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A lidar return from a homogeneous atmosphere is used to examine the sensitivity of an inversion algorithm to uncertainties in parameters in a power law relating backscatter to extinction.
INTRODUCTION

In an earlier paper, Klett\(^1\) presented a stable solution to the single-scattering lidar equation

\[
S(r) \equiv \ln[P(r)r^2] = \ln[C_1 c_2] + k \ln \sigma(r) - 2 \int_0^r \sigma(r) \, dr.
\] (1)

In this equation \(P(r)\) is the power received at a range \(r\) and \(C_1\) is \(P_0^{\text{STAP}}\)
where \(P_0\) is the transmitted power, \(c\) is the speed of light, \(\tau\) is the pulse width, \(A\) is the receiver area and \(F\) is the overall system response. The constants \(C_2\) and \(k\) are atmospheric dependent parameters which relate the coefficients of backscatter, \(\beta_\pi(r)\), and extinction, \(\sigma(r)\), according to the power law relationship\(^1\)

\[
\beta_\pi(r) = C_2 \sigma(r)^k.
\] (2)

From Equation (1) a differential equation (independent of \(C_1\) and \(C_2\))

\[
\frac{dS(r)}{dr} = k \frac{d\sigma(r)}{\sigma(r)} - 2\sigma(r)
\] (3)

can be obtained which has the well known solution

\[
\sigma(r) = \frac{\exp\left\{[S(r) - S(r_0)]/k\right\}}{\sigma(r_0)^{-1} - \frac{2}{k} \int_0^r \exp\left\{[S(r) - S(r_0)]/k\right\} \, dr},
\] (4)

where \(\sigma(r_0)\) is the extinction coefficient at a range \(r_0\). This range is normally chosen where the transmitted beam and receiver field-of-view overlap. In this equation the extinction is determined by the ratio of two
numbers which each become smaller as \( r \) increases. In particular, for good
visibilities the denominator becomes the difference between large numbers and
small fluctuations in \( S(r) \) can cause large oscillations in \( \sigma(r) \). Klett\(^1\) also
discusses the instabilities in \( \sigma(r) \) caused by uncertainties in estimating
\( \sigma(r_0) \). He proposed a more stable solution to Equation (3) for \( r<r_f \) rather
than \( r>r_o \). This new solution is

\[
\sigma(r) = \frac{\exp\left[\frac{S(r) - S(r_f)}{k}\right]}{\sigma(r_f)^{-1} + \frac{2}{k} \int_{r_f}^{r} \exp\left[\frac{S(r) - S(r_f)}{k}\right] dr}
\]

(5)

where \( \sigma(r_f) \) is the extinction coefficient at range \( r_f \). In this case it is not
possible for the denominator to become zero or negative and \( \sigma(r) \) is
determined by the ratio of two numbers which become larger as \( r \) decreases from
\( r_f \).

Recently, Ferguson and Stephens\(^2\) described a novel approach in deter-
mining \( \sigma(r) \) using both Equations (4) and (5). In their algorithm an iterative
scheme is used to select the appropriate value of \( \sigma(r) \) using Equation (5). A
value of \( \sigma(r) \) is determined such that the value of \( S_c(r) \) calculated from
Equation (1) matches the measured value \( S_m(r) \). The iteration procedure is
initiated at a close-in range where the returned signal is well above the
system noise and requires a minimum amount of computational time. To
demonstrate the utility of their algorithm, Ferguson and Stephens\(^2\) presented
examples of calculated and measured values of \( S(r) \) normalized to the product
\( C_1C_2 \). RMS differences of less than \( 10^{-4} \) over the range \( r_o \) to \( r_f \) indicate the
value of \( \sigma(r_f) \) to be very accurate. This value was then used as \( \sigma(r_o) \) in
Equation (4) to determine \( \sigma(r) \) for \( r>r_f \). In this case it is possible to
extend \( \sigma(r) \) out to ranges where \( S_m(r) \) disappears into system noise. However,
it must be realized that the solution for \( \sigma(r) \) is not unique unless the values of \( C_2 \) and \( k \) along the propagation path are known. The proper choice of these parameters is a critical problem in interpretation of lidar returns. In his original paper, Klett\(^1\) showed Equation (5) to be rather insensitive to changes in \( k \) for highly turbid atmospheres. Little attention has been given to the sensitivity of the inversion algorithms to changes in \( C_2 \) which is usually assumed to be invariant. However, from the work of Barteneva\(^3\), a change greater than an order of magnitude can be inferred in the value of \( C_2 \) between clear air and fog conditions. Fitzgerald\(^4\) has also pointed out that a power-law relationship between backscatter and extinction is only valid for relative humidities greater than about 80 percent and, even then, it is dependent upon the air mass characteristics.

In this paper, we investigate the sensitivity of the inversion algorithm to uncertainties in \( C_2 \) and \( k \). For this study we utilized data acquired in September 1982 at a research site located near the Baltic Sea. The data were obtained using a handheld battery operated lidar, termed a "visioceilometer," developed by the U.S. Army for measuring atmospheric visibility and cloud heights.\(^5\) The optical unit is a modified AN/GVS-5 laser rangefinder which emits a 10-m J, 6-nsec pulse at 1.06\( \mu \)m. The signal processing unit clocks the return signal through a transient recorder at a 20-MHz rate giving a 7.5 m sampling interval. The digitized results are transferred to a microprocessor and then to a tape recorder for off-line processing by a Hewlett-Packard 9845B computer.

**INVERSION EXAMPLES AND ANALYSIS**

For this study we have chosen a lidar return obtained during reduced visibility conditions for which the Klett inversion (Equation 5) rapidly
converges. Figure 1 is a plot of $S(r)$ versus range determined from a lidar return. The initial slope of the curve shows a nearly monotonic decrease with range and indicates a homogeneous atmosphere out to nearly 500 m. In this case, (from Equation 1) the extinction coefficient is given by $-\frac{1}{2} \frac{dS(r)}{dr}$. Beyond 500 m the atmosphere appears less homogeneous and may represent a variation with altitude as the lidar was elevated 3°. Here we will limit the analysis to the homogeneous portion of the return between ranges $r_o = 112.5$ m and $r_f = 412.5$ m. For this portion of the curve, a least squares straight line fit (with a correlation coefficient of 0.956) gives

$$S(r) = -1.68r - 4.98$$

(6)

from which the homogeneous extinction coefficient $\sigma_0$ is determined to be $0.84$ km$^{-1}$. By equating Equation (6) to Equation (1) for the homogeneous case we find (for $k = 1$)

$$\ln[C_1C_2] = -4.8.$$  

(7)

If the system constant $C_1$ is known, then $C_2$ is determined for this special case. However, since the overall system response of the visioceilometer is not known, we will in fact determine the sensitivity of the algorithm to uncertainties in the parameter $\ln[C_1^'C_2^']$ which are directly related to variations in $C_2$ (for fixed values of $k$) by the expression

$$\ln[C_1^'C_2^'] = \ln[C_2/C_2] - 4.8$$

(8)
Figure 1. $S(r)$ versus range determined from a lidar return.

$S(r) = -1.68r - 4.98$

($\rho = .956$)
where \( C_2' \) differs from the correct value, \( C_2 \). In Figure 2 the value of \( C_2' \) is allowed to change by \( \pm 40\% \) from \( C_2 \) with resulting changes in \( \ln[C_1C_2'] \) from +7\% to -10\%.

Figures 3A to I are plots of the extinction coefficients calculated using the values of \( \ln[C_1C_2'] \) shown in each figure and the \( S(r) \) values from Figure 1 with \( k = 1 \). For these calculations the RMS differences between the measured and calculated values of \( S(r) \) ranged between \( 10^{-2} \) and \( 10^{-4} \). The extinction coefficient in Figure 3E \( (\ln[C_1C_2'] = -4.79) \) is nearly constant out to 500 m and is in close agreement with that determined from the slope of \( S(r) \) in Figure 1. However, for the cases where \( C_2'/C_2 < 1 \) (Figures 3A to D), the calculated extinctions increase with range to a value near 4 km\(^{-1} \) in Figure 3A. When \( C_2'/C_2 > 1 \) (Figures 3F to I) the extinction coefficients decrease with range and tend towards zero in Figure 3I. These tendencies are similar to the singularities and zeros discussed by Klett\(^1 \) when the inversion is started at the beginning of the return signal.

Also listed in the figures are the visibilities determined from the relation

\[
\text{VIS} = \frac{3.912 (r_f - r_o)}{\int_{r_o}^{r_f} \sigma(r) \, dr}.
\]  

(9)

In Figure 4 the calculated visibilities are plotted for the relative changes in \( C_2 \) and differing values of \( k \). In general, these calculations show the visibility to be rather insensitive to \( k \). However, we have found the extinction coefficients to be sensitive to small changes in \( k \) when \( \sigma(r) \) is at or approaching a singularity as in Figures 3A and 3B. The sensitivity of
Figure 2. Changes in the parameter $\ln[c_1c_2']$ with changes in the value of $C_2$ appropriate for Equation 7.
Figure 3. Calculated values of extinction coefficient $\sigma(x)$ as a function of range for different values of the parameter $\ln[C_{1}/C_{2}]$. 
Figure 4. Calculated visibilities for relative changes in $C_2$ for differing values of $k$. 
visibility to relative uncertainties in $C_2$ is seen in Figure 4 where a 20 percent uncertainty can cause nearly a 30 percent change in visibility.

The variations of extinction coefficients with range can be explained for a uniform atmosphere [$\sigma(r) = \sigma_o$] if we use values of $C_2$ and $\sigma^*(r)$ to evaluate the closeness of the measured and calculated values of $S(r)$, i.e.,

$$S_m(r) \equiv S_c(r)$$

or

$$\ln[C_1 C_2] + k \ln(\sigma_o) - 2 \sigma_o r = \ln[C_1 C_2'] + k \ln\sigma^*(r)$$

$$- 2 \sigma^*(r_o) r_o - 2 \int \sigma^*(r) dr.$$  \hspace{1cm} (11)

Rearranging gives

$$-\ln[C_2/C_2'] + k \ln[\sigma_o/\sigma^*(r)] - 2 \sigma_o r + 2 \sigma^*(r_o) r_o$$

$$+ 2 \int \sigma^*(r) dr = 0.$$  \hspace{1cm} (12)

For a uniform atmosphere,

$$S_m(r) - S_m(r_o) = -2 \sigma_o (r - r_o)$$

and $\sigma^*(r)$ are then determined from Equation (4) to be (for $k = 1$)

$$\sigma^*(r) = \frac{\exp[-2 \sigma_o (r - r_o)]}{\sigma^*(r_o)^{-1} - 2 \int \exp[-2 \sigma_o (r - r_o)] dr}$$  \hspace{1cm} (14)
from which $\sigma'(r_o)$ is determined to be

$$\sigma'(r_o) = \frac{\sigma_o}{[\sigma_o/\sigma'(r) - 1] \exp[-2\sigma_o(r - r_o)] + 1}.$$  \hfill (15)

The integral term in Equation (12) is also readily determined from tables to be

$$2 \int_r^{r_o} \sigma'(r)dr =$$

$$\frac{\sigma_o}{[\sigma_o/\sigma'(r) - 1] \exp[-2\sigma_o(r - r_o)] + 1} \left[ (\sigma_o/\sigma'(r) - 1) \exp[-2\sigma_o(r - r_o)] + 1 \right].$$  \hfill (16)

and upon substitution of Equations (15) and (16) into (12) we arrive at the equation

$$\ln \chi = 2 \sigma_o (1 - \frac{1}{\chi}) + \ln[\sigma_o'^{2}/C_2^{2}]$$  \hfill (17)

where

$$\chi = [\sigma_o/\sigma'(r) - 1] \exp[-2\sigma_o(r - r_o)] + 1.$$  \hfill (18)

and

$$\sigma'(r) = \frac{\sigma_o}{1 + (\chi - 1) \exp [2\sigma_o(r - r_o)]}.$$  \hfill (19)

It is then seen that for $\chi < 1$, $\sigma'(r)$ becomes infinite at a range

$$r - r_o = -\frac{1}{2\sigma_o} \ln(1 - \chi).$$  \hfill (20)
For \( \chi > 1 \) we find that \( \sigma'(r) + 0 \) as \( r \to \infty \). Denoting the left and right sides of Equation (17) as \( f_1(\chi) \) and \( f_2(\chi) \), respectively, a graphical solution to the equation is presented in Figure 5 for \( \sigma = 0.85 \text{ km}^{-1} \). From the curves it is seen that \( \chi > 1 \) when \( C_2'/C_2 > 1 \) and \( \chi < 1 \) for \( C_2'/C_2 < 1 \). An example of a numerical solution of Equation (17) is shown in Figure 6 which illustrates the behavior of \( \sigma'(r) \) with range for uncertainties in \( C_2 \). If \( C_2 \) is underestimated, \( \sigma'(r) \) tends to increase without bound. If \( C_2 \) is underestimated by 20 percent, \( \sigma'(r) \) differs from the known \( \sigma \) by a factor of 4 at a range less than 1 km. If \( C_2 \) is over estimated, \( \sigma'(r) \) tends to zero. These trends in \( \sigma'(r) \) are identical to those illustrated from the measured data in Figure 3.
Figure 5. Graphical solution of the parameter $\chi$ in Equation (17) for relative changes in $C_2'$. 

$\sigma_0 = 0.85 \text{ km}^{-1}$

$k = 1.0$

$f_1(\chi) = \chi$

$f_2(\chi) = \begin{cases} 
\frac{C_2'}{C_2} = 1.2 \\
\frac{C_2'}{C_2} = 0.8 
\end{cases}$
Figure 6. Numerical solution of Equation (17) for $\sigma'(r)$ versus range for relative changes in $C_2'$.
CONCLUSIONS

This analysis has determined that range dependent extinction coefficients inferred from lidar returns are extremely sensitive to the value of the constant relating backscatter and extinction for a reduced visibility condition. The instabilities will also occur for better visibilities but at longer ranges. A correct interpretation of atmospheric structure deduced from lidar data requires precise knowledge of $C_2$ along the path. It is also important to realize that changes in extinction coefficients occur with changes in the product $C_1 C_2$. Uncertainties or instabilities in the system constant $C_1$ can result in similar instabilities in the extinction coefficient.
REFERENCES


