ON A THEOREM OF SZEGÖ ON UNIVALENT
CONVEX MAPS OF THE UNIT CIRCLE

I. J. Schoenberg
For a positive constant \( \lambda \) we denote by \( K(\lambda) = \{ f(z) \} \) the class of function \( f(z) \) which are regular and univalent in \( |z| < \lambda \) and map this circle on a convex domain. In 1928 G. Szegö [3] proved the

**Theorem 1.** If \( f(z) = \sum_0^\infty c_n z^n \in K(1) \), then all its sections

\[ S_n(z) = \sum_0^n c_n z^n \in K(\frac{1}{4}) \text{ for } n = 1, 2, \ldots. \]

An evident consequence is

**Corollary 1.** Since the geometric series

\[ f_0(z) = \sum_0^\infty z^n \in K(1) \]

we have

\[ \sum_0^n z^n \in K(\frac{1}{4}), \text{ (n = 1, 2, \ldots)} \]

and therefore, on replacing \( z \) by \( z/4 \), we have

\[ \sum_0^n \frac{1}{4^n} z^n \in K(1) \text{ for } n = 1, 2, \ldots. \]

In the present paper we prove directly Corollary 1, and derive from it Szegö’s Theorem 1. This is done by appealing to Theorem 2 which was conjectured by Polya and Schoenberg [1] in 1958, but only proved in 1973 by St. Ruscheweyh and T. Sheil-Small [2].

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SIGNIFICANCE AND EXPLANATION

There is a fine interplay between two fundamental notions of geometry: Convexity and Conformal Mapping. The subject belongs to Geometric Function Theory. In 1928 Gabor Szegő showed that if a power series converges in the unit circle $|z| < 1$ and maps it onto a convex domain, then all its finite sections map the circle $|z| < \frac{1}{4}$ onto convex domains. The present paper shows that Szegő's theorem reduces to a study of the finite sections of the geometric series

$$1 + \frac{1}{4} z + \frac{1}{8} z^2 + \ldots = \sum_{n=0}^{\infty} \frac{1}{2^n} z^n + \ldots$$

The main tool is a result conjectured in 1958 by Polya and Schoenberg, but only established in 1973 by St. Ruscheweyh and T. Sheil-Small.
ON A THEOREM OF SZEGÖ ON UNIVALENT CONVEX MAPS OF THE UNIT CIRCLE

I. J. Schoenberg

1. INTRODUCTION. There is an interesting interplay between two fundamental notions of geometry: Convexity and Conformal Mapping. The subject belongs to Geometric Function Theory. We use the following notation: If \( r > 0 \), we denote \( K(r) = \{ f(z) \} \) the class of functions \( f(z) \) which are univalent in the circle \( |z| < r \), and map it onto a convex domain.

In 1958 Polya and Schoenberg conjectured that for \( r = 1 \) the class \( K(1) \) form a semi-group with respect to Hadamard multiplication of power series. This was established in 1973 by St. Ruscheweyh and T. Sheil-Small in \([2]\) by the following

Theorem 1. (Ruscheweyh and Sheil-Small). If

\[
\sum_{0}^{\infty} a_{\nu}z^{\nu} \in K(1) \quad \text{and} \quad b_{\nu}z^{\nu} \in K(1)
\]

then

\[
\sum_{0}^{\infty} a_{\nu}b_{\nu}z^{\nu} \in K(1).
\]

Before we pass to the work of Szegö, let us use Theorem 1 to establish a simple and well known

Proposition. If

\[
f(z) = \sum_{0}^{\infty} a_{\nu}z^{\nu} \in K(1)
\]

and \( 0 < \lambda < 1 \), then

\[
f(z) = \sum_{0}^{\infty} a_{\nu}z^{\nu} \in K(\lambda).
\]

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Proof: We start from the geometric series

\begin{equation}
(1.3) \quad f_0(z) = \sum_{v=0}^\infty \frac{z^v}{1 - z} \in K(1),
\end{equation}

because it maps \(|z| < 1\) onto the half-plane \(\text{Re } z > \frac{1}{2}\). However, \(w = f_0(z) = 1/(1 - z)\) clearly maps \(|z| < \lambda\) onto a circle, and so

\begin{equation}
\sum_{v=0}^\infty z^v \in K(\lambda).
\end{equation}

Replacing \(z\) by \(\lambda z\), we obtain that

\begin{equation}
(1.4) \quad \sum_{v=0}^\infty \lambda^v z^v \in K(1),
\end{equation}

However, from (1.1) and (1.4), by Theorem 1 we obtain

\begin{equation}
\sum_{v=0}^\infty \lambda^v a_\nu z^v \in K(1),
\end{equation}

or, replacing \(\lambda z\) by \(z\), we have

\begin{equation}
\sum_{v=0}^\infty a_\nu z^v \in K(1),
\end{equation}

which is the desired conclusion (1.2).

Remark. Our derivation of the conclusion (1.2) from the special case of the geometric series (1.3) justifies Polya's statement that within the class \(K(1)\) the geometric series (1.3) "sets the fashion" (in German: "tonangebend").

Szegő's question: If

\begin{equation}
(1.5) \quad f(z) = \sum_{v=0}^\infty a_\nu z^v \in K(1),
\end{equation}

what can we say about its sections

\begin{equation}
S_n(z) = \sum_{v=0}^n a_\nu z^v?
\end{equation}

In 1928 G. Szegő [3, Satz II', page 204] established
Theorem 2 (Szegő). If (1.5) holds, then

$$S_n(z) = \sum_{0}^{n} c_v z^v \in K(\frac{1}{4}) \text{ for } n = 1, 2, \ldots .$$

Using Theorem 1 we will show in §2 that we can reduce Theorem 2 to the question concerning the section of the geometric series

$$f_0(z) = 1 + \frac{1}{4} z + \ldots + \frac{1}{4^n} z^n + \ldots .$$

Theorem 3. Let \( \lambda \) be a constant satisfying

$$0 < \lambda < 1 .$$

If (1.5) holds, then

$$S_n(z) = \sum_{0}^{n} c_v z^v \in K(\lambda) \text{ for all } n \geq m(\lambda) ,$$

where

$$m(\lambda) \text{ is the least integer such that (1.7) holds.}$$

Evidently

$$m(\lambda) = 1 \text{ if } \lambda \leq \frac{1}{4} .$$

by Theorem 2. It is equally evident that

$$\lambda_1 < \lambda_2 \text{ implies that } m(\lambda_1) < m(\lambda_2) ,$$

for if \( S_n(z) \in K(\lambda_2) \), then also \( S_n(z) \in K(\lambda_1) \).

The main difficulty in Theorem 3 is an explicit determination of \( m(\lambda) \), if \( \lambda > \frac{1}{4} \), and we will do that for the value

$$\lambda = \frac{1}{3}$$

only, and find that

$$m\left(\frac{1}{3}\right) = 4 .$$

We state this result as

\[\square\]
Theorem 4. If

\[(1.13) \quad f(z) = \sum_0^\infty c_v z^v \in K(1)\]

then

\[(1.14) \quad S_n(z) = \sum_0^n c_v z^v \in K(\frac{1}{4}) \quad \text{for} \quad n \geq 4,\]

but not necessarily for \( n = 2 \) or 3.

In establishing Theorems 3 and 4 I gratefully acknowledge the help of Fred W. Sauer, of the Computing Staff of the Mathematics Research Center.

2. THE NEW APPROACH TO SZEGÖ'S THEOREM 2. Since evidently

\[(2.1) \quad f_0(z) = \sum_0^\infty z^v \in K(1)\]

we conclude by Szegö's Theorem 2 that

\[(2.2) \quad \sum_0^n z^v \in K(\frac{1}{4}) \quad \text{for all} \quad n = 1, 2, \ldots .\]

Replacing \( z \) by \( z/4 \) we may restate (2.2) as

Corollary 1. We have

\[(2.3) \quad S_n^{(0)}(z) = \sum_0^n \frac{1}{4} z^v \in K(1) \quad \text{for} \quad n = 1, 2, \ldots ,\]

The new approach to Theorem 1 is to establish Corollary 1 directly. If now

\[(2.4) \quad f(z) = \sum_0^\infty c_v z^v \in K(1)\]

is an arbitrary element of \( K(1) \) we argue as follows: Applying Theorem 1 to (2.3) and

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(2.4) we conclude that

\[ \sum_{v=0}^{n} \frac{1}{4^v} c_v z^v \in X(1) \]

and therefore

(2.5)

\[ \sum_{v=0}^{n} c_v z^v \in X(\frac{1}{4}) \text{ for } n = 1, 2, \ldots, \]

which is the conclusion (1.6) of Theorem 2.

Szegő's theorem is therefore made to depend on the proof that all sections of the geometric series

(2.6)

\[ f(z) = \sum_{v=0}^{\infty} \frac{1}{4^v} z^v, \quad (|z| < 4) \]

are in \( X(1) \).

3. A DIRECT PROOF OF COROLLARY 1. We begin with

Lemma 1. All sections

(3.1)

\[ f_n(z) = \sum_{v=0}^{n} \frac{1}{4^v} z^v \quad (n = 1, 2, \ldots) \]

are univalent in the closed unit circle \( \bar{U} = \{|z| \leq 1\} \)

Proof: We are to show that

(3.2)

\[ |z| \leq 1, \quad |z_2| \leq 1, \quad z_1 \neq z_2 \text{ imply that } f_n(z_1) \neq f(z_2). \]

Observe that

(3.3)

\[ |f_n(z_1) - f_n(z_2)| = \left| \sum_{v=0}^{n} \frac{1}{4^v} (z_1^v - z_2^v) \right| \]

\[ = \frac{|z_1 - z_2|}{4} \cdot \left| 1 + \frac{z_1 + z_2}{4} + \frac{z_1^2 + z_2^2}{4^2} + \ldots + \frac{z_1^{n-1} + z_2^{n-2} z_2 + \ldots + z_2^{n-1}}{4^{n-1}} \right| \]

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However, by (3.2),
\[
\left|\frac{z_1 + z_2}{4} + \cdots + \frac{z_1^{n-1} + \cdots + z_2^{n-1}}{4^{n-1}}\right| \leq \frac{2}{4} + \frac{3}{4^2} + \cdots + \frac{n}{4^{n-1}} = \frac{7}{9} - \frac{3n + 4}{9 \cdot 4^{n-1}},
\]
the last equality being easily obtained by summation, or by complete induction. The last member being \(< 1\). We obtain from (3.3) that
\[
|f_n(z_1) - f_n(z_2)| > \frac{|z_1 - z_2|}{4} \left|1 - \left(\frac{7}{9} - \frac{3n + 4}{9 \cdot 4^{n-1}}\right)\right| = \frac{|z_1 - z_2|}{4} \left|\frac{2}{9} + \frac{3n + 4}{9 \cdot 4^{n-1}}\right| > 0,
\]
and (3.2) is established.

Lemma 1 shows that the polynomial (3.1) maps the circle \(|z| = 1\) onto a closed Jordan curve \(C_n\). Separating real and imaginary parts by
\[
\epsilon_n(\alpha t) = x_n(t) + iy_n(t)
\]
we obtain for \(C_n\) the parametric representation
\[
C_n : x = x_n(t), \quad y = y_n(t), \quad (0 \leq t \leq \pi) .
\]
Wishing to study its curvature
\[
R = \frac{1}{\epsilon''(\alpha t)} = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}
\]
we first establish

**Lemma 2.** Defining
\[
T_n(t) = x_n'(t)y_n''(t) - y_n'(t)x_n''(t),
\]
we have
\[
T_n(t) \geq 0 \quad \text{for} \quad -\pi \leq t \leq \pi, \quad \text{and} \quad n = 1, 2, \ldots .
\]

**Proof:** From (3.1) and (3.4) we have
\[
x_n(t) = 1 + \sum_{a=1}^{n} \frac{\cos at}{4^a}, \quad y_n(t) = \sum_{a=1}^{n} \frac{\sin at}{4^a}
\]
and (3.7) becomes
\[
\text{(3.7')} \quad x_n'(t)y_n''(t) - y_n'(t)x_n''(t) \geq 0
\]
By splitting the columns we rewrite the determinant as a sum of $n^2$ determinants obtaining

$$T_n(t) = \frac{n}{\alpha} \sum_{\beta=1}^{n} \sin \alpha t \cos \beta t$$

and finally

$$(3.10) \quad T_n(t) = \frac{n}{\alpha} \sum_{\alpha,\beta=1}^{n} \frac{a_\beta}{a_\alpha} \cos(t - \beta t),$$

which is a cosine polynomial of order $n - 1$.

Establishing the non-negativity of a cosine polynomial is in general a difficult problem. Fortunately, in our case it is easy due to the structure of the infinite matrix

$$(3.11) \quad \begin{vmatrix} \frac{a_\beta}{a_\alpha} \\ \frac{4^{\alpha+\beta}}{a_\alpha} \end{vmatrix}$$

of the coefficients of all (3.10). For instance, we obtain $T_3(t)$ as the sum of the elements of the $3 \times 3$ matrix (3.11) provided with appropriate cosine factors:

$$T_3(t) = \begin{cases} \frac{1.1^2}{4^2} \cos t + \frac{1.2^2}{4^3} \cos 2t + \frac{1.3^2}{4^4} \cos 3t \\ + \frac{2.1^2}{4^3} \cos t + \frac{2.2^2}{4^4} \cos 2t + \frac{2.3^2}{4^5} \cos 3t \\ + \frac{3.1^2}{4^4} \cos 2t + \frac{3.2^2}{4^5} \cos 3t + \frac{3.3^2}{4^6} \cos 3t \end{cases}$$

Since $T_n(t)$ is a cosine polynomial, we may restrict $t$ to $0 \leq t \leq \pi$. 

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For \( n = 2 \) (3.8) presents no difficulty since
\[
(3.13) \quad T_2(t) = \frac{3}{32} (1 + \cos t),
\]
which is non-negative. I owe to Fred Sauer the positivity of \( T_3(t) \), \( T_4(t) \) and \( T_5(t) \)
who provided the following (positive) minima
\[
\begin{align*}
\min_{t} T_3(t) &= .01309 \\
\min_{t} T_4(t) &= .01221 \\
\min_{t} T_5(t) &= .01600.
\end{align*}
\]
(3.14)

I claim that the last result (3.14) allows us to show that
\[
(3.15) \quad \min_{t} T_n(t) > 0 \text{ for all } n \geq 6,
\]
but this requires some elementary Algebraic Analysis.

The sum of all elements of the infinite matrix (3.11) is
\[
(3.16) \quad \sum_{a,b=1}^{\infty} \frac{a b^2}{4^a 4^b} = \left( \sum_{a=1}^{\infty} \frac{a}{4^a} \right) \left( \sum_{b=1}^{\infty} \frac{b^2}{4^b} \right).
\]

From this sum we subtract the sum of the elements of the principal \( n \times n \) minor of (3.11),
and define the new sequence
\[
(3.17) \quad N_n = \left( \sum_{a=1}^{n} \frac{a}{4^a} \right) \left( \sum_{b=1}^{n} \frac{b^2}{4^b} \right) - \left( \sum_{a=1}^{\infty} \frac{a}{4^a} \right) \left( \sum_{b=1}^{\infty} \frac{b^2}{4^b} \right).
\]

By iteration of \( z(d/dz) \) applied to \( \sum_{1}^{n} (z/4)^v \) we obtain the identities
\[
(3.18) \quad \frac{n}{4^n} = \frac{4.4^n - 3n - 4}{9.4^n}, \quad \frac{n^2}{4^n} = \frac{20.4^n - 9n^2 - 24n - 20}{27.4^n},
\]
and letting \( n \to \infty \) we obtain
Now the sequence (3.17) may be explicitly written as

\[
N_n = \frac{4}{9} n - \frac{4 \cdot 4^n - 3n - 4}{27} - \frac{20 \cdot 4^n - 9n^2 - 24n - 20}{27 \cdot 4^n},
\]

which yield the numerical values

\[
N_4 = 0.0216009195
\]

and

\[
N_5 = 0.0073673303 < 0.0074.
\]

I claim that this last inequality completes our proof of Lemma 2: Indeed, from (3.21) and the definition (3.17) of \( N_5 \) we conclude that all \( T_n(t) \) are positive for all \( n > 5 \), for in view of the relations (3.14) and (3.21) we have for \( n > 5 \) and all real \( t \)

\[
T_n(t) > T_5(t) - N_5 > \min_t T_5(t) - N_5 > 0.0160 - 0.0074 = 0.0086 > 0.
\]

4. PROOF OF THEOREM 3. Our previous discussion makes it clear that it suffices to consider the geometric series

\[
f(z) = \sum_{n=0}^{\infty} \lambda^n z^n = \frac{1}{1 - \lambda z} \quad (|z| < \lambda^{-1}),
\]

and prove

Lemma 3. For its partial sums we have

\[
S_n(z) = \sum_{0}^{n} \lambda^n z^n \in K(1) \quad \text{for all} \quad n \geq m(\lambda),
\]

where

\[
m(\lambda) \quad \text{is the least integer such that (4.2) holds.}
\]
Proof of Lemma 3. Notice that

\[(4.4) \quad f(e^{it}) = \frac{1}{1 - \lambda e^{it}} = x(t) + iy(t)\]

traces out a circle \( C \) having the interval of reals \([1/(1 + \lambda), 1/(1 - \lambda)]\) as diameter and therefore the radius

\[(4.5) \quad R = \frac{\lambda}{1 - \lambda^2} .\]

Setting

\[(4.6) \quad s_n(e^{it}) = x_n(t) + iy_n(t) ,\]

and observing that \(|z| = 1\) is well within the circle of convergence of (4.1), it should be clear that the real periodic functions

\[x_n(t), y_n(t), x(t), y(t)\]

are regular in a neighborhood of the real \( t \)-axis. Also that \( y_n(t) \) and \( y_n(t) \), as well as their derivatives, converge uniformly to the corresponding derivatives of \( x(t) \) and \( y(t) \), respectively. It follows that the closed curve \( C_n = (x_n(t), y_n(t)) \) converges to the circle \( C \) and that its curvature

\[\frac{x'_n(t)y''_n(t) - y'_n(t)x''_n(t)}{(x'_n(t)^2 + y'_n(t)^2)^{3/2}}\]

converges uniformly in \( t \), as \( n \to \infty \), to the curvature \( 1/R \) of \( C \). This establishes Lemma 3.

The determination of \( m(\lambda) \), satisfying (4.3), is difficult and will be solved for \( \lambda = 1/3 \) only.

5. THE CASE \( \lambda = \frac{1}{3} \): PROOF OF THEOREM 4. As an analogue of Lemma 1 we should prove that the image of \(|z| = 1\) by

\[(5.1) \quad f_n(z) = \sum_{0}^{n} \frac{1}{3^V} z^V \quad (n = 1, 2, \ldots)\]
are all simple closed curves. However, our simple proof of Lemma 1 does not generalize. Rather we consider

\[(5.2)\]

\[w = f_n(z) - 1 = \sum_{i=1}^{n} \frac{1}{3^i} z^i = \frac{1}{3 - (z/3)^n}\]

and show that it maps \(z = e^{it}\) into a curve which is star-shaped with respect to the origin. This requires two facts: 1. That as \(t\) varies from \(-\pi\) to \(\pi\), the argument of the function \((5.2)\) increases by \(2\pi\). 2. That we have

\[(5.3)\]

\[\text{Im}(\frac{d}{dt}/w) > 0\] for all \(t\).

However, we omit the tedious calculations.

We rather pass to considering the curvature of the curve; here matters are very close to those of \(\S 3\) and obtained from them by replacing \(\frac{1}{4}\) by \(\frac{1}{3}\). We shall also use the same notations.

Setting

\[(5.4)\]

\[f_n(e^{it}) = x_n(t) + iy_n(t),\]

we have as an analogue of Lemma 2 the

**Lemma 3.**

Defining

\[(5.5)\]

\[T_n(t) = x_n^2(t)y_n^2(t) - y_n^2(t)x_n^2(t),\]

we have

\[(5.6)\]

\[T_n(t) > 0\] for \(-\pi < t < \pi\) and \(n \geq 4,\]

but not for \(n = 2\) and \(n = 3.\)

**Proof:** For the analogues of \((3.10)\) and \((3.17)\) we find

\[(5.7)\]

\[T_n(t) = \sum_{\alpha \neq \beta} \frac{\alpha^2}{3^{\alpha+\beta}} \cos(\alpha - \beta)t\]

and

\[(5.8)\]

\[N_n = \left(\sum_{\alpha} \frac{\alpha}{3^\alpha}\right)\left(\frac{6^2}{1}\right) - \left(\sum_{\alpha} \frac{\alpha}{3^\alpha}\right)\left(\frac{6^2}{1}\right)\]

and explicitly

\[\text{---}\]
From Fred Sauer's values to six decimal places, we have

\[
\begin{align*}
\min T_2(t) &= -0.012346 \\
\min T_3(t) &= -0.002057 \\
\min T_4(t) &= 0.004268 \\
\min T_5(t) &= 0.013733 \\
\min T_6(t) &= 0.014480, \quad N_6 = 0.036836 \\
\min T_7(t) &= 0.017663, \quad N_7 = 0.015400.
\end{align*}
\]

The first two minima being negative shows that the curves \( C_2 = (x_2(t), y_2(t)) \) and \( C_3 = (x_3(t), y_3(t)) \) are not convex. However, from (5.10) we can conclude that (5.6) holds: From the above data we see that \( T_4(t), T_5(t), T_6(t), \) and \( T_7(t) \) are everywhere positive. Now (5.10) shows that if \( n > 7 \) then

\[
T_n(t) \geq T_7(t) - N_7 \geq \min T_7(t) - N_7 > .0176 - .0155 = .0021 > 0.
\]

Thus \( T_n(t) \geq 0 \) for all \( t \) and \( n = 4, 5, ... \), proving Theorem 4.
REFERENCES


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Convexity, Geometric Function Theory

For a positive constant \( \lambda \) we denote by \( K(\lambda) = \{ f(z) \} \) the class of function \( f(z) \) which are regular and univalent in \( |z| < \lambda \) and map this circle on a convex domain. In 1928 G. Szegö [3] proved the
ABSTRACT (cont.)

Theorem 1. If \( f(z) = \sum_{0}^{\infty} C_{0} z^{n} \in K(1) \), then all its sections

\[ S_{n}(z) = \sum_{0}^{n} C_{0} z^{n} \in K\left(\frac{1}{4}\right) \text{ for } n = 1,2,\ldots. \]

An evident consequence is

Corollary 1. Since the geometric series

\[ f_{0}(z) = \sum_{0}^{\infty} z^{n} \in K(1) \]

we have

\[ \sum_{0}^{n} z^{n} \in K\left(\frac{1}{4}\right), \text{ (n = 1,2,\ldots) } \]

and therefore, on replacing \( z \) by \( z/4 \), we have

\[ \sum_{0}^{n} \frac{1}{4} z^{n} \in K(1) \text{ for } n = 1,2,\ldots. \]

In the present paper we prove directly Corollary 1, and derive from it Szegő's Theorem 1. This is done by appealing to Theorem 2 which was conjectured by Polya and Schoenberg [1] in 1958, but only proved in 1973 by St. Ruscheweyh and T. Sheil-Small [2].