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THE CAUCHY PROBLEM IN ONE-DIMENSIONAL NONLINEAR VISCOELASTICITY

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ABSTRACT

We study the initial value problem for a nonlinear hyperbolic Volterra equation which models the motion of an unbounded viscoelastic bar. Under physically motivated assumptions, we establish the existence of a unique, globally defined, classical solution provided the initial data are sufficiently smooth and small. We also discuss boundedness and asymptotic behavior. Our analysis is based on energy estimates in conjunction with properties of strongly positive definite kernels.

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SIGNIFICANCE AND EXPLANATION

We study the initial value problem for a nonlinear integrodifferential equation of Volterra type which models the motion of an unbounded viscoelastic bar. If the kernel in the integral term vanishes identically (which corresponds to the case of an elastic bar), the equation reduces to an undamped quasilinear wave equation which does not generally have globally defined smooth solutions - no matter how smooth and small the initial data are - due to the formation of shock waves. Under physically natural assumptions on the kernel (which exclude the trivial case), the integral term has a damping effect and prevents the development of shocks if the initial data and forcing function are suitably small.

We here establish the existence of a unique, globally defined, classical solution provided the given data are sufficiently smooth and small. Moreover, we show that first and second order partial derivatives of the solution decay to zero uniformly as time tends to infinity. Analogous results are already known for bounded bars with appropriate boundary conditions, but the proofs make crucial use of certain Poincaré inequalities which are not valid for an unbounded bar. As far as local existence of solutions is concerned, it is easy to circumvent this difficulty. However, some substantial modifications are needed to show that local solutions can be continued globally. Our analysis is based on energy estimates in conjunction with properties of Volterra integral kernels.

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W. J. Hrusa¹,² and J. A. Nohel¹

1. Introduction

The aim of this paper is to establish global existence and decay of classical solutions to the Cauchy problem

\[ u_{tt}(x,t) = \phi(u_x(x,t)) + \int_0^t a'(t-\tau) \phi(u_x(x,\tau)) \, d\tau + f(x,t), \quad -\infty < x < \infty, \quad t > 0 , \quad (1.1) \]

\[ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad -\infty < x < \infty , \quad (1.2) \]

for suitably smooth and small data \( u_0, u_1, f \). Here \( \phi, \Psi, \) and \( a \) are assigned smooth functions, subscripts \( x \) and \( t \) indicate partial differentiation, and a prime denotes the derivative of a function of a single variable. Throughout this paper all derivatives should be interpreted in the distributional sense. Moreover, when we speak of a solution we always mean a classical solution.

The above problem serves as a model for the motion of an unbounded, homogeneous, viscoelastic bar. On physical grounds, it is natural to assume that \( a \) is positive, decreasing, and convex, with \( a(0) = 0 \) as \( t \to \infty \), and that

\[ \phi(0) = \Psi(0) = 0, \quad \phi'(0) > 0, \quad \Psi'(0) > 0, \quad a(0)\psi'(0) > 0 . \quad (1.3) \]

We refer to our survey paper [5] for a discussion of the physical interpretation of (1.1) and a much more complete summary of previous related work. (In addition, [5] contains a proof of Theorem 1.1 in the (much simpler) special case of an exponential kernel.)

Observe that if \( a' \equiv 0 \), then (1.1) reduces to the quasilinear wave equation

\[ u_{tt} = \phi(u_x) + f . \quad (1.4) \]

It is well known that (1.4), (1.2) does not generally have a global (in time) smooth

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Since \( a' \) (rather than \( a \)) appears in the equation of motion, a constant can be added to the kernel \( a \) without affecting (1.1). The normalization \( a(0) = 0 \) is convenient for our purposes. The reader is cautioned that other normalizations are also used.
solution, no matter how smooth and small the data are. (See, for example, [8] and [11].)
As explained in [5], the memory term in (1.1) has a damping effect if \( a' \neq 0 \) and the
appropriate sign conditions are satisfied. However, this damping mechanism is quite subtle
and globally defined smooth solutions should be expected only if the data are suitably
small.

Our main interest here is in global phenomena. If the kernel is sufficiently regular,
it is a more or less routine matter to establish local existence of solutions to (1.1),
(1.2). However, the question of securing suitable global estimates is considerably more
delicate, especially for an unbounded bar.

Dafermos and Nohel [2] have established small-data global existence theorems for
analogous initial-boundary value problems corresponding to the motions of bounded
viscoelastic bodies. (They treat Neumann, Dirichlet, and mixed conditions.) However,
their argument makes crucial use of various Poincaré inequalities and consequently is not
applicable to (1.1), (1.2).

Equations of the form (1.1) with \( \psi \equiv \phi \) have been studied by MacCamy [10], Dafermos
and Nohel [1], Staffans [16], and Hattori [4]. Small-data global existence theorems (for
bounded and unbounded bodies) are given in [10], [1], and [16]. Nonexistence of global
solutions for certain large data (of arbitrary smoothness) is established in [4]. (See
also [3], [12], and [14] for some related nonexistence results.) If \( \psi \equiv \phi \), equation
(1.1) admits certain estimates which do not carry over to the general case with \( \psi 
\) different from \( \phi \). However, we know of no physical motivation for the restriction \( \psi \equiv \phi \).

We here establish global existence and decay of classical solutions to (1.1), (1.2)
(with \( \psi \) different from \( \phi \)), under assumptions quite similar to those used in [2] for the
case of a bounded bar. The proof combines certain estimates of Dafermos and Nohel [2]
(which remain valid for unbounded bars) with a variant of a procedure introduced by MacCamy
in [9], [10].

As in [2], we assume
\[
a, a', a'' \in L^1(0, \infty), \ a \text{ is strongly positive definite.} \tag{1.5}
\]
(We note that twice continuously differentiable \( a \) with \((-1)^j a^{(j)}(t) > 0 \) for all \( t > 0 \),
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\[ j = 0, 1, 2; \ a' \neq 0 \text{ are automatically strongly positive definite. See } [13]. \] In addition, we require
\[ \int_{0}^{\infty} t|s(t)|dt < \infty, \quad \widehat{s'(z)} \neq 0 \quad \forall \ z \in \Pi, \tag{1.6} \]
where \( \widehat{\cdot} \) denotes the Laplace transform and \( \Pi := \{ z \in \mathbb{C} : \text{Re} \ z > 0 \} \). The additional conditions (1.6) are not terribly restrictive. (This will be discussed further at the end of Section 2.)

Regarding \( u_0, u_1, \) and \( f, \) we assume
\[ u_0 \in L^2_{\text{loc}}(\mathbb{R}), \ u_0', u_0''', u_0'''', u_1', u_1'' \in L^2(\mathbb{R}) \tag{1.7} \]
\[ f, f_x, f_t \in C([0,\infty); L^2(\mathbb{R})) \cap L^\infty([0,\infty); L^2(\mathbb{R})) \tag{1.8} \]
\[ f \in L^1([0,\infty); L^2(\mathbb{R})), \ f_x, f_t, f_{xt} \in L^2([0,\infty); L^2(\mathbb{R})) \tag{1.9} \]
In order to keep things reasonably simple, we have made our hypotheses on \( f \) slightly stronger than necessary. Several similar conditions can be used in place of (1.9). (In fact, we could assume that (1.8) holds and that \( f \) is a sum of several functions each of which satisfies a condition in the spirit of (1.9).) Finally, to measure the size of the data we define
\[ U_0(u_0, u_1) := \int_{-\infty}^{\infty} [u_0''(x)^2 + u_0''''(x)^2 + u_1'(x)^2 + u_1''(x)^2 + u_1'''(x)^2]dx \tag{1.10} \]
and
\[ F(f) := \sup_{t \geq 0} \left[ \int_{-\infty}^{\infty} (x^2 + f_x^2 + f_t^2)(x,t)dx \right. \]
\[ + \left. \int_{0}^{\infty} \int_{-\infty}^{\infty} (x^2 + f_t^2 + f_{xt}^2)(x,t)dxdt + \left( \int_{0}^{\infty} f(x,t)^2 dx \right)^{1/2} dt \right]^2. \tag{1.11} \]

Our main result is

**Theorem 1.1**: Assume that \( \phi, \psi \in C^3(\mathbb{R}) \) and that (1.3), (1.5), and (1.6) hold. Then, there exists a constant \( \mu > 0 \) such that for every \( u_0, u_1, \) and \( f \) satisfying (1.7), (1.8), (1.9), and

\[ \text{The symbol } := \text{ indicates an equality in which the left hand side is defined by the right hand side.} \]
U_0(u_0, u_1) + P(f) < u^2 \quad (1.12)

the initial value problem (1.1), (1.2) has a unique solution \( u \in C^2(\mathbb{R} \times [0,\infty)) \) with

\[
\begin{align*}
& u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, \\
& u_{ttt} \in C([0,\infty); L^2(\mathbb{R})) \cap L^\infty([0,\infty); L^2(\mathbb{R})).
\end{align*}
\]

In addition,

\[
\begin{align*}
& u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt} \in L^2([0,\infty); L^2(\mathbb{R})) \quad (1.14) \\
& u_{xx}, u_{xt}, u_{tt} \to 0 \text{ in } L^2(\mathbb{R}) \text{ as } t \to \infty \quad (1.15)
\end{align*}
\]

and

\[
\begin{align*}
& u_x, u_{xti}, u_{tt}, u_{ttt} \to 0 \text{ uniformly on } \mathbb{R} \text{ as } t \to \infty \quad (1.16)
\end{align*}
\]

Remark 1.1: Assumption (1.5) implies that \( a' \in AC(0,\infty) \). There are indications that for certain viscoelastic materials \( a'(t) \sim t^{\alpha-1} \) as \( t \to 0 \), with \( 0 < \alpha < 1 \). Recently, Hrusa and Renardy [6] have studied equations of the form (1.1) under assumptions on \( a \) which permit such singularities in \( a' \). For the case of a bounded bar, they establish local as well as global existence theorems. For (1.1), (1.2), they have local results, but no global results. (Again, this is due to the lack of Poincaré inequalities on all of space.) Unfortunately, the techniques which we employ here to estimate lower order derivatives make essential use of the assumption \( a'' \in L^1(0,\infty) \), and consequently we cannot handle the case when \( a' \) is singular.

Remark 1.2: Dafermos and Nohel [2] mention possible extensions of their results to problems involving motions of multidimensional viscoelastic bodies. The same comments apply here. In particular, if the kernel is a scalar multiple of the identity, then a straightforward (but tedious) modification of the proof of Theorem 1.1 can be used to establish global existence of solutions for small data. For a three dimensional problem, estimates on derivatives of \( u \) through order 4 (rather than through order 3 as in one dimension) would be required. However, the case of a general matrix-valued kernel \( A \) is
considerably more complicated. It is not very hard to state implicit assumptions on $A$ under which global existence could be established. The difficulty lies in determining simple and direct conditions on $A$ which would guarantee that these assumptions are satisfied.

The remaining two sections of this paper are devoted to the proof of Theorem 1.1. Section 2 contains some preliminary material on local solutions as well as properties of the kernel $a$ and several related resolvent kernels. The actual proof is presented in Section 3.

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2. Preliminaries

We begin by stating a local existence result for (1.1), (1.2).

Lemma 2.1: Assume that $\phi, \psi \in C^3(\mathbb{R})$, $a, a', a'' \in L^1_{\text{loc}}([0,\infty))$, and that there exists a constant $\hat{a} > 0$ such that
\[
\phi'(t) > \hat{a} \quad \forall t \in \mathbb{R}.
\] (2.1)

Let $u_0, u_1, \text{ and } f$ satisfying (1.7), (1.8), and $f_{xt} \in L^1_{\text{loc}}([0,\infty); L^2(\mathbb{R}))$ be given.

Then, the initial value problem (1.1), (1.2) has a unique local solution $u$, defined on a maximal time interval $[0,T_0)$, with
\[
\{u, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{ttt}\} \in C([0,T_0); L^2(\mathbb{R})).
\] (2.2)

Moreover, if
\[
\sup_{t \in [0,T_0]} \int \left( u_x^2 + u_t^2 + u_{xx}^2 + u_{xt}^2 + u_{tt}^2 + u_{xxx}^2 + \right. \\
\left. u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2 \right) (x,t) \, dx < \infty,
\] (2.3)
then $T_0 = \infty$. 

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Remark 2.1: The Sobolev embedding theorem and (2.2) imply \( u \in C^2(\mathbb{R} \times [0,T_0]) \).

The proof of Lemma 2.1 is almost identical to the proof of Theorem 2.1 of [2]. Therefore, we omit the details. The only significant difference is that certain additional estimates are needed for lower order derivatives. As far as local existence is concerned, this causes no difficulties; one simply expresses the lower order derivatives in terms of initial conditions and time integrals of higher order derivatives. (Of course, such a procedure yields time-dependent bounds and cannot be used to obtain global estimates.)

The integrability properties of several resolvent kernels associated with \( a \) are crucial to our analysis of (1.1), (1.2). Therefore, we briefly recall a few basic concepts. Let \( b \in L^1_{\text{loc}}[0,\infty) \) be given and consider the linear (scalar) Volterra equation
\[
y(t) + \int_0^t b(t-s)y(s)\,ds = g(t), \quad t > 0.
\] (2.4)
For each \( g \in L^1_{\text{loc}}[0,\infty) \), equation (2.4) has a unique solution \( y \in L^1_{\text{loc}}[0,\infty) \). Moreover, this solution is given by
\[
y(t) = g(t) + \int_0^t p(t-s)g(s)\,ds, \quad t > 0,
\] (2.5)
where \( p \) is the unique solution of the resolvent equation
\[
p(t) + \int_0^t b(t-s)p(s)\,ds = -b(t), \quad t > 0.
\] (2.6)
A classical theorem of Paley and Wiener states that if \( h \) belongs to \( L^1(0,\infty) \), then the resolvent kernel \( p \) belongs to \( L^1(0,\infty) \) if and only if \( 1 + \dot{b}(z) \) does not vanish for any \( z \in \mathbb{T} \).

We shall also make use of several basic properties of strongly positive definite kernels. A function \( a \in L^1_{\text{loc}}[0,\infty) \) is said to be positive definite if
\[
\int_0^t y(s)\int_0^s a(s-r)y(r)\,dr\,ds > 0 \quad \forall t > 0,
\] (2.7)
for every \( y \in C(0,\infty) \); \( a \) is called strongly positive definite if there exists a constant \( C > 0 \) such that the function defined by \( a(t) - Ce^{-t}, \, t > 0 \), is positive definite. As the terminology suggests, strongly positive definite implies positive definite.

These definitions are generally not very easy to check directly. For our purposes here, it is useful to know that if \( a \) belongs to \( L^1(0,\infty) \) then \( a \) is strongly positive.
definite if and only if there exists a constant \( C > 0 \) such that
\[
\Re a(iw) > \frac{C}{w^2 + 1} \quad \forall w \in \mathbb{R}.
\] (2.8)

Moreover, if a positive definite function is sufficiently regular then statements can be made concerning its pointwise behavior near zero. In particular, (1.5) implies
\[
a(0) > 0, \quad a'(0) < 0.
\] (2.9)

(That (1.5) implies \( a(0) > 0 \) follows easily from \( a(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \Re a(iw) dw \) and (2.8). To see that (1.5) implies \( a'(0) < 0 \), observe that \( \lim_{w \to \infty} \Re a(iw) = -a'(0) \), as can be verified using two integrations by parts and the Riemann-Lebesgue lemma. This limit must be strictly positive by (2.8)). See, for example, [13] and [15] for more information on these matters.

The kernel \( k \) defined by
\[
\phi'(0)k(t) + \int_0^t a'(t-\tau)\phi'(0)k(\tau)d\tau = -\phi'(0)a'(t), \quad t > 0,
\] (2.10)
can be used to express \( u_{xx} \) (or \( u_{xxx} \)) in terms of \( u_{tt} \) (or \( u_{xtt} \)) and small correction terms through an equation quite similar to (2.5). (The fact that the leading coefficient in (2.10) is different from one does not affect the representation formulas (2.4), (2.5), (2.6) in a significant way. We can simply divide through by \( \phi'(0) \) since \( \phi'(0) > 0 \).) Thus, if \( k \in L^1(0,\infty) \), bounds on \( u_{xx} \) (or \( u_{xxx} \)) can be inferred from bounds on \( u_{tt} \) (or \( u_{xtt} \)). Using the Paley-Wiener theorem, (1.3), (1.5), and properties of strongly positive definite kernels, one can establish

**Lemma 2.2:** Assume that (1.3) and (1.5) hold. Then, the solution \( k \) of (2.10) belongs to \( L^1(0,\infty) \).

This lemma was used previously by Dafermos and Nohel. See Lemma 3.2 of [2] for the proof.

In order to simplify the formulas in Lemmas 2.3 and 2.4 below, we assume, without loss of generality, that
\[
a'(0) = -1.
\] (2.11)

(Since \( a'(0) < 0 \), we can multiply \( \phi \) by \(-a'(0)\) and divide \( a \) by \(-a'(0)\) to achieve

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Such a change does not affect the assumptions of Theorem 1.1.)

Let \( r \) denote the resolvent kernel associated with \(-a''\), i.e., the solution of
\[
 r(t) = \int_0^t a''(t-r)r(t)\,dr = a''(t), \quad t > 0.
\]
(2.12)

It is not hard to see that \( r \notin L^1(0,\infty) \), since \( a'(0) = -1 \). However, it follows from (1.5) and (1.6) that \( r \) is the sum of a positive constant and an \( L^1 \)-function. More precisely, we have

**Lemma 2.3:** Assume that (1.5), (1.6), and (2.11) hold. Then, the solution \( r \) of (2.12) satisfies
\[
 r(t) = \frac{1}{a(0)} + R(t) \quad \forall \, t > 0,
\]
where \( R \in L^1(0,\infty) \).

**Proof:** Formally taking Laplace transforms in (2.12) and using (2.13), we find, after a simple computation that
\[
 R(z) = \frac{1}{a(0)} + \hat{a}'(z), \quad z \in \mathbb{N}.
\]
(2.14)

Since \( \hat{a}' \) does not vanish on \( \mathbb{N} \), it is clear that \( R \) is locally analytic on \( \mathbb{N} \) in the sense of Definition 2.1 of [7] (with \( \rho(t) \equiv 1 \)). Observe that for \( z \) near infinity, we have
\[
 R(z) = \frac{\hat{a}'(z)}{1 - \hat{a}'(z)} - \frac{1}{a(0)z}.
\]
(2.15)

Thus \( R \) is locally analytic at infinity and \( R(\infty) = 0 \). Therefore, Proposition 2.3 of [7] implies that \( \hat{R} \) is the Laplace transform of a function \( R \in L^1(0,\infty) \), and the lemma follows easily.

It is convenient to define another kernel \( M \) by
\[
 M(t) := \int_0^t R(s)\,ds \quad \forall \, t > 0.
\]
(2.16)

For our proof of Theorem 1.1, it is essential to know that \( M \in L^1(0,\infty) \) and \( M(0) < 1 \).

**Lemma 2.4:** Assume that (1.5), (1.6), and (2.11) hold. Then, the kernel \( M \) defined by
(2.12), (2.13), and (2.16) satisfies
\[ M \in AC(0,\infty) \cap L^1(0,\infty), \quad M(0) = 1 - \frac{a(0)}{a(0)} < 1 \quad (2.17) \]

**Proof:** That \( M \in AC(0,\infty) \) is immediate. The claim concerning \( M(0) \) follows from (2.14) and the facts that \( M(0) = R(0), \quad a'(0) = -a(0), \quad \text{and} \quad a(0) > 0 \). To show that \( M \in L^1(0,\infty) \), we proceed as in the proof of Lemma 2.3. Formally, we have
\[ M(z) = \frac{a(z) - a(0)}{za'(z)} + \frac{1}{z} \frac{a(0)}{za(0)^2}, \quad z \in \mathbb{R} \setminus \{0\} \quad (2.18) \]
and it is clear that \( \hat{M} \) is locally analytic on \( \mathbb{R} \setminus \{0\} \). To study the behavior of \( \hat{M} \) near zero, we rewrite (2.18) as
\[ \hat{M}(z) = \frac{a(z) - a(0)}{za'(z)} - \frac{a(0)a(z)}{za(0)^2a'(z)} \quad z \in \mathbb{R} \setminus \{0\} \quad (2.19) \]
Using Lemma 4.3 of [7] and the second part of (1.6), we find that \( a(0) - a(z) \) has a locally analytic zero of order at least one at \( z = 0 \). Therefore, \( \hat{M}(0) \) can be defined in such a way that \( \hat{M} \) is locally analytic on \( \mathbb{R} \). Finally, for \( z \) near infinity we have
\[ \hat{M}(z) \sim \frac{-a(z)a(0)}{za'(z)} \quad z \to \infty \quad (2.20) \]
from which we conclude that \( \hat{M} \) is locally analytic at infinity and \( \hat{M}(\infty) = 0 \). Therefore, by Proposition 2.3 of [7], \( \hat{M} \) is the Laplace transform of a function \( M \in L^1(0,\infty) \), and the desired result follows easily.

Before stating the next lemma, we introduce some notation which will also be used in the next section. For \( b \in L^1_{loc}(0,\infty) \), we set
\[ Q(w, t, b) := \int_0^T \int_0^T w(x, s) \int_0^T b(s-t)w(x, t) dt dx ds, \quad w \in C[0, T], \quad (2.21) \]
for every \( T > 0 \) and every \( w \in C([0, T]; L^2(\mathbb{R})) \). Moreover, for \( T > 0 \) and \( 0 < h < T \), we define the forward difference operator \( \Delta_h \) of stepsize \( h \) (in the time variable) by
\[ \Delta_h w(x, t) := w(x, t+h) - w(x, t), \quad w \in C([0, T-h]), \quad (2.22) \]
for every \( w \in C([0, T]; L^2(\mathbb{R})) \).
Lemma 2.5: Assume that (1.5) holds. Then, there exists a constant $\kappa > 0$ such that

$$
\int_0^T \int_{R^d} w_t(x,t)^2 \, dx < \int_0^T \int_{R^d} w_t(x,0)^2 \, dx + \kappa Q(w_t, t, s)
$$

\[ + \kappa \lim_{h \to 0} \frac{1}{h^2} Q(\Delta_h w_t, t, s) \quad \forall t \in [0, T], \]

for every $T > 0$ and every $w \in C^1([0,T], L^2(\mathbb{R}))$.

Proof: We first note that (1.5) implies $a(0) > 0$, $a \in L^2(0, \infty)$, and that there exists a constant $C > 0$ such that

$$
0 < Q(v, t, s) \leq C Q(v, t, s) \quad \forall t \in [0, T]
$$

for every $T > 0$ and every $v \in C([0,T]; L^2(\mathbb{R}))$, where $a(t) := e^{\alpha t}$, $t > 0$. Let $T > 0$, $h \in (0, T)$, and $w \in C^1([0,T], L^2(\mathbb{R}))$ be given. The identity

$$
a(0)\Delta_h w(x,t) = a(t)\Delta_h w(x,0) + \int_0^t a(s-t)\Delta_h w(x,t) \, dt
$$

\[ + \int_0^t a(s-t)\Delta_h w(x,t) \, dt \]

can easily be checked via integration by parts. Taking square $L^2(\mathbb{R})$ norms in (2.25) and integrating the result from 0 to $t$, we see that

$$
a(0) \int_0^T \int_{R^d} [\Delta_h w(x,s)]^2 \, dx \, ds < 3(a(t)^2 \cdot \int_0^T a(s-t)^2 \, \Delta_h w(x,0)^2 \, dx)
$$

\[ + 3 \int_0^T \int_{R^d} \left( \int_0^t a(s-t)\Delta_h w(x,t) \, dt \right)^2 \, dx \, ds
$$

\[ + 3 \int_0^T \int_{R^d} \left( \int_0^t a(s-t)\Delta_h w(x,t) \, dt \right)^2 \, dx \, ds .
$$

It follows from (2.24), (2.26), and Lemma 4.2 of [16] that

$$
a(0) \int_0^T \int_{R^d} [\Delta_h w(x,s)]^2 \, dx \, ds < 3(a(t)^2 \cdot \int_0^T a(s-t)^2 \, \Delta_h w(x,0)^2 \, dx)
$$

\[ + C \Delta_h w_t, t, s) + C \Delta_h w, t, s) \]

where $C$ is a constant which depends only on properties of $a$. To obtain the desired conclusion, we divide both sides of (2.27) by $h^2$ and let $h \to 0$.

We close this section with a few remarks concerning the class of kernels which satisfy
(1.5), (1.6). As noted in Section 1, twice continuously differentiable $a$ which satisfy 
$(-1)^j a^{(j)}(t) > 0$ for all $t > 0$, $j = 0,1,2$, $a' 
eq 0$ are strongly positive definite.
(Corollary 2.2 of [13].) The interpretation of the integrability conditions in (1.5) and
(1.6) is clear. It is not difficult to impose assumptions directly on $a$ which will
guarantee that $\Re a'$ does not vanish on $\mathbb{R}$.

Kernels of the form
\begin{equation}
\frac{\partial}{\partial t} (t) := \sum_{j=1}^{N} \alpha_j e^{u_j t}, \quad t > 0 \quad (2.28)
\end{equation}
with $\alpha_j u_j > 0$ for $j = 1,\ldots,N$, which are commonly employed in applications of visco-
elasticity theory, satisfy (1.5), (1.6). In fact, it is not hard to show that if $a$
satisfies
\begin{equation}
a \in C^3(\mathbb{R}), \quad (-1)^j a^{(j)}(t) > 0 \quad \forall t > 0, \quad j = 0,1,2,3;
\end{equation}
\begin{equation}
a' \neq 0, \quad \int_0^\infty t a(t) dt < \infty, \quad (2.29)
\end{equation}
then (1.5) and (1.6) hold. We remark, however, that (2.29) is by no means necessary for
(1.5) and (1.6) to hold. Indeed, one readily verifies that kernels of the form
$a(t) := e^{-\mu t} \cos \beta t$, $t > 0$, with $\mu$ positive satisfy (1.5), (1.6).

3. Proof of Theorem 1.1.

Let us define $\chi \in C^3(\mathbb{R})$ by
\begin{equation}
\chi(\xi) := \phi(\xi) - a(0)\psi(\xi), \quad \forall \xi \in \mathbb{R} \quad (3.1)
\end{equation}
We choose a sufficiently small positive number $\delta$ and modify $\phi$ and $\psi$ (and also $\chi$
accordingly by (3.1)) smoothly outside the interval $[-\delta,\delta]$ in such a way that $\phi''$, $\phi'$(and hence also $\chi''$) vanish outside $[-2\delta,2\delta]$ and
\begin{equation}
\phi'(\xi) > \psi(\xi) > \phi', \quad \chi(\xi) > \chi \quad \forall \xi \in \mathbb{R}, \quad (3.2)
\end{equation}
where $\phi$, $\psi$, and $\chi$ are positive constants. (This can always be accomplished in view of
(1.3).) There is no harm in making this modification because we will show a posteriori
that $|u_n(x,t)| < \delta$ for all $x \in \mathbb{R}$, $t > 0$. 

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By Lemma 2.1, (1.1), (1.2) has a unique local solution \( u \) which satisfies (2.2) on a maximal time interval \((0, T_0)\). We want to show that if (1.12) holds with \( u \) sufficiently small, then (2.3) must also hold, and hence \( T_0 = \infty \).

Estimates for the \( L^2(\mathbb{R}) \) norms of certain derivatives of \( u \) can be derived via energy identities. Due to the nonlinear nature of (1.1), these identities generally contain remainder terms involving integrals over time and space of quantities which are of higher algebraic order in derivatives of \( u \). To draw useful conclusions from such an energy identity we must also obtain estimates for the remainder terms. The estimation of a remainder term often introduces new remainder terms. Therefore, the trick is to develop a closed (or self sufficient) chain of estimates. This is an especially delicate matter here due to the failure of Poincaré inequalities on unbounded intervals. The quantity \( E \) defined below has been carefully constructed for this purpose.

For \( t \in [0, T_0) \), we set

\[
E(t) := \max_{u \in [0, t]} \int_{\mathbb{R}} \left( u^2 + u^2 + u^2 + u^2 + u^2 + u^2 + u^2 + u^2 \right) \frac{1}{2} (x, s) dx
\]

\[
+ \int_{0}^{t} \int_{\mathbb{R}} \left[ u^2 + u^2 + u^2 + u^2 + u^2 \right] (x, s) dx ds
\]

(3.3)

Our objective is to show that if (1.12) holds with \( u \) sufficiently small then \( E \) remains bounded on \([0, T_0)\). For this purpose, it is convenient to define

\[
\nu(t) := \sup_{x \in \mathbb{R}} \left( u^2 + u^2 + u^2 \right)^{1/2} (x, s) \quad \forall t \in [0, T_0)
\]

(3.4)

To simplify the notation, we write \( U_0 \) and \( F \) in place of \( U_0(u_0, u_1) \) and \( F(f) \), and we use \( \Gamma \) to denote a (possibly large) generic positive constant which can be chosen independently of \( u_0, u_1, f, \) and \( T_0 \).

As noted in Section 1, the procedure of [2] can be used to obtain estimates for certain higher order derivatives. In fact, an inequality of the form (3.27) below can essentially be inferred from a careful examination of [2] and a few simple computations.
For the sake of completeness (and because a few changes are needed), we repeat the procedure of Dafeoros and Nohel to derive (3.27).

An integration by parts in (1.1) produces

\[ u_{tt}(x,t) = \chi(u_x(x,t))_x + \int_0^t a(t-\tau) \psi(u_x)_{xt}(x,t) \, d\tau \]

\[ + a(t) \psi(u_{0x}(x))_x + f(x,t) \]  

We multiply (3.5) by \( \psi(u_x)_{xt} \) and integrate over \( \mathbb{R} \times [0,t], t \in [0,T_0] \). After several integrations by parts, this yields

\[
\frac{1}{2} \int_0^t \int_0^t \left( \psi'(u_x) u_{xt}^2 + \chi'(u_x) \psi'(u_x) u_{xx}^2 \right) (x,t) \, dx + G(\psi(u_x)_{xt}, t, s) \\
+ \frac{1}{2} \int_0^t \int_0^t \left( \psi''(u_x) u_{xt}^3 - \chi''(u_x) \psi''(u_x) u_{xt} u_{xx}^2 \right) (x,t) \, dxds \\
+ \int_0^t \int_0^t a(s) \psi(u_{0x}(x))_x \psi(u_x)_{xt} (x,s) \, dxds \\
- \int_0^t \int_0^t f(x,t) \psi(u_x)_{xt}(x,s) \, dxds \quad \forall t \in [0,T_0),
\]

where \( O \) is defined by (2.21).

To obtain the next identity, we apply the forward difference operator \( \Delta_h \) to (3.5) and multiply by \( \Delta_h \psi(u_x)_{xt} \). We then integrate the resulting expression over \( \mathbb{R} \times [0,t], t \in [0,T_0] \). After various integrations by parts, we divide by \( h^2 \) and let \( h \to 0 \). The result of this computation is
\[
\frac{1}{2} \int_{-\infty}^{\infty} (\psi'(u_x)u_{xxt} + \psi'(u_x)u_{xxt}^2)(x,t)dx + \lim_{h \to 0} \frac{1}{2} Q(\psi(u_x_{xxt}, t, s)
\]
\[- \frac{1}{2} \int_{-\infty}^{\infty} (\psi'(u_x)u_{xxt}^2 + \psi'(u_x)u_{xxt}^2)(x,0)dx
\]
\[+ \int_0^t \int_{-\infty}^{\infty} \left( \frac{1}{2} \psi''(u_x)u_{xxt}^2 - 2\psi''(u_x)u_{xxt} u_{xxtt} \right) dx dt
\]
\[+ \psi'''(u_x)u_{xxtt} u_{xxtt} + x'''(u_x)\psi'(u_x)u_{xxt} u_{xxtt}
\]
\[+ x'(u_x)\psi'(u_x)u_{xxt} u_{xxtt} + x'''(u_x)\psi'(u_x)u_{xxt} u_{xxtt}
\]
\[-2x''(u_x)\psi''(u_x)u_{xxt} u_{xxtt} - x'''(u_x)\psi'''(u_x)u_{xxt} u_{xxtt}
\]
\[- \frac{3}{2} x'(u_x)\psi'''(u_x)u_{xxt} u_{xxtt} - x'(u_x)\psi'''(u_x)u_{xxt} u_{xxtt}
\]
\[\quad \text{for } t \in [0, T_0).
\]

It is not a priori evident that \(\lim_{h \to 0} Q(\psi(u_x_{xxt}, t, s))\) exists for \(t \in [0, T_0]\). However, the limit of each of the other terms involved in the derivation of (3.7) exists. Consequently, the limit in question exists (and is, in fact, nonnegative.) We add (3.6) to (3.7), and use Lemma 2.5 (with \(w = \psi(u_{x_x})\)) and (3.2) to obtain a lower bound for the left hand side. After some routine estimations on the right hand side, this yields
\[
\int_{-\infty}^{\infty} (u_x^2 + u_{xxt}^2 + u_{xxtt}^2)(x,t)dx + \int_0^t \int_{-\infty}^{\infty} u_{xxtt}^2(x,s)dxds
\]
\[< \Gamma(U_0 + F) + \Gamma(v(t) + v(t)^3)\tilde{E}(t) + \Gamma(\sqrt{U_0} + \sqrt{F})\sqrt{E(t)}
\]
\[\quad \text{for } t \in [0, T_0).
\]
To give an indication of the steps involved in deriving (3.8) from (3.6) and (3.7), we show the detailed estimation of several typical terms. The reader is cautioned that there are many possible ways to carry out these and the numerous estimations which follow. We note that derivatives of $\phi$, $\psi$, and $\chi$ of orders one through three are bounded on $\mathbb{R}$ by virtue of our modification of these functions outside $[-\delta, \delta]$.

Many of the terms from (3.6) and (3.7) can be handled in a very simple manner, e.g.

$$
|\int_0^t \int_0^\infty \phi''(u_{x_t})u_{x_{tt}}u_{ttt}(x,s)dxds| \\
< \sup_{s \in (0,t)} \left| \phi''(u_{x_t})u_{x_{tt}}(x,s) \right| \int_0^t \int_0^\infty |u_{x_{tt}}u_{ttt}(x,s)| dxds \\
< \Gamma \left( t \int_0^\infty \phi''(u_{x_t})u_{x_{tt}}(x,s)dxds \right) \\
< \Gamma \left( t \int_0^\infty \phi''(u_{x_t})u_{x_{tt}}(x,s)dxds \right) \\
< \Gamma \left( t \int_0^\infty \phi''(u_{x_t})u_{x_{tt}}(x,s)dxds \right)
$$

(3.9)

A similar computation yields

$$
|\int_0^t \int_0^\infty \phi'(u_{x_t})u_{x_{tt}}(x,s)dxds| \\
< \int_0^t \int_0^\infty \phi'(u_{x_t})u_{x_{tt}}(x,s)dxds + \int_0^t \int_0^\infty \phi'(u_{x_t})u_{x_{tt}}^2(x,s)dxds \\
< \Gamma \left( t \int_0^\infty \phi'(u_{x_t})u_{x_{tt}}(x,s)dxds \right)^{1/2} + \Gamma \left( t \int_0^\infty \phi'(u_{x_t})u_{x_{tt}}^2(x,s)dxds \right)^{1/2} \\
+ \Gamma \left( t \int_0^\infty \phi'(u_{x_t})u_{x_{tt}}^4(x,s)dxds \right)^{1/2} \\
< \Gamma \sqrt{\Gamma} + \Gamma \sqrt{\Gamma} + \Gamma \sqrt{\Gamma} + \Gamma \sqrt{\Gamma}
$$

(3.10)

To estimate terms such as the first integral on the right hand side of (3.7), we observe that the initial values of derivatives of $u$ can be expressed in terms of $u_0$ and $u_1$ by using (1.1) if necessary. For example,

$$
u_{x_{tt}}(x,0) = \phi''(u_{x})u_{x_{xx}}(x,0) + \phi''(u_{x})u_{x_{xx}}^2(x,0) + f_x(x,0).
$$

(3.11)

Therefore,
\begin{equation}
\int_0^\infty u_{xxt}^2(x,t)dx \leq \Gamma \int_0^\infty u_{xxxx}^2(x)dx + \Gamma \int_0^\infty u_{x}^4(x,0)dx + \Gamma \int_0^\infty f^2(x,0)dx
\end{equation}

Of course, we could use $U_0^2$ in place of $\Gamma v(t)^2E(t)$ in (3.12). However, we already have a $\Gamma v(t)^2E(t)$ term in (3.10), so there is no harm in including it here. Moreover, it simplifies matters slightly to avoid terms involving $U_0^2$.

The other calculations used to derive (3.8) (and our subsequent estimates) are in the same spirit as those shown above. It is useful to note that (1.5) implies $a,a' \in L^2(0,\infty)$, and that (1.8), (1.9) imply $f \in L^2([0,\infty), L^2(0,\infty))$. Moreover, we have $\int_0^\infty |f(x,t)|^2 dx < \frac{1}{2} \Gamma$, and clearly $v(t) < v(t) + v(t)^3$ for all $t \in [0,T_0)$.

Taking $L^2(\mathbb{R})$ norms in (1.1) and squaring the result, we see that

\begin{equation}
\int_0^\infty u_{tt}^2(x,t)dx < 3 \int_0^\infty (\phi''(u_x)^2 u_x^2 + f^2)(x,t)dx
\end{equation}

from which it follows easily that

\begin{equation}
\int_0^\infty u_{tt}^2(x,t)dx \leq \Gamma \max_{s \in [0,t]} \int_0^\infty u_{xx}^2(x,s)dx \quad \forall t \in [0,T_0)
\end{equation}

A similar argument gives bounds on $u_{ttt}$. Differentiation of (1.1) with respect to $t$ yields

\begin{equation}
u_{ttt}(x,t) = \phi''(u_x)u_{xxt}(x,t) + \phi''(u_x)u_{xxt}(x,t) + a'(t)\psi(u_{xx}(x)) + f_t(x,t)
\end{equation}

Squaring (3.15) and integrating over $\mathbb{R}$ and $\mathbb{R} \times [0,t]$, we obtain

\begin{equation}
\int_0^\infty u_{ttt}^2(x,t)dx \leq \Gamma [U_0 + F] + \Gamma v(t)^2E(t) + \Gamma \max_{s \in [0,t]} \int_0^\infty u_{xx}^2(x,s)dx \quad \forall t \in [0,T_0)
\end{equation}

and

\begin{equation}
\int_0^t \int_0^\infty u_{ttt}^2(x,s)dsdx \leq \Gamma [U_0 + F] + \Gamma v(t)^2E(t) + \Gamma \int_0^t \int_0^\infty u_{xx}^2(x,s)dsdx \quad \forall t \in [0,T_0)
\end{equation}

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Combining (3.8), (3.14), (3.16), and (3.17), we now have the estimate

$$\int_{-\infty}^{t}((u_x^2 + u_t^2 + u_{xx}^2 + u_{xt}^2 + u_{tt}^2 + u_{ttt}^2)(x,t)dx + \int_{0}^{t}\int_{-\infty}^{\infty}((u_{xxx}^2 + u_{ttt}^2)(x,s)dxds < \Gamma(U_0 + \delta) + \Gamma(U(t) + \delta(t)^2)\delta(t) + \Gamma(\sqrt{U_0} + \sqrt{\delta})\sqrt{\delta(t)}$$

$$\forall t \in [0,T_0).$$

We can obtain a bound for $\int_{0}^{t}\int_{-\infty}^{\infty}u_{xxx}^2$ by interpolation. The identity

$$\int_{0}^{t}\int_{-\infty}^{\infty}v_{xxx}(x,s)dxds = \int_{0}^{t}\int_{-\infty}^{\infty}v_{xxx}v_{xxx}(x,s)dxds + \int_{0}^{t}\int_{-\infty}^{\infty}v_{xxx}v_{xxx}(x,s)dxds \forall t \in [0,T_0),$$

holds for all functions $v$ having the regularity (2.2). (It is easy to give a formal derivation of (3.19) via integration by parts. It can be established rigorously using difference operators or a simple density argument.) Employing (3.19) with $v = u$, we see that

$$\int_{0}^{t}\int_{-\infty}^{\infty}u_{xxx}^2(x,s)dxds < \Gamma(U_0 + \delta) + \Gamma(U(t) + \delta(t)^2)\delta(t) + \Gamma(\sqrt{U_0} + \sqrt{\delta})\sqrt{\delta(t)}$$

$$\forall t \in [0,T_0).$$

To obtain an estimate for $u_{xxx}$, we set

$$G(x,t) := u_{xxx}(x,t) - f(x,t) + [\phi'(0) - \phi'(u_x)]u_{xxx}(x,t)$$

and observe that (1.1) can be rewritten as

$$\phi'(0)u_{xxx}(x,t) + \int_{0}^{t}a'(t-\tau)\phi'(0)u_{xxx}(x,\tau)d\tau = G(x,t).$$

Using the resolvent kernel $k$ defined by (2.10), we solve (3.22) for $u_{xxx}$ to get

$$\phi'(0)u_{xxx}(x,t) = G(x,t) + \int_{0}^{t}k(t-\tau)G(x,\tau)d\tau.$$  

Differentiation of (3.23) with respect to $x$ yields

$$\phi'(0)u_{xxx}(x,t) = G_x(x,t) + \int_{0}^{t}k(t-\tau)G_x(x,\tau)d\tau.$$  

Since $k \in L^1(0,\infty)$ (by Lemma 2.2), it follows from (3.21), (3.24), and a routine computa-
that
\[
\int_0^t \int_a^b u_{xxx}^2(x,t) \, dx \, dt \leq \|L\| + \|v(t)\|^2 \, v(t) + \max_{s \in [0,t]} \int_a^b u_{xtt}^2(x,s) \, dx
\]
\[
\forall \, t \in (0,T_0)
\]

and
\[
\int_0^t \int_a^b u_{xxx}^2(x,t) \, dx \, dt \leq \|L\| + \|v(t)\|^2 \, v(t) + \max_{s \in [0,t]} \int_a^b u_{xtt}^2(x,s) \, dx
\]
\[
\forall \, t \in (0,T_0)
\]

Combining (3.18), (3.20), (3.25), and (3.26), we see that
\[
\int_0^t \int_a^b u_{xxx}^2 + u_{xtt}^2 + u_{xtt}^2 + u_{xxx}^2 + u_{xxx}^2 + u_{xxx}^2 \, dx \, dt
\]
\[
\leq \|L\| + \|v(t)\|^2 \, v(t)
\]
\[
+ \max_{s \in [0,t]} \int_a^b u_{xtt}^2(x,s) \, dx
\]
\[
\forall \, t \in (0,T_0)
\]

It remains to obtain a similar estimate for
\[
\int_0^t \int_a^b (u_{xxx}^2 + u_{xtt}^2 + u_{xtt}^2 + u_{xxx}^2) \, dx \, dt
\]
\[
\forall \, t \in (0,T_0)
\]

In particular, the remainder terms must be estimable in terms of \( U_0, F, v(t) \), and \( E(t) \).

The time integral in (3.28) causes the most difficulty. As can be seen by examining the derivation of (3.27), an estimate for this term is essential.

To proceed further, we transform (1.1) to a more convenient form involving the resolvent kernel \( r \) defined by (2.12). This transformation was motivated by an idea of MacCamy [9], [10]. As explained in Section 2, we assume without loss of generality that \( a'(0) = -1 \).

Differentiating (1.1) with respect to \( t \), we get
\[
u_{ttt}(x,t) = \Phi(u_t(x,t)) - \Psi(u_x(x,t)) + \int_0^t a''(t-\tau)\Phi(u_t(x,\tau)) \, d\tau + f_t(x,t)
\]

Solving (3.29) for \( \Phi(u_t) \) and rearranging the terms, we obtain
\[
u_{ttt}(x,t) = \Phi(u_t(x,t)) - \Psi(u_x(x,t)) + f_t(x,t)
\]
\[
+ \int_0^t r(t-\tau)[\Phi(u_t) - u_{ttt} + f_t] \, d\tau,
\]

\[
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\]
or, setting \( a := a(0) \) and using (2.13),
\[
\begin{align*}
\tau_{ttt} + \alpha \tau_{tt} &= \phi(u_x)_{xt} + \alpha x(u_x^2 + f_t + \alpha f + R^*(\phi(u_x)_{xt} - u_{ttt} + f_t) ,
\end{align*}
\]
(3.31)
or finally,
\[
\begin{align*}
\tau_{ttt} + \alpha \tau_{tt} &= \phi(u_x)_{xt} + \alpha x(u_x^2 + f_t + \alpha f
\end{align*}
\]
+ [R^*(\phi(u_x^2) - u_{ttt} + f) ,
\]
(3.32)
where the * denotes convolution with respect to the time variable on \([0,t] \), i.e.
\[
(R^*v)(x,t) := \int_0^t R(t- \tau)v(x, \tau) d\tau .
\]
(3.33)
(In the above calculations, we have made use of the fact that \([\phi(u_x) - u_{tt} + f](x,0) \leq 0\)
which follows from (1.1).) Recall that \( a := a(0) > 0 \) and that \( R \in L^1(0, \infty) \) by Lemma 2.3.

Let us set
\[
W(\xi) := \int_0^T \chi(x) ds \quad \forall \xi \in \mathbb{R} ,
\]
(3.34)
and note that
\[
W(\xi) > \frac{1}{2} \chi^2 \quad \forall \xi \in \mathbb{R}
\]
(3.35)
by virtue of (1.3), (3.2), and (3.34). We multiply (3.32) by \( u_t \) and integrate over \( \mathbb{R} \times [0,t] \), as before, to get
\[
\begin{align*}
\int_0^T \int_{\mathbb{R}} & \left\{ \frac{1}{2} u_{ttt}^2 + W(u_x) \right\} dx + \int_0^T \int_{\mathbb{R}} \phi'(u_x) u_{ttt}^2(x,s) dx ds \\
& - \int_0^T \int_{\mathbb{R}} u_{tt}^2(x,s) ds + \int_0^T \int_{\mathbb{R}} u_t [R^*(\phi(u_x^2) - u_{ttt} + f)](x,s) ds dx \\
& \quad = \int_0^T \left\{ \frac{1}{2} u_{ttt}^2 + \alpha W(u_x) + uu_{tt} - fu_t \right\}(x,0) dx \\
& \quad + \int_0^T \left\{ fu_t - u_{ttt} \right\}(x,t) dx \\
& \quad + \int_0^T \int_{\mathbb{R}} [afu_t - fu_{ttt}](x,s) dx ds \quad \forall t \in [0,T_0] .
\end{align*}
\]
(3.36)
Next, we multiply (3.5) by \( u_{tt} \) and integrate over \( \mathbb{R} \times [0,t] \). This yields
\[ \int_0^t \int_0^\infty u_{tt}^2(x,s)dxds - \int_0^t \int_0^\infty \chi'(u_x)u_{xt}(x,s)dxds \]

\[ = \int_0^\infty \chi'(u_x)u_x(x,0)dx - \int_0^\infty \chi'(u_x)u_x(x,t)dx \]

\[ + \int_0^t \int_0^\infty [f u_{tt} + \chi''(u_x)u_x^2]u_{xt}(x,s)dxds \]

\[ + \int_0^t \int_0^\infty [a(s)\psi(u_{0x}(x))]u_{tt}(x,s)dxds \]

\[ + \int_0^t \int_0^\infty u_{tt}[a^*\psi(u_x)](x,s)dxds \quad \forall t \in (0,T_0) . \]

Adding (3.36) to (3.37), we find that

\[ a \int_0^\infty \left( \frac{1}{2} u_t^2 + W(u_x) \right)(x,t)dx + a(0) \int_0^t \int_0^\infty \psi'(u_x)u_{xt}(x,s)dxds \]

\[ + \int_0^t \int_0^\infty u_{tt}[R^*\psi(u_x) - u_{tt} + f](x,s)dxds \]

\[ = \int_0^\infty \left( \frac{1}{2} u_t^2 + aW(u_x) + u_t u_{tt} + \chi'(u_x)u_x - fu_t \right)(x,0)dx \]

\[ + \int_0^t \int_0^\infty \left( fu_t - u_t u_{tt} - \chi'(u_x)u_{xxt} \right)(x,t)dx \]

\[ + \int_0^t \int_0^\infty u_{tt}[R^*\psi(u_x) - u_{tt} + f](x,t)dx \]

\[ + \int_0^t \int_0^\infty [a(\psi(u_{0x}(x))]u_{tt}(x,s)dxds \]

\[ + \int_0^t \int_0^\infty u_{tt}[a^*\psi(u_x)](x,s)dxds \quad \forall t \in (0,T_0) . \]

The crucial term to analyze is

\[ \psi(t) := \int_0^t \int_0^\infty u_{tt}[R^*\psi(u_x) - u_{tt} + f](x,s)dxds . \]

(The other terms in (3.38) are favorable or can be estimated routinely.) We see from (3.1) and (3.5) that

\[ \psi(u_x) - u_{tt} + f = a(0)\psi(u_x) - a(t)\psi(u_{0x}) - a^*\psi(u_x) \quad \text{and substitution into (3.39) yields} \]

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\( \Theta(t) = a(0) \int_0^t \int_\Omega u_{tt}(x,s) \psi(u_x(x)) (x,s) \, dx \, ds \\
- \int_0^t \int_\Omega \psi'(u_x(x)) u_{tt}(x,s) \, dx \, ds \\
- \int_0^t \int_\Omega u_{tt}(x,s) \psi(u_x(x)) (x,s) \, dx \, ds \quad (3.41) \)

Since \( R, a \in L^1(0,\infty) \) and we already have estimates on third order derivatives of \( u \), the last two terms in (3.41) cause no difficulties. However, the first term on the right hand side of (3.41) requires special attention.

Employing the kernel \( M \) defined by (2.16), we find that

\[ R^* \psi(u_x(x)) = -M(0) \psi(u_x(x)) + M(u_{xx}(x)) + M(u_{xt}(x)) \quad (3.42) \]

Recall that \( M \in L^1(0,\infty) \) and \( M(0) < 1 \). We observe further that

\[ \int_0^t \int_\Omega \psi(u_x(x)) u_{xt}(x,s) \, dx \, ds = \int_0^t \int_\Omega \psi(u_x(x)) u_{xt}(x,s) \, dx \, ds \\
+ \int_0^t \int_\Omega \psi(u_x(x)) u_{xt}(x,0) \, dx - \int_0^t \int_\Omega \psi(u_x(x)) u_{xt}(x,t) \, dx \quad (3.43) \]

using integration by parts. Combining (3.41), (3.42), and (3.43), we arrive at the following expression for \( \Theta \):

\[ \Theta(t) = -a(0)M(0) \int_0^t \int_\Omega \psi(u_x(x)) u_{xt}(x,s) \, dx \, ds \\
+ a(0) \int_0^t \int_\Omega \psi(u_x(x)) u_{xt}(x,0) \, dx - a(0) \int_0^t \int_\Omega \psi(u_x(x)) u_{xt}(x,t) \, dx \\
+ \int_0^t \int_\Omega [M - (R^a)](x) \psi(u_{xx}(x)) u_{xt}(x,s) \, dx \, ds \\
+ \int_0^t \int_\Omega u_{tt}(x,s) \psi(u_x(x)) (x,s) \, dx \, ds \quad (3.44) \]

Since \( M(0) < 1 \), the first integral in the above expression can be absorbed by the second integral on the left hand side of (3.38). Moreover, the remaining terms can be handled rather easily. (Note that \( [M - (R^a)] \in L^1(0,\infty) \), since \( M, R, a \in L^1(0,\infty) \).) After substitution of (3.44) into (3.38), and a long computation, we obtain
\[
\int_0^\infty (u_t^2 + u_x^2)(x,t)\,dx + \int_0^\infty \int_0^\infty u_{ttt}(x,s)\,dx\,ds
\]
\[
< \Gamma[U_0 + F] + \Gamma v(t)\Xi(t)
\]
\[
+ \Gamma \left( \int_0^t \int_0^s u_{ttt}^2(x,s)\,dx\,ds \right)^{1/2} \cdot \left( \int_0^t \int_0^s u_{xtt}^2(x,s)\,dx\,ds \right)^{1/2}
\]
\[
+ \Gamma \max_{s \in [0,t]} \int_0^\infty u_{ttt}^2(x,s)\,dx
\]
\[
\forall t \in [0,T_0).
\]

In the derivation of (3.45), we have used the simple algebraic inequality
\[
|AB| < \varepsilon A^2 + \frac{1}{4\varepsilon} B^2 \quad \forall \varepsilon > 0,
\]
to handle several terms. For example, observe that
\[
|\int_0^\infty x'(u_t\,u_x)(x,t)\,dt| < \frac{\varepsilon}{t} \int_0^\infty u_x u_t x(x,t)\,dx
\]
\[
< \frac{\varepsilon}{t} \int_0^\infty u_x^2(x,t)\,dx + \frac{\varepsilon}{4t} \int_0^\infty u_{xt}^2(x,t)\,dx \quad \forall t \in [0,T_0)
\]
for every \( \varepsilon > 0 \), where \( \overline{x} := \sup |x'(t)| \). On account of (3.35), \( \varepsilon \int_0^\infty u_x^2 \) can be absorbed by the first integral on the left hand side of (3.38) if \( \varepsilon \) is sufficiently small. The size of the coefficient \( \frac{\varepsilon}{4t} \) is unimportant because we already have an estimate for \( \int_0^\infty u_{xt}^2 \). Moreover, we have made essential use of the assumption \( f \in L^1((0,\infty); L^2(\mathbb{R})) \) to estimate \( \int_0^t \int_0^s f u_t \) since it does not seem possible to obtain a time independent bound for \( \int_0^t \int_0^\infty u_{tt}^2 \).

It follows from (3.37) and a simple computation that
\[
\int_0^t \int_0^\infty u_{tt}^2(x,s)\,dx\,ds < \Gamma[U_0 + F] + \Gamma v(t)\Xi(t)
\]
\[
+ \Gamma \left( \int_0^t \int_0^s u_{tt}^2(x,s)\,dx\,ds \right)^{1/2} \cdot \left( \int_0^t \int_0^s u_{xt}^2(x,s)\,dx\,ds \right)^{1/2}
\]
\[
+ \Gamma \max_{s \in [0,t]} \int_0^\infty u_{ttt}^2(x,s)\,dx
\]
\[
\forall t \in [0,T_0).
\]

Combining (3.45) and (3.48), we thus obtain
\[\int_0^1 (u_x^2 + u_t^2)(x,s)dx + \int_0^t \int_0^1 (u_{xt}^2 + u_{tt}^2)(x,s)dxds\]
\[< \Gamma (u_0 + F) + \Gamma v(t)B(t) + \Gamma (\frac{\Gamma}{\sqrt{u_0 + F}}) \frac{1}{\sqrt{B(t)}} \quad \forall t \in [0,T_0) .\]

Using (3.49) with \( \epsilon \) sufficiently small, and (3.27),
\[\int_0^1 (u_x^2 + u_t^2)(x,t)dx + \int_0^t \int_0^1 (u_{xx}^2 + u_{xt}^2 + u_{tt}^2)(x,s)dxds\]
\[< \Gamma (u_0 + F) + \Gamma (v(t) + v(t)^3)B(t) + \Gamma (\sqrt{u_0 + F}) \frac{1}{\sqrt{B(t)}} \quad \forall t \in [0,T_0) .\]

To obtain our last estimate, we go back to (3.23). Using (3.23), (3.21), and the fact that \( k \in L^1(0,\infty) \), we deduce that
\[\int_0^t \int_0^1 u_{xx}^2(x,s)dxds < \frac{\Gamma}{2} + \Gamma^2 B(t) + \Gamma \int_0^t \int_0^1 u_{xx}^2(x,s)dxds\]
\[\forall t \in [0,T_0) .\]

Combining (3.50) and (3.51), and adding the result to (3.27), we conclude that
\[E(t) < \Gamma (u_0 + F) + \Gamma (v(t) + v(t)^3)B(t) + \Gamma (\sqrt{u_0 + F}) \frac{1}{\sqrt{B(t)}} \quad \forall t \in [0,T_0) .\]

We choose \( \mu > 0 \) such that
\[\mu < \delta^2, \quad \Gamma ((2\mu)^{3/2} + (2\mu^{3/2})^2) < \frac{1}{2}, \quad \Gamma u^2 < \frac{1}{4} \mu .\]

(Here \( \delta \) is the constant that was introduced in the first paragraph of this section.)
\begin{equation}
v(t) \leq \sqrt{2E(t)} \quad \forall t \in [0,T_0).
\end{equation}

We therefore conclude from (3.53), (3.54), and (3.55) that for any \( t \in [0,T_0) \) with \( E(t) < \bar{E} \), we actually have \( E(t) < \frac{1}{2} \bar{E} \). Consequently, by continuity,

\begin{equation}
E(t) < \frac{1}{2} \bar{E} \quad \forall t \in [0,T_0)
\end{equation}

provided that \( E(0) < \frac{1}{2} \bar{E} \).

We can always choose a smaller \( \mu > 0 \) (if necessary such that (1.12) implies \( E(0) < \frac{1}{2} \bar{E} \). (Observe that (3.54) still holds if the size of \( \mu \) is reduced.) Thus, if (1.12) is satisfied with our revised choice of \( \mu \) than (3.56) holds. This implies \( T_0 = \) by Lemma 2.1. In addition, it immediately yields (1.13) and (1.14) from which (1.15) and (1.16) follow by standard embedding inequalities. Finally, we note that

\begin{equation}
|u_\lambda(x,t)| \leq v(t) \leq (E)^{1/2} \leq \delta \quad \forall x \in \mathbb{R}, t > 0.
\end{equation}

by (3.4), (3.55), (3.56), and (3.54). The proof of Theorem 1.1 is complete.
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We study the initial value problem for a nonlinear hyperbolic Volterra equation which models the motion of an unbounded viscoelastic bar. Under physically motivated assumptions, we establish the existence of a unique, globally defined, classical solution provided the initial data are sufficiently smooth and small. We also discuss boundedness and asymptotic behavior. Our analysis is based on energy estimates in conjunction with properties of strongly positive definite kernels.