of A_ε are not analytic functions of ε, is it possible to derive "weak" approximations to (3.1)-(3.3) using the weak limits of the eigenvectors of A_ε? The answer is in the affirmative, but in doing so "something" ought to be lost. The major thrust of the present chapter will clarify this loss for each of the boundary value problems (3.1)-(3.3). For elliptic problems (i.e., (3.1)), by appropriately modifying the weak limits of the eigenvectors, one may be able to calculate a "strong" asymptotic expansion of the solution of (3.1). In so doing, the formal results derived in [24] are complemented. For evolution problems (i.e., (3.2)-(3.3)), the concept of weak solutions [23,30] is used to define weak asymptotic approximation of the solutions of (3.2)-(3.3). Hence, extensions of the results in [23] are accomplished.

This chapter is organized as follows. In section 3.2, the solution of (3.1) is derived. The weak limits of $\{\varphi_\varepsilon^k\}_{k=1}^\infty$ and $\{\lambda_\varepsilon^k\}_{k=1}^\infty$ are modified by adding to them some appropriately selected functions. The rationale behind such modification is that the modified limits become elements of V and hence can be used to derive an asymptotic expansion of the solution of (3.1).

In Section 3.3, the convergence of the solution of (3.2) is investigated and a weak asymptotic approximation is constructed for it. In Section 3.4, an analysis similar to that of Section 3.3 for a class of hyperbolic problems (i.e., (3.3)) is undertaken. In Section 3.5, some concluding remarks and extensions are presented.

3.2. Elliptic Boundary Value Problems

As in Section 2.2, suppose two Hilbert spaces V and H are given. The same notation and the same assumptions are kept.

First, the modifications of the weak limits of the eigenvectors are considered. Then, using these modified limits, an asymptotic expansion of the solution y_ε of the following boundary value problem is constructed

$$A_\varepsilon y_\varepsilon = f \ , \ f \in H \tag{3.4}$$

or equivalently

$$a_0(y_\varepsilon,\varphi) + \varepsilon a_1(y_\varepsilon,\varphi) = (f,\varphi) \ , \ y_\varepsilon \in V \ , \ \forall\varphi\in V \ . \tag{3.5}$$

From the preceding chapter, the weak limit of ψ_ε^k in H and the weak limit of φ_ε^k in V satisfy, respectively:

$$a_0(\psi^k,\chi) = \mu_0^k(\psi^k,\chi), \ \psi^k\in H_1, \ \forall\chi\in V \tag{3.6}$$

$$a_1(\varphi^k,\chi) = \lambda_1^k(\varphi^k,\chi) \ , \ \varphi^k\in V_0 \ , \ \forall\chi\in V_0 \tag{3.7}$$

$$k=1,2,3,\ldots$$

(respectively, V_0) is a subspace of H (respectively, V).

Since $\psi^k \in H_1$, one may add to it a function

$$\xi_\varepsilon^k = \xi_0^k + \varepsilon\,\xi_1^k + \varepsilon^2\,\xi_2^k + \dots \tag{3.8}$$

where

$$\xi_0^k \in H \text{ such that } \psi^k + \xi_0^k \in V \tag{3.9}$$

$$\xi_\ell^k \in V\ ,\quad \ell=1,2,\dots$$

It was noted in the preceding chapter that $\{\psi^k\}_{k=1}^\infty$ and $\{\varphi^k\}_{k=1}^\infty$ form an orthonormal system in H. Therefore, one requires that ξ_ε^k satisfy

$$a_0(\psi^k + \xi_\varepsilon^k,\chi) + \varepsilon a_1(\psi^k + \xi_\varepsilon^k,\chi) = \mu_0^k(\psi^k,\chi),\ \forall\chi\in V \tag{3.10}$$

substitute (3.8) into (3.10) and identify formally equal powers of ε, to get:

$$\left.\begin{aligned} & a_0(\xi_0^k,\chi) = 0\ ,\ \forall\chi\in V \\ & a_0(\xi_\ell^k,\chi) + a_1(\xi_{\ell-1}^k,\chi) = 0\ ,\ \forall\chi\in V \\ & \ell=1,2,\dots \end{aligned}\right\} \tag{3.11}$$

from which one finds that $\{\xi_\ell^k\}_{\ell=0,k=1}^\infty$ obey

$$a_1(\xi_0^k,\chi) = 0\ ,\quad \psi^k + \xi_0^k \in V\ ,\ \forall\chi\in V_0 \tag{3.12}$$

$$\dots$$

$$\left.\begin{aligned} & \left.\begin{aligned} & a_0(\xi_\ell^k,\chi) = -\,a_1(\xi_{\ell-1}^k,\chi)\ ,\ \forall\chi\in V \\ & a_1(\xi_\ell^k,\chi) = 0\ ,\ \forall\chi\in V_0 \end{aligned}\right\} \\ & \ell=1,2,\dots \\ & k=1,2,\dots \end{aligned}\right\} \tag{3.13}$$

Remark 3.1: It is worthy of mention that the iterative process described by (3.11)-(3.13) appears to average the oscillatory behavior of φ_ε^k.

Similarly, one adds to φ^k

$$\theta_\varepsilon^k = \varepsilon\, \theta_1^k + \varepsilon^2 \theta_2^k + \dots \tag{3.14}$$

where

$$\theta_\ell^k \in V \ , \ \ell=1,2,\dots \quad .$$

The function θ_ε^k is chosen such that

$$a_0(\varphi^k + \theta_\varepsilon^k, \chi) + \varepsilon a_1(\varphi^k + \theta_\varepsilon^k, \chi) = \varepsilon \lambda_1^k(\varphi^k, \chi) \ , \ \forall \chi \in V$$

from which one concludes (using the fact that $\varphi^k \in V_0$) that $\{\theta_\ell^k\}_{\ell=1,k=1}^{\infty}$ satisfy

$$\left.\begin{aligned}
&\left.\begin{aligned}
&a_0(\theta_1^k, \chi) = -\, a_1(\varphi^k, \chi) \ , \ \forall \chi \in V \\
&a_1(\theta_1^k, \chi) = 0 \ , \ \forall \chi \in V_0
\end{aligned}\right\} \\
&\dots \\
&\left.\begin{aligned}
&a_0(\theta_\ell^k, \chi) = -\, a_1(\theta_{\ell-1}^k, \chi) \ , \ \forall \chi \in V \\
&a_1(\theta^k, \chi) = 0 \ , \ \forall \chi \in V_0
\end{aligned}\right\} \\
&\ell=2,3,\dots \\
&k=1,2,\dots
\end{aligned}\right\} \tag{3.15}$$

Remark 3.2: The zeroth term in (3.14) is zero because $\varphi^k \in V$.

Remark 3.3: The iterative process presented above appears to average the flattening (and attenuation) of $\{\varphi_\varepsilon^k\}_{k=1}^\infty$.

Before proceeding further, consider the following two examples to examine what (3.11)-(3.13), (3.15) yield:

Example 3.1: (Example 2.1 continued) Let $H = L^2(\Omega)$, $V = H_0^1(\Omega)$

$$a_i(\varphi,\psi) = \sum_{j=1}^n \int_{\Omega_i} \frac{\partial\varphi}{\partial x_j} \frac{\partial\psi}{\partial x_j}\, dx$$

then (3.11)-(3.13), (3.15) become

$$\left.\begin{aligned}
&\xi_{00}^k = 0 \text{ on } \Omega_0 \\
&-\Delta\xi_{01}^k = 0 \text{ on } \Omega_1 \\
&\xi_{01}^k\big|_{\Gamma_1} = 0 \ , \ \xi_{01}^k\big|_S = \varphi_0^k\big|_S
\end{aligned}\right\}$$

$$\cdots$$

$$\left.\begin{aligned}
&-\Delta\xi_{\ell 0}^k = 0 \text{ on } \Omega_0 \\
&\xi_{\ell 0}^k\big|_{\Gamma_0} = 0 \ , \ \frac{\partial\xi_{\ell 0}^k}{\partial\nu}\bigg|_S = \frac{\partial\xi_{\ell-1\,1}^k}{\partial\nu}\bigg|_S \\
&-\Delta\xi_{\ell 1}^k = 0 \text{ on } \Omega_1 \\
&\xi_{\ell 1}^k\big|_{\Gamma_1} = 0 \ , \ \xi_{\ell 1}^k\big|_S = \xi_{\ell 0}^k\big|_S
\end{aligned}\right\}$$

$\ell=1,2,\ldots$

$k=1,2,\ldots$

$$\left.\begin{aligned}
&-\Delta\theta_{10}^{k} = 0 \text{ on } \Omega_0 \\
&\theta_{10}^{k}\Big|_{\Gamma_1} = 0\ ,\ \frac{\partial\theta_{10}^{k}}{\partial\nu}\Big|_S = \frac{\partial\varphi_1^{k}}{\partial\nu}\Big|_S \\
&-\Delta\theta_{11}^{k} = 0 \text{ on } \Omega_1 \\
&\theta_{11}^{k}\Big|_{\Gamma_1} = 0\ ,\ \theta_{11}^{k}\Big|_S = \theta_{10}^{k}\Big|_S
\end{aligned}\right\}$$

$$\cdots$$

$$\left.\begin{aligned}
&-\Delta\theta_{\ell 0}^{k} = 0 \text{ on } \Omega_0 \\
&\theta_{\ell 0}^{k}\Big|_{\Gamma_0} = 0\ ,\ \frac{\partial\theta_{\ell 0}^{k}}{\partial\nu}\Big|_S = \frac{\partial\theta_{\ell-1\,1}^{k}}{\partial\nu}\Big|_S \\
&-\Delta\theta_{\ell 1}^{k} = 0 \text{ on } \Omega_1 \\
&\theta_{\ell 1}^{k}\Big|_{\Gamma_1} = 0\ ,\ \theta_{\ell 1}^{k}\Big|_S = \theta_{\ell 0}^{k}\Big|_S
\end{aligned}\right\}$$

$$\ell=2,3,\ldots$$

$$k=1,2,\ldots$$

The above equations represent boundary value problems for the Laplacian operator for each region Ω_i, $i=0,1$ to be solved sequentially.

Example 3.2: (Cf. Example 2.2) Let $H = L^2(\Omega)$, $V = H_0^2(\Omega)$

$$a_i(\varphi,\psi) = \int_{\Omega_i} \Delta\varphi\,\Delta\psi\ dx$$

then (3.11)-(3.13), (3.15) become

$$
\left.
\begin{array}{l}
\xi^k_{00} = 0 \text{ on } \Omega_0 \\[2ex]
\Delta^2 \xi^k_{01} = 0 \text{ on } \Omega_1 \\[2ex]
\xi^k_{01} = \dfrac{\partial \xi^k_{01}}{\partial \nu} = 0 \text{ on } \Gamma_1 \\[2ex]
\left.
\begin{array}{l}
\xi^k_{01} = \psi^k_0 \\[2ex]
\dfrac{\partial \xi^k_{01}}{\partial \nu} = \dfrac{\partial \psi^k_0}{\partial \nu}
\end{array}
\right\} \text{ on } S
\end{array}
\right\}
$$

$$\cdots$$

$$
\left.
\begin{array}{l}
\Delta^2 \xi^k_{\ell 0} = 0 \text{ on } \Omega_0 \\[2ex]
\xi^k_{\ell 0} = \dfrac{\partial \xi_{\ell 0}}{\partial \nu} = 0 \text{ on } \Gamma_0 \\[2ex]
\left.
\begin{array}{l}
\Delta \xi^k_{\ell 0} = \Delta \xi^k_{\ell-1\ 1} \\[2ex]
\dfrac{\partial \Delta \xi^k_{\ell 0}}{\partial \nu} = \dfrac{\partial \Delta \xi^k_{\ell-1\ 1}}{\partial \nu}
\end{array}
\right\} \text{ on } S \\[2ex]
\Delta^2 \xi^k_{\ell 1} = 0 \text{ on } \Omega_1 \\[2ex]
\xi^k_{\ell 1} = \dfrac{\partial \xi^k_{\ell 1}}{\partial \nu} = 0 \text{ on } \Gamma_1 \\[2ex]
\left.
\begin{array}{l}
\xi^k_{\ell 1} = \xi^k_{\ell 0} \\[2ex]
\dfrac{\partial \xi^k_{\ell 1}}{\partial \nu} = \dfrac{\partial \xi^k_{\ell 0}}{\partial \nu}
\end{array}
\right\} \text{ on } S
\end{array}
\right\}
$$

$\ell=1,2,\ldots$

$k=1,2,\ldots$

$$
\left.
\begin{aligned}
&\Delta^2\theta^k_{10} = 0 \text{ on } \Omega_0 \\
&\theta^k_{10} = \frac{\partial\theta^k_{10}}{\partial\nu} = 0 \text{ on } \Gamma_0 \\
&\left.
\begin{aligned}
\Delta\theta^k_{10} &= \Delta\varphi^k_1 \\
\frac{\partial\Delta\theta^k_{10}}{\partial\nu} &= \frac{\partial\Delta\varphi^k_1}{\partial\nu}
\end{aligned}
\right\} \text{ on } S \\
&\Delta^2\theta^k_{11} = 0 \text{ on } \Omega_1 \\
&\theta^k_{11} = \frac{\partial\theta^k_{11}}{\partial\nu} = 0 \text{ on } \Gamma_1 \\
&\left.
\begin{aligned}
\theta^k_{11} &= \theta^k_{10} \\
\frac{\partial\theta^k_{11}}{\partial\nu} &= \frac{\partial\theta^k_{10}}{\partial\nu}
\end{aligned}
\right\} \text{ on } S
\end{aligned}
\right\}
$$

$$\cdots$$

$$
\left.
\begin{aligned}
&\Delta^2\theta^k_{\ell 0} = 0 \text{ on } \Omega_0 \\
&\theta^k_{\ell 0} = \frac{\partial\theta^k_{\ell 0}}{\partial\nu} = 0 \text{ on } \Gamma_0 \\
&\left.
\begin{aligned}
\Delta\theta^k_{\ell 0} &= \Delta\theta^k_{\ell-1\ 1} \\
\frac{\partial\Delta\theta^k_{\ell 0}}{\partial\nu} &= \frac{\partial\Delta\theta^k_{\ell-1\ 1}}{\partial\nu}
\end{aligned}
\right\} \text{ on } S \\
&\Delta^2\theta^k_{\ell 1} = 0 \text{ on } \Omega_1 \\
&\theta^k_{\ell 1} = \frac{\partial\theta^k_{\ell 1}}{\partial\nu} = 0 \text{ on } \Gamma_1 \\
&\left.
\begin{aligned}
\theta^k_{\ell 1} &= \theta^k_{\ell 0} \\
\frac{\partial\theta^k_{\ell 1}}{\partial\nu} &= \frac{\partial\theta^k_{\ell 0}}{\partial\nu}
\end{aligned}
\right\} \text{ on } S
\end{aligned}
\right\}
$$

$$\ell=2,3,\ldots\ ,\quad k=1,2,\ldots$$

The above equations represent boundary value problems for the biharmonic operator in each region Ω_i, i=0,1.

Remark 3.4: The computation of these modifications is unquestionably easy, because the dependence on the parameter ε is eliminated.

Now attention is focused into obtaining an asymptotic expansion of the solution of the boundary value problem (3.1) using the modified weak limits of the eigenvectors of A_ε.

It is noteworthy to indicate that (3.4) (or equivalently (3.5)) has been solved in a more general context than here in [24]. Therefore, only the details pertaining to the present approach are given.

In the next theorem, the usage of the modified limits is shown.

Theorem 3.1: For sufficiently small ε, the solution of (3.4) is given by

$$y_\varepsilon = \sum_{k=1}^{\infty} \frac{c^k}{\mu_0^k} (\psi^k + \xi_\varepsilon^k) + \sum_{k=1}^{\infty} \frac{d^k}{\varepsilon\lambda_1^k} (\varphi^k + \theta_\varepsilon^k) \tag{3.16}$$

where

$\{\mu_0^k, \psi^k\}_{k=1}^\infty$ and $\{\lambda_1^k, \varphi^k\}_{k=1}^\infty$ satisfy (3.6)-(3.7)

$\{\xi_\varepsilon^k\}_{k=1}^\infty$ and $\{\theta_\varepsilon^k\}_{k=1}^\infty$ are given by (3.8), (3.14)

$\{c^k\}_{k=1}^\infty$ and $\{d^k\}_{k=1}^\infty$ are the Fourier coefficients of f, i.e.,

$$c^k = (f, \psi^k)_H \tag{3.17}$$

$$d^k = (f, \varphi^k)_H \tag{3.18}$$

Proof: The eigenvectors $\{\varphi^k\}_{k=1}^{\infty}$ and $\{\psi^k\}_{k=1}^{\infty}$ form a complete orthonormal[1] basis of H and $\varphi^k + \theta_\varepsilon^k \in V$, $\psi^k + \xi_\varepsilon^k \in V$ by construction. Now it is a matter of verification that (3.16) is the unique asymptotic expansion of y_ε.

Remark 3.5: Truncate ξ_ε^k and θ_ε^k to the pth term and denote these truncated series, respectively, by $\xi_\varepsilon^{k,p}$ and $\theta_\varepsilon^{k,p}$. Define e_ε^p by

$$e_\varepsilon^p = y_\varepsilon - y_\varepsilon^p$$

where

$$y_\varepsilon^p = \sum_{k=1}^{\infty} \frac{c^k}{\mu_0^k} (\psi^k + \xi_\varepsilon^{k,p}) + \sum_{k=1}^{\infty} \frac{d^k}{\varepsilon\lambda_1^k} (\varphi^k + \theta_\varepsilon^{k,p}) \quad .$$

Then it can be shown ([24], p. 13) that

$$\| e_\varepsilon^p \|_V \leq C \varepsilon^p \quad .$$

where C is a constant independent of ε.

Remark 3.6: Note, that if the forcing f equals φ^k (respectively, ψ^k) then y_ε becomes $\frac{1}{\varepsilon\lambda_1^k}(\varphi^k + \theta_\varepsilon^k)$ (respectively, $\frac{1}{\mu_0^k}(\psi^k + \xi_\varepsilon^k)$). This clarifies Remarks 3.1, 3.3.

Example 3.3: (Examples 3.1-3.2) In this case, (3.16)-(3.18) become

$$\left.\begin{aligned} y_{\varepsilon 0} &= \sum_{k=1}^{\infty} \frac{c^k}{\mu_0^k} (\psi_0^k + \xi_{\varepsilon 0}^k) + \sum_{k=1}^{\infty} \frac{d^k}{\varepsilon\lambda_1^k} \theta_{\varepsilon 0}^k \\ y_{\varepsilon 1} &= \sum_{k=1}^{\infty} \frac{c^k}{\mu_0^k} \xi_{\varepsilon 1}^k + \sum_{k=1}^{\infty} \frac{d^k}{\varepsilon\lambda_1^k} (\varphi_1^k + \theta_{\varepsilon 1}^k) \end{aligned}\right\}$$

---

[1] Note, $\{\psi^k\}_{k=1}^{\infty}$ are renormalized in H.

where

$$c^k = (f,\psi^k)_{L^2(\Omega)} = (f_0,\psi_0^k)_{L^2(\Omega_0)}$$

$$d^k = (f,\psi^k)_{L^2(\Omega)} = (f_1,\psi_1^k)_{L^2(\Omega_1)}$$

ξ_ε^k , θ_ε^k are computed in Example 3.1 or 3.2.

Remark 3.7: Note that $\lim_{\varepsilon\to 0} \|y_{\varepsilon 0}\|_{L^2(\Omega_0)}$ is finite but not $\lim_{\varepsilon\to 0} \|y_{\varepsilon 1}\|_{L^2(\Omega_1)}$.

Remark 3.8: By examining (3.16), one concludes that the group of eigenvalues that accumulate at zero as $\varepsilon \to 0$, i.e., $\{\lambda_\varepsilon^k\}_{k=1}^\infty$ causes

$$\lim_{\varepsilon\to 0} \|y_{\varepsilon 1}\|_{L^2(\Omega_1)} \to +\infty \quad .$$

Remark 3.9: The eigenvalue problems considered in the preceding chapter can be interpreted as the boundary value problem (3.4) with f depending nonanalytically on ε, i.e., $f = f(x,\varepsilon) = \gamma_\varepsilon^k \chi_\varepsilon^k$. However, it can readily be seen that the process by which (3.4) is solved, is not applicable in this case.

Remark 3.10: If f_ε were analytic in ε and hence expandable as

$$f_\varepsilon = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + \ldots$$

then using the above procedure, one can solve

$$A_\varepsilon y_\varepsilon^r = f^r \ , \ r=0,1,2,\ldots$$

to get an approximation of the solution of

$$A_\varepsilon y_\varepsilon = f_\varepsilon$$

which would be given by

$$y_\varepsilon = y_\varepsilon^0 + \varepsilon y_\varepsilon^1 + \varepsilon^2 y_\varepsilon^2 + \ldots \qquad \square$$

This section is concluded with a simpler one-dimensional version of Example 3.1.

<u>Example 3.4</u>: Recall that $\Omega_0 = (-1,0)$, $\Omega_1 = (0,1)$, $S = \{0\}$, $\Gamma_0 = \{-1\}$, $\Gamma_1 = \{1\}$. It was shown in Example 2.4 that

$$\lambda_1^k = (k\pi)^2, \quad \varphi^k = \begin{cases} 0 & \text{on } \Omega_0 \\ \sqrt{2}\sin\sqrt{\lambda_1^k}\,x & \text{on } \Omega_1 \end{cases}$$

$$\mu_0^k = \left((2k-1)\tfrac{\pi}{2}\right)^2, \quad \psi^k = \begin{cases} \sqrt{2}\cos\sqrt{\mu_0^k}\,x & \text{on } \Omega_0 \\ 0 & \text{on } \Omega_1 \end{cases}$$

$$k=1,2,\ldots$$

The modifications outlined in section 3.2.1 are

$$\theta_\ell^k = \begin{cases} (-1)^{k+\ell}\sqrt{2}\sqrt{\lambda_1^k}\,(1+x) & \text{on } \Omega_0 \\ (-1)^{k+\ell}\sqrt{2}\sqrt{\lambda_1^k}\,(1-x) & \text{on } \Omega_1 \end{cases}$$

$$\xi_0^k = \begin{cases} 0 & \text{on } \Omega_0 \\ (-1)^{k-1}\sqrt{2}\,(1-x) & \text{on } \Omega_1 \end{cases}$$

$$\xi_\ell^k = \begin{cases} (-1)^{k+\ell-1}\sqrt{2}\,(1+x) & \text{on } \Omega_0 \\ (-1)^{k+\ell-1}\sqrt{2}\,(1-x) & \text{on } \Omega_1 \end{cases}$$

$$\ell=1,2,\ldots$$

Let $f = 1$ in (3.4). Then y_ε, as given by (3.16) is

$$\left.\begin{aligned}
y_{\varepsilon 0} &= \sum_{k=1}^{\infty} \frac{2(-1)^{k-1}}{(\mu_0^k)^{3/2}} \left[\cos\sqrt{\mu_0^k}\,x - \frac{\varepsilon}{1+\varepsilon}(1+x)\right] \\
&+ \sum_{k=1}^{\infty} 2\,\frac{1-(-1)^k}{\varepsilon(\lambda_1^k)^{3/2}} \frac{\varepsilon}{1+\varepsilon} \sqrt{\lambda_1^k}\,(1+x) \\
y_{\varepsilon 1} &= \sum_{k=1}^{\infty} \frac{2(-1)^{k-1}}{(\mu_0^k)^{3/2}} \frac{1}{1+\varepsilon}(1-x) \\
&+ \sum_{k=1}^{\infty} 2\,\frac{1-(-1)^k}{\varepsilon(\lambda_1^k)^{3/2}} \left[\sin\sqrt{\lambda_1^k}\,x + \frac{\varepsilon}{1+\varepsilon}\sqrt{\lambda_1^k}\,(1-x)\right.
\end{aligned}\right\} \tag{3.19}$$

where no simplification is made to allow the origin of each term to be identified. The exact solution can be computed directly and is given by

$$\left.\begin{aligned}
y_{\varepsilon 0} &= -\frac{x^2}{2} + \frac{1}{2}\frac{1-\varepsilon}{1+\varepsilon}\,x + \frac{1}{1+\varepsilon} \\
y_{\varepsilon 1} &= -\frac{x^2}{2\varepsilon} + \frac{1}{2}\frac{1-\varepsilon}{\varepsilon(1+\varepsilon)}\,x + \frac{1}{1+\varepsilon}
\end{aligned}\right\} \tag{3.20}$$

It can be easily shown that (3.19) is identical to (3.20).

Remark 3.11: The generalization of the techniques presented in this section to include p+1 (p>1) bilinear forms is straightforward (Cf. Section 2.4).

Remark 3.12: Similarly, the techniques of this section can be applied to boundary value problems involving some of the operators discussed in section 2.5.

3.3. Parabolic Boundary Value Problems

In this section, an evolution problem of parabolic type is considered. Hence let the variable t denote time. It is assumed that $t \in (0,T)$, $T < \infty$ and that all the assumptions made in Section 2.2 hold. Let $L^2(0,T,V)$, $L^2(0,T;H)$,

$L^2(0,T;V^*)$ denote the Hilbert spaces of Lebesgue square integrable functions with values in V, H, V^*, respectively (Cf. Appendix). Let prime denote the distributional derivative with respect to time [30] . In the sequel, the following parabolic boundary value problem is analyzed

$$(y'_\varepsilon,\varphi) + a_0(y_\varepsilon,\varphi) + \varepsilon a_1(y_\varepsilon,\varphi) = (f,\varphi) \ , \ \forall\varphi\in V \tag{3.21}$$

$$y_\varepsilon(0) = h \ , \ h \text{ given in } H \tag{3.22}$$

$$y_\varepsilon \in L^2(0,T;V) \ , \ f \in L^2(0,T;H) \quad . \tag{3.23}$$

Under the assumptions made, problem (3.21)-(3.23) admits a unique solution $y_\varepsilon \in L^2(0,T;V)$ [30] . Using the results derived in Chapter 2, the convergence of y_ε as $\varepsilon \to 0$ is studied. Then an asymptotic approximation of y_ε is constructed and an asymptotic error estimate is derived.

3.3.1. Convergence of y_ε as $\varepsilon \to 0$

As in section 2.2, let $\{\lambda^k_\varepsilon,\varphi^k_\varepsilon\}^\infty_{k=1}$ and $\{\mu^k_\varepsilon,\psi^k_\varepsilon\}^\infty_{k=1}$ be the exact eigenvalue-eigenvector pairs of A_ε with the eigenvectors normalized to one in H.

Let

$$y_\varepsilon = \sum_{k=1}^{\infty} c^k_\varepsilon(t)\psi^k_\varepsilon + \sum_{k=1}^{\infty} d^k_\varepsilon(t)\varphi^k_\varepsilon \tag{3.24}$$

$$f = \sum_{k=1}^{\infty} (f,\psi^k_\varepsilon)\psi^k_\varepsilon + \sum_{k=1}^{\infty} (f,\varphi^k_\varepsilon)\varphi^k_\varepsilon \tag{3.25}$$

$$h = \sum_{k=1}^{\infty} (h,\psi^k_\varepsilon)\psi^k_\varepsilon + \sum_{k=1}^{\infty} (h,\varphi^k_\varepsilon)\varphi^k_\varepsilon \quad . \tag{3.26}$$

Substitute (3.24)-(3.26) into (3.21)-(3.23) to find that $\{c^k_\varepsilon\}^\infty_{k=1}$, $\{d^k_\varepsilon\}^\infty_{k=1}$ satisfy the following ordinary differential equations:

$$\frac{dc_\varepsilon^k}{dt} + \mu_\varepsilon^k c_\varepsilon^k = (f,\psi_\varepsilon^k) \ , \ c_\varepsilon^k(0) = (h,\psi_\varepsilon^k) \tag{3.27}$$

$$\frac{dd_\varepsilon^k}{dt} + \lambda_\varepsilon^k d_\varepsilon^k = (f,\varphi_\varepsilon^k) \ , \ d_\varepsilon^k(0) = (h,\varphi_\varepsilon^k) \tag{3.28}$$

$k=1,2,\ldots$

whose solutions are

$$c_\varepsilon^k(t) = e^{-\mu_\varepsilon^k t}(h,\psi_\varepsilon^k) + \int_0^t e^{-\mu_\varepsilon^k(t-\tau)}(f,\psi_\varepsilon^k)\, d\tau \tag{3.29}$$

$$d^k(t) = e^{-\lambda_\varepsilon^k t}(h,\varphi_\varepsilon^k) + \int_0^t e^{-\lambda_\varepsilon^k(t-\tau)}(f,\varphi_\varepsilon^k)\, d\tau \ . \tag{3.30}$$

Using Theorem 2.1, one concludes with no difficulty that

$$y_\varepsilon \to y \text{ weakly in } L^2(0,T;H)$$

where

$$y = \sum_{k=1}^\infty c^k\psi^k + \sum_{k=1}^\infty d^k\varphi^k \tag{3.31}$$

$\{\varphi^k\}_{k=1}^\infty$, $\{\psi^k\}_{k=1}^\infty$ are the weak limits of $\{\varphi_\varepsilon^k\}_{k=1}^\infty$ (respectively, $\{\psi_\varepsilon^k\}_{k=1}^\infty$) in V, (respectively, in H) given by (2.22) (respectively, (2.23))

$$c^k(t) = e^{-\mu_0^k t}(h,\psi^k) + \int_0^t e^{-\mu_0^k(t-\tau)}(f,\psi^k)\, d\tau \tag{3.32}$$

$$d^k(t) = (h,\varphi^k) + \int_0^t (f,\varphi^k)\, d\tau \ . \tag{3.33}$$

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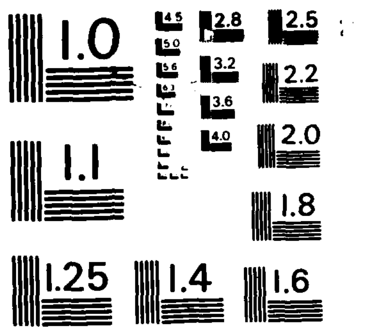

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The preceding discussion is summarized in

Theorem 3.2: Let y_ε denote the solution of (3.21)-(3.23). Given a sequence of ε converging to zero,

$$y_\varepsilon \to y \text{ weakly in } L^2(0,T;H) \tag{3.34}$$

where y is given by (3.31). Moreover,

$$\sqrt{\varepsilon}\, \| y_\varepsilon \|_{L^2(0,T;H)} \leq C \tag{3.35}$$

C is a constant independent of ε.

Proof: Use the eigenvalue-eigenvector pairs of the operator A_ε and their properties, as described by Theorem 2.1, to obtain (3.34). The estimate (3.35) is then readily derived by employing (2.21).

Remark 3.13: The convergence of y_ε in Theorem 3.2 cannot be improved (cf. Remark 2.7). Also, it should be emphasized that $\{\psi^k\}_{k=1}^\infty$ in (3.31) are not renormalized in H.

3.3.2. Asymptotic approximation of y_ε

It should be noted that the method by which stiff elliptic boundary value problems were solved in section 3.2 does not yield an iterative process for evolution problems in general. Therefore, one would be content to obtain a "weak" approximation of the solution of (3.21)-(3.23) using only the weak limits of the eigenvectors of A_ε.

In the sequel, only a more general version of Example 2.1 is considered, i.e., the operator A_ε is written as

$$A_\varepsilon = \begin{bmatrix} A_o & 0 \\ 0 & \varepsilon A_1 \end{bmatrix} , \quad 0 < \varepsilon << 1$$

where

$$A_k = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} a_{ij}^k(x) \frac{\partial}{\partial x_j} , \; k=0,1$$

$a_{ij}^k(x)$ satisfy the conditions of Remark 2.9. The chief reason for this digression, is that one can explicitly specify the regularity conditions of the functions involved in the construction of the approximation of y_ε. However, the concepts involved herein are equally applicable to the general case.

First, the following notation is adopted in the sequel.

$$Q_i = \Omega_i \times (0,T) \; , \; i=0,1$$

$$\Sigma_i = \Gamma_i \times (0,T) \; , \; i=0,1$$

$$R = S \times (0,T) \quad .$$

Now, let the zeroth order approximation be denoted by $y_\varepsilon^0 = (y_0^0, y_{\varepsilon 1}^0)$ and defined as follows. Let $\varepsilon \to 0$ formally in (3.21) and retain the following part

$$\frac{\partial y_0^0}{\partial t} + A_0 y_0^0 = f_0 \text{ on } Q_o \tag{3.36}$$

$$\left.\begin{aligned} & y_0^0 = 0 \text{ on } \Sigma_0 \\ & \frac{\partial y_0^0}{\partial \nu_{A_0}} = 0 \text{ on } R \end{aligned}\right\} \tag{3.37}$$

$$y_0^0(0) = h_0 \text{ on } \Omega_0 \quad . \tag{3.38}$$

The solution of y_0^0 of (3.36)-(3.38) is regular. Actually, $y_0^0 \in L^2(0,T;H^1(\Omega_0;\Gamma_0)) \subset L^2(Q_0)$. Hence $y_0^0\big|_R \in L^2(0,T;H^{1/2}(S)) \subset L^2(R)$.

Let the second part of y_ε^0 satisfy

$$\frac{\partial y_{\varepsilon 1}^0}{\partial t} + \varepsilon A_1 y_{\varepsilon 1}^0 = f_1 \text{ on } Q_1 \tag{3.39}$$

$$\left.\begin{aligned} &y_{\varepsilon 1}^0 = 0 \text{ on } \Sigma_1 \\ &y_{\varepsilon 1}^0 = y_0^0 \text{ on } R \end{aligned}\right\} \tag{3.40}$$

$$y_{\varepsilon 1}^0(0) = h_1 \text{ on } \Omega_0 \quad . \tag{3.41}$$

Problem (3.39)-(3.41) is a nonhomogeneous boundary value problem. Consequently, $y_{\varepsilon 1}^0$ has meaning in a weak sense using transposition [30] .

Since the zeroth order approximation y_ε^0 is weak, one rewrites (3.36)-(3.53) in the proper form, using transposition as follows. Let

$$\Phi_0 = \{\chi_0 : \chi_0 \in H^{2,1}(Q_0) \ , \ \chi_0\big|_{\Sigma_0} = 0, \ \frac{\partial \chi_0}{\partial \nu_{A_0}}\bigg|_R = 0 \ , \ \chi_0(T) = 0\}$$

$$\Phi_1 = \{\chi_1 : \chi_1 \in H^{2,1}(Q_1) \ , \ \chi_1\big|_{\Sigma_1} = 0, \ \chi_1\big|_R = 0 \ , \ \chi_1(T) = 0\} \quad .$$

Now consider the following isomorphisms

$$\chi_0 \xmapsto{\mathcal{L}_0} -\frac{\partial \chi_0}{\partial t} + A_0\chi_0 \text{ from } \Phi_0 \text{ to } L^2(Q_0) \tag{3.42}$$

$$\chi_1 \overset{\mathcal{L}_1}{\longmapsto} -\frac{\partial\chi_1}{\partial t} + \varepsilon A_1\chi_1 \text{ from } \Phi_1 \text{ to } L^2(Q_1) \ . \tag{3.43}$$

By transposition, one concludes that $\chi_0 \longmapsto M^0(\chi_0)$, being a continuous linear form on Φ_0 (endowed with the topology induced by $H^{2,1}(Q_0)$), there exists a unique $y_0^0 \in L^2(Q_0)$ such that

$$\int_{Q_0} y_0^0\left(-\frac{\partial\chi_0}{\partial t} + A_o\chi_o\right) d_{Q_0} = M^0(\chi_0) \ , \ \forall\chi_0\in\Phi_0 \tag{3.44}$$

and

$M^0 \longmapsto y_0^0$ is a continuous linear mapping of

$$\Theta^*_{\Gamma_0} \longmapsto L^2(Q_0) \text{ (Cf. Remark 3.14).} \tag{3.45}$$

Similarly,

$\chi_1 \longmapsto M^1_\varepsilon(\chi_1)$ being a continuous linear form on Φ_1 (endowed with the topology of $H^{2,1}(Q_1)$), there exists a unique $y^0_{\varepsilon 1} \in L^2(Q_1)$ such that

$$\int_{Q_1} y^0_{\varepsilon 1}\left(-\frac{\partial\chi_1}{\partial t} + \varepsilon A_1\chi_1\right) dQ_0 = M^1_\varepsilon(\chi_1) \ , \ \forall\chi_1\in\Phi_1 \tag{3.46}$$

and

$$M^1_\varepsilon \longmapsto y^0_{\varepsilon 1} \text{ is a continuous linear mapping of } \Phi_1^* \longmapsto L^2(Q_1). \tag{3.47}$$

<u>Remark 3.14</u>: Since $C_0^\infty(Q_0) \not\subset \Phi_0$, the dual of Φ_0 is not a space of distributions. Therefore, the introduction of Θ_{Γ_0} is required to interpret duality [30] .

Select $M^0(\chi_0)$ and $M^1(\chi_1)$ as

$$M^0(\chi_0) = \int_{Q_0} f_0\chi_0 dQ_0 + \int_{\Omega_1} h_0\chi_0(x,0)d\Omega_0 \tag{3.48}$$

$$M_\varepsilon^1(\chi_1) = \int_{Q_1} f_1\chi_1 dQ_0 + \int_{\Omega_1} h_1\chi_1(x,0)d\Omega_1 - \varepsilon \int_R y_0^0 \frac{\partial\chi_1}{\partial\nu_{A_1}} dR \tag{3.49}$$

where

$$f_i \in L^2(Q_i) \ , \ h_i \in L^2(Q_i) \ , \ i=0,1 \quad . \tag{3.50}$$

<u>Theorem 3.3</u>: Given f , h as in (3.50), M^0, M_ε^1 as in (3.48)-(3.49), there exists a unique $y_\varepsilon^0 = (y_0^0, y_{\varepsilon 1}^0)$ with $y_0^0 \in L^2(Q_0)$, $y_{\varepsilon 1}^0 \in L^2(Q_1)$ such that (3.44), (3.46) are satisfied.

<u>Proof</u>: It is easy to verify that the solution of (3.44), (3.46) is identical to that of (3.36)-(3.41). Let $\chi_0 \in C_0^\infty(Q_0)$ in (3.60) to obtain

$$M^0(\chi_o) = \int_{Q_0} f_0\chi_0 dQ_0$$

and hence

$$\int_{Q_0} y_0^0\left(-\frac{\partial\chi_0}{\partial t} + A_0\chi_0\right)dQ_0 = \int_{Q_0} f_0\chi_0 dQ_0 \ , \ \forall\chi_0 \in C_0^\infty(Q_0)$$

which yields (3.44). Now multiply (3.36) by $\chi_0 \in \Phi_0$ and apply Green's formula to get

$$\int_{Q_0} f_0\chi_0 dQ_0 = -\int_{\Omega_0} y_0^0\chi_0(x,0)d\Omega_0 + \int_R \frac{\partial y_0^0}{\partial\nu_{A_0}}\chi_0 \, dR + \int_{Q_0} y_0^0\left(-\frac{\partial\chi_0}{\partial t} + A_0\chi_0\right)dQ_0 \quad .$$

Hence by comparing it with (3.44), (3.48), one can obtain (3.37), (3.38).

Likewise, one can easily verify that the solutions of (3.39)-(3.41) and (3.46) are identical. Now an error estimate between the exact solution of (3.21)-(3.23) and y_ε^0 is derived.

<u>Theorem 3.4</u>: Let y_ε be the solution of (3.21)-(3.23) and y_ε^0 the solution of (3.44), (3.46). Then, for sufficiently small ε, one has

$$\| y_\varepsilon - y_\varepsilon^0 \|_{L^2(Q)} \leq C\ \varepsilon^{1/2} \tag{3.51}$$

where

$$Q = Q_0 \times Q_1$$

C is a positive constant, independent of ε.

<u>Proof</u>: First, rewrite (3.21)-(3.23) in the weak form, i.e.,

$$\int_{Q_0} y_{\varepsilon 0}\left(-\frac{\partial \chi_0}{\partial t} + A_0 \chi_0\right) dQ_0 = \int_{Q_0} f_0 \chi_0 dQ_0 + \int_{\Omega_0} h_0 \chi_0(x,0)\, d\Omega_0 - \int_R \frac{\partial y_{\varepsilon 0}}{\partial \nu_{A_0}} \chi_0\ dR,\ \forall \chi_0 \in \Phi_0 \tag{3.52}$$

$$\int_{Q_1} y_{\varepsilon 1}\left(-\frac{\partial \chi_1}{\partial t} + \varepsilon A_1 \chi_1\right) dQ_0 = \int_{Q_1} f_1 \chi_1 dQ_1 + \int_{\Omega_1} h_1 \chi_1(x,0)\, d\Omega_0 - \varepsilon \int_R y_{\varepsilon 1} \frac{\partial \chi_1}{\partial \nu_{A_1}}\ dR,\ \forall \chi_1 \in \Phi_1. \tag{3.53}$$

Subtract (3.44) from (3.52) and (3.46) from (3.53) to obtain

$$\int_{Q_0} (y_{\varepsilon 0} - y_0^0)\left(-\frac{\partial \chi_0}{\partial t} + A_0 \chi_0\right) dQ_0 = -\ \varepsilon \int \frac{\partial y_{\varepsilon 1}}{\partial \nu_{A_1}} \chi_0 dR,\ \forall \chi_0 \in \Phi_0 \tag{3.54}$$

$$\int_{Q_1} (y_{\varepsilon 1} - y_{\varepsilon 1}^0)\left(-\frac{\partial \chi_1}{\partial t} + \varepsilon A_1 \chi_1\right) dQ_1 = -\ \varepsilon \int_R (y_{\varepsilon 0} - y_0^0) \frac{\partial \chi_1}{\partial \nu_{A_1}}\ dR,\ \forall \chi_1 \in \Phi_1 \tag{3.55}$$

where the following interface condition is used

$$\left.\begin{aligned} y_{\varepsilon 0} &= y_{\varepsilon 1} \\ \frac{\partial y_{\varepsilon 0}}{\partial \nu_{A_0}} &= \varepsilon \frac{\partial y_{\varepsilon 1}}{\partial \nu_{A_1}} \end{aligned}\right\} \quad \text{on } R \quad .$$

Let

$$N_\varepsilon^0(\chi_0) = -\int_R \frac{\partial y_{\varepsilon 1}}{\partial \nu_{A_1}} \chi_0 \tag{3.56}$$

$$N_\varepsilon^1(\chi_1) = -\int_R (y_{\varepsilon 0} - y_0^0) \frac{\partial \chi_1}{\partial \nu_{A_1}} dR \quad . \tag{3.57}$$

Since $y_\varepsilon \to y$ weakly in $L^2(0,T;L^2(\Omega))(= L^2(Q))$

$$\|y_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq C$$

then

$$\left.\frac{\partial y_{\varepsilon 1}}{\partial \nu_{A_1}}\right|_R \in L^2(0,T;H^{-3/2}(S)) \tag{3.58}$$

since $y_0^0 \in L^2(0,T;H^1(\Omega_0;\Gamma_0)) \subset L^2(Q_0)$

then $\left.y_0^0\right|_R \in L^2(0,T;H^{1/2}(S)) \subset L^2(0,T;H^{-1/2}(S))$. (3.59)

From (3.58), (3.59), one deduces that $\chi_i \mapsto N_\varepsilon^i(\chi_i)$ is a continuous linear form on Φ_i , $i=0,1$ (3.60)

and

$N_\varepsilon^0 \mapsto y_{\varepsilon 0} - y_0^0$ is a continuous linear mapping of $\left.\Theta_{\Gamma_0}^*\right| \to L^2(Q_0)$ (3.61)

$$N_\varepsilon^1 \mid\!\rightarrow y_{\varepsilon 1} - y_{\varepsilon 1}^0 \text{ is a continuous linear mapping of } \Phi_1^* \mid\!\rightarrow L^2(Q_1) \tag{3.62}$$

Now consider the following equations:

$$\left.\begin{aligned} & -\frac{\partial \chi_{\varepsilon 0}}{\partial t} + A_0 \chi_{\varepsilon 0} = y_{\varepsilon 0} - y_0^0 \text{ on } Q_0 \\ & \chi_{\varepsilon 0} = 0 \text{ on } \Sigma_0 \\ & \frac{\partial \chi_{\varepsilon 0}}{\partial \nu_{A_0}} = 0 \text{ on } R \\ & \chi_{\varepsilon 0}(x,T) = 0 \end{aligned}\right\} \tag{3.63}$$

$$\left.\begin{aligned} & -\frac{\partial \chi_{\varepsilon 1}}{\partial t} + \varepsilon A_1 \chi_{\varepsilon 1} = y_{\varepsilon 1} - y_{\varepsilon 1}^0 \text{ on } Q_1 \\ & \chi_{\varepsilon 1} = 0 \text{ on } \Sigma_1 \\ & \chi_{\varepsilon 1} = 0 \text{ on } R \\ & \chi_{\varepsilon 1}(x,T) = 0 \end{aligned}\right\} \tag{3.64}$$

Since the coefficients of A_i , $i=0,1$ are assumed to be sufficiently smooth (Cf. Remark 2.9) and $y_{\varepsilon 0} - y_0^0 \in L^2(0,T;L^2(\Omega_0))$, $y_{\varepsilon 1} - y_{\varepsilon 1}^0 \in L^2(0,T;L^2(\Omega_1))$, one concludes that

$$\chi_{\varepsilon i} \in \Phi_i \quad , \quad i=0,1$$

Let $\chi_0 = \chi_{\varepsilon 0}$ in (3.54) and $\chi_1 = \chi_{\varepsilon 1}$ in (3.55) and use (3.63), (3.64) to obtain

$$\| y_{\varepsilon 0} - y_0^0 \|_{L^2(Q_0)} \leq C \; \varepsilon^{1/2}$$

$$\|y_{\varepsilon 1} - y^0_{\varepsilon 1}\|_{L^2(Q_1)} \leq C_2\, \varepsilon^{1/2}$$

where C_1, C_2 are some positive constants, independent of ε. Hence,

$$\|y_\varepsilon - y^0_\varepsilon\|_{L^2(Q)} = \|y_{\varepsilon 0} - y^0_0\|_{L^2(Q_0)} + \|y_{\varepsilon 1} - y^0_{\varepsilon 1}\|_{L^2(Q_1)} \leq C\, \varepsilon^{1/2}.$$

Remark 3.15: One ought to mention that (3.51)holds for ε small, but strictly positive, because of the heavy reliance upon the fact that the solution of (3.64) belongs to Φ_1. It is easy to see that if one sets formally $\varepsilon = 0$ in the first equation of (3.64), its right-hand side would be in $L^2(0,T;\, L^2(\Omega_1))$ but $\chi_{\varepsilon 1}$ for $\varepsilon = 0$ would not be in Φ_1.

Recall, that it was shown in Section 2.3 that the eigenvalue-eigenvector pairs of A_ε, i.e., $\{\gamma^k_\varepsilon, \chi^k_\varepsilon\}^\infty_{k=1}$ are decomposable into two groups $\{\lambda^k_\varepsilon, \varphi^k_\varepsilon\}^\infty_{k=1}$ and $\{\mu^k_\varepsilon, \psi^k_\varepsilon\}^\infty_{k=1}$.

The normalized weak limits (in $H^1_0(\Omega)$) of $\{\varphi^k_\varepsilon\}^\infty_{k=1}$ satisfy

$$\left.\begin{array}{l} \left.\begin{array}{l} \varphi^k_0 = 0 \text{ on } \Omega_0 \\ A_1\varphi^k_1 = \lambda^k_1\varphi^k_1 \text{ on } \Omega_1 \end{array}\right\} \\ \varphi^k_1\Big|_{\Gamma_1} = 0,\ \varphi^k_1\Big|_S = 0 \end{array}\right\} \qquad (3.65)$$

$k=1,2,\ldots$

The normalized weak limits (in $L^2(\Omega)$) of $\{\psi^k_\varepsilon\}^\infty_{k=1}$ obey

$$\left.\begin{array}{l} \left.\begin{array}{l} A_0\varphi_0^k = \mu_0^k\varphi_0^k \text{ on } \Omega_0 \\ \varphi_0^k = 0 \text{ on } \Omega_1 \end{array}\right\} \\ \varphi_0^k\Big|_{\Gamma_0} = 0, \quad \dfrac{\partial\varphi_0^k}{\partial\nu_{A_0}}\Big|_S = 0 \end{array}\right\} \tag{3.66}$$

$k=1,2,\ldots$

Then the solution of (3.44), (3.46) may be represented uniquely by

$$\left.\begin{array}{l} y_0^0 = \sum_{k=1}^{\infty} c^k(t)\varphi_0^k \\ c^k \in L^2(0,T), \quad \sum_{k=1}^{\infty}\int_0^T |c^k(t)|^2 dt < \infty \end{array}\right\} \tag{3.67}$$

$$\left.\begin{array}{l} y_{\varepsilon 1}^0 = \sum_{k=1}^{\infty} d_\varepsilon^k(t)\varphi_1^k \\ d_\varepsilon^k \in L^2(0,T), \quad \sum_{k=1}^{\infty}\int_0^T |d_\varepsilon^k(t)|^2 dt < \infty \end{array}\right\} \tag{3.68}$$

In order to determine $c^k(t)$, $d_\varepsilon^k(t)$, let

$$\chi_0(x,t) = \theta(t)\varphi_0^k(x), \ \theta \in C^1([0,T]), \ \theta(T) = 0 \ (\text{ so that } \chi_0 \in \Phi_0)$$

$$\chi_1(x,t) = \nu(t)\varphi_1^k(x), \ \nu \in C^1([0,T]), \ \nu(T) = 0 \ (\text{so that } \chi_1 \in \Phi_1)$$

in (3.44), (3.46) to get

$$\int_0^T c^k\left(-\frac{d\theta}{dt} + \mu_0^k\theta\right)dt = \int_0^T (f_0,\varphi_0^k)\theta dt - (h_0,\varphi_0^k)\theta(0) \tag{3.69}$$

$$\int_0^T d_\varepsilon^k\left(-\frac{dv}{dt} + \lambda_1^k \varepsilon v\right)dt = \int_0^T (f_1,\varphi_1^k)v\,dt - (h_1,\varphi_1^k)v(0) - \varepsilon \int_0^T \left(\sum_{\ell=1}^{\infty} c^\ell \int_S \psi^\ell \frac{\partial \varphi_1^k}{\partial \nu_{A_1}} dS\right) v\,dt, \quad (3.70)$$

which are equivalent to

$$\left.\begin{aligned} &\frac{dc^k}{dt} + \mu_0^k c^k = (f_0,\psi_0^k) \\ &c^k(0) = (h_0,\psi_0^k) \end{aligned}\right\} \quad (3.71)$$

$$\left.\begin{aligned} &\frac{d\,d_\varepsilon^k}{dt} + \lambda_1^k \varepsilon d_\varepsilon^k = (f_1,\varphi_1^k) - \varepsilon \sum_{\ell=1}^{\infty} c^\ell \int_S \psi_0^\ell \frac{\partial \varphi_1^k}{\partial \nu_{A_1}} dS \\ &d_\varepsilon^k(0) = (h_1,\varphi_1^k) \quad . \end{aligned}\right\} \quad (3.72)$$

Remark 3.16: Using the terminology of singular perturbation, one concludes that (3.67) represents the "fast" subsystem, while (3.68) represents the "slow" subsystem driven by the fast one.

Remark 3.17: Note that $y_{\varepsilon 1}^0\big|_S = y_0^0\big|_S$ has meaning in an "average" sense. Hence, there is no contradiction in representing $y_{\varepsilon 1}^0$ on S with the aid of φ_1^k, which are null on S [23].

Remark 3.18: If $f = (0,f_1)$ and $h = (0,h_1)$ then $y_0^0 \equiv 0$ and hence $y_\varepsilon^0 \in L^2(0,T;H_0^1(\Omega))$. In conjunction with this remark, one adds that if Ω_0 is the empty set, then problem (3.21)-(3.23) degenerates to a problem analyzed in [24].

Remark 3.19: At this point, it may not be clear how better convergence of $y_\varepsilon - y_\varepsilon^0$ (as evidenced by the estimate (3.51)) is achieved. Due to the weak convergence of $\{\psi_\varepsilon^k\}_{k=1}^\infty$ in H, by taking the limit as $\varepsilon \to 0$ in (3.21)-(3.23), something of O(1) (in $L^2(0,T;H)$) is lost from y_ε. However, by renormalizing the weak limits of $\{\psi_\varepsilon^k\}_{k=1}^\infty$, i.e., $\{\psi^k\}_{k=1}^\infty$, this deficiency is corrected. Intuitively speaking, this action amounts to saying: since $\psi_{\varepsilon 1}^k$ converges to zero weakly in $L^2(\Omega_1)$, keep all of the energy associated with ψ_ε^k in region Ω_0. It is noteworthy to report that this phenomenon does not occur with $\{\varphi_\varepsilon^k\}_{k=1}^\infty$, because they converge weakly in V (hence, strongly in H). Therefore, their weak limits are automatically normalized. However, $\lambda_\varepsilon^k \to 0$ linearly as $\varepsilon \to 0$. Hence, λ_ε^k must be replaced by its asymptotic equivalent, i.e., $\lambda_1^k \varepsilon$, and not its limit, to get a better estimate. At present, consider the following example:

<u>Example 3.6</u>: (Cf. Example 3.4). Let $f_0 = 1$, $f_1 = 0$, $h = 0$. Then (3.21)-(3.23) become

$$\left.\begin{array}{l} \left.\begin{array}{l} \dfrac{\partial y_{\varepsilon 0}}{\partial t} - \dfrac{\partial^2 y_{\varepsilon 0}}{\partial x^2} = 1 \text{ on } (-1,0)\times(0,T) \\[2ex] \dfrac{\partial y_{\varepsilon 1}}{\partial t} - \varepsilon \dfrac{\partial^2 y_{\varepsilon 1}}{\partial x^2} = 0 \text{ on } (0,1)\times(0,T) \end{array}\right\} \\[2ex] y_{\varepsilon 0}(-1,t) = y_{\varepsilon 1}(1,t) = 0 \\[1ex] \left.\begin{array}{l} y_{\varepsilon 0}(0,t) = y_{\varepsilon 1}(0,t) \\[1ex] \dfrac{\partial y_{\varepsilon 0}}{\partial x}(0,t) = \varepsilon \dfrac{\partial y_{\varepsilon 1}}{\partial x}(0,t) \end{array}\right\} \\[2ex] y_{\varepsilon 0}(x,0) = y_{\varepsilon 1}(x,0) = 0 \quad . \end{array}\right\} \tag{3.73}$$

Recall that

$$\left.\begin{aligned} \mu_0^k &= ((2k-1)\,\tfrac{\pi}{2})^2 \\ \psi_0^k &= \sqrt{2}\,\cos\sqrt{\mu_0^k}\,x \end{aligned}\right\}$$

$$\left.\begin{aligned} \lambda_1^k &= (k\pi)^2 \\ \varphi_1^k &= \sqrt{2}\,\sin\sqrt{\lambda_1^k}\,x \quad . \end{aligned}\right\}$$

The solutions to (3.71)-(3.72) are easily computed and are given by

$$c^k(t) = \frac{\sqrt{2}\,(-1)^{k-1}}{(\mu_0^k)^{3/2}}\,(1 - e^{-\mu_0^k t})$$

$$d_\varepsilon^k(t) = \sum_{\ell=1}^{\infty} \frac{2\sqrt{2}(-1)^{\ell-1}}{(\mu_0^\ell)^{3/2}}\,\frac{1}{(\lambda_1^k)^{1/2}}\,\Big(1 - \frac{\mu_0^\ell}{\mu_0^\ell-\lambda_1^k\varepsilon}\,e^{-\lambda_1^k\varepsilon t}\Big)$$
$$+\ \varepsilon \sum_{\ell=1}^{\infty} \frac{2\sqrt{2}(-1)^{\ell-1}}{(\mu_0^\ell)^{3/2}}\,\frac{(\lambda_1^k)^{1/2}}{\mu_0^\ell-\lambda_1^k\varepsilon}\,e^{-\mu_0^\ell t} \quad .$$

Consequently, y_ε^0 is written as

$$y_0^0 = \sum_{k=1}^{\infty} \frac{2(-1)^{k-1}}{(\mu_0^k)^{3/2}}\,(1 - e^{-\mu_0^k t})\,\cos\sqrt{\mu_0^k}\,x \qquad (3.74)$$

$$y_{\varepsilon 1}^0 = \sum_{k=1}^{\infty}\left[\sum_{\ell=1}^{\infty} \frac{4(-1)^{\ell-1}}{(\mu_0^\ell)^{3/2}}\,\frac{1}{(\lambda_1^k)^{1/2}}\,\Big(1 - \frac{\mu_0^\ell}{\mu_0^\ell-\lambda_1^k\varepsilon}\,e^{-\lambda_1^k\varepsilon t}\Big)\right]\sin\sqrt{\lambda_1^k}\,x$$
$$+ \sum_{k=1}^{\infty}\left[\varepsilon\sum_{\ell=1}^{\infty} \frac{4(-1)^{\ell-1}}{(\mu_0^\ell)^{3/2}}\,\frac{(\lambda_1^k)^{1/2}}{\mu_0^\ell-\lambda_1^k\varepsilon}\,e^{-\mu_0^\ell t}\right]\sin\sqrt{\lambda_1^k}\,x \quad . \qquad (3.75)$$

Remark 3.20: Observe that y_0^0 contains exponential functions, decaying with rates μ_0^k. Therefore, y_0^0 represents the fast subsystem. In contrast, $y_{\varepsilon 1}^0$ contains one slow component and a fast one.

Remark 3.21: Note that $y_{\varepsilon 1}^0 \to 0$ in $L^2(Q_1)$ in this case, which makes sense. See numerical analysis in Chapter 5. □

One may proceed further along the same lines, to derive higher order approximations of y_ε in $L^2(Q)$. However, it would be difficult to justify a better estimate than (3.51). For example, in order to ameliorate the counterpart of (3.54), one has to show that

$$\left\| \frac{\partial y_{\varepsilon 1}}{\partial \nu_{A_1}} - \frac{\partial y_{\varepsilon 1}^0}{\partial \nu_{A_1}} \right\|_{L^2(0,T;\, H^{-3/2}(S))} \to 0 \quad \text{as } \varepsilon \to 0$$

which is, by no means, trivial.

Consequently, no attempt will be made to pursue this any further.

3.4. Hyperbolic Boundary Value Problems

In this section, an evolution problem of hyperbolic type is investigated, namely, the following boundary value problem:

$$(y_\varepsilon'', \chi) + a_0(y_\varepsilon, \chi) + \varepsilon\, a_1(y_\varepsilon, \chi) = (f, \chi), \ \forall \chi \in V \tag{3.76}$$

$$y_\varepsilon(0) = h, \ h \text{ given in } V \tag{3.77}$$

$$y_\varepsilon'(0) = g, \ g \text{ given in } H \tag{3.78}$$

$$y_\varepsilon \in L^2(0,T;V),\ y_\varepsilon' \in L^2(0,T;H),\ f \in L^2(0,T;H) \quad . \tag{3.79}$$

The present analysis would be parallel to that of Section 3.3. Hence, the convergence of y_ε as $\varepsilon \to 0$ is studied. Then a zeroth order approximation of y_ε is constructed, using the weak limits of the eigenvectors of A_ε (i.e., the operator associated with the bilinear form $a_\varepsilon(\varphi,\psi) = a_0(\varphi,\psi) + \varepsilon\, a_1(\varphi,\psi)$ (Cf. Chapter 2). It is well-known that hyperbolic boundary value problems are more complex than their parabolic counterparts. Moreover, not all eigenvectors of A_ε converge in V, as proved in Section 2.3. Therefore, (3.76) has to be specialized to second order operators A_ε, so that one can specify exactly what is needed for the present analysis to hold true.

In the sequel, the following problem is considered

$$\left.\begin{aligned} &\frac{\partial^2 y_{\varepsilon 0}}{\partial t^2} + A_0\, y_{\varepsilon 0} = f_0 \text{ on } Q_0 \\ &\frac{\partial^2 y_{\varepsilon 1}}{\partial t^2} + \varepsilon\, A_1\, y_{\varepsilon 1} = f_1 \text{ on } Q_1 \end{aligned}\right\} \tag{3.80}$$

$$y_{\varepsilon 0} = 0 \text{ on } \Sigma_0,\; y_{\varepsilon 1} = 0 \text{ on } \Sigma_1 \tag{3.81}$$

$$\left.\begin{aligned} &y_{\varepsilon 0} = y_{\varepsilon 1} \\ &\frac{\partial y_{\varepsilon 0}}{\partial \nu_{A_0}} = \varepsilon\, \frac{\partial y_{\varepsilon 1}}{\partial \nu_{A_1}} \end{aligned}\right\} \text{ on } R \tag{3.82}$$

$$y_\varepsilon(0) = h(=(h_0,h_1)),\; h_i \in H_0^1(\Omega_i),\; i=0,1 \tag{3.83}$$

$$\frac{\partial y_\varepsilon}{\partial t}(0) = g \text{ on } \Omega,\; g \in L^2(\Omega) \tag{3.84}$$

$$y_\varepsilon \in L^2(0,T;H_0^1(\Omega)),\; \frac{\partial y_\varepsilon}{\partial t} \in L^2(0,T;L^2(\Omega)) \tag{3.85}$$

$$f \in L^2(0,T;L^2(\Omega)) \tag{3.86}$$

where the operator A_ε is as in Section 3.3.

3.4.1. Convergence of y_ε as $\varepsilon \to 0$

This subsection commences with the statement of the convergence theorem, and then a constructive proof of it follows as in Section 3.3.

Theorem 3.5: Let y_ε be the solution of (3.80)-(3.86). Then, given a sequence of ε converging to zero,

$$y_\varepsilon \to y \text{ weakly in } L^2(0,T;L^2(\Omega)) \tag{3.87}$$

$$\frac{\partial y_\varepsilon}{\partial t} \to \frac{\partial y}{\partial t} \text{ weakly in } L^2(0,T;L^2(\Omega)) \quad . \tag{3.88}$$

Moreover,

$$\sqrt{\varepsilon}\, \|y_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \leq C \tag{3.89}$$

C is a constant independent of ε.

Proof: Let $\{\lambda_\varepsilon^k,\varphi_\varepsilon^k\}_{k=1}^\infty$ and $\{\mu_\varepsilon^k,\psi_\varepsilon^k\}_{k=1}^\infty$ be the exact eigenvalue-eigenvector pairs of A_ε, with the eigenvectors normalized (in $L^2(\Omega)$) for fixed ε. Using a finite dimensional approximation of y_ε such as

$$y_\varepsilon^m = \sum_{k=1}^m c_\varepsilon^k \psi_\varepsilon^k + \sum_{k=1}^m d_\varepsilon^k \varphi_\varepsilon^k \tag{3.90}$$

it is shown in [23] that

$$y_\varepsilon^m \to y_\varepsilon \text{ strongly in } L^2(0,T;H_0^1(\Omega)) \text{ as } m \to +\infty \tag{3.91}$$

$$\frac{\partial y_\varepsilon^m}{\partial t} \to \frac{\partial y_\varepsilon}{\partial t} \text{ strongly in } L^2(0,T;L^2(\Omega)) \text{ as } m \to +\ \infty \tag{3.92}$$

where

$$y_\varepsilon = \sum_{k=1}^{\infty} c_\varepsilon^k \psi_\varepsilon^k + \sum_{k=1}^{\infty} d_\varepsilon^k \varphi_\varepsilon^k \tag{3.93}$$

$\{c_\varepsilon^k\}_{k=1}^\infty$, $\{d_\varepsilon^k\}_{k=1}^\infty$ satisfy the following set of ordinary differential equations

$$\left.\begin{aligned} &\frac{d^2 c_\varepsilon^k}{dt^2} + \mu_\varepsilon^k c_\varepsilon^k = (f,\psi_\varepsilon^k) \\ &c_\varepsilon^k(0) = (h,\psi_\varepsilon^k) \\ &\frac{dc_\varepsilon^k}{dt}(0) = (g,\psi_\varepsilon^k) \end{aligned}\right\} \tag{3.94}$$

$$\left.\begin{aligned} &\frac{d^2 d_\varepsilon^k}{dt^2} + \lambda_\varepsilon^k d_\varepsilon^k = (f,\varphi_\varepsilon^k) \\ &d_\varepsilon^k(0) = (h,\varphi_\varepsilon^k) \\ &\frac{d\, d_\varepsilon^k}{dt}(0) = (g,\varphi_\varepsilon^k) \end{aligned}\right\} \tag{3.95}$$

$k=1,2,\ldots$

whose solutions are given respectively by

$$c_\varepsilon^k(t) = (h,\psi_\varepsilon^k)\cos\sqrt{\mu_\varepsilon^k}\,t + \frac{(g,\psi_\varepsilon^k)}{\sqrt{\mu_\varepsilon^k}}\sin\sqrt{\mu_\varepsilon^k}\,t + \int_0^T \frac{\sin\sqrt{\mu_\varepsilon^k}(t-\tau)}{\sqrt{\mu_\varepsilon^k}}(f,\psi_\varepsilon^k)\,d\tau \tag{3.96}$$

$$d_\varepsilon^k(t) = (h,\varphi_\varepsilon^k)\cos\sqrt{\lambda_\varepsilon^k}\,t + \frac{(g,\varphi_\varepsilon^k)}{\sqrt{\lambda_\varepsilon^k}}\sin\sqrt{\lambda_\varepsilon^k}\,t + \int_0^T \frac{\sin\sqrt{\lambda_\varepsilon^k}(t-\tau)}{\sqrt{\lambda_\varepsilon^k}}(f,\varphi_\varepsilon^k)\,d\tau\ . \tag{3.97}$$

<u>Remark 3.22</u>: Since $f \in L^2(0,T;L^2(\Omega))$, $h = (h_0,h_1) \in H_0^1(\Omega_0) \times H_0^1(\Omega_1)$ and $g \in L^2(\Omega)$, the following inequalities hold

$$\sum_{k=1}^{\infty} \int_0^T ((f,\psi_\varepsilon^k)^2 + (f,\varphi_\varepsilon^k)^2)dt < \infty \tag{3.98}$$

$$\sum_{k=1}^{\infty} (h,\psi_\varepsilon^k)^2 \mu_\varepsilon^k < \infty , \sum_{k=1}^{\infty} (h,\varphi_\varepsilon^k)^2 \frac{\lambda_\varepsilon^k}{\varepsilon} < \infty \tag{3.99}$$

$$\sum_{k=1}^{\infty} ((g,\psi_\varepsilon^k)^2 + (g,\varphi_\varepsilon^k)^2) < \infty \quad . \tag{3.100}$$

Now, let ε be a sequence converging to zero. It was shown in Theorem 2.1 that

$$\text{1)} \quad \mu_\varepsilon^k \to \mu_0^k > 0, \ \psi_\varepsilon^k \to \psi^k \text{ weakly in } L^2(\Omega) \tag{3.101}$$

$$\text{2)} \quad \lambda_\varepsilon^k \to 0, \text{ linearly in } \varepsilon, \ \varphi_\varepsilon^k \to \varphi^k \text{ strongly in } H_0^1(\Omega) \tag{3.102}$$

from which (3.87) is deduced.

Differentiate (3.93) with respect to time and take the limit as $\varepsilon \to 0$ using (3.101)-(3.102) to get (3.99). This limit is well-defined because of (3.98)-(3.100), which hold as $\varepsilon \to 0$.

Consequently, y can be written as

$$y = \sum_{k=1}^{\infty} c^k \psi^k + \sum_{k=1}^{\infty} d^k \varphi^k \tag{3.103}$$

where $\{c^k\}_{k=1}^{\infty}$, $\{d^k\}_{k=1}^{\infty}$ satisfy (3.96)-(3.97) after letting $\varepsilon \to 0$, i.e,

$$c^k(t) = (h,\psi^k) \cos \sqrt{\mu_0^k} t + \frac{(g,\psi^k)}{\sqrt{\mu_0^k}} \sin \sqrt{\mu_0^k} t + \int_0^t \frac{\sin \sqrt{\mu_0^k}(t-\tau)}{\sqrt{\mu_0^k}} (f,\psi^k)d\tau \tag{3.104}$$

$$d^k(t) = (h,\varphi^k) + (g,\varphi^k)t + \int_0^t (t-\tau)(f,\varphi^k)d\tau \quad . \tag{3.105}$$

Now using (3.93) and Theorem 2.1, one obtains (3.89).

Remark 3.23: Note that $\{\psi^k\}_{k=1}^{\infty}$ in (3.103) are not renormalized. □

3.4.2. Asymptotic approximation of y_ε

In the sequel, the zeroth order approximation y_ε^0 of y_ε is constructed using the same approach as in Section 3.3. First, renormalize $\{\psi^k\}_{k=1}^{\infty}$ so that what is lost by taking the weak limit as $\varepsilon \to 0$ in (3.91) is regained. Then an error estimate is derived. An outline on how to solve for y_ε^0 using the weak limits of the eigenvectors of A_ε is also given. At the end of this subsection, an example is solved in detail.

Let $y_\varepsilon^0 = (y_0^0, y_{\varepsilon 1}^0)$ be defined by

$$\left.\begin{aligned}
&\frac{\partial^2 y_0^0}{\partial t^2} + A_0\, y_0^0 = f_0 \text{ on } Q_0 \\
&y_0^0 = 0 \text{ on } \Sigma_0 \\
&\frac{\partial y_0^0}{\partial \nu_{A_0}} = 0 \text{ on } R \\
&y_0^0(0) = h_0 \text{ on } \Omega_0 \\
&\frac{\partial y_0^0}{\partial t}(0) = g_0 \text{ on } \Omega_0
\end{aligned}\right\} \tag{3.106}$$

$$\left.\begin{aligned}
&\frac{\partial^2 y^0_{\varepsilon 1}}{\partial t^2} + \varepsilon A_1 y^0_{\varepsilon 1} = f_1 \text{ on } Q_1 \\
&y^0_{\varepsilon 1} = 0 \text{ on } \Sigma_1 \\
&y^0_{\varepsilon 1} = y^0_0 \text{ on } R \\
&y^0_{\varepsilon 1}(0) = h_1 \text{ on } \Omega_0 \\
&\frac{\partial y^0_\varepsilon}{\partial t}(0) = g_1 \text{ in } \Omega_1 \quad .
\end{aligned}\right\} \tag{3.107}$$

Problem (3.107) is a nonhomogeneous boundary value problem. Using transposition, $y^0_{\varepsilon 1}$ is defined in a weak sense as in Section 3.3.

In order to derive an asymptotic error estimate between y_ε and y^0_ε, they need to be redefined using transposition. For this purpose, let

$$\Phi_0 = \{\chi_0 : \chi_0 \in L^2(0,T;H^1(\Omega_0)),\ \frac{\partial \chi_0}{\partial t} \in L^2(Q_0),\ \frac{\partial^2 \chi_0}{\partial t^2} + A_0\chi_0 \in L^2(Q_0),\ \chi_0 = 0 \text{ on } \Sigma_0,$$

$$\frac{\partial \chi_0}{\partial \nu_{A_0}} = 0 \text{ on } R,\quad \chi_0(x,T) = 0,\ \frac{\partial \chi_0}{\partial t}(x,T) = 0\}^2 \tag{3.108}$$

$$\Phi_1 = \{\chi_1 : \chi_1 \in L^2(0,T;H^1(\Omega_1)),\ \frac{\partial \chi_1}{\partial t} \in L^2(Q_1),\ \frac{\partial^2 \chi_1}{\partial t^2} + \varepsilon A_1\chi_1 \in L^2(Q_1), \chi_1 = 0 \text{ on } \Sigma_1 \cup R,$$

$$\chi_1(x,T) = 0,\ \frac{\partial \chi_1}{\partial t}(x,T) = 0\}^2 \tag{3.109}$$

It can be easily verified that y_ε, y^0_ε satisfy

---

[2] Φ_i is endowed with the topology carried over by the mapping $\varphi_i \in L^2(0,T;L^2(\Omega_i)) \mapsto \chi_i$, i=0,1.

$$\int_{Q_0} y_{\varepsilon 0}\left(\frac{\partial^2 \chi_0}{\partial t^2} + A_0\chi_0\right)dQ_0 = \int_{Q_0} f_0\chi_0 dQ_0 - \int_{\Omega_0} h_0 \frac{\partial \chi_0}{\partial t}(x,0)d\Omega_0$$

$$+ \int_{\Omega_0} g_0\chi_0(x,0)d\Omega_0 + \varepsilon \int_R \frac{\partial y_{\varepsilon 1}}{\partial \nu_{A_1}}\chi_0 \, dR, \ \forall \chi_0 \in \Phi_0 \tag{3.110}$$

$$\int_{Q_1} y_{\varepsilon 1}\left(\frac{\partial^2 \chi_1}{\partial t^2} + \varepsilon A_1\chi_1\right)dQ_1 = \int_{Q_1} f_1\Omega_1 dQ_1 - \int_{\Omega_1} h_1 \frac{\partial \chi_1}{\partial t}(x,0)d\Omega_1$$

$$+ \int_{\Omega_1} g_1\chi_1(x,0)d\Omega_1 - \varepsilon \int_R y_{\varepsilon 0} \frac{\partial \chi_1}{\partial \nu_{A_1}} dR, \ \forall \chi_1 \in \Phi_1 \tag{3.111}$$

$$\int_{Q_0} y_0^0\left(\frac{\partial^2 \chi_0}{\partial t^2} + A_0\chi_0\right)dQ_0 = \int_{Q_0} f_0\chi_0 dQ_0 - \int_{\Omega_0} h_0 \frac{\partial \chi_0}{\partial t}(x,0)d\Omega_0$$

$$+ \int_{\Omega_0} g_0\chi_0(x,0)d\Omega_0, \ \forall \chi_0 \in \Phi_0 \tag{3.112}$$

$$\int_{Q_1} y_{\varepsilon 1}^0\left(\frac{\partial^2 \chi_1}{\partial t^2} + \varepsilon A_1\chi_1\right)dQ_1 = \int_{Q_1} f_1\chi_1 dQ_1 - \int_{\Omega_1} h_1 \frac{\partial \chi_1}{\partial t}(x,0)d\Omega_1$$

$$+ \int_{\Omega_1} g_1\chi_1(x,0)d\Omega_1 - \varepsilon \int_R y_0^0 \frac{\partial \chi_1}{\partial \nu_{A_1}} dR, \ \forall \chi_1 \in \Phi_1 \quad . \tag{3.113}$$

<u>Theorem 3.6</u>: Let y_ε, y_ε^0 be the solutions of (3.110)-(3.113). Then for sufficiently small ε the following estimate holds

$$\| y_\varepsilon - y_\varepsilon^0 \|_{L^2(Q)} \leq C\, \varepsilon^{1/4} \tag{3.114}$$

Proof: Subtract (3.112) from (3.110) and (3.113) from (3.111) to get:

$$\int_{Q_0} (y_{\varepsilon 0} - y_0^0)\left(\frac{\partial^2 \chi_0}{\partial t^2} + A_0\chi_0\right)dQ_0 = -\varepsilon \int_R \frac{\partial y_{\varepsilon 1}}{\partial \nu_{A_1}} \chi_0 \, dR \tag{3.115}$$

$$\int_{Q_1} (y_{\varepsilon 1} - y_{\varepsilon 1}^0)\left(\frac{\partial^2 \chi_1}{\partial t^2} + \varepsilon A_1\chi_1\right)dQ_1 = -\varepsilon \int_R (y_{\varepsilon 0} - y_0^0)\frac{\partial \chi_1}{\partial \nu_{A_1}} dR \quad . \tag{3.116}$$

Now consider the following equations

$$\left.\begin{aligned}
&\frac{\partial^2 \chi_{\varepsilon 0}}{\partial t^2} + A_0\chi_{\varepsilon 0} = y_{\varepsilon 0} - y_0^0 \text{ on } Q_0 \\
&\chi_{\varepsilon 0} = 0 \text{ on } \Sigma_0 \\
&\frac{\partial \chi_{\varepsilon 0}}{\partial \nu_{A_0}} = 0 \text{ on } R \\
&\chi_{\varepsilon 0}(x,T) = 0 \text{ on } \Omega_0 \\
&\frac{\partial \chi_{\varepsilon 0}}{\partial t}(x,T) = 0 \text{ on } \Omega_0
\end{aligned}\right\} \tag{3.117}$$

$$\left.\begin{aligned}
&\frac{\partial^2 \chi_{\varepsilon 1}}{\partial t^2} + \varepsilon A_1\chi_{\varepsilon 1} = y_{\varepsilon 1} - y_{\varepsilon 1}^0 \text{ on } Q_1 \\
&\chi_{\varepsilon 1} = 0 \text{ on } \Sigma_1 \\
&\chi_{\varepsilon 1} = 0 \text{ on } R \\
&\chi_{\varepsilon 1}(x,T) = 0 \text{ on } \Omega_1 \\
&\frac{\partial \chi_{\varepsilon 1}}{\partial t}(x,T) = 0 \text{ on } \Omega_1 \quad .
\end{aligned}\right\} \tag{3.118}$$

It can be shown [30] that

$$\chi_{\varepsilon i} \in \Phi_i \ , \ i=0,1 \quad . \tag{3.119}$$

Consequently,

$$\chi_{\varepsilon 0}\Big|_R \in L^2(0,T;H^{1/2}(S)) \tag{3.120}$$

$$\frac{\partial \chi_{\varepsilon 1}}{\partial \nu_{A_1}}\Big|_R \in L^2(0,T;H^{-1/2}(S)) \quad . \tag{3.121}$$

From (3.89), one concludes that

$$\sqrt{\varepsilon}\ \| y_{\varepsilon 0}\Big|_R \|_{L^2(0,T;H^{1/2}(S))} \leq C_1 \tag{3.122}$$

$$\sqrt{\varepsilon}\ \| \frac{\partial y_{\varepsilon 1}}{\partial \nu_{A_1}}\Big|_R \|_{L^2(0,T;H^{-1/2}(S))} \leq C_2 \quad . \tag{3.123}$$

Let $\chi_0 = \chi_{\varepsilon 0}$ in (3.115) and $\chi_1 = \chi_{\varepsilon 1}$ in (3.116) and use (3.117)-(3.118), (3.122)-(3.123), to get

$$\| y_{\varepsilon 0} - y_0^0 \|_{L^2(Q_0)} \leq C_3\ \varepsilon^{1/4} \tag{3.124}$$

$$\| y_{\varepsilon 1} - y_{\varepsilon 1}^0 \|_{L^2(Q_1)} \leq C_4\ \varepsilon^{1/4} \tag{3.125}$$

and hence one obtains (3.114) using (3.124)-(3.125). □

Now the weak limits of the eigenvectors of A_ε, i.e., (3.65)-(3.66) are employed to solve (3.112)-(3.113). First, renormalize $\{\psi^k\}_{k=1}^{\infty}$. Then, the solution of (3.112)-(3.113) may be represented by:

$$\left.\begin{aligned} y_0^0 &= \sum_{k=1}^{\infty} c^k \psi_0^k \\ c^k &\in L^2(0,T), \quad \sum_{k=1}^{\infty} \int_0^T |c^k(t)|^2 dt < \infty \end{aligned}\right\} \tag{3.126}$$

$$\left.\begin{aligned} y_{\varepsilon 1}^0 &= \sum_{k=1}^{\infty} d_\varepsilon^k \varphi_1^k \\ d_\varepsilon^k &\in L^2(0,T), \quad \sum_{k=1}^{\infty} \int_0^T |d_\varepsilon^k(t)|^2 dt < \infty \quad . \end{aligned}\right\} \tag{3.127}$$

In order to obtain $c^k(t)$, $d_\varepsilon^k(t)$, let

$$\chi_0(x,t) = \theta(t)\psi_0^k(x), \ \theta \in C^2([0,T]), \ \theta(T) = \frac{d\theta}{dt}(T) = 0$$

$$\chi_1(x,t) = \nu(t)\varphi_1^k(x), \ \nu \in C^2([0,T]), \ \nu(T) = \frac{d\nu}{dt}(T) = 0$$

in (3.112)-(3.113) to get

$$\int_0^T c^k\left(\frac{d^2\theta}{dt^2} + \mu_0^k\theta\right)dt = \int_0^T (f_0,\psi_0^k)\theta dt - (h_0,\psi_0^k)\frac{d\theta}{dt}(0) + (g_0,\psi_0^k)\theta(0) \tag{3.128}$$

$$\int_0^T d_\varepsilon^k\left(\frac{d^2\nu}{dt^2} + \lambda_1^k\varepsilon\nu\right)dt = \int_0^T (f_1,\varphi_1^k)\theta dt - (h_1,\varphi_1^k)\frac{d\nu}{dt}(0)$$

$$+ (g_1,\varphi_1^k)\nu(0) - \varepsilon\int_0^T \left(\sum_{\ell=1}^{\infty} c^\ell \int_S \psi_0^\ell \frac{\partial\varphi_1}{\partial\nu_{A_1}} dS\right)\nu dt \tag{3.129}$$

which are equivalent to

$$\left.\begin{aligned} &\frac{d^2c^k}{dt^2} + \mu_0^k c^k = (f_0,\psi_0^k) \\ &c^k(0) = (h_0,\psi_0^k) \\ &\frac{dc^k}{dt}(0) = (g_0,\psi_0^k) \end{aligned}\right\} \tag{3.130}$$

$$\left.\begin{aligned}
&\frac{d^2 d_\varepsilon^k}{dt^2} + \lambda_1^k \varepsilon\, d_\varepsilon^k = (f_1,\varphi_1^k) - \varepsilon \sum_{\ell=1}^{\infty} c^\ell \int_S \psi_0^\ell \frac{\partial \varphi_1^k}{\partial \nu_{A_1}}\, dS \\
&d_\varepsilon^k(0) = (h_1,\varphi_1^k) \\
&\frac{d\, d_\varepsilon^k}{dt}(0) = (g_1,\varphi_1^k) \quad .
\end{aligned}\right\} \tag{3.131}$$

<u>Remark 3.24</u>: Remarks (3.16)-(3.17) also apply in this case. □

<u>Remark 3.25</u>: If $h = (h_0,h_1) \notin H_0^1(\Omega_0)\times H_0^1(\Omega_1)$ but belongs to $H_0^1(\Omega)$, some technical difficulties would be encountered in defining (3.106).

<u>Remark 3.26</u>: The difference between the estimates (3.63) and (3.129) is due to the fact that the solutions of (3.63)-(3.64) are more regular than the solutions of (3.117)-(3.118). For further inquiry, the interested reader is referred to [23,30]. □

An example is now given to illustrate the computational aspects of the approximation of y_ε. This example is also considered in applications in Chapter 5.

<u>Example 3.7</u>: Let $f_0 = 1$, $f_1 = 0$, $h = 0$, $g = 0$. Then (3.80)-(3.84) become

$$\left.\begin{aligned}
&\frac{\partial^2 y_{\varepsilon 0}}{\partial t^2} - \frac{\partial^2 y_{\varepsilon 0}}{\partial x^2} = 1 \text{ on } (-1,0)\times(0,T) \\
&\frac{\partial^2 y_{\varepsilon 1}}{\partial t^2} - \varepsilon\,\frac{\partial^2 y_{\varepsilon 1}}{\partial x^2} = 0 \text{ on } (0,1)\times(0,T) \\
&y_{\varepsilon 0}(-1,t) = y_{\varepsilon 1}(1,t) = 0 \\
&\left.\begin{aligned} &y_{\varepsilon 0}(0,t) = y_{\varepsilon 1}(0,t) \\ &\frac{\partial y_{\varepsilon 0}}{\partial x}(0,t) = \varepsilon\,\frac{\partial y_{\varepsilon 1}}{\partial x}(0,t) \end{aligned}\right\} \\
&y_{\varepsilon 0}(x,0) = y_{\varepsilon 1}(x,0) = 0 \\
&\frac{\partial y_{\varepsilon 0}}{\partial t}(x,0) = \frac{\partial y_{\varepsilon 1}}{\partial t}(x,0) = 0 \quad .
\end{aligned}\right\} \tag{3.132}$$

Recall that

$$\left.\begin{aligned} &\mu_0^k = \left((2\,k-1)\,\frac{\pi}{2}\right)^2 \\ &\psi_0^k = \sqrt{2}\,\cos\sqrt{\mu_0^k}\,x \end{aligned}\right\}$$

$$\left.\begin{aligned} &\lambda_1^k = (k\pi)^2 \\ &\varphi_1^k = \sqrt{2}\,\sin\sqrt{\lambda_1^k}\,x \quad . \end{aligned}\right\}$$

The solutions to (3.130)-(3.131) are readily calculated and are given by:

$$c^k(t) = \frac{\sqrt{2}\,(-1)^{k-1}}{(\mu_0^k)^{3/2}}\left(1 - \cos\sqrt{\mu_0^k}\,t\right)$$

$$d_\varepsilon^k(t) = \frac{2\sqrt{2}}{\sqrt{\lambda_1^k}}\left(1 - \cos\sqrt{\lambda_1^k \varepsilon}\,t\right)\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell-1}}{(\mu_0^\ell)^{3/2}}$$
$$+ 2\sqrt{2}\,\sqrt{\lambda_1^k}\,\varepsilon\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell-1}}{(\mu_0^\ell)^{3/2}(\mu_0^\ell - \lambda_1^k\varepsilon)}\left(\cos\sqrt{\lambda_1^k \varepsilon}\,t - \cos\sqrt{\mu_0^\ell}\,t\right) \quad .$$

Therefore, y_ε^0 is written as

$$y_0^0 = \sum_{k=1}^{\infty} \frac{2(-1)^{k-1}}{(\mu_0^k)^{3/2}} (1 - \cos\sqrt{\mu_0^k}t) \cos\sqrt{\mu_0^k}x \tag{3.133}$$

$$\begin{aligned} y_1^0 = \sum_{k=1}^{\infty} \frac{4}{\sqrt{\lambda_1^k}} \Bigg((1 - \cos\sqrt{\lambda_1^k \varepsilon t}) \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{(\mu_0^\ell)^{3/2}} \\ + \varepsilon\, \lambda_1^k \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{(\mu_0^\ell)^{3/2}(\mu_0^\ell - \lambda_1^k \varepsilon)} (\cos\sqrt{\mu_0^\ell}t - \cos\sqrt{\lambda_1^k \varepsilon t}) \Bigg) \sin\sqrt{\lambda_1^k}x \ . \end{aligned} \tag{3.134}$$

<u>Remark 3.27</u>: The counterparts of Remarks 3.19-3.20 are also applicable in this example.

<u>3.5. Concluding Remarks</u>

In this chapter, the convergence of the solution of three classical boundary value problems (namely elliptic, parabolic and hyperbolic) as $\varepsilon \to 0$ has been analyzed, using the spectral analysis undertaken in Chapter 2. For elliptic boundary value problems, it is found, that by modifying the weak limits appropriately, a strong Laurent series expansion of y_ε can be derived. For parabolic and hyperbolic boundary value problems, zeroth order approximations in $L^2(0,T;L^2(\Omega))$ were easily constructed, using only the weak limits of the eigenvectors of A_ε.

Several examples were solved to illustrate and clarify the aspects of the problems at hand. Table 3.1 summarizes the properties of boundary value problems investigated in this chapter, in their simplest form.

TABLE 3.1. SUMMARY OF BOUNDARY VALUE PROBLEMS INVESTIGATED IN THIS CHAPTER.

Boundary Value Problem	Convergence and Asymptotic Expansion as $\varepsilon \to 0$	Section
$-\Delta y_{\varepsilon 0} = f_0$ on Ω_0, $-\varepsilon\Delta y_{\varepsilon 1} = f_1$ on Ω_1 $y_{\varepsilon 0}\big\|_{\Gamma_0} = y_{\varepsilon 1}\big\|_{\Gamma_1} = 0$ $y_{\varepsilon 0} = y_{\varepsilon 1}$ on S, $\frac{\partial y_{\varepsilon 0}}{\partial \nu} = \varepsilon \frac{\partial y_{\varepsilon 1}}{\partial \nu}$ on S	• $y_\varepsilon \to +\infty$ unless $f_1 = 0$ • Laurent series in ε, i.e., $y_\varepsilon^P = \frac{y^{-1}}{\varepsilon} + y^0 + \varepsilon y^1 + \ldots + \varepsilon^P y^P$ • $\\|y_\varepsilon - y_\varepsilon^P\\|_{H_0^1(\Omega)} \le C\,\varepsilon^P$	3.2
$\frac{\partial y_{\varepsilon 0}}{\partial t} - \Delta y_{\varepsilon 0} = f_0$ on Q_0, $\frac{\partial y_{\varepsilon 0}}{\partial t} - \varepsilon\Delta y_{\varepsilon 1} = f_1$ on Q_1 $y_{\varepsilon 0} = 0$ on Σ_0, $y_{\varepsilon 1} = 0$ on Σ_1 $y_{\varepsilon 0} = y_{\varepsilon 1}$ on R, $\frac{\partial y_{\varepsilon 0}}{\partial \nu} = \varepsilon \frac{\partial y_{\varepsilon 1}}{\partial \nu}$ on R $y_\varepsilon(0) = h$ on Ω	• $y_\varepsilon \to y$ weakly in $L^2(0,T;L^2(\Omega))$ • Zeroth order approximation • $\\|y_\varepsilon - y_\varepsilon^0\\|_{L^2(0,T;L^2(\Omega))} \le C\,\varepsilon^{1/2}$	3.3
$\frac{\partial^2 y_{\varepsilon 0}}{\partial t^2} - \Delta y_{\varepsilon 0} = f_0$ on Q_0, $\frac{\partial^2 y_{\varepsilon 1}}{\partial t^2} - \varepsilon\Delta y_{\varepsilon 1} = f_1$ on Q_1 $y_{\varepsilon 0} = 0$ on Σ_0, $y_{\varepsilon 1} = 0$ on Σ_1 $y_{\varepsilon 0} = y_{\varepsilon 1}$ on R, $\frac{\partial y_{\varepsilon 0}}{\partial \nu} = \varepsilon \frac{\partial y_{\varepsilon 1}}{\partial \nu}$ on R $y_\varepsilon(0) = h$ on Ω, $\frac{\partial y_\varepsilon}{\partial t}(0) = g$ on Ω	• $y_\varepsilon \to y$ weakly in $L^2(0,T;L^2(\Omega))$ • $\frac{\partial y_\varepsilon}{\partial t} \to \frac{\partial y}{\partial t}$ weakly in $L^2(0,T;L^2(\Omega))$ • Zeroth order approximation • $\\|y_\varepsilon - y_\varepsilon^0\\|_{L^2(0,T;L^2(\Omega))} \le C\,\varepsilon^{1/4}$	3.4

The approach followed in the present exposition is general enough to encompass many stiff operators. As an example, many boundary value problems involving the stiff operators considered in Sections 2.3-2.5 can be approximated using the same concepts developed herein.

CHAPTER 4

SUBOPTIMAL CONTROL OF STIFF SYSTEMS

4.1. Introduction

In Chapter 2 the spectral decomposition of a class of stiff operators A_ε, including the convergence of their eigenvalue-eigenvector pairs as $\varepsilon \to 0$, has been analyzed. Using the results of this investigation, some classical boundary value problems involving the aforementioned operators were studied in Chapter 3. The convergence of their solutions as $\varepsilon \to 0$ were analyzed. Then, asymptotic approximations of these solutions were constructed, using the weak limits of the eigenvectors of A_ε as $\varepsilon \to 0$. Asymptotic error estimates were also obtained.

In the present chapter, some control problems with quadratic cost functionals are considered. The results derived in the previous two chapters are used to investigate these problems. The objectives of this chapter are:

1. to obtain information about the behavior of the optimality system as $\varepsilon \to 0$,
2. to "approximate" the state and the control of the system for small values of ε.

The control problem of distributed parameter systems is formulated in many books and manuscripts such as [2,8,23-30], to name a few. In [8], a semi-group approach is followed. However, a variational approach is chosen in [23] and the subsequent references. Recently, several results about Dirichlet boundary control in parabolic and hyperbolic systems have appeared in the literature [7,20,21]. However, the assumptions made therein, concerning the coefficients of the operator A_ε, are more restrictive. Hence, a control problem with Neumann boundary control is considered in the sequel.

The formulation of [23] seems to be adequate for the presentation herein. Consequently, the control problems to be considered are adapted from there. In this chapter, the control of a class of parabolic systems is investigated. Two types of control are considered. In Section 4.2, the control is distributed. In Section 4.3, the control is of Neumann type, exercised through the boundary. In each section, the problem formulation is first presented. Then the convergence of the state and the costate as $\varepsilon \to 0$ is studied. Their asymptotic approximations are then constructed, using the approach developed in Section 3.3. In Section 4.4, some concluding remarks are given.

4.2. A Parabolic Problem with Distributed Control

4.2.1. Problem formulation

Let $H = L^2(\Omega)$, $V = H_0^1(\Omega)$ where $\Omega = \Omega_0 \cup \Omega_1 \cup S \subset \mathbb{R}^n$, with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ as depicted in Figure 2.1. Let

$$a_k(\varphi,\psi) = \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\Omega_k} a_{ij}^k(x) \frac{\partial\varphi}{\partial x_i} \frac{\partial\psi}{\partial x_j}\, dx, \qquad k = 0,1 \tag{4.1}$$

where $a_{ij}^k(x)$ satisfy the conditions of Remark 2.9. Consider the following control problem

$$\inf_{v \in L^2(0,T;H)} J_\varepsilon(v) = \frac{1}{2} \|y_\varepsilon(T)-z_f\|_H^2 + \frac{1}{2} \|y_\varepsilon - z_d\|^2_{L^2(0,T;H)} + \frac{1}{2} (Nv,v)_{L^2(0,T;H)} \tag{4.2}$$

subject to

- $(y'_\varepsilon,\varphi) + a_0(y_\varepsilon,\varphi) + \varepsilon a_1(y_\varepsilon,\varphi) = (f,\varphi) + (v,\varphi), \quad \forall\varphi \in V$ (4.3)
- $y_\varepsilon(0) = h$, h and z_f are given H (4.4)
- f and z_d are given in $L^2(0,T;H)$ (4.5)
- $y_\varepsilon \in L^2(0,T;V)$ (4.6)
- N is a given operator in $\mathcal{L}(L^2(0,T;H);\ L^2(0,T;H))$, which is hermitian and positive definite. (4.7)

Under these assumptions, the above control problem admits a unique optimal solution [23] $\{y_\varepsilon,u_\varepsilon\} \in L^2(0,T;V) \times L^2(0,T;H)$ for fixed ε, characterized by the following optimality system

$$\left.\begin{aligned} &(y'_\varepsilon,\varphi) + a_0(y_\varepsilon,\varphi) + \varepsilon a_1(y_\varepsilon,\varphi) = (f,\varphi) + (u_\varepsilon,\varphi), && \forall\varphi \in V \\ &(-p'_\varepsilon,\psi) + a_0(p_\varepsilon,\psi) + \varepsilon a_1(p_\varepsilon,\psi) = (y_\varepsilon - z_d,\psi), && \forall\psi \in V \end{aligned}\right\} \tag{4.8}$$

$$y_\varepsilon(0) = h, \qquad p_\varepsilon(T) = y_\varepsilon(T) - z_f \tag{4.9}$$

$$u_\varepsilon = -N^{-1} p_\varepsilon \tag{4.10}$$

$$y_\varepsilon, p_\varepsilon \in L^2(0,T;V). \tag{4.11}$$

The optimality system (4.7)-(4.10) can be decoupled through the affine map

$$p_\varepsilon = P_\varepsilon y_\varepsilon + r_\varepsilon \tag{4.12}$$

where P_ε, r_ε are defined by

$$\left.\begin{aligned} &-P'_\varepsilon + P_\varepsilon A_\varepsilon + A_\varepsilon P_\varepsilon + P_\varepsilon N^{-1} P_\varepsilon = I \\ &P_\varepsilon(T) = I \end{aligned}\right\} \tag{4.13}$$

$$\left.\begin{aligned} &-r'_\varepsilon + A_\varepsilon r_\varepsilon + P_\varepsilon N^{-1} r_\varepsilon = P_\varepsilon f - z_d \\ &r_\varepsilon(T) = -z_f \end{aligned}\right\} \tag{4.14}$$

where A_ε is the operator associated with the bilinear form $a_\varepsilon(\varphi,\psi) = a_0(\varphi,\psi) + \varepsilon a_1(\varphi,\psi)$. Consequently, the optimal control u_ε given by (4.10) can be written in the feedback form as

$$u_\varepsilon = -N^{-1}(P_\varepsilon y_\varepsilon + r_\varepsilon). \tag{4.15}$$

The properties of P_ε and r_ε are summarized in [23], Theorem 4.4, p. 148, some of which are

$$P_\varepsilon(t) \in \mathcal{L}(H;H) \tag{4.16}$$

$$P^*_\varepsilon(t) = P_\varepsilon(t), \quad (P_\varepsilon(t)s,s) \geq 0, \quad \forall s \in H \tag{4.17}$$

$$r_\varepsilon \in L^2(0,T;V). \tag{4.18}$$

Now let $\{\gamma^k_\varepsilon, \chi^k_\varepsilon\}^\infty_{k=1}$ be the eigenvalue-eigenvector pairs of the operator A_ε, i.e.,

$$a_0(\chi^k_\varepsilon,\varphi) + \varepsilon a_1(\chi^k_\varepsilon,\varphi) = \gamma^k_\varepsilon(\chi^k_\varepsilon,\varphi), \qquad \forall \varphi \in V \tag{4.19}$$

such that

$$(\chi_\varepsilon^i, \chi_\varepsilon^j) = \delta^{ij} \text{ (Kronecker delta)}. \tag{4.20}$$

For simplicity, assume $N = \rho I$, $\rho > 0$ and the eigenvalues of A_ε are not repeated. Then the operator $P_\varepsilon(t)$ can be expressed as

$$P_\varepsilon(t)\varphi = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_\varepsilon^{ij} \chi_\varepsilon^i (\varphi, \chi_\varepsilon^j), \qquad \forall \varphi \in H \tag{4.21}$$

where $\{P_\varepsilon^{ij}\}_{i,j=1}^{\infty}$ satisfy

$$\left.\begin{aligned} & -\frac{dp_\varepsilon^{ij}}{dt} + (\gamma_\varepsilon^i + \gamma_\varepsilon^j) p_\varepsilon^{ij} + \frac{1}{\rho} \sum_{k=1}^{\infty} p_\varepsilon^{ki} p_\varepsilon^{kj} = \delta^{ij} \\ & p_\varepsilon^{ij}(T) = \delta^{ij} \\ & i,j = 1,2,\ldots \end{aligned}\right\} \tag{4.22}$$

It can easily be verified [8] in this case, that

$$p_\varepsilon^{ij} = 0 \qquad \text{if} \quad i \neq j \tag{4.23}$$

is a solution of (4.22). Therefore, (4.22) reduces to

$$\left.\begin{aligned} & -\frac{dp_\varepsilon^{ii}}{dt} + 2\gamma_\varepsilon^i p_\varepsilon^{ii} + \frac{1}{\rho} (p_\varepsilon^{ii})^2 = 1 \\ & p_\varepsilon^{ii}(T) = 1 \\ & i = 1,2,\ldots \end{aligned}\right\} \tag{4.24}$$

The function r_ε can be expressed as

$$r_\varepsilon = \sum_{i=1}^{\infty} s_\varepsilon^i \chi_\varepsilon^i \tag{4.25}$$

where $\{s_\varepsilon^i\}_{i=1}^\infty$ satisfy

$$\left.\begin{aligned} &-\frac{ds_\varepsilon^i}{dt} + (\gamma_\varepsilon^i + \frac{1}{\rho}\, p_\varepsilon^{ii}) s_\varepsilon^i = (f,\chi_\varepsilon^i)p_\varepsilon^{ii} - (z_d,\chi_\varepsilon^i) \\ &s_\varepsilon^i(T) = -(z_f,\chi_\varepsilon^i) \\ &i = 1,2,\ldots \end{aligned}\right\} \tag{4.26}$$

4.2.2. Convergence of the state and the costate as $\varepsilon \to 0$

The convergence of y_ε and p_ε as $\varepsilon \to 0$ is summarized in

<u>Theorem 4.1</u>: Let y_ε and p_ε be the solution of the optimality system (4.8)-(4.11). Then as $\varepsilon \to 0$,

$$y_\varepsilon \to y \quad \text{weakly in } L^2(0,T;H)$$

$$p_\varepsilon \to p \quad \text{weakly in } L^2(0,T;H)$$

$$\sqrt{\varepsilon}\, \|y_\varepsilon\|_{L^2(0,T;V)} \le C_1$$

$$\sqrt{\varepsilon}\, \|p_\varepsilon\|_{L^2(0,T;V)} \le C_2$$

$$J_\varepsilon(u_\varepsilon) \to J(u)$$

where y and p satisfy

$$\left.\begin{aligned} &\left.\begin{aligned} &(y',\varphi) + a_0(y,\varphi) + \frac{1}{N}\,(p,\varphi) = (f,\varphi), && \forall \varphi \in V \\ &(-p',\psi) + a_0(p,\psi) - (y,\psi) = -(z_d,\psi), && \forall \psi \in V \end{aligned}\right\} \\ &y(0) = h, \qquad p(T) = y(T) - z_f. \end{aligned}\right\} \tag{4.27}$$

Proof: Let

$$y_\varepsilon = \sum_{k=1}^{\infty} c_\varepsilon^k \psi_\varepsilon^k + \sum_{k=1}^{\infty} d_\varepsilon^k \varphi_\varepsilon^k$$

$$p_\varepsilon = \sum_{k=1}^{\infty} a_\varepsilon^k \psi_\varepsilon^k + \sum_{k=1}^{\infty} b_\varepsilon^k \varphi_\varepsilon^k$$

where $\{c_\varepsilon^k\}_{k=1}^{\infty}$, $\{a_\varepsilon^k\}_{k=1}^{\infty}$, $\{d_\varepsilon^k\}_{k=1}^{\infty}$, and $\{b_\varepsilon^k\}_{k=1}^{\infty}$ satisfy

$$\left.\begin{array}{l}
\left.\begin{array}{l}
\dfrac{dc_\varepsilon^k}{dt} + \mu_\varepsilon^k c_\varepsilon^k + \dfrac{1}{N} a_\varepsilon^k = (f,\psi_\varepsilon^k) \\
-\dfrac{da_\varepsilon^k}{dt} + \mu_\varepsilon^k a_\varepsilon^k - c_\varepsilon^k = -(z_d,\psi_\varepsilon^k)
\end{array}\right\} \\
c_\varepsilon^k(0) = (h,\psi_\varepsilon^k), \quad a_\varepsilon^k(T) = c_\varepsilon^k(T) - (z_f,\psi_\varepsilon^k)
\end{array}\right\}$$

$$\left.\begin{array}{l}
\left.\begin{array}{l}
\dfrac{dd_\varepsilon^k}{dt} + \lambda_\varepsilon^k d_\varepsilon^k + \dfrac{1}{N} b_\varepsilon^k = (f,\varphi_\varepsilon^k) \\
-\dfrac{db_\varepsilon^k}{dt} + \lambda_\varepsilon^k b_\varepsilon^k - d_\varepsilon^k = -(z_d,\varphi_\varepsilon^k)
\end{array}\right\} \\
d_\varepsilon^k(0) = (h,\varphi_\varepsilon^k), \quad b_\varepsilon^k(T) = d_\varepsilon^k(T) - (z_f,\varphi_\varepsilon^k)
\end{array}\right\}$$

$k = 1,2,\ldots$

The above coupled equations have unique solutions for any value of ε. As $\varepsilon \to 0$, using the results of Theorem 2.1, take the limit in the above equations to derive easily the conclusions of Theorem 4.1.

For a more general approach that is applicable when N is a function of x or the control is exercised on the boundary, consult [24]. As a result of this, the Riccati operator $P_\varepsilon(t)$ and the function $r_\varepsilon(t)$ also converge in some sense.

<u>Theorem 4.2</u>: Let $P_\varepsilon(t)$, $r_\varepsilon(t)$ be the solutions of (4.13)-(4.14). Then as $\varepsilon \to 0$,

$$P_\varepsilon(t) \to P(t) \quad \text{in } \mathcal{L}(H;H) \tag{4.28}$$

$$r_\varepsilon(t) \to r(t) \quad \text{weakly in } L^2(0,T;H) \tag{4.29}$$

$P(t)$, $r(t)$ can be written as

$$P(t) = \begin{pmatrix} P_0(t) \\ P_1(t) \end{pmatrix} \tag{4.30}$$

$$r(t) = \begin{pmatrix} r_0(t) \\ r_1(t) \end{pmatrix} \tag{4.31}$$

where

$$P_0(t)\chi = \sum_{i=1}^{\infty} p_0^{ii} \psi^i (\chi, \psi^i), \qquad \forall \chi \in H \tag{4.32}$$

$$P_1(t)\chi = \sum_{i=1}^{\infty} p_1^{ii} \varphi^i (\chi, \varphi^i), \qquad \forall \chi \in H \tag{4.33}$$

$$r_0(t) = \sum_{i=1}^{\infty} s_0^i \psi^i \tag{4.34}$$

$$r_1(t) = \sum_{i=1}^{\infty} s_1^i \varphi^i \tag{4.35}$$

$\{p_k^i\}_{i=1}^\infty$, $\{s_k^i\}_{i=1}^\infty$, $k = 0,1$ satisfy

$$\left.\begin{aligned} &\frac{dp_0^{ii}}{dt} + 2\mu_0^i p_0^{ii} + \frac{1}{\rho}(p_0^{ii})^2 = 1 \\ &p_0^{ii}(T) = 1 \end{aligned}\right\} \tag{4.36}$$

$$\left.\begin{aligned} &-\frac{dp_1^{ii}}{dt} + \frac{1}{\rho}\,(p_1^{ii})^2 = 1 \\ &p_1^i(T) = 1 \end{aligned}\right\} \qquad (4.37)$$

$$\left.\begin{aligned} &-\frac{ds_0^i}{dt} + (\mu_0^i + \frac{1}{\rho}\,p_0^{ii})s_0^i = (f,\psi^i)p_0^{ii} - (z_d,\psi^i) \\ &s_0^i(T) = -(z_f,\psi^i) \end{aligned}\right\} \qquad (4.38)$$

$$\left.\begin{aligned} &-\frac{ds_1^i}{dt} + \frac{1}{\rho}\,p_1^{ii}s_1^i = (f,\varphi^i)p_1^{ii} - (z_d,\varphi^i) \\ &s_1^i(T) = -(z_f,\varphi^i) \end{aligned}\right\} \qquad (4.39)$$

$$i = 1,2,\ldots$$

Proof: Decompose the eigenvalue-eigenvector pairs of A_ε as in Theorem 2.1 and use the limits therein and (4.21)-(4.26) to get (4.28)-(4.39). □

Remark 4.1: It is noteworthy to mention that the pair $\{P(t),r(t)\}$ decouples the optimality system limit, i.e., (4.27).

4.2.3. Asymptotic approximation of y_ε and p_ε

Using the same approach as in Section 3.3.2, let the zeroth order approximations in $L^2(0,T;H)$ of y_ε and p_ε be denoted by $y_\varepsilon^0 = (y_0^0, y_{\varepsilon 1}^0)$ and $p_\varepsilon^0 = (p_0^0, p_{\varepsilon 1}^0)$ and defined by (for $0 < \varepsilon << 1$)

$$\left.\begin{aligned} &\frac{\partial y_0^0}{\partial t} + A_0 y_0^0 = f_0 - \frac{1}{N}\,p_0^0 \quad \text{on} \quad Q_0 \\ &-\frac{\partial p_0^0}{\partial t} + A_0 p_0^0 = y_0^0 - z_{d0} \quad \text{on} \quad Q_0 \end{aligned}\right\} \qquad (4.40)$$

$$\left.\begin{aligned} &y_0^0 = 0, \qquad p_0^0 = 0 \quad \text{on } \Sigma_0 \\ &\frac{\partial y_0^0}{\partial \nu_{A_0}} = 0, \qquad \frac{\partial p_0^0}{\partial \nu_{A_0}} = 0 \quad \text{on } R \end{aligned}\right\} \tag{4.41}$$

$$y_0^0(0) = h_0, \quad p_0^0(T) = y_0^0(T) - z_{f0} \quad \text{on } \Omega_0 \tag{4.42}$$

$$\left.\begin{aligned} &\frac{\partial y_{\varepsilon 1}^0}{\partial t} + \varepsilon A_1 y_{\varepsilon 1}^0 = f_1 - \frac{1}{N} p_{\varepsilon 1}^0 \quad \text{on} \quad Q_1 \\ &-\frac{\partial p_{\varepsilon 1}^0}{\partial t} + \varepsilon A_1 p_{\varepsilon 1}^0 = y_{\varepsilon 1}^0 - z_d \quad \text{on} \quad Q_1 \end{aligned}\right\} \tag{4.43}$$

$$\left.\begin{aligned} &y_{\varepsilon 1}^0 = 0, \quad p_{\varepsilon 1}^0 = 0 \quad \text{on} \quad \Sigma_1 \\ &y_{\varepsilon 1}^0 = y_0^0, \quad p_{\varepsilon 1}^0 = p_0^0 \quad \text{on} \quad R \end{aligned}\right\} \tag{4.44}$$

$$\left.\begin{aligned} &y_{\varepsilon 1}^0(0) = h_1 \quad \text{on} \quad \Omega_1 \\ &p_{\varepsilon 1}^0(T) = y_{\varepsilon 1}^0(T) - z_{f1} \quad \text{on} \quad \Omega_1 \end{aligned}\right\} \tag{4.45}$$

where the solution of (4.43)-(4.45) is defined as in Section 3.3.2 by transposition.

Theorem 4.3: Let $\{y_\varepsilon, p_\varepsilon\}$ be the solution of (4.8)-(4.10) and $\{y_\varepsilon^0, p_\varepsilon^0\}$ be the solution of (4.40)-(4.45). Then the following estimates hold for $0 < \varepsilon << 1$

$$\|y_\varepsilon - y_\varepsilon^0\|_{L^2(0,T;H)} \le C_1 \varepsilon^{1/2} \tag{4.46}$$

$$\|p_\varepsilon - p_\varepsilon^0\|_{L^2(0,T;H)} \le C_2 \varepsilon^{1/2} \tag{4.47}$$

$$|J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_\varepsilon^0)| \le C_3 \varepsilon. \tag{4.48}$$

Proof: The proof of (4.46)-(4.47) is practically identical to that of Theorem 3.4. Since

$$u_\varepsilon^0 \equiv -N^{-1} p_\varepsilon^0 \tag{4.49}$$

using (4.46)-(4.47), one gets

$$J_\varepsilon(u_\varepsilon) = J_\varepsilon(u_\varepsilon^0) + O(\varepsilon). \quad \square \tag{4.50}$$

Now let $\{\varphi^k\}_{k=1}^\infty$ be renormalized. Then the optimality system (4.40)-(4.42) can be decoupled by the map

$$p_0^0 = P_0 y_0^0 + r_0 \tag{4.51}$$

where $P_0(t)$ and $r_0(t)$ are written as (4.32) and (4.34), respectively, with $\{p_0^i\}_{i=1}^\infty$ and $\{s_0^i\}_{i=1}^\infty$ satisfying (4.36) and (4.38). The optimality system (4.43)-(4.45) can also be decoupled by the map

$$p_{\varepsilon 1}^0 = P_{\varepsilon 1} y_{\varepsilon 1}^0 + r_{\varepsilon 1} \tag{4.52}$$

where $P_{\varepsilon 1}(t)$, $r_{\varepsilon 1}(t)$ are written as

$$P_{\varepsilon 1}(t)\chi = \sum_{i=1}^\infty p_{\varepsilon 1}^{ii} \varphi^i (\chi, \varphi^i), \qquad \forall \chi \in H \tag{4.53}$$

$$r_{\varepsilon 1}(t) = \sum_{i=1}^{\infty} s_{\varepsilon 1}^{i} \varphi^{i} \tag{4.54}$$

with $\{p_{\varepsilon 1}^{ii}\}_{i=1}^{\infty}$ and $\{s_{\varepsilon 1}^{i}\}_{i=1}^{\infty}$ satisfying, respectively,

$$\left.\begin{aligned} & -\frac{dp_{\varepsilon 1}^{ii}}{dt} + 2\lambda_1^i \varepsilon p_{\varepsilon 1}^{ii} + \frac{1}{\rho}\,(p_{\varepsilon 1}^{ii})^2 = 1 \\ & p_{\varepsilon 1}^{ii}(T) = 1 \end{aligned}\right\} \tag{4.55}$$

$$\left.\begin{aligned} & -\frac{ds_{\varepsilon 1}^{i}}{dt} + (\lambda_1^i \varepsilon + \frac{1}{\rho}\, p_{\varepsilon 1}^{ii}) s_{\varepsilon 1}^{i} = ((f,\varphi^i) - \varepsilon \int_S y_0^0 \frac{\partial \varphi^i}{\partial \nu_{A_1}}\, dS) p_{\varepsilon 1}^{ii} \\ & \qquad\qquad -\varepsilon \int_S p_0^0 \frac{\partial \varphi^i}{\partial \nu_{A_1}}\, dS \\ & s_{\varepsilon 1}^{i}(T) = -(z_f,\varphi^i). \end{aligned}\right\} \tag{4.56}$$

The presence of y_0^0 and p_0^0 in (4.56) implies that the optimality systems given by (4.40)-(4.42) and (4.43)-(4.45) have to be solved sequentially. However, it is clear from the analysis of Section 3.3 that setting y_0^0 and p_0^0 to zero in (4.44) induces errors in $y_{\varepsilon 1}^0$ and $p_{\varepsilon 1}^0$ no more than $O(\sqrt{\varepsilon})$. Consequently, it may be desirable to set them to zero. Another possibility may be that no control is exercised on Ω_0. In such a case, it may be advantageous to synthesize a feedback law of the form (4.52), where $\{s_{\varepsilon 1}^{i}\}_{i=1}^{\infty}$ are computed from (4.56) with p_0^0 set to zero.

Example 4.1: (Cf. Example 3.6) In this example, the following control problem is analyzed:

$$\inf_{v \in L^2(Q)} J_\varepsilon(v) = \frac{1}{2}\, \|y_\varepsilon\|_{L^2(Q)}^2 + \frac{\rho}{2}\, \|v\|_{L^2(Q)}^2 \tag{4.57}$$

subject to

$$\left.\begin{aligned}&\frac{\partial y_{\varepsilon 0}}{\partial t}-\frac{\partial^2 y_{\varepsilon 0}}{\partial x^2}=v_0 \qquad \text{on } Q_0\\ &\frac{\partial y_{\varepsilon 1}}{\partial t}-\varepsilon\frac{\partial^2 y_{\varepsilon 1}}{\partial x^2}=v_1 \qquad \text{on } Q_1\end{aligned}\right\} \tag{4.58}$$

$$y_{\varepsilon 0}(-1,t)=0, \quad y_{\varepsilon 1}(1,t)=0 \tag{4.59}$$

$$\left.\begin{aligned}&y_{\varepsilon 0}(0,t)=y_{\varepsilon 1}(0,t)\\ &\frac{\partial y_{\varepsilon 0}}{\partial x}(0,t)=\varepsilon\frac{\partial y_{\varepsilon 1}}{\partial x}(0,t)\end{aligned}\right\} \tag{4.60}$$

$$y_{\varepsilon 0}(x,0)=1, \quad y_{\varepsilon 1}(x,0)=1. \tag{4.61}$$

A suboptimal feedback control law as outlined previously would be given by

$$\left.\begin{aligned}&u_0^0=-\frac{1}{\rho}P_0y_0^0\\ &u_{\varepsilon 1}^0=-\frac{1}{\rho}P_{\varepsilon 1}y_{\varepsilon 1}^0.\end{aligned}\right\} \tag{4.62}$$

Recall that

$$\mu_0^k=\left((2k-1)\frac{\pi}{2}\right)^2, \qquad \psi_0^k=\sqrt{2}\cos(2k-1)\frac{\pi}{2}x$$

$$\lambda_1^k=(k\pi)^2, \qquad \varphi_1^k=\sqrt{2}\sin k\pi x.$$

Hence, (4.62) can be rewritten as

$$\left.\begin{aligned} u_0^0 &= -\frac{1}{\rho}\sum_{i=1}^{\infty} p_0^i(y_0^0,\varphi_0^i)\varphi_0^i \\ u_{\varepsilon 1}^0 &= -\frac{1}{\rho}\sum_{i=1}^{\infty} p_{\varepsilon 1}^i(y_{\varepsilon 1}^0,\varphi_1^i)\varphi_1^i \end{aligned}\right\} \tag{4 63}$$

where $\{p_0^i\}_{i=1}^{\infty}$ and $\{p_{\varepsilon 1}^i\}_{i=1}^{\infty}$ satisfy

$$\left.\begin{aligned} &-\frac{dp_0^i}{dt} + 2\mu_0^i p_0^i + \frac{1}{\rho}(p_0^i)^2 = 1 \\ &p_0^i(T) = 0 \end{aligned}\right\} \tag{4.64}$$

$$\left.\begin{aligned} &-\frac{dp_{\varepsilon 1}^i}{dt} + 2\lambda_1^i \varepsilon p_{\varepsilon 1}^i + \frac{1}{\rho}(p_{\varepsilon 1}^i)^2 = 1 \\ &p_{\varepsilon 1}^i(T) = 0. \end{aligned}\right\} \tag{4.65}$$

In this case, $\{p_0^i\}_{i=1}^{\infty}$ and $\{p_{\varepsilon 1}^i\}_{i=1}^{\infty}$ can be computed in closed form as

$$p_0^i = \rho\,\frac{B_0^i A_0^i(e^{-D_0^i(t-T)}-i)}{A_0^i e^{-D_0^i(t-T)}-B_0^i} \tag{4.66}$$

$$p_{\varepsilon 1}^i = \rho\,\frac{B_{\varepsilon 1}^i A_{\varepsilon 1}^i(e^{-D_{\varepsilon 1}^i(t-T)}-1)}{A_{\varepsilon 1}^i e^{-D_{\varepsilon 1}^i(t-T)}-B_{\varepsilon 1}^i} \tag{4.67}$$

where

$$\left.\begin{aligned} A_0^i &= -\mu_0^i - \sqrt{(\mu_0^i)^2 + \frac{1}{\rho}} \\ B_0^i &= -\mu_0^i + \sqrt{(\mu_0^i)^2 + \frac{1}{\rho}} \\ D_0^i &= 2\sqrt{(\mu_0^i)^2 + \frac{1}{\rho}} \end{aligned}\right\} \tag{4.68}$$

$$\left.\begin{aligned} A_{\varepsilon 1}^i &= -\lambda_1^i \varepsilon - \sqrt{(\lambda_1^i \varepsilon)^2 + \frac{1}{\rho}} \\ B_{\varepsilon 1}^i &= -\lambda_1^i \varepsilon + \sqrt{(\lambda_1^i \varepsilon)^2 + \frac{1}{\rho}} \\ D_{\varepsilon 1}^i &= 2\sqrt{(\lambda_1^i \varepsilon)^2 + \frac{1}{\rho}} \; . \end{aligned}\right\} \tag{4.69}$$

The numerical results are discussed in Chapter 5 for various values of ε and T.

4.3. A Parabolic Problem with Neumann Boundary Control and Boundary Observation

4.3.1. Problem formulation

Let $H = L^2(\Omega)$, $V = H^1(\Omega)$ where $\Omega = \Omega_0 \cup \Omega_1 \cup S \subset \mathbb{R}^n$ with smooth boundary Γ_0 and interface S as indicated in Figure 2.1b. Let

$$A_\varepsilon = \begin{bmatrix} A_0 & 0 \\ 0 & \varepsilon A_1 \end{bmatrix}$$

where

$$A_k = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}^k(x) \frac{\partial}{\partial x_j}, \qquad k = 0,1$$

a_{ij}^k satisfy the conditions of Remark 2.9. Let

$$Q_i = \Omega_i \times (0,T), \qquad i = 0,1$$

$$\Sigma = \Gamma \times (0,T)$$

$$R = S \times (0,T).$$

Now consider the following control problem

$$\inf_{v \in L^2(\Sigma)} J_\varepsilon(v) = \int_\Sigma (y_\varepsilon - z_d)^2 d\Sigma + (Nv,v)_{L^2(\Sigma)} \tag{4.70}$$

subject to

- $(y_\varepsilon', \varphi) + a_0(y_\varepsilon, \varphi) + \varepsilon a_1(y_\varepsilon, \varphi) = (f,\varphi) + (v,\varphi)_{L^2(\Gamma)}, \quad \forall \varphi \in V$ (4.71)
- $y_\varepsilon(0) = h, \quad h$ given in $L^2(\Omega)$ (4.72)
- $f = (f_0, f_1)$, given in $L^2(0,T;H)$ such that

$$(f_i, 1)_{L^2(\Omega_i)} = 0, \qquad i = 0,1 \tag{4.73}$$

- z_d given in $L^2(\Sigma)$ (4.74)
- N is a given operator in $\mathcal{L}(L^2(\Sigma); L^2(\Sigma))$, which is hermitian and positive definite. (4.75)

Under these assumptions, the above control problem admits a unique optimal solution $\{y_\varepsilon, u_\varepsilon\} \in L^2(0,T;H) \times L^2(\Sigma)$ for fixed ε [23], characterized by the following optimality system

$$\left.\begin{aligned} &(y'_\varepsilon,\varphi) + a_0(y_\varepsilon,\varphi) + \varepsilon a_1(y_\varepsilon,\varphi) = (f,\varphi)+(u_\varepsilon,\varphi)_{L^2(\Gamma)}, \quad \forall\varphi\in V \\ &-(p'_\varepsilon,\psi) + a_0(p_\varepsilon,\psi) + \varepsilon a_1(p_\varepsilon,\psi) = (y_\varepsilon - z_d,\psi)_{L^2(\Gamma)}, \quad \forall\psi\in V \end{aligned}\right\} \tag{4.76}$$

$$y_\varepsilon(0) = h, \qquad p_\varepsilon(T) = 0 \tag{4.77}$$

$$y_\varepsilon, p_\varepsilon \in L^2(0,T;V) \tag{4.78}$$

where

$$u_\varepsilon = -N^{-1} p_\varepsilon\big|_\Sigma . \tag{4.79}$$

This optimality system can be decoupled as in Section 4.2 by the affine map

$$p_\varepsilon = P_\varepsilon y_\varepsilon + r_\varepsilon, \qquad P^*_\varepsilon = P_\varepsilon \tag{4.80}$$

where P_ε and r_ε satisfy, respectively,

$$\left.\begin{aligned} &(-P'_\varepsilon\varphi,\psi) + a_0(\varphi,P_\varepsilon\psi) + \varepsilon a_1(\varphi,P_\varepsilon\psi) + a_0(P_\varepsilon\varphi,\psi) + \varepsilon a_1(P_\varepsilon\varphi,\psi) \\ &\qquad + (N^{-1}P_\varepsilon\varphi,P_\varepsilon\psi)_{L^2(\Gamma)} = (\varphi,\psi)_{L^2(\Gamma)}, \quad \forall\varphi,\psi\in V \\ &P_\varepsilon(T) = 0 \end{aligned}\right\} \tag{4.81}$$

$$\left.\begin{aligned} &(-r'_\varepsilon,\varphi) + a_0(r_\varepsilon,\varphi) + \varepsilon a_1(r_\varepsilon,\varphi) = (P_\varepsilon f,\varphi) - (z_d,\varphi)_{L^2(\Gamma)}, \quad \forall\varphi\in V \\ &r_\varepsilon(T) = 0. \end{aligned}\right\} \tag{4.82}$$

Now let $\{\gamma^k_\varepsilon, \chi^k_\varepsilon\}^\infty_{k=1}$ be the eigenvalue-eigenvector pairs of A_ε, i.e.,

$$a_0(\chi^k_\varepsilon,\varphi) + \varepsilon a_1(\chi^k_\varepsilon,\varphi) = \gamma^k_\varepsilon(\chi^k_\varepsilon,\varphi), \quad (\chi^k_\varepsilon,\chi^j_\varepsilon) = \delta^{ij}, \qquad \forall\varphi\in V. \tag{4.83}$$

For simplicity, let $N = \rho I$, $\rho > 0$, then P_ε and r_ε can be written as

$$P_\varepsilon \varphi = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_\varepsilon^{ij} \chi_\varepsilon^i (\varphi, \chi_\varepsilon^j), \qquad \forall \varphi \in V \tag{4.84}$$

$$r_\varepsilon = \sum_{i=1}^{\infty} s_\varepsilon^i \chi_\varepsilon^i \tag{4.85}$$

where $\{p_\varepsilon^{ij}\}_{i,j=1}^{\infty}$ and $\{s_\varepsilon^i\}_{i=1}^{\infty}$ satisfy

$$\left.\begin{aligned} & -\frac{dp_\varepsilon^{ij}}{dt} + (\gamma_\varepsilon^i + \gamma_\varepsilon^j) p_\varepsilon^{ij} + \frac{1}{\rho} \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} p_\varepsilon^{ki} p_\varepsilon^{\ell j} \int_\Gamma \chi_\varepsilon^k \chi_\varepsilon^\ell d\Gamma = \int_\Gamma \chi_\varepsilon^i \chi_\varepsilon^j d\Gamma \\ & p_\varepsilon^{ij}(T) = 0, \qquad i,j = 1,2,\ldots \end{aligned}\right\} \tag{4.86}$$

where

$$p_\varepsilon^{ij} = p_\varepsilon^{ji}, \qquad \forall i,j$$

$$\left.\begin{aligned} & -\frac{ds_\varepsilon^i}{dt} + \gamma_\varepsilon^i s_\varepsilon^i = \sum_{\ell=1}^{\infty} p_\varepsilon^{i\ell} (f, \chi_\varepsilon^\ell) - (z_d, \chi_\varepsilon^i)_{L^2(\Gamma)} \\ & s_\varepsilon^i(T) = 0. \end{aligned}\right\} \tag{4.87}$$

Remark 4.2: Since $V = H^1(\Omega)$, the operator A_ε has an eigenvalue which is zero and its corresponding eigenvector is a constant on Ω. By assumption (Cf., (4.73)), they are excluded from (4.83).

Remark 4.3: Observe that (4.86) is not decoupled as in Section 4.2.

4.3.2. Convergence of y_ε and p_ε as $\varepsilon \to 0$

It is straightforward to prove, as in Section 4.2, the following theorem.

Theorem 4.4: Let y_ε and p_ε be the solution of the optimality system (4.76)-(4.79). Then as $\varepsilon \to 0$,

$$y_\varepsilon \to y \quad \text{weakly in } L^2(0,T;H)$$

$$p_\varepsilon \to p \quad \text{weakly in } L^2(0,T;H)$$

$$J_\varepsilon(u_\varepsilon) \to J(u)$$

where y,p satisfy

$$\left.\begin{aligned}
&(y',\varphi) + a_0(y,\varphi) + \frac{1}{N}(p,\varphi)_{L^2(\Gamma)} = (f,\varphi), && \forall \varphi \in V \\
&-(p',\psi) + a_0(p,\psi) - (y,\psi)_{L^2(\Gamma)} = -(z_d,\psi)_{L^2(\Gamma)}, && \forall \psi \in V \\
&y(0) = h, \quad p(T) = 0.
\end{aligned}\right\} \tag{4.88}$$

Proof: See proof of Theorem 4.1.

Remark 4.4: As in Section 4.2,

$$P_\varepsilon(t) \to P(t) \quad \text{in } \mathcal{L}(H;H)$$

$$r_\varepsilon(t) \to r(t) \quad \text{weakly in } L^2(0,T;H)$$

where the pair $\{P,r\}$ satisfy (4.81)-(4.82), respectively, after letting $\varepsilon \to 0$. Note that $\{P,r\}$ decouples the optimality system limit, i.e., (3.88). Furthermore, P(t) and r(t) may split, as in Section 4.2, in some instances (depending on the forcing term f), because the influence of the null eigenvalue of A_ε is excluded by assumption (Cf., Remark 4.2).

4.3.3. Asymptotic approximation of y_ε and p_ε

Let $y_\varepsilon^0 = (y_0^0, y_{\varepsilon 1}^0)$ and $p_\varepsilon^0 = (p_0^0, p_{\varepsilon 1}^0)$ denote the zeroth approximations in $L^2(0,T;H)$ of y_ε and p_ε, respectively. For $0 < \varepsilon << 1$, they are defined by

$$\left.\begin{aligned} &\frac{\partial y_0^0}{\partial t} + A_0 y_0^0 = f_0 \quad \text{on} \quad Q_0 \\ -&\frac{\partial p_0^0}{\partial t} + A_0 p_0^0 = 0 \quad \text{on} \quad Q_0 \end{aligned}\right\} \tag{4.89}$$

$$\left.\begin{aligned} &\frac{\partial y_0^0}{\partial \nu_{A_0}} = u_0^0, \qquad \frac{\partial p_0}{\partial \nu_{A_0}} = y_0^0 - z_d \quad \text{on} \quad \Sigma \\ &\frac{\partial y_0^0}{\partial \nu_{A_0}} = 0, \qquad \frac{\partial p_0^0}{\partial \nu_{A_0}} = 0 \quad \text{on } R \end{aligned}\right\} \tag{4.90}$$

$$y_0^0(0) = h_0, \qquad p_0^0(0) = 0 \quad \text{on} \quad \Omega_0 \tag{4.91}$$

$$u_0^0 = -\frac{1}{N} p_0^0\Big|_\Sigma \tag{4.92}$$

$$\left.\begin{aligned} &\frac{\partial y_{\varepsilon 1}^0}{\partial t} + \varepsilon A_1 y_{\varepsilon 1}^0 = f_1 \quad \text{on} \quad Q_1 \\ -&\frac{\partial p_{\varepsilon 1}^0}{\partial t} + \varepsilon A_1 p_{\varepsilon 1}^0 = 0 \quad \text{on} \quad Q_1 \end{aligned}\right\} \tag{4.93}$$

$$y_{\varepsilon 1}^0 = y_0^0, \qquad p_{\varepsilon 1}^0 = p_0^0 \quad \text{on} \quad R \tag{4.94}$$

$$y_{\varepsilon 1}^0(0) = h_1, \qquad p_{\varepsilon 1}^0(T) = 0 \quad \text{on} \quad \Omega_1 \tag{4.95}$$

where the solution of (4.93)-(4.95) is defined using transposition.

Theorem 4.5: Let $\{y_\varepsilon, p_\varepsilon\}$ be the solution of (4.76)-(4.79) and $\{y_\varepsilon^0, p_\varepsilon^0\}$ be the solution of (4.89)-(4.95). Then the following estimates hold for $0 < \varepsilon << 1$.

$$\|y_\varepsilon - y_\varepsilon^0\|_{L^2(0,T;H)} \le C_1 \varepsilon^{1/2} \tag{4.96}$$

$$\|p_\varepsilon - p_\varepsilon^0\|_{L^2(0,T;H)} \le C_2 \varepsilon^{1/2} \tag{4.97}$$

$$|J_\varepsilon(u_\varepsilon) - J(u_0^0)| \le C_3 \varepsilon \tag{4.98}$$

C_1, C_2, C_3 are some constants independent of ε.

Proof: See proof of Theorem 4.3.

As before, (4.89)-(4.95) can be solved, using the weak limits of the eigenvectors of the operator A_ε. It can be easily shown as in Section 2.5.1 that the weak limits of $\{\gamma_\varepsilon^k, \chi_\varepsilon^k\}_{k=1}^\infty$ can be decomposed into $\{\lambda_\varepsilon^k, \varphi_\varepsilon^k\}_{k=0}^\infty$, $\{\mu_\varepsilon^k, \psi_\varepsilon^k\}_{k=1}^\infty$ whose weak limits satisfy

$$\left.\begin{array}{l} \left.\begin{array}{lll} \varphi_0^k = \text{constant on} & & \Omega_0 \\ A_1 \varphi_1^k = \lambda_1^k \varphi_1^k & \text{on} & \Omega_1 \end{array}\right\} \\ \left.\dfrac{\partial \varphi_1^k}{\partial \nu_{A_1}}\right|_S = 0 \\ k = 0,1,2,\ldots \end{array}\right\} \tag{4.99}$$

$$\left.\begin{array}{l} \left.\begin{array}{ll} A_0\psi_0^k = \mu_0^k\psi_0^k & \text{on } \Omega_0 \\ \psi_1^k = 0 & \text{on } \Omega_1 \end{array}\right\} \\ \left.\dfrac{\partial\psi_1^k}{\partial\nu_{A_0}}\right|_\Gamma = 0, \quad \left.\dfrac{\partial\psi_1^k}{\partial\nu_{A_0}}\right|_S = 0 \\ k = 1,2,\ldots \end{array}\right\} \tag{4.100}$$

where the null eigenvalue and its corresponding eigenvector are included in (4.99). The constant in (4.99) is chosen such that $\varphi^k \in H^1(\Omega)$, k=0,1,2,... However, because of (4.73), $\varphi_0^k = 0$ and, therefore, (4.99) has to be modified into

$$\left.\begin{array}{l} \left.\begin{array}{ll} \varphi_0^k = 0 & \text{on } \Omega_0 \\ A_1\varphi_1^k = \lambda_1^k\varphi_1^k & \text{on } \Omega_1 \end{array}\right\} \\ \left.\varphi_1^k\right|_S = 0 \\ k = 1,2,\ldots \end{array}\right\} \tag{4.101}$$

Now (4.89)-(4.91) can be decoupled by the following map

$$p_0^0 = P_0 y_0^0 + r_0 \tag{4.102}$$

with P_0 and r_0 written as

$$P_0\chi = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_0^{ij}(\chi,\psi^i)\psi^j \tag{4.103}$$

$$r_0 = \sum_{i=1}^{\infty} s_0^i\psi^i \tag{4.104}$$

where $\{p_0^{ij}\}_{i,j=1}^{\infty}$ and $\{s_0^i\}_{i=1}^{\infty}$ satisfy

$$\left.\begin{aligned} & -\frac{dp_0^{ij}}{dt} + (\mu_0^i + \mu_0^j)\, p_0^{ij} + \frac{1}{\rho}\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} p_0^{ki} p_0^{\ell j} \int_\Gamma \psi^k \psi^\ell d\Gamma = \int_\Gamma \psi^i \psi^j d\Gamma \\ & p_0^{ij}(T) = 0, \qquad p_0^{ij} = p_0^{ji}, \quad i,j = 1,2,\ldots \end{aligned}\right\} \tag{4.105}$$

$$\left.\begin{aligned} & -\frac{ds_0^i}{dt} + \mu_0^i s_0^i = \sum_{\ell=1}^{\infty} p_0^{i\ell}(f,\psi^\ell) - (z_d,\psi^i)_{L^2(\Gamma)} \\ & s_0^i(T) = 0 \\ & i = 1,2,\ldots \end{aligned}\right\} \tag{4.106}$$

<u>Remark 4.5</u>: Note that (4.93)-(4.95) are decoupled. Hence there is no need to compute p_0^0 for control purposes. As before, (4.89)-(4.92) and (4.93)-(4.95) have to be calculated sequentially in time because of (4.94). However, by setting $y_{\varepsilon 1}^0 = 0$ on R, a cost no larger than $O(\sqrt{\varepsilon})$ is incurred since the solution of (4.93)-(4.95) does not influence (4.92). Therefore, it is rational to do exactly that to avoid the sequential computation.

<u>Remark 4.6</u>: If the condition (4.73) were read $(f,1)_{L^2(\Omega)} = 0$, then, by letting $f_0 = \frac{1}{\text{meas. } \Omega_0}$, $f_1 = -\frac{1}{\text{meas. } \Omega_1}$ in (4.71), some "controllability" difficulty would be encountered.

<u>Example 4.2</u>: (Cf., Example 3.6) Let

$$z_d = 0, \quad f = \begin{cases} \sin 2\pi x & \text{for } x \in (-1,0) \\ 0 & \text{for } x \in (0,1). \end{cases}, \quad h = 0$$

Then the control problem (4.70)-(4.77) becomes

$$\inf_{v\in L^2(0,T)} J_\varepsilon(v) = \int_0^T (y_\varepsilon(-1,t))^2 dt + \rho \int_0^T (v(t))^2 dt \tag{4.107}$$

subject to

$$\left.\begin{aligned} &\frac{\partial y_{\varepsilon 0}}{\partial t} - \frac{\partial^2 y_{\varepsilon 0}}{\partial x^2} = f_0 \quad \text{on} \quad (-1,0)\times(0,T) \\ &\frac{\partial y_{\varepsilon 1}}{\partial t} - \varepsilon \frac{\partial^2 y_{\varepsilon 1}}{\partial x^2} = 0 \quad \text{on} \quad (0,1)\times(0,T) \end{aligned}\right\} \tag{4.108}$$

$$\left.\begin{aligned} &y_{\varepsilon 0}(0,t) = y_{\varepsilon 1}(0,t) \\ &\frac{\partial y_{\varepsilon 0}}{\partial x}(0,t) = \varepsilon \frac{\partial y_{\varepsilon 1}}{\partial x}(0,t) \end{aligned}\right\} \tag{4.109}$$

$$\frac{\partial y_{\varepsilon 0}}{\partial x}(-1,t) = v \tag{4.110}$$

$$y_\varepsilon(x,0) = 0. \tag{4.111}$$

A suboptimal feedback control law can be synthesized as

$$u_0^0 = -\frac{1}{\rho}(P_0 y_0^0 + r_0)\Big|_{x=-1} \tag{4.112}$$

where P_0 and r_0 are given by (4.103)-(4.104) where $\{\psi^i\}_{i=1}^\infty$ are $\{\sqrt{2}\cos i\pi x\}_{i=1}^\infty$ and $\{p_0^{ij}\}_{i,j=1}^\infty$, $\{s_0^i\}_{i=1}^\infty$ satisfy

$$\left.\begin{aligned} &-\frac{dp_0^{ij}}{dt} + ((i\pi)^2 + (j\pi)^2)p_0^{ij} + \frac{2}{\rho}\sum_{k=1}^\infty \sum_{\ell=1}^\infty (-1)^{k+\ell} p_0^{ki} p_0^{\ell j} = (-1)^{i+j} \\ &p_0^{ij}(T) = 0 \end{aligned}\right\} \tag{4.113}$$

$$\left.\begin{aligned} &-\frac{ds_0^i}{dt} + (i\pi)^2 s_0^i = \sum_{\ell=1}^{\infty} p_0^{i\ell} \frac{2(1-(-1)^{\ell})}{\pi(\ell^2-4)} \\ &s_0^i(T) = 0. \end{aligned}\right\} \tag{4.114}$$

Remark 4.7: The present methodology for approximating control problems cannot be applied blindly because it may yield erroneous results. Many factors can limit its applicability, depending on the control problem at hand. Two prominent factors, which are central in any control problem, are the type of control and the type of observation. One problem where this methodology fails is the elliptic stiff control problem with Neumann boundary control and Dirichlet boundary observation considered in [29], page 323.

4.4. Conclusion

In this chapter, the control of two stiff systems was considered. Using the concepts developed in the previous chapter, suboptimal feedback control laws were derived for these problems for small values of the parameter ε.

It was shown, that the approximations of the state and the costate are easy to obtain, provided some care is taken, depending upon the specific problem at hand. It is safe to claim, based on the present results, that these approximations alleviate stiffness for most problems with meaningful disturbances. Control problems with Dirichlet boundary control are more complex. For practical classes of control inputs, the state is not "sufficiently" regular [22,23]. Therefore, the control space has to be restricted to obtain meaningful results. As a consequence, the feedback synthesis of the control is involved. One remedy for this dilemma is to assume that the coefficients of the operator A_ε, as well as the boundary where the control is exercised, are more regular. In turn, this assumption restricts the number of problems that can be considered. For example, a possible class of stiff control problems, with Dirichlet boundary control that may be investigated, is the class of problems when the operator A_ε is as given in Section 3.5.3.

CHAPTER 5

APPLICATIONS TO HEAT TRANSFER AND ELECTROMAGNETICS

5.1. Introduction

There are numerous dynamical systems whose evolution can be modeled better by partial differential operators such as heat transfer, electromagnetic wave propagation, chemical processes, elasticity, just to name a few.

The introduction of a (or several) small parameter ε may have a physical meaning of a small conduction (or convection) coefficient in heat transfer or a small permittivity in electromagnetics. It may also be completely artificial, such as in penalized and regularized problems.

In the previous chapters, the theoretical implications of letting $\varepsilon \to 0$ in some of these models have been studied. In the present chapter, two specific examples of such models are considered. The first example describes the heat conduction in a one-dimensional rod, made of two interfaced media, having heat conduction coefficients of $O(1)$ and $O(\varepsilon)$, respectively. The second example considers the propagation of an electric field in a one-dimensional waveguide, consisting of two interfaced media, having permittivities of $O(1)$ and $O(\varepsilon)$, respectively. Most of the interpretations given in the sequel are of general nature and hence applicable in many other related problems.

This chapter is organized as follows. In Section 5.2, physical interpretations of the results obtained in Sections 2.2 and 2.5 concerning the convergence of the eigenvalue-eigenvector pairs of stiff operators are given.

In Section 5.3, the asymptotic approximations of the solution of the boundary value problems of Examples 3.6-3.7 are compared with finite-dimensional approximations of these problems for different values of ε. In Section 5.4, the control problem of Example 4.1 is solved numerically. The last section contains some concluding remarks.

5.2. Physical Interpretation of the Limits of the Eigenvectors of Stiff Operators

Physical interpretations of the convergence of the eigenvectors of stiff operators as $\varepsilon \to 0$ are given within the framework of the examples discussed in Chapter 2.

The operator A_ε in Example 2.1 may represent the heat diffusion in a slab occupying the space $\Omega \in \mathbb{R}^n$ $(n \leq 3)$, composed of two interfaced media having diffusivities $O(1)$ and $O(\varepsilon)$, respectively. The complement of the set Ω in $\mathbb{R}^n$ represents the surrounding. There are many possible boundary conditions on the interface between the slab and its surroundings.

1. One possibility is to assume that the slab is insulated from its surroundings. This condition would be fulfilled if the normal derivative of the temperature (outward relative to the set Ω) is set to zero on the boundary Γ of Ω.
2. Another possibility is to suppose that the surrounding is an infinite sink, i.e., its temperature is not affected by the heat diffusion in the slab. This state would be indicated by setting the temperature of the slab on the boundary Γ to a constant, which may be assumed to be zero by translating it.

Many other possibilities may occur such as a combination of 1 and 2. In the sequel, condition 2 is assumed to fix the ideas.

The smallness of ε means physically that the relative diffusivity of medium 1 is small with respect to the diffusivity of medium 0. Letting $\varepsilon \to 0$, e.g., in Example 2.1, signifies that medium 1 is less and less conductive. In the limit, it becomes an insulator. This situation is symbolized mathematically by the normal derivative (outward relative to Ω_o) of the temperature of medium 0 going to zero on the interface S of the two media. Consequently, some of the eigenvectors of A_ε, i.e., $\{\varphi_\varepsilon^k\}_{k=1}^\infty$ do reflect this behavior as indicated by their weak limits (in $L^2(\Omega)$) given by (2.32).

From the viewpoint of medium 1, medium 0 is so conductive that it may be considered an extension of the surrounding for small values of ε. If medium 0 is insulated from (respectively connected to) the surrounding, it becomes an insulator (respectively a sink) in the limit. These situations are clearly depicted by the limits of some of the eigenvectors, i.e., $\{\varphi_\varepsilon^k\}_{k=1}^\infty$ in Example 2.3 (respectively Example 2.1).

Now consider the eigenvalue problem Example 2.6. In this case, the conductivities of both media are of the same order of magnitude. However, the convection coefficients are of $O(1)$ (respectively $O(\varepsilon)$) in medium 0 (respectively medium 1), i.e., although the heat diffuses in the slab with comparable rates, the internal heat exchange with the surrounding in medium 0 is much greater than in medium 1 and this causes stiffness. Consequently, medium 0 is a better heat dissipator than medium 1.

From this discussion, it seems logical to expect that the eigenvalue-eigenvector pairs of A_ε would reflect this behavior as $\varepsilon \to 0$.

It was shown that the eigenvalues of A_ε can be decomposed into two groups $\{\lambda_\varepsilon^k\}_{k=1}^\infty$ and $\{\mu_\varepsilon^k\}_{k=1}^\infty$ depending on how they converge as $\varepsilon \to 0$. Exactly as previously indicated, medium 0 becomes an extension of the surrounding as $\varepsilon \to 0$ and eventually a sink in the limit. By contrast, medium 1 loses more and more of its ability to dissipate energy as $\varepsilon \to 0$, which becomes negligible for small values of ε. This is clearly demonstrated by the limits of the eigenvectors given by (2.65a-b), which are completely decoupled.

Identical interpretations can be advanced in the field of electromagnetics, provided diffusivity, sink, insulator, etc., are replaced by appropriate terminology.

5.3. Numerical Analysis of Parabolic and Hyperbolic Boundary Value Problems

In this section, the boundary value problems of Examples 3.6-3.7 are are revisited. The exact solution of each problem is not available for the reasons previously discussed. In the sequel, the zeroth order approximations obtained in the aforementioned examples are compared with the finite-dimensional approximations of these boundary value problems.

The set $\Omega = (-1,1)$ is divided into N equal intervals of length $h = \frac{2}{N}$. The roof functions $\{\varphi_h^i\}_{i=1}^{N-1}$ are selected as a basis for the finite-dimensional approximation of $H_0^1(\Omega)$ [2,20,39].

5.3.1. Parabolic problem

It is straightforward to show that the solution of the boundary value problem (3.73) can be approximated by

$$y_\varepsilon^h = \sum_{i=1}^{N-1} c_\varepsilon^i(t)\varphi_h^i \tag{5.1}$$

where $c_\varepsilon^t = [c_\varepsilon^1\ c_\varepsilon^2\ \ldots\ c_\varepsilon^{N-1}]$ (t = transpose) satisfies

$$M^h\dot{c}_\varepsilon + K_\varepsilon^h c_\varepsilon = f^h \tag{5.2}$$

$$c_\varepsilon(0) = a^h \tag{5.3}$$

M_ε^h is given by (2.89)

K_ε^h is given by (2.90)

$$f_i^h = (f,\varphi_i^h), \qquad i = 1,2,\ldots,N-1 \tag{5.4}$$

$$a_i^h = (a,\varphi_i^h), \qquad i = 1,2,\ldots,N-1. \tag{5.5}$$

Remark 5.1: Note the notation change from Chapter 3, i.e., $y_\varepsilon(0) = g$ on Ω instead of $y_\varepsilon(0) = h$ on Ω because h designates the mesh size. □

Remark 5.2: The solution of (5.2)-(5.3) for $t \in (0,10)$ is obtained by using the integration routine DGEAR from the IMSL library. □

Remark 5.3: In the forthcoming plots, the broken lines represent $y_\varepsilon^0(x,t)$ as computed in Example 3.6 and the solid lines depict $y_\varepsilon^h(x,t)$. □

For all computer runs, N was chosen to be 60. Table 5.1 summarizes the computer runs. It is noteworthy to mention that these plots are both finite-dimensional approximations of y_ε because $y_\varepsilon^0(x,t)$ is also approximated by a finite summation (large enough to obtain a smooth plot!).

It is evident that these approximations are close. For small ε, they almost coincide with each other. Note $y_\varepsilon^0(x,t)$ and $y_\varepsilon^h(x,t)$ for $t = 4$, $\varepsilon = 0.1$ in Figure 5.2a are at steady state. The temperature distribution on

TABLE 5.1. PARABOLIC PROBLEM

Figure	Plots of $y_\varepsilon^0(x,t)$ and $y_\varepsilon^h(x,t)$ for
5.1a	$t = 2$, $x \in (-1,1)$, $\varepsilon = 0.1$
5.1b	$t = 2$, $x \in (-1,1)$, $\varepsilon = 0.001$
5.2a	$t = 4$, $x \in (-1,1)$, $\varepsilon = 0.1$
5.2b	$t = 4$, $x \in (-1,1)$, $\varepsilon = 0.001$
5.3a	$t \in (0,10)$, $x = -0.5$, $\varepsilon = 0.1$
5.3b	$t \in (0,10)$, $x = -0.5$, $\varepsilon = 0.001$
5.4a	$t \in (0,10)$, $x = 0.5$, $\varepsilon = 0.1$
5.4b	$t \in (0,10)$, $x = 0.1$, $\varepsilon = 0.001$

Ω_1 is a straight line, i.e., because it is due to a point source on the interface $x = 0$.

The temperature at $x = 0.5$ as a function of time for $\varepsilon = 0.001$ is very small. Hence, the temperature at $x = 0.1$ is plotted in Figure 5.4b.

Remark 5.4: As a general rule, y_ε^0 approximates y_ε pointwise much better in the interior of Ω, away from the interface. To substantiate this claim, plots 5.3a–5.4b are provided. In plots 5.4b, even though ε decreased by a factor of 100, the error between y_ε^0 and y_ε^h at $x = 0.1$ is comparable to the error at $x = 0.5$ for $\varepsilon = 0.1$.

5.3.2. Hyperbolic problem

As with the parabolic problem, the solution of the hyperbolic boundary value problem (3.132) can be approximated by

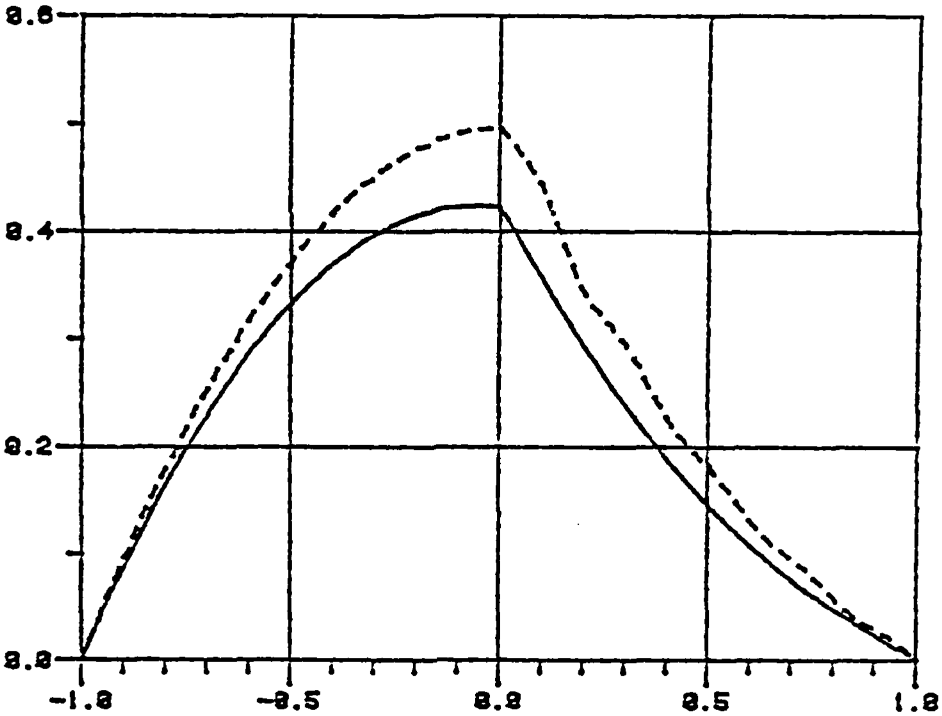

Figure 5.1a. y_ε^0 and y_ε^h for $t = 2$, $\varepsilon = 0.1$.

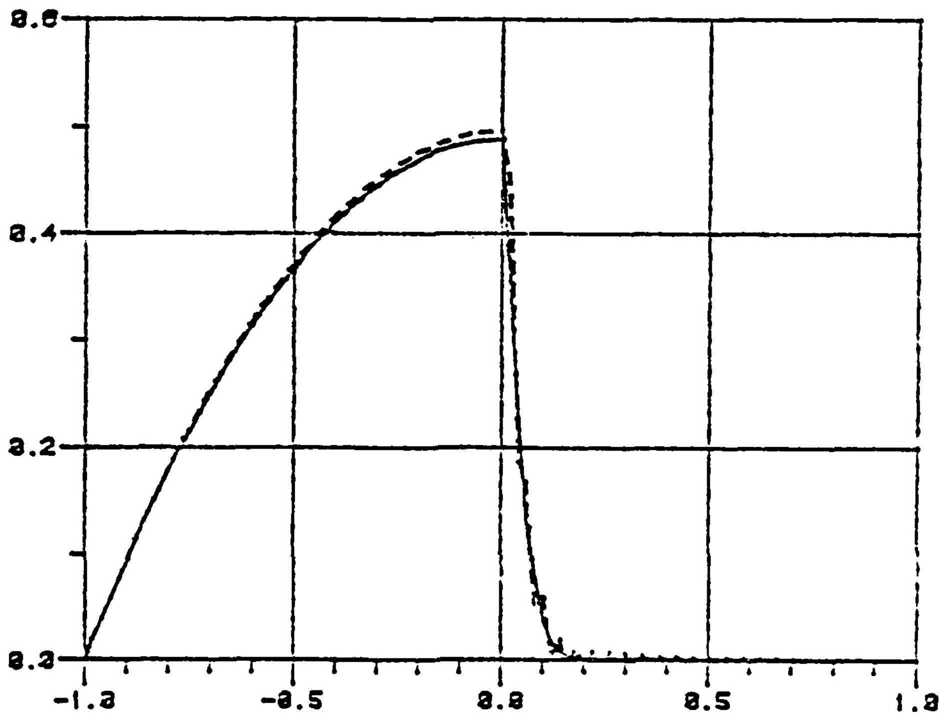

Figure 5.1b. y_ε^0 and y_ε^h for $t = 2$, $\varepsilon = 0.001$.

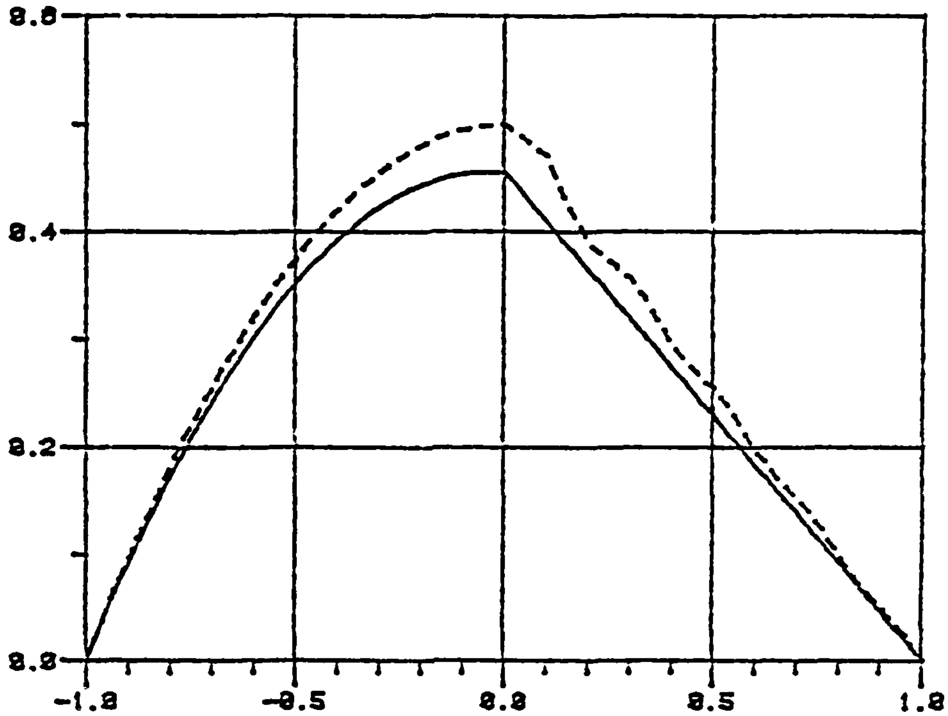

Figure 5.2a. y_ε^0 and y_ε^h for $t = 4$, $\varepsilon = 0.1$.

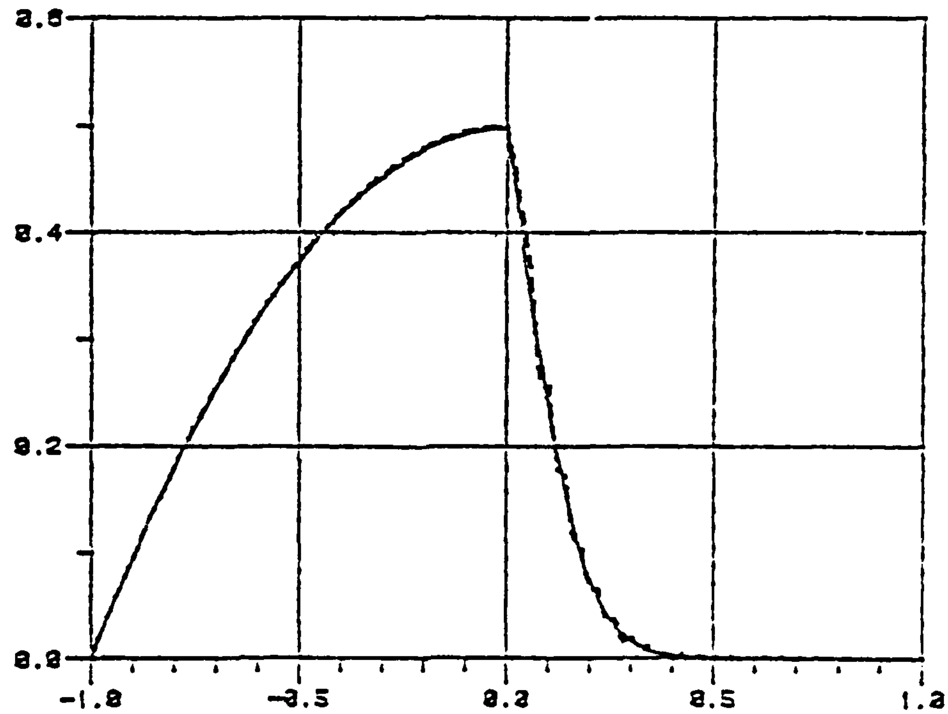

Figure 5.2b. y_ε^0 and y_ε^h for $t = 4$, $\varepsilon = 0.001$.

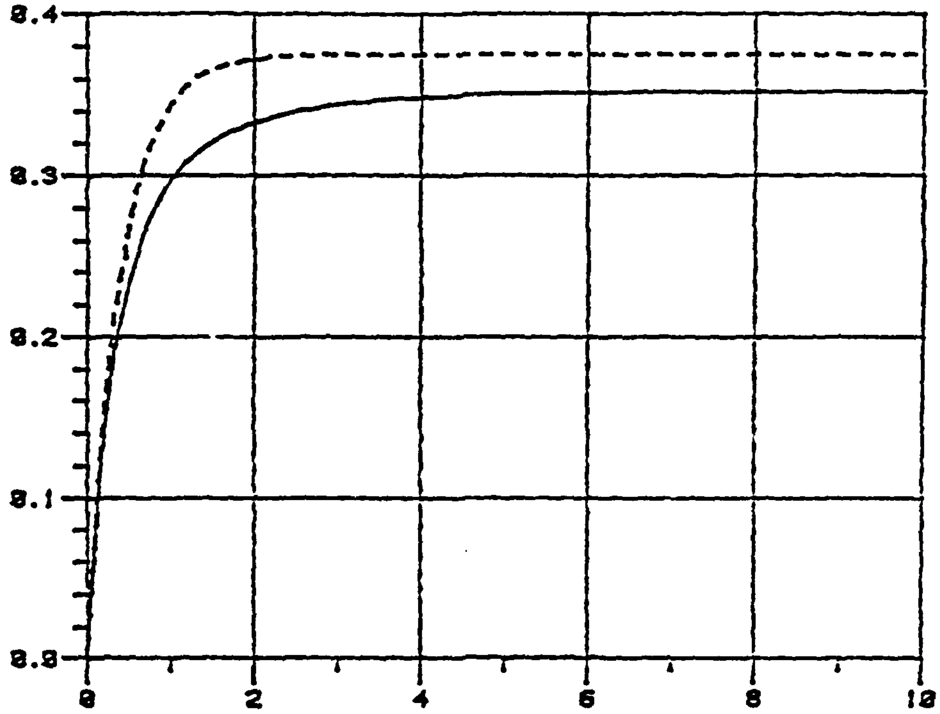

Figure 5.3a. y_ε^0 and y_ε^h for $x = -0.5$, $\varepsilon = 0.1$.

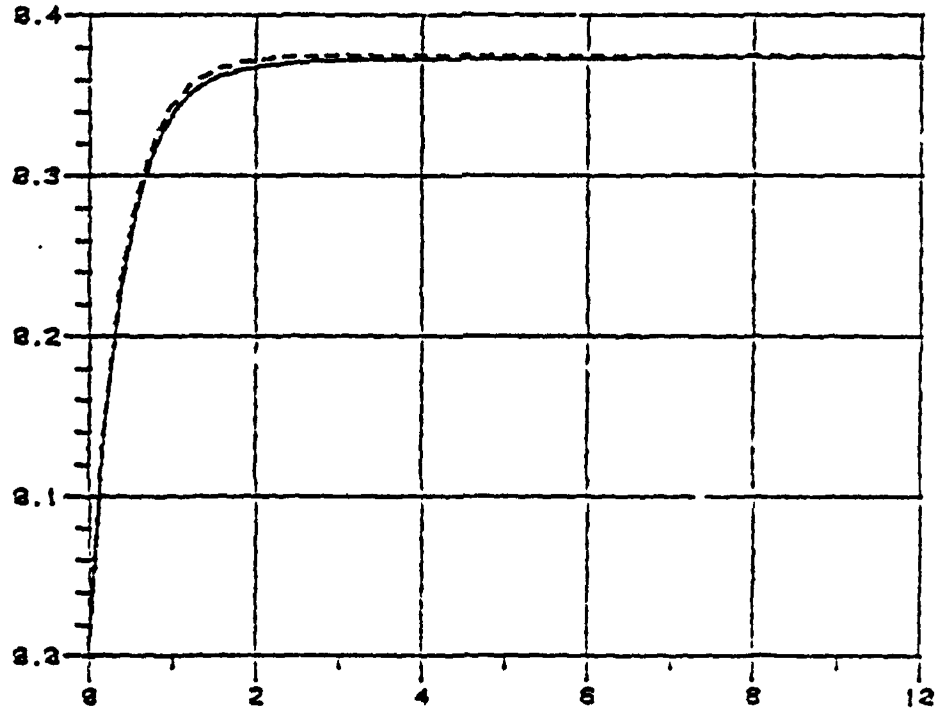

Figure 5.3b. y_ε^0 and y_ε^h for $x = -0.5$, $\varepsilon = 0.001$.

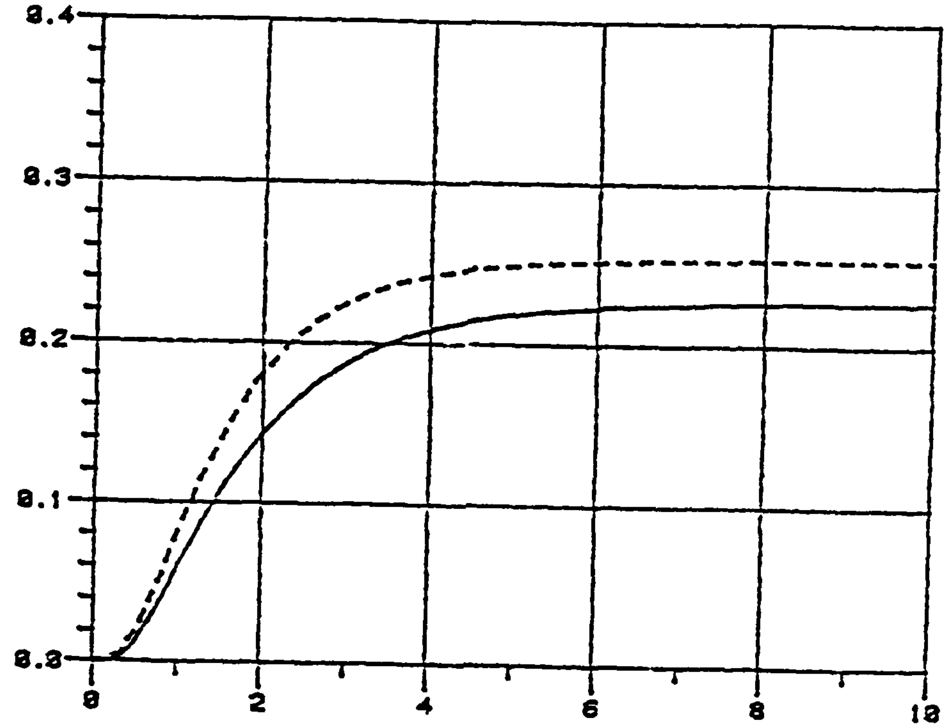

Figure 5.4a. y_{ε}^{0} and y_{ε}^{h} for $x = 0.5$, $\varepsilon = 0.1$.

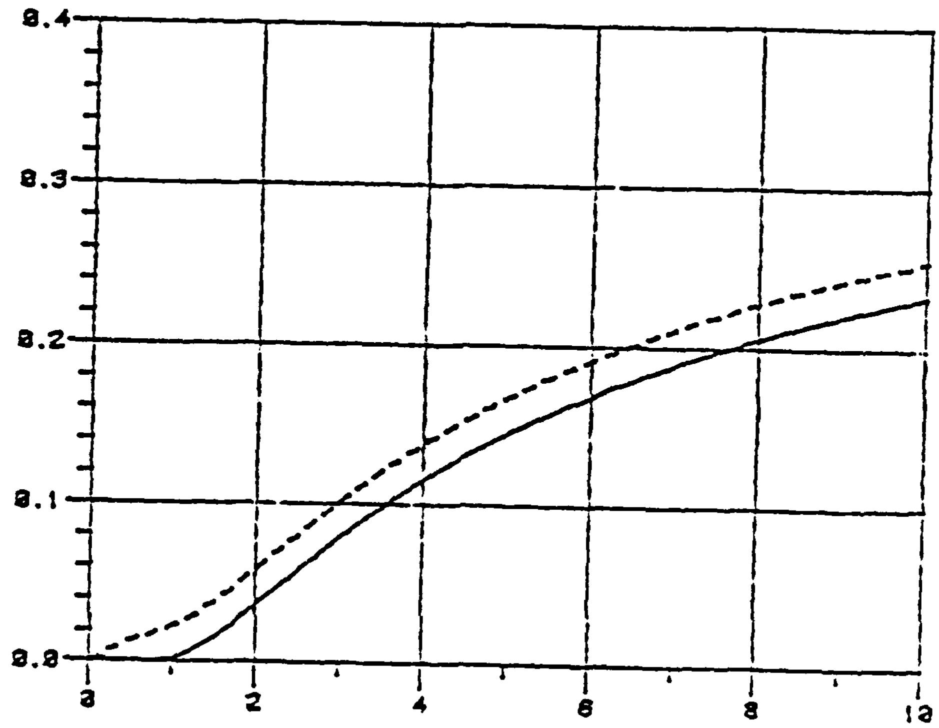

Figure 5.4b. y_{ε}^{0} and y_{ε}^{h} for $x = 0.1$, $\varepsilon = 0.001$.

$$y_\varepsilon^h = \sum_{i=1}^{N-1} c_\varepsilon^i(t)\varphi_h^i \tag{5.6}$$

where $c_\varepsilon^t = [c_\varepsilon^1 \; c_\varepsilon^2 \; \ldots \; c_\varepsilon^{N-1}]$ satisfies

$$M^h\ddot{c}_\varepsilon + K_\varepsilon^h c_\varepsilon = f^h \tag{5.7}$$

$$c_\varepsilon(0) = a^h \tag{5.8}$$

$$\dot{c}_\varepsilon(0) = b^h \tag{5.9}$$

M^h is given by (2.89)

K_ε^h is given by (2.90)

f^h is given by (5.4)

a^h is given by (5.5)

$$b^h = (h,\varphi_i^h), \qquad i = 1,2,\ldots,N-1. \tag{5.10}$$

Remark 5.5: Cf., Remarks 5.1-5.3. □

For all computer runs, N was selected to be 60. Table 5.2 summarizes the computer runs. Due to the asymptotic error estimate (3.114), one would expect that y_ε^0 and y_ε^h would not be as "close" as in parabolic systems. Nevertheless, the two approximations "approach" each other as $\varepsilon \to 0$. These facts are clearly illustrated by the plots in Figures 5.5a-5.10b.

Remark 5.6: At $t = 1$, y_0^0 is growing up until it reaches its maximum at $t = 2$. At $t = 4$, it attains the minimum and this process is repeated periodically every four units of time. However, $y_{\varepsilon 1}^0$ behaves differently because it is

TABLE 5.2. HYPERBOLIC PROBLEM

Figure	Plots of $y_\varepsilon^0(x,t)$ and $y_\varepsilon^h(x,t)$ for
5.5a	$t=1$, $x\in(-1,1)$, $\varepsilon=0.1$
5.5b	$t=1$, $x\in(-1,1)$, $\varepsilon=0.001$
5.6a	$t=2$, $x\in(-1,1)$, $\varepsilon=0.1$
5.6b	$t=2$, $x\in(-1,1)$, $\varepsilon=0.001$
5.7a	$t=4$, $x\in(-1,1)$, $\varepsilon=0.1$
5.7b	$t=4$, $x\in(-1,1)$, $\varepsilon=0.001$
5.8a	$t=8$, $x\in(-1,1)$, $\varepsilon=0.1$
5.8b	$t=8$, $x\in(-1,1)$, $\varepsilon=0.001$
5.9a	$t\in(0,10)$, $x=-0.5$, $\varepsilon=0.1$
5.9b	$t\in(0,10)$, $x=-0.5$, $\varepsilon=0.001$
5.10a	$t\in(0,10)$, $x=0.5$, $\varepsilon=0.1$
5.10b	$t\in(0,10)$, $x=0.1$ $\varepsilon=0.001$

the "transmitted wave" from region 0 to region 1. Since the "velocity of propagation" in medium 1 is $O(\sqrt{\varepsilon})$, it takes longer for $y_{\varepsilon 1}^0$ to reach its maximum. □

Remark 5.7: Note that the system given by (5.7)-(5.9) is of order 2N. For $N=60$, it took approximately four hours of CPU time on the VAX computer system to solve for y_ε^h. Consequently, this is not an economical approach. See Section 5.5 for a better procedure. □

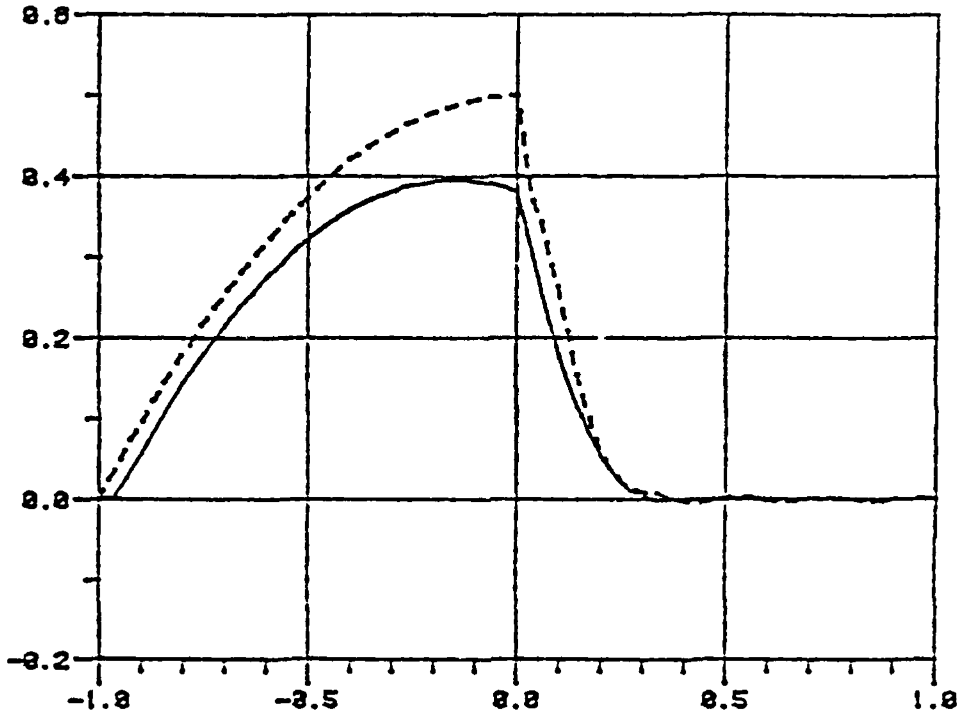

Figure 5.5a. y_ε^0 and y_ε^h for $t = 1$, $\varepsilon = 0.1$.

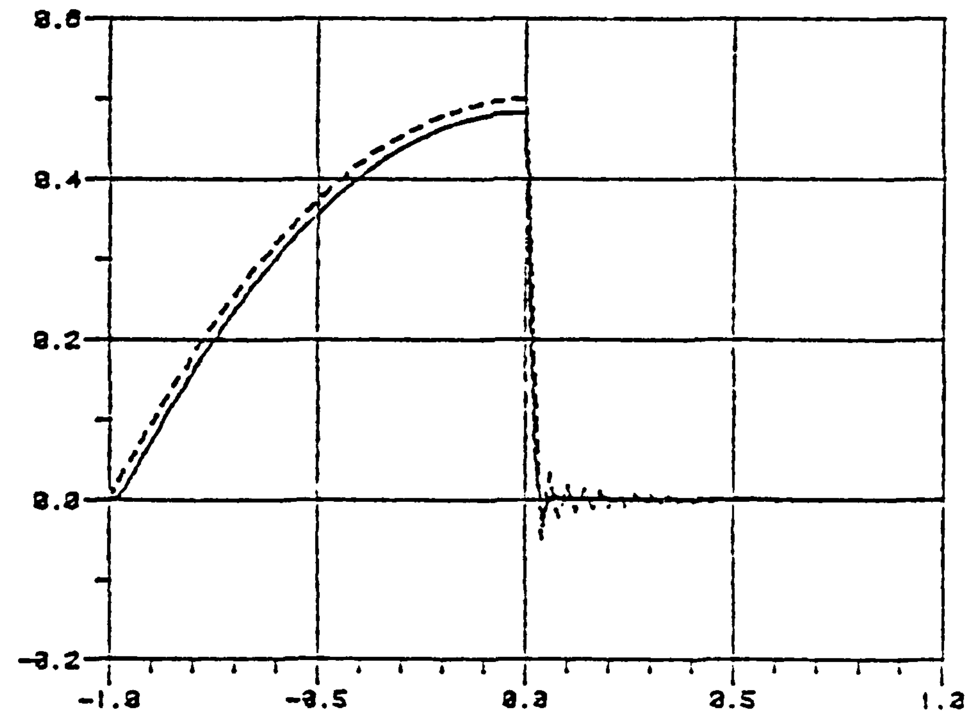

Figure 5.5b. y_ε^0 and y_ε^h for $t = 1$, $\varepsilon = 0.001$.

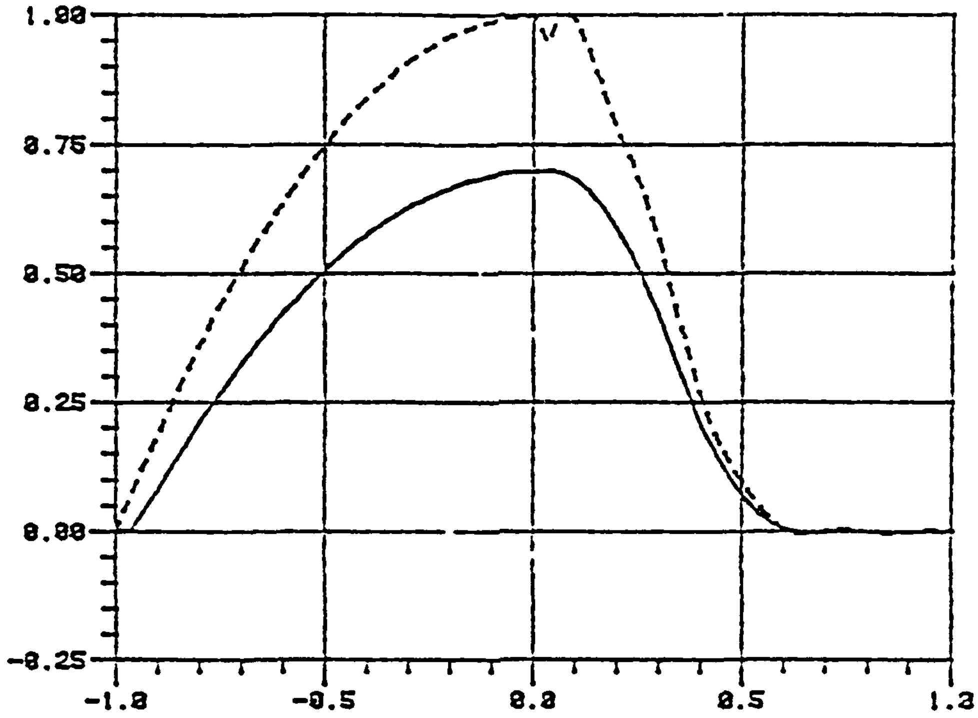

Figure 5.6a. y_ε^0 and y_ε^h for $t=2$, $\varepsilon=0.1$.

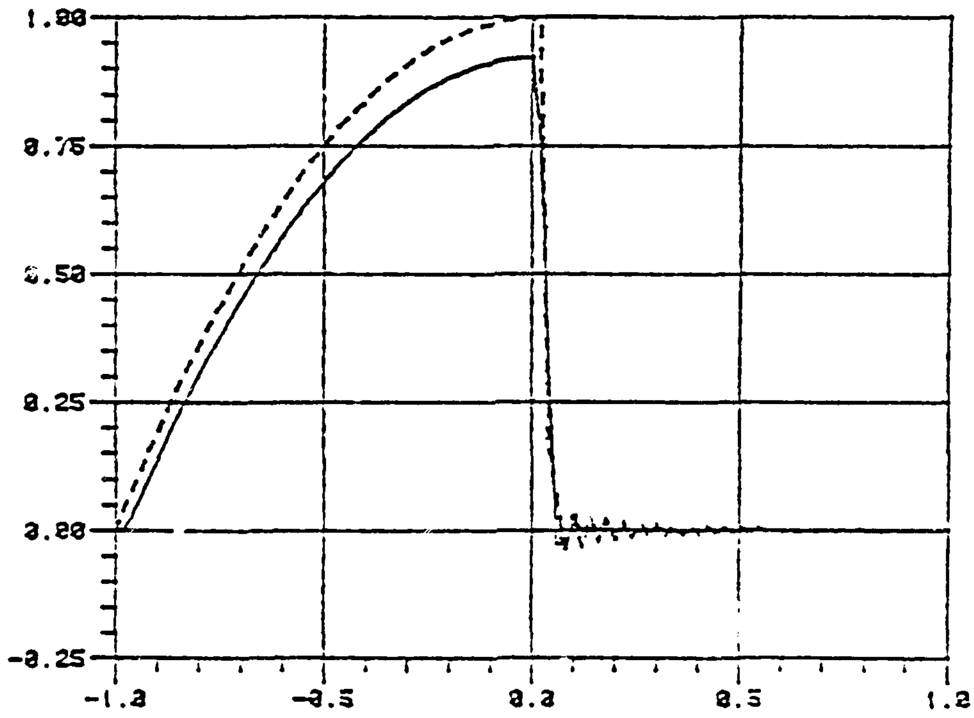

Figure 5.6b. y_ε^0 and y_ε^h for $t=2$, $\varepsilon=0.001$.

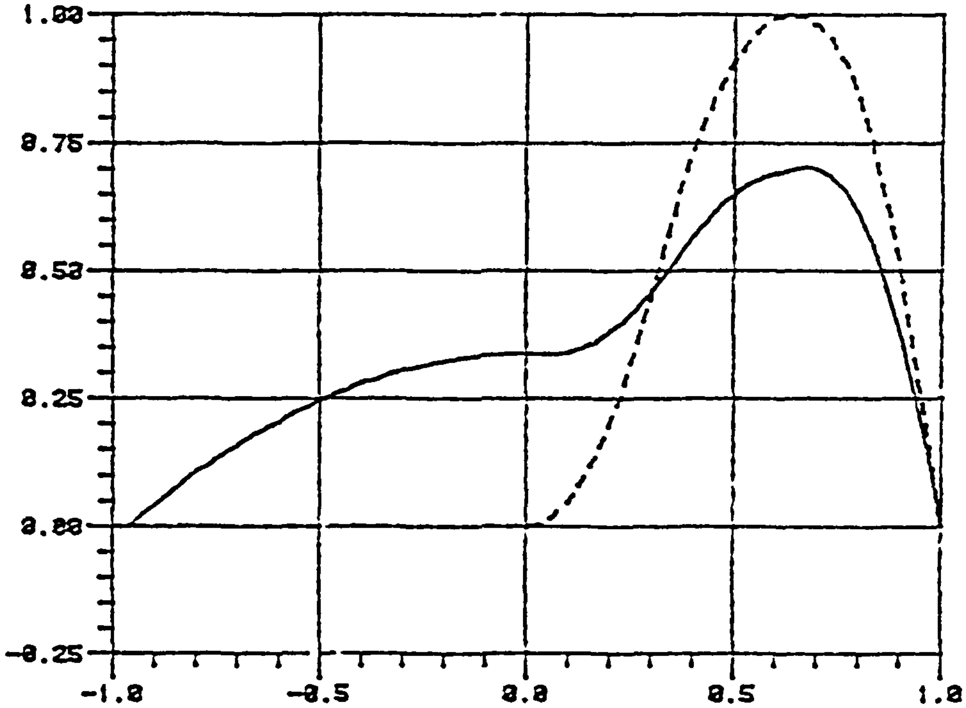

Figure 5.7a. y_ε^0 and y_ε^h for $t = 4$, $\varepsilon = 0.1$.

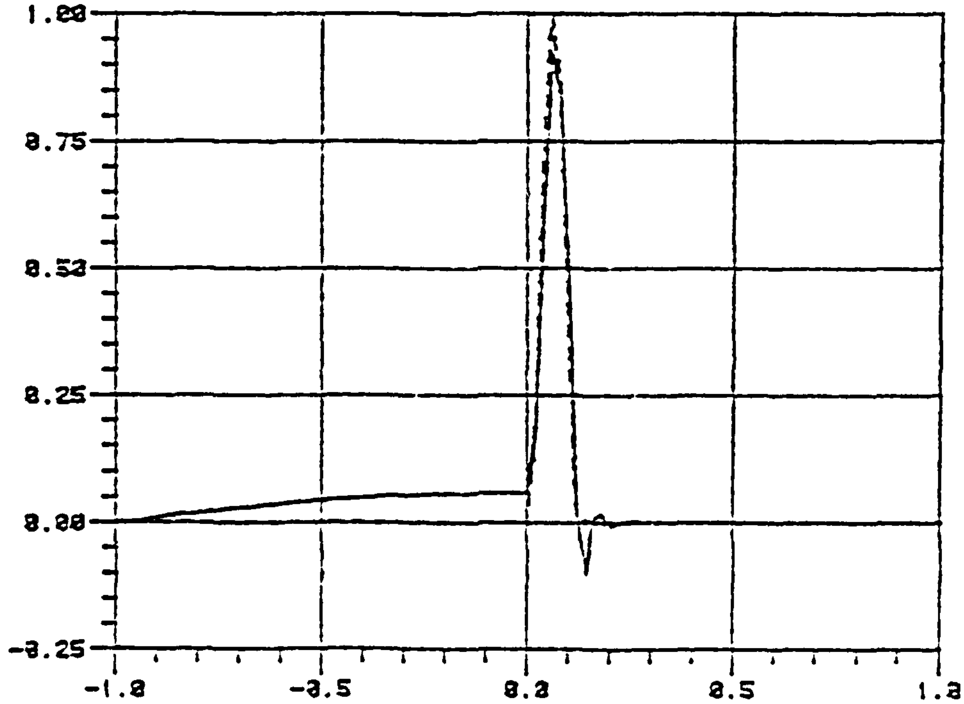

Figure 5.7b. y_ε^0 and y_ε^h for $t = 4$, $\varepsilon = 0.001$.

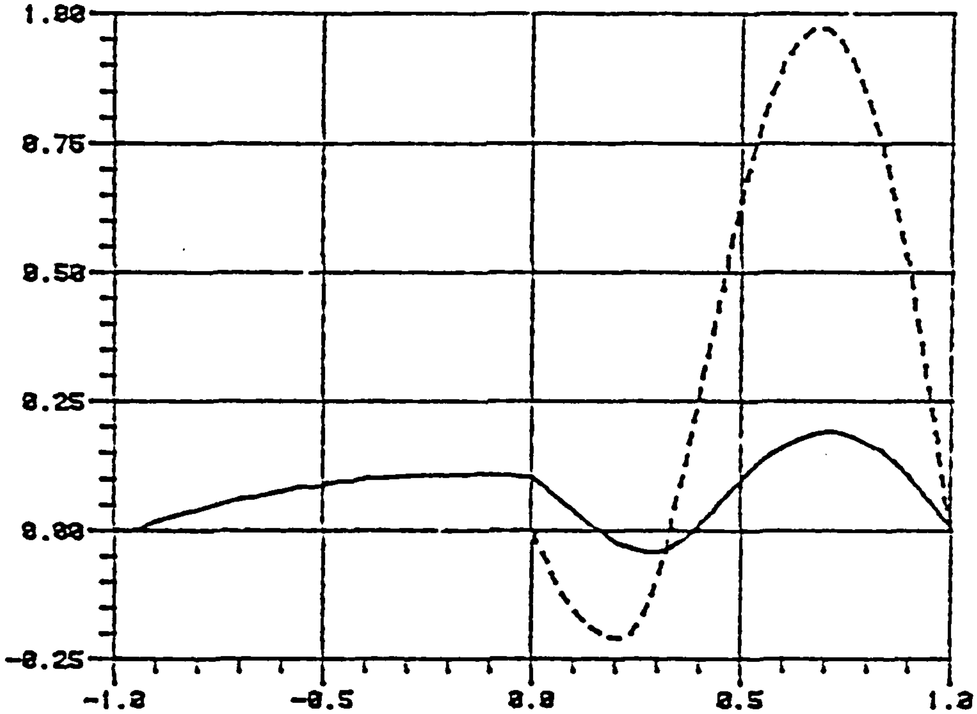

Figure 5.8a. y_ε^0 and y_ε^h for $t = 8$, $\varepsilon = 0.1$.

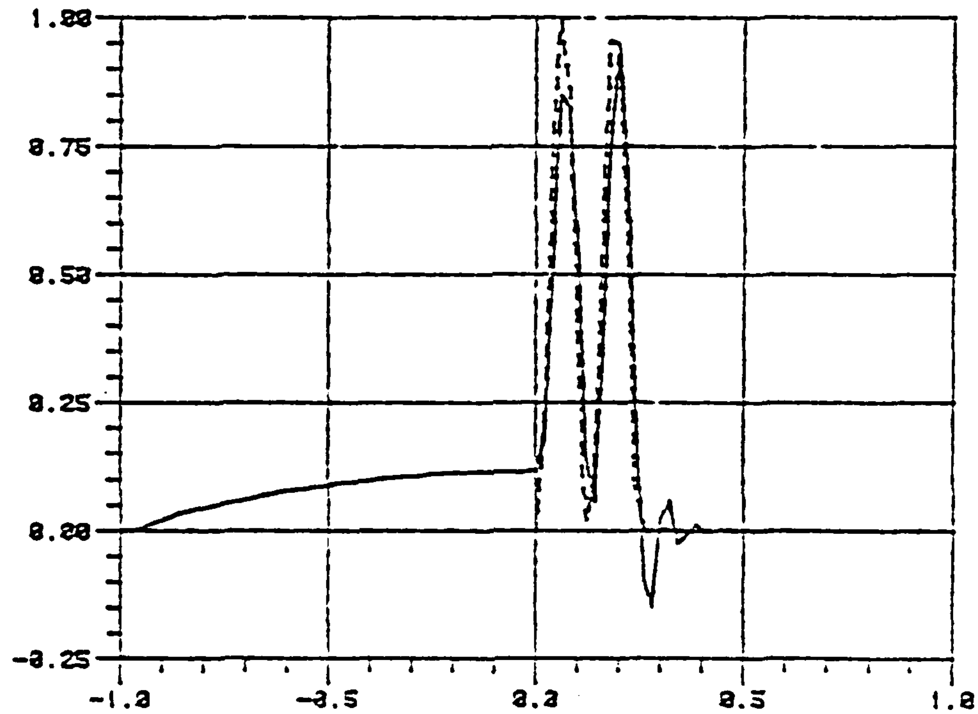

Figure 5.8b. y_ε^0 and y_ε^h for $t = 8$, $\varepsilon = 0.001$.

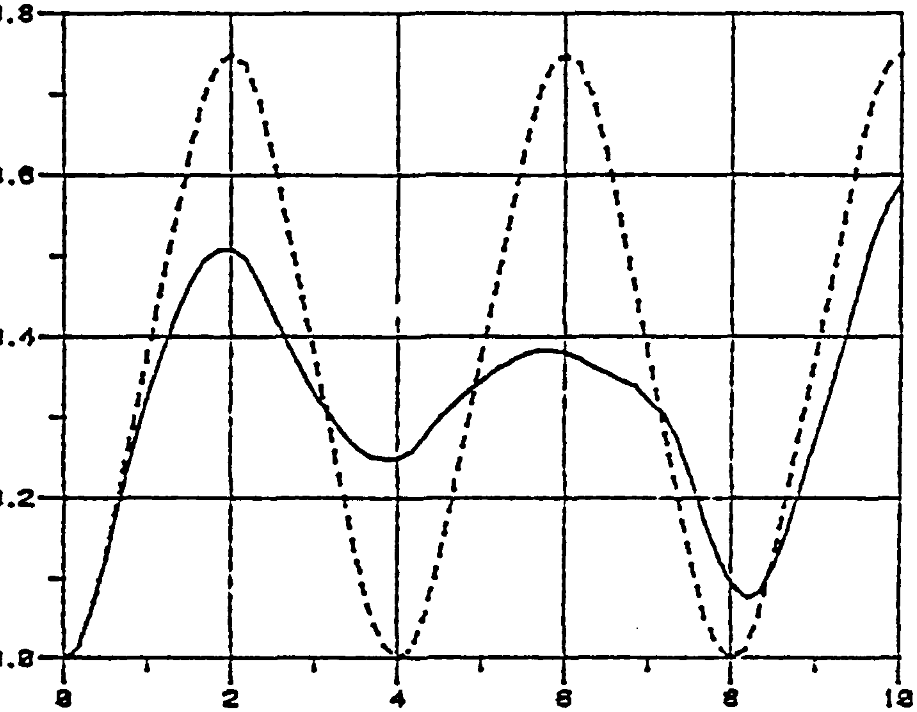

Figure 5.9a. y_{ε}^{0} and y_{ε}^{h} for $x = -0.5$, $\varepsilon = 0.1$.

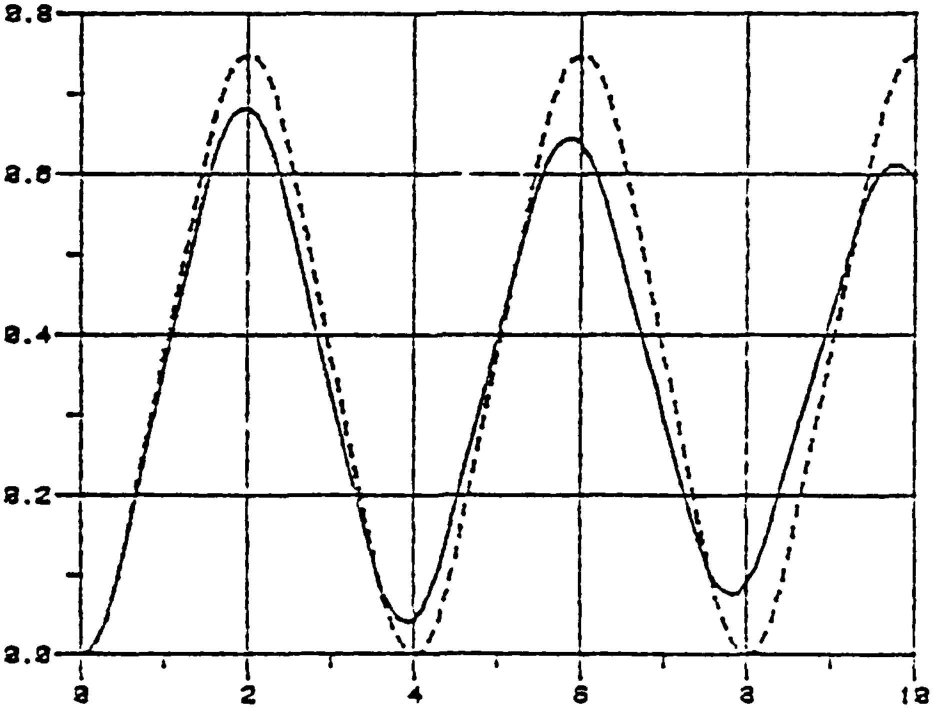

Figure 5.9b. y_{ε}^{0} and y_{ε}^{h} for $x = -0.5$, $\varepsilon = 0.001$.

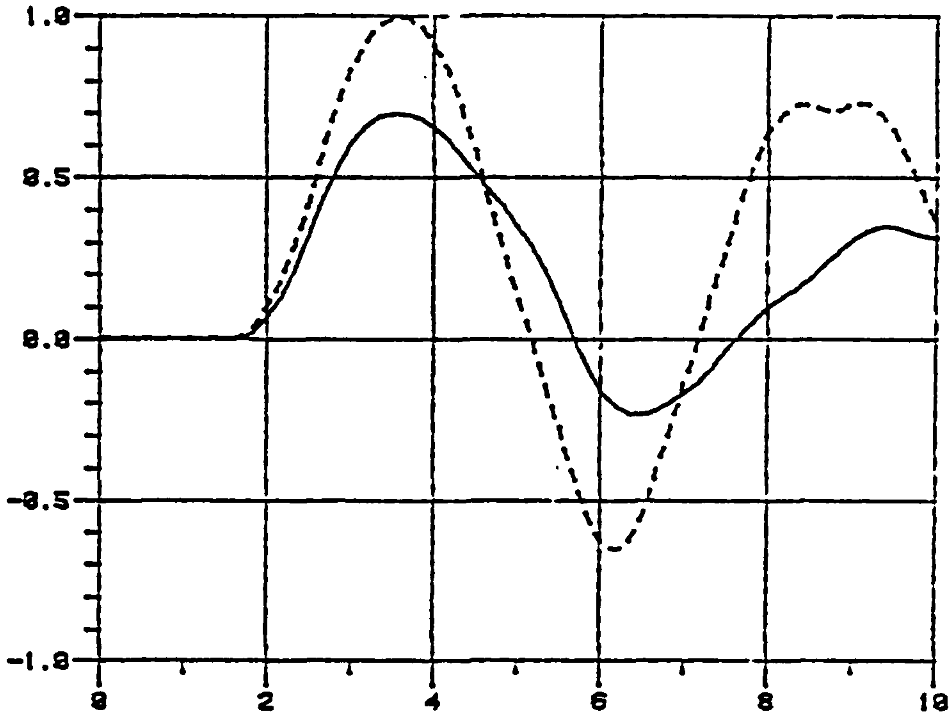

Figure 5.10a. y_ε^0 and y_ε^h for $x = 0.5$, $\varepsilon = 0.1$.

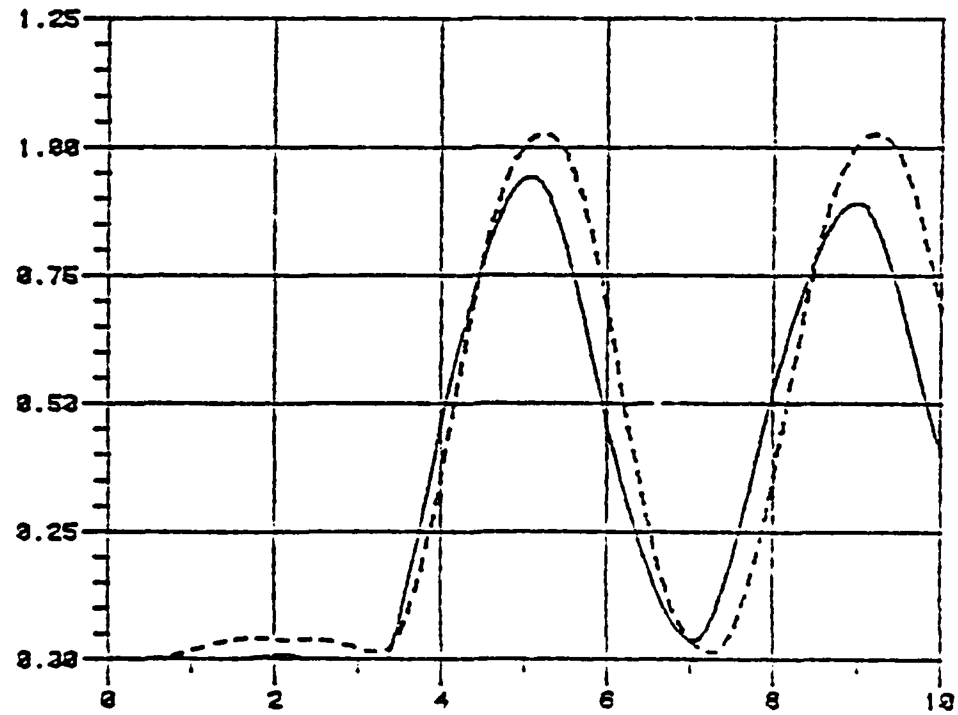

Figure 5.10b. y_ε^0 and y_ε^h for $x = 0.1$, $\varepsilon = 0.001$.

5.4. A Parabolic Control Problem

In this section, the control problem of Example 4.1 is solved numerically. Basically, the zeroth order approximation y_ε^0 of the solution of (4.58)-(4.61) for $v = 0$ is compared with the controller zeroth order approximation $y_\varepsilon^{0^c}$ of (4.58)-(4.61) for v given (4.62). The various results are summarized in Table 5.3.

TABLE 5.3. PARABOLIC CONTROL PROBLEM

Plot	Plot of $y_\varepsilon^0(x,t)$ and $y_\varepsilon^{0^c}(x,t)$ for			
5.11a	$t = 0.1$,	$x \in (-1,1)$,	$\varepsilon = 0.1$,	$\rho = 0.1$
5.11b	$t = 0.1$,	$x \in (-1,1)$,	$\varepsilon = 0.001$,	$\rho = 0.1$
5.12a	$t = 0.5$,	$x \in (-1,1)$,	$\varepsilon = 0.1$,	$\rho = 0.1$
5.12b	$t = 0.5$,	$x \in (-1,1)$,	$\varepsilon = 0.001$,	$\rho = 0.1$
5.13a	$t \in (0,1)$,	$x = -0.5$,		$\rho = 0.1$
5.13b	$t \in (0,1)$,	$x = -0.5$,		$\rho = 0.01$
5.14a	$t \in (0,4)$,	$x = 0.5$,	$\varepsilon = 0.1$,	$\rho = 0.1$
5.14b	$t \in (0,4)$,	$x = 0.5$,	$\varepsilon = 0.001$,	$\rho = 0.1$

Now some general observations are in order. First, if no control is applied, the time constants associated with region 1 become larger and larger as ε decreases. Hence it takes longer and longer for the state to decay to zero. This fact is clearly demonstrated by Figures 5.11b-5.13b.

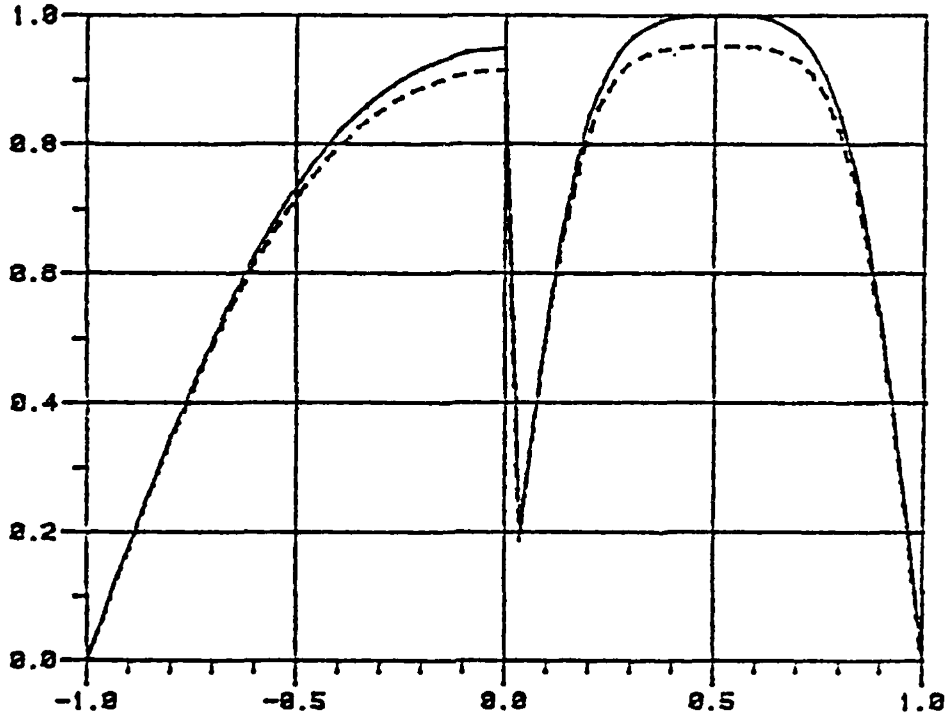

Figure 5.11a. y_ε^0 and $y_\varepsilon^{0^c}$ for $t = 0.1$, $\varepsilon = 0.1$, $\rho = 0.1$.

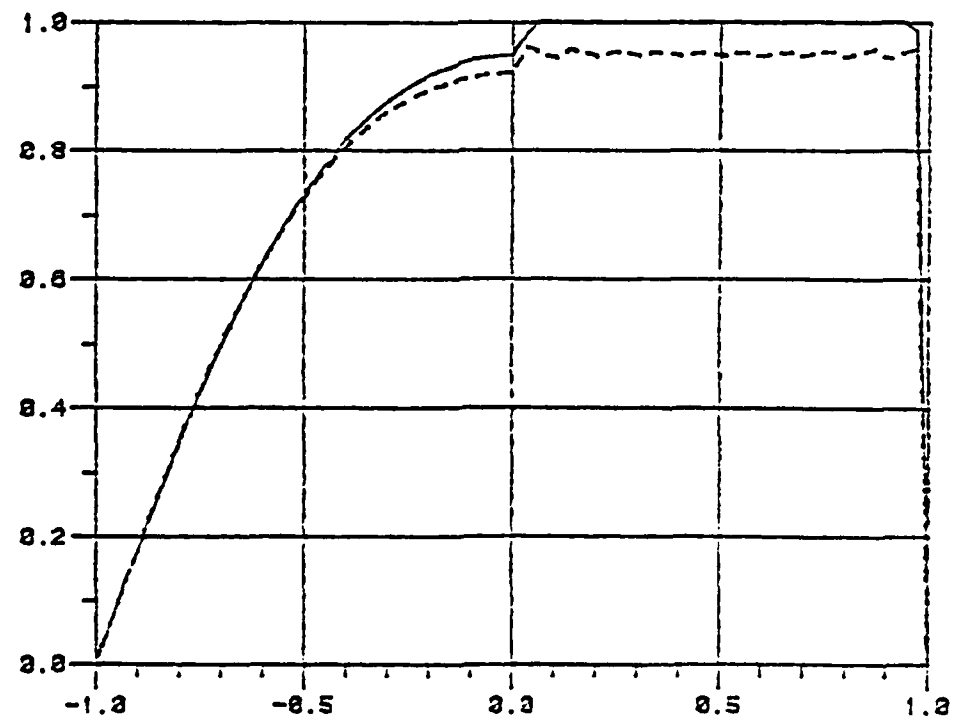

Figure 5.11b. y_ε^0 and $y_\varepsilon^{0^c}$ for $t = 0.1$, $\varepsilon = 0.001$, $\rho = 0.1$.

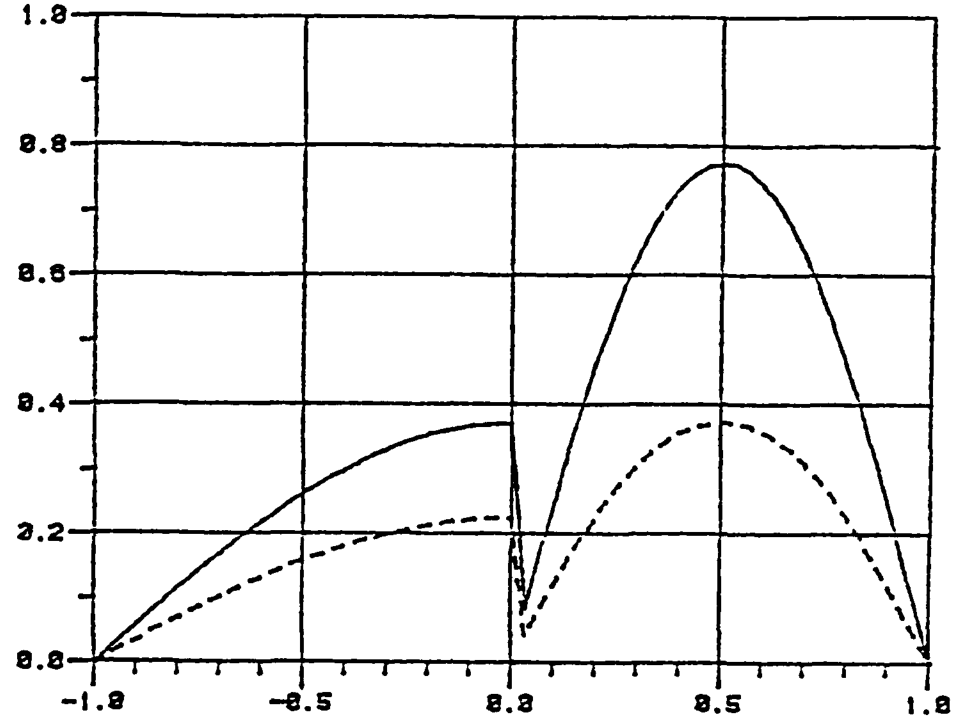

Figure 5.12a. y_ε^0 and $y_\varepsilon^{0^c}$ for $t = 0.5$, $\varepsilon = 0.1$, $\rho = 0.1$.

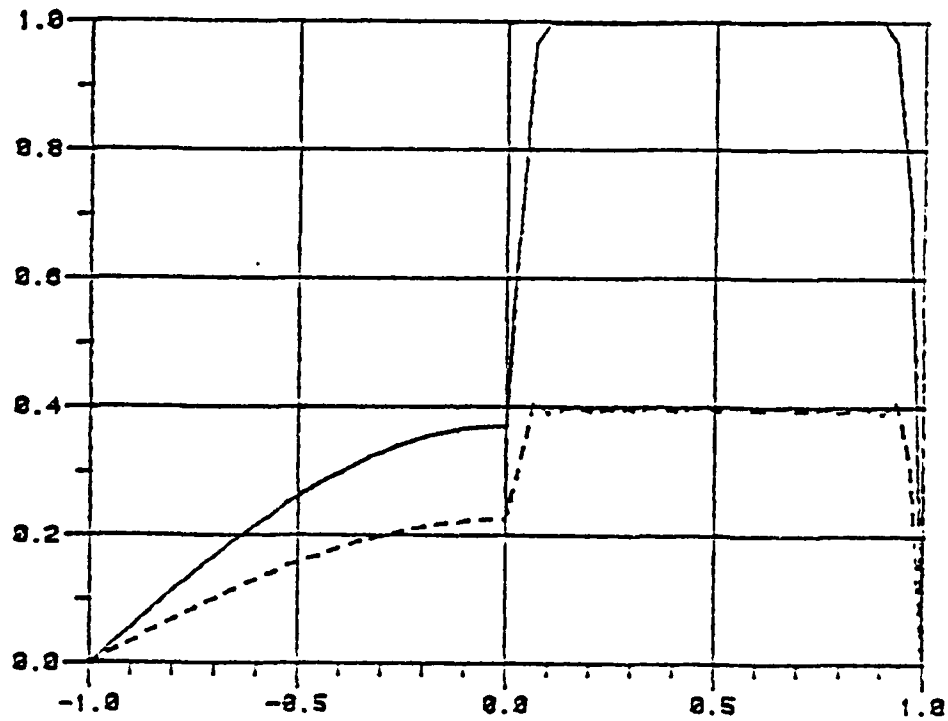

Figure 5.12b. y_ε^0 and $y_\varepsilon^{0^c}$ for $t = 0.5$, $\varepsilon = 0.001$, $\rho = 0.1$.

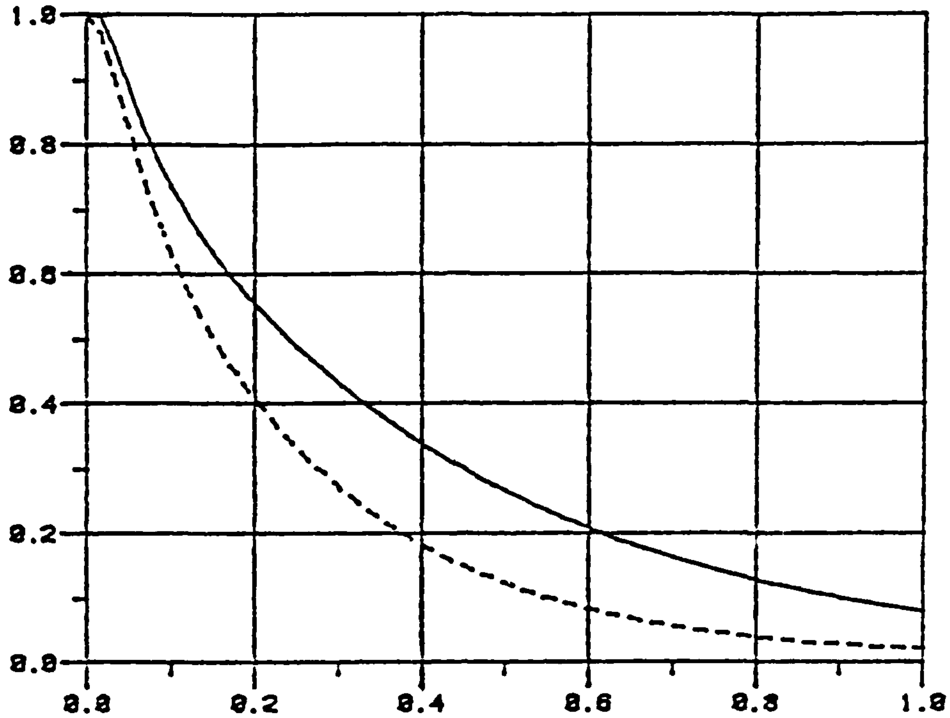

Figure 5.13a. y_0^0 and $y_0^{0^c}$ for $x = 0.5$, $\rho = 0.1$.

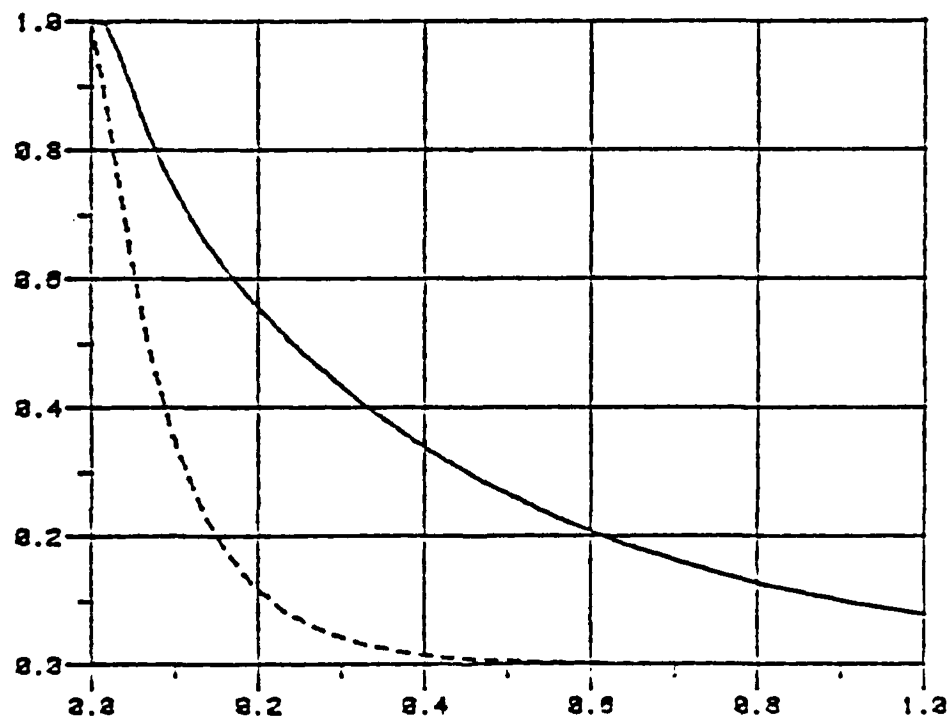

Figure 5.13b. y_0^0 and $y_0^{0^c}$ for $x = -0.5$, $\rho = 0.01$.

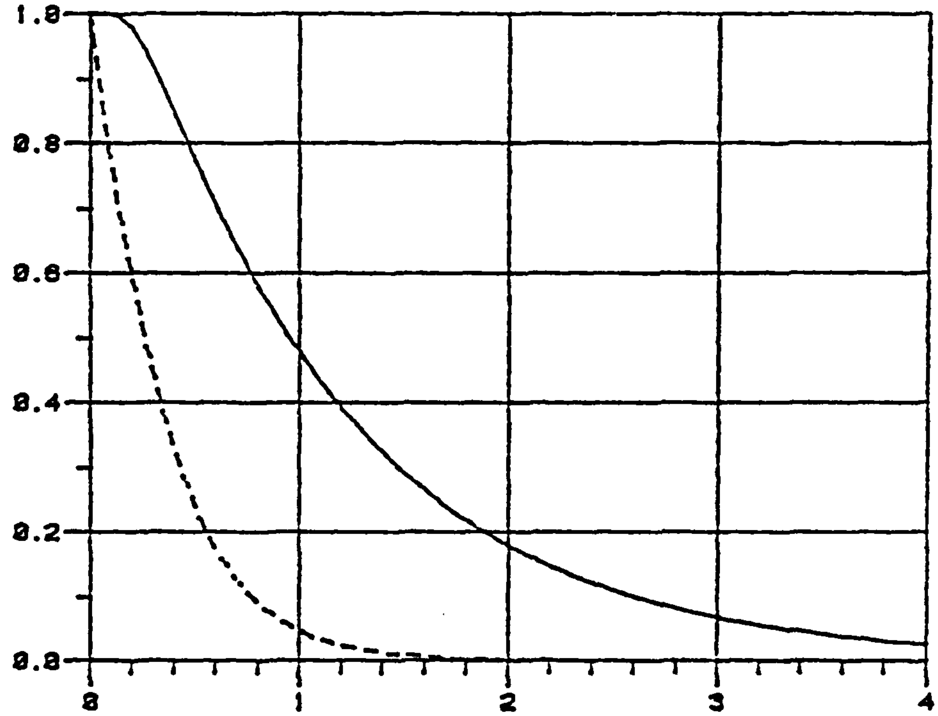

Figure 5.14a. y_ε^0 and $y_\varepsilon^{0^c}$ for $x = 0.5$, $\varepsilon = 0.1$, $\rho = 0.1$.

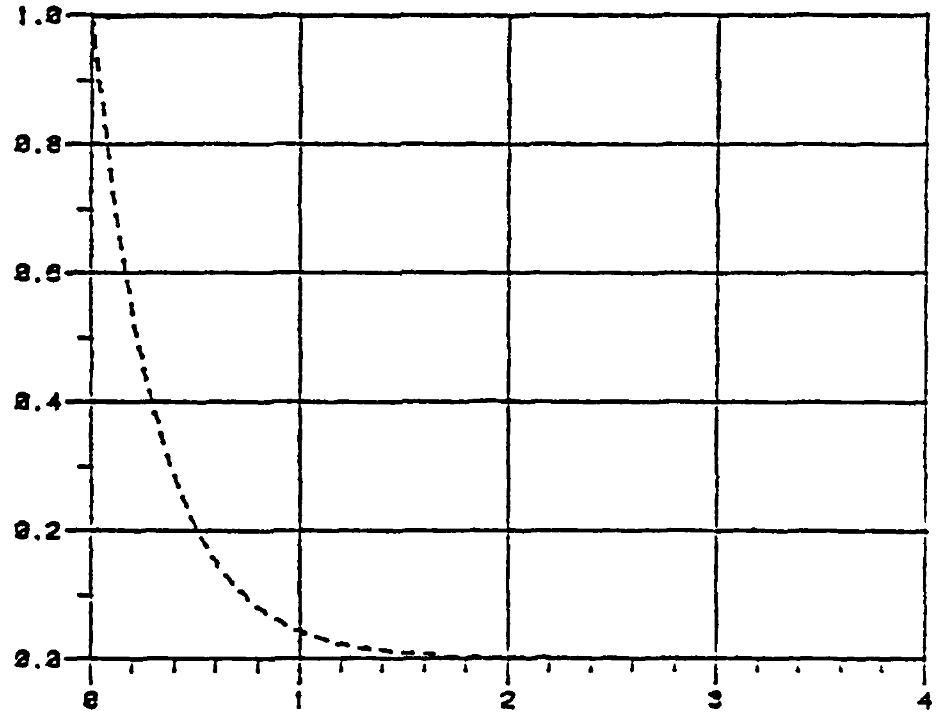

Figure 5.14b. y_ε^0 and $y_\varepsilon^{0^c}$ for $x = 0.5$, $\varepsilon = 0.001$, $\rho = 0.1$.

Second, no matter how small ε is, the effectiveness of control is not appreciably diminished. This is readily visible in Figures 5.12b, 5.14b and may be substantiated by inspecting the Riccati gains given by (4.67),(4.69).

Third, the control is regular in the interior of Ω, away from the interface as shown in Figures 5.13a-5.14b.

Finally, the effectiveness of the control is enhanced as ρ decreases, i.e., if the control becomes "cheaper," its action is more effective as seen in Figures 5.13a-b. This is not a pecularity of this example, but a general principle of control theory.

Remark 5.8: The effect of setting $y^0_{\varepsilon 1}$ to zero on R produces the dip in Figures 5.11a, 5.12a. As previously indicated, such action simplifies the feedback control synthesis and induces an error no larger than $O(\sqrt{\varepsilon})$.

5.5. Concluding Remarks

In this chapter, the numerical analysis of Examples (3.6)-(3.7), (4.1) is undertaken. The conclusion is that the numerical results agree quite well with the theoretical ones obtained in Chapters 3 and 4.

The simplicity of the examples investigated conceals many aspects. For example, if the geometry of Ω is more complicated, e.g., n-dimensional $(n > 1)$ and polygonal, the direct computation of even the limits of the eigenvectors is very elaborate. Hence, how can the approach of this thesis be extended? It was indicated in Remark 5.7 that a direct approach using a finite element method is hopeless. Moreover, very complex integration routines have to be used because of the inherent stiffness of the problem at hand.

The following approach seems to be the logical alternative, which consists of the combination of the direct approach and the approach pursued in this thesis, i.e.,

Step 1: Use a finite element method to find the limits of the eigenvectors of A_ε as $\varepsilon \to 0$. In so doing, the limits of $\{\mu_\varepsilon^k\}_{k=1}^\infty$ and the rate of convergence of $\{\lambda_\varepsilon^k\}_{k=1}^\infty$ are obtained as a by-product of this computation.

Step 2: Use the zeroth order approximations derived in Chapter 3, where $\{\lambda_1^k, \varphi^k\}_{k=1}^\infty$ and $\{\mu_0^k, \psi^k\}_{k=1}^\infty$ are replaced by those computed in Step 1.

In the first step, the stiffness of the problem is alleviated. In the second step, the use of expensive integration routines is eliminated. However, some integration is still required to get the desired approximation.

CHAPTER 6

CONCLUSION

6.1. Concluding Remarks

In this report, the spectral decomposition of some stiff partial differential operators is undertaken. One class of such operators has coefficients that are $O(1), O(\varepsilon), \ldots, O(\varepsilon^p)$ in $\Omega_0, \Omega_1, \ldots, \Omega_p$ ($\Omega_i \cap \Omega_j = \phi$ if $i \neq j$), whose union constitutes the open connected set $\Omega \subset R^n$. It is found that the eigenvalues of these operators can be separated into p+1 groups, depending on how they converge as $\varepsilon \to 0$. Each group is associated with a region Ω_i, $i = 0, 1, \ldots, p$. The eigenvectors can also be classified accordingly. Their convergence as $\varepsilon \to 0$ is intimately related to the order of the operators and the location of the coefficients in question. A general rule has emerged out of the present investigation. Suppose, without loss of generality, that the coefficients in question occur in the kth derivatives ($|k| \leq$ operator order). Then the higher the $|k|$, the weaker is the convergence of these eigenvectors as $\varepsilon \to 0$. This conclusion is not surprising because in this case the eigenvectors are more regular, in general [30]. The details for some typical cases are summarized in Table 2.1.

In this thesis, the value of the parameter ε is assumed to be small. Consequently, the behavior of the spectrum of stiff operators is analyzed as $\varepsilon \to 0$. However, similar results can be stated as $\varepsilon \to +\infty$, as evidenced by Corollary 2.1.

In this thesis, it is also shown that approximations of solutions of stiff boundary value problems (including some control problems) can be derived using the weak limits of the eigenvectors of stiff operators as $\varepsilon \to 0$. These approximations are "readily" computable. However, in general, they are not as regular as the exact solutions, which may not be accessible at all.

6.2. Topics for Future Research

Many unsettled issues directly related to this thesis merit further research, some of which are:

1. The analyticity of $\{\varphi_\varepsilon^k\}_{k=1}^\infty$ (Cf. Remark 2.26).

2. When Ω_0 is not connected, e.g., $\Omega_0 = \Omega_{00} \cup \Omega_{01}$ (Cf. Remark 2.10), the limits of $\{\mu_\varepsilon^k, \psi_\varepsilon^k\}_{k=1}^\infty$, i.e., $\{\mu_0^k, \psi^k\}_{k=1}^\infty$ can be decomposed further into $\{\mu_{00}^k, \psi_{00}^k\}_{k=1}^\infty$ and $\{\mu_{01}^k, \psi_{01}^k\}_{k=1}^\infty$, each pair associated with a subset of Ω_0, as indicated by the subscripts. It is not clear if there exists one or two eigenvalue-eigenvector pairs for $\varepsilon > 0$, which correspond to each of the above pairs.

Finally, it would be worthwhile to examine if the present methodology can be used to approximate the solutions of the following problems:

3. Semilinear boundary value problems [4],

4. Unilateral problems [4,23],

5. Inverse problems [6],

6. Games [3,35],

when they involve stiff operators.

APPENDIX

In this appendix, some of the mathematical tools needed in this thesis are discussed and the inquiring reader is referred to the appropriate references.

The following subjects are very briefly reviewed.

1. Definition of $O(\cdot)$ and $o(\cdot)$
2. Weak convergence
3. Distributional derivatives and functional spaces
4. Definition of the space $\mathcal{L}(X;Y)$

1. Definition of $O(\cdot)$ and $o(\cdot)$

Let $f_0(\varepsilon)$ and $f_1(\varepsilon)$ be real, positive, continuous functions of ε in $0 < \varepsilon \le \varepsilon_0$ such that $\lim_{\varepsilon \to 0} f_i(\varepsilon)$, $i = 0,1$ exist.

1.1. Definition of $O(\cdot)$

$f_0(\varepsilon) = O(f_1(\varepsilon))$ if there exists a constant C such that

$$f_0(\varepsilon) < Cf_1(\varepsilon) \qquad \text{for} \quad 0 < \varepsilon \le \varepsilon_0.$$

1.2. Definition of $o(\cdot)$

$$f_0(\varepsilon) = o(f_1(\varepsilon)) \quad \text{if} \quad \lim_{\varepsilon \to 0} \frac{f_0(\varepsilon)}{f_1(\varepsilon)} = 0$$

For more details, consult [11].

2. Weak Convergence

Let H be a real Hilbert space with scalar product $(x,y)_H$. A sequence $\{x_n\}$ converges weakly to an element $x \in H$ if

$$(x_n, y)_H \to (x, y)_H, \qquad \forall y \in H.$$

For further inquiry, consult [42].

3. Distributional Derivatives and Function Spaces

Let $x = \{x_1, x_2, \ldots, x_n\}$ denote the space variable; x ranges over an open set $\Omega \subset \mathbb{R}^n$, with boundary Γ; t denotes time, $t \in (0,T)$, $T < \infty$. Let

$C^k(\Omega)$ = space of k-times continuously differentiable functions on Ω, $k \in \mathbb{N}$

$C_0^k(\Omega)$ = space of k-times differentiable functions in Ω, with compact support in Ω.

$\mathcal{D}(\Omega) = C_0^\infty(\Omega)$.

$\mathcal{D}^*(\Omega)$ = dual space of $\mathcal{D}(\Omega)$, i.e., the space of distributions on Ω.

$L^2(\Omega)$ = space of functions, which are square integrable on Ω, which is a Hilbert space ($L^2(\Omega)$ is identified with its dual).

3.1. Distributional derivatives

Let

$$p = \{p_1, p_2, \ldots, p_n\}, \quad |p| = p_1 + p_2 + \cdots + p_n$$

$$D^p = D_1^{p_1} D_2^{p_2} \ldots D_n^{p_n}, \quad D_i = \frac{\partial}{\partial x_i}$$

$$f \in L^2(\Omega)$$

$D^q f$ is said to be the qth-distributional derivative of f if

$$\langle D^q f, \varphi \rangle = (-1)^{|q|} \langle f, D^q \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega)$$

where $\langle \cdot\, , \cdot \rangle$ designates duality between $\mathcal{D}(\Omega)$ and $\mathcal{D}^*(\Omega)$.

3.2. Sobolev spaces

One may define the Sobolev space of order m as

$$H^m(\Omega) = \{\varphi : \varphi \in L^2(\Omega),\ D^q\varphi \in L^2(\Omega),\ \forall q,\ |q| \le m\}$$

which is a Hilbert space if endowed with the scalar product

$$(\varphi,\psi)_{H^m(\Omega)} = \sum_{|q| \le m} (D^q\varphi, D^q\psi)_{L^2(\Omega)} .$$

For any $\varphi \in H^m(\Omega)$, one can define uniquely its traces on the boundary, i.e.,

$$\varphi\Big|_\Gamma,\ \frac{\partial\varphi}{\partial\nu}\Big|_\Gamma, \ldots,\ \frac{\partial^{m-1}\varphi}{\partial\nu^{m-1}}\Big|_\Gamma$$

where $\varphi \longmapsto \{\frac{\partial^k\varphi}{\partial\nu^k}\Big|_\Gamma,\ 0 \le k \le m-1\}$ is a continuous linear surjective map of $H^m(\Omega)$ onto $\prod_{k=0}^{m-1} H^{m-k-1/2}(\Gamma)$. Using the trace map, one can define several subspaces of $H^m(\Omega)$ such as

$$H^m(\Omega;\Gamma_0) = \{\varphi : \varphi \in H^m(\Omega),\ \frac{\partial^k\varphi}{\partial\nu^k}\Big|_{\Gamma_0} = 0,\ \Gamma_0 \subset \Gamma,\ 0 \le k \le m-1\}$$

$$H_0^1(\Omega) = H^m(\Omega;\Gamma).$$

If time is involved, many Hilbert spaces can be defined in a similar way, e.g., $L^2(0,T;V)$ = space of functions defined on $(0,T)$ with values in a space V such that

$$\int_0^T \|\varphi\|_V^2 dt < \infty$$

where V may be any of the above Sobolev spaces or their duals.

If $V = L^2(\Omega)$, then

$$L^2(0,T;V) = L^2(0,T;L^2(\Omega)) = L^2(\Omega \times (0,T))$$

A systematic study of these spaces, as well as many of their subspaces, is found in [2,30].

4. Definition of $\mathcal{L}(X;Y)$

$\mathcal{L}(X;Y)$ is the vector space of continuous linear operators from X to Y. The norm of an operator $A \in \mathcal{L}(X;Y)$ is defined by

$$\|A\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Ax\|_Y}{\|x\|_X} .$$

If Y is a Hilbert space and X is a pre-Hilbert space, the space $\mathcal{L}(X;Y)$ is a Banach space. See [2,15,42] for more details.

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