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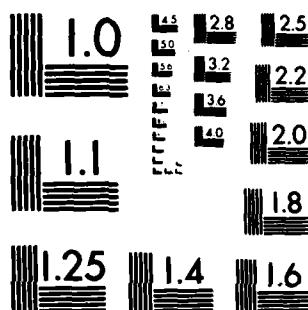UNCLASSIFIED   JHU/APL/TG-1344 N00024-83-C-5301   F/G 20/14   NL
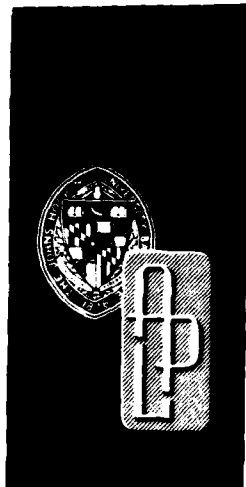
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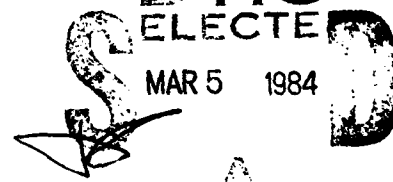NATIONAL BUREAU OF STANDARDS-1963-A

Technical Memorandum

CALCULATION PROCEDURES FOR VARIATIONAL, BORN, AND EXACT SOLUTIONS FOR ELECTROMAGNETIC SCATTERING FROM TWO RANDOMLY SEPARATED DIELECTRIC RAYLEIGH CYLINDERS

J. A. KRILL, R. H. ANDREO, and R. A. FARRELL

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The exact solution, the Born approximation, and its variational improvement are obtained for the scattering of electromagnetic waves from a random ensemble of systems, each consisting of two Rayleigh cylinders. The cylinders are parallel, of infinite length, and of equal radius. Their separation varies randomly among ensemble members except that the cylinders cannot overlap. The intent is to test a recently developed vector stochastic variational principle. The exact solutions are obtained for the average differential scattering cross sections of both the transverse electric (TE) and transverse magnetic (TM) fields relative to the cylinder axes with normal plane wave incidence. The corresponding variational approximations are obtained using a recent reported computational alternative to the more familiar dyadic Green's function solution. They are in essential agreement with the exact TE and TM solutions, whereas the Born results are not. In particular, the variational results accurately account for multiple scattering, which is significant in the exact TE, but not TM, solution, and also account for the difference in geometric polarizability between the two solutions.

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Technical Memorandum

CALCULATION PROCEDURES FOR VARIATIONAL, BORN, AND EXACT SOLUTIONS FOR ELECTROMAGNETIC SCATTERING FROM TWO RANDOMLY SEPARATED DIELECTRIC RAYLEIGH CYLINDERS

J. A. KRILL, R. H. ANDREO, and R. A. FARRELL

ABSTRACT

The exact solution, the Born approximation, and its variational improvement are obtained for the scattering of electromagnetic waves from a random ensemble of systems, each consisting of two Rayleigh cylinders. The cylinders are parallel, of infinite length, and of equal radius. Their separation varies randomly among ensemble members except that the cylinders cannot overlap. The intent is to test a recently developed vector stochastic variational principle. The exact solutions are obtained for the average differential scattering cross sections of both the transverse electric (TE) and transverse magnetic (TM) fields relative to the cylinder axes with normal plane wave incidence. The corresponding variational approximations are obtained using a recently reported computational alternative to the more familiar dyadic Green's function solution. They are in essential agreement with the exact TE and TM solutions, whereas the Born results are not. In particular, the variational results accurately account for multiple scattering, which is significant in the exact TE, but not TM, solution, and also account for the difference in geometric polarizability between the two solutions.

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ILLUSTRATIONS

1.0 INTRODUCTION

The potential advantages of variational approximations in scattering problems are well known.[1-4] First, by virtue of variational invariance, errors in the initial trial approximation for the fields on or within the scatterer do not lead to corresponding first-order errors in the calculated far field scattering amplitude T. Thus, good initial approximations can lead to improved results, and estimates of the accuracy of variational results can be made in some cases.[3,4] Because the usual Schwinger variational form for the scattering amplitude is a ratio of integrals,[2] difficulties in computing the statistical moments of this ratio resulted in very limited application of this method to random scattering problems until the mid-seventies. In 1977, Hart and Farrell[4] demonstrated that, for arbitrary scatterer statistics, the integrals whose ratio comprises the deterministic variational expression can be averaged individually and recombined to form an invariant expression for the average scattering amplitude $\langle T \rangle$. This form, i.e., $\langle T \rangle = \langle N_1 \rangle \times \langle N_2 \rangle / \langle D \rangle$, is inherently simpler to evaluate than the direct average of the deterministic expression $\langle T \rangle = \langle N_1 N_2 / D \rangle$, where N_1, N_2, and D are integrals involving the fields at the scatterer.

Hart and Farrell demonstrated this result for $\langle T \rangle$ and $\langle |T|^2 \rangle$ in the case of scalar wave scattering from objects or surfaces with homogeneous boundary conditions. An extension was made recently[5,6] to vector wave scattering and inhomogeneous boun-

dary conditions. Specifically, we have derived[5,6] general vector stochastic variational expressions for the statistical moments and probability density functions of arbitrary polarization components of the vector scattering amplitude, $\hat{e}_\lambda \cdot T$ and the differential scattering cross section $|\hat{e}_\lambda \cdot T|^2$. These invariant expressions are applicable to random scatterers with arbitrary inhomogeneous and anisotropic permittivity, conductivity, and permeability.

Of present interest is the extent to which the general vector stochastic variational principle, by virtue of first-order error cancellation, can approximate polarization-dependent interactive phenomena, i.e., interference and multiple scattering, even if the initial field approximation does not. This question is addressed in this report by calculating the exact solution to a random scattering problem exhibiting these effects and then comparing the result with the corresponding Born approximation and its variational improvement.

Our approach is based on a similar earlier analysis.[7,8] In an effort to examine the accuracy and efficacy of the original scalar stochastic variational principle,[4] Gray, Hart, and Farrell[7] calculated closed-form variational and first-order perturbational approximations to transverse magnetic (TM) wave scattering from a classic, random, perfectly conducting surface[9] on which Dirichlet boundary conditions are satisfied. The surface consisted of many parallel, nonoverlapping, hemicylindrical protrusions on an infinite plane. The hemicylinders were of equal radii, infinite length, and random separation. The variational approximation for $\langle |T|^2 \rangle$ was found to include the sum of independent hemicylinder scattering contributions as well as a correction term proportional to the fractional area of the plane

[1] L. Cairo and T. Kahan, *Variational Techniques in Electromagnetism*, Gordon and Breach, Science Publishers, Inc., New York (1965).

[2] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics, Parts I and II*, McGraw-Hill, New York (1953).

[3] D. S. Jones, "A Critique of the Variational Method in Scattering Problems," *IRE Trans. Antennas Propag.* AP-4, 297-301 (1956).

[4] R. W. Hart and R' A. Farrell, "A Variational Principle for Scattering from Rough Surfaces," *IEEE Trans. Antennas Propag.* AP-25, 708-710 (1977).

[5] J. A. Krill and R. H. Andreo, "Vector Stochastic Variational Principles for Electromagnetic Wave Scattering," *IEEE Trans. Antennas Propag.* AP-28, 770-776 (1980).

[6] R. H. Andreo and J. A. Krill, "Vector Stochastic Variational Expressions for Scatterers with Dielectric, Conductive, and Magnetic Properties," *J. Opt. Soc. Am.* 71, 978-982 (1981).

[7] E. P. Gray, R. W. Hart, and R. A. Farrell, "An Application of a Variational Principle for Scattering by Random Rough Surfaces," *Radio Sci.* 13, 333-348 (1978).

[8] J. A. Krill and R. A. Farrell, "Comparisons Between Variational, Perturbational, and Exact Solutions for Scattering from a Random Rough Surface Model," *J. Opt. Soc. Am.* 68, 768-774 (1978).

[9] V. Twersky, "Multiple Scattering of Radiation by an Arbitrary Planar Configuration of Parallel Cylinders and by Two Parallel Cylinders," *J. Appl. Phys.* 23, No. 4, 407-414 (1952).

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covered by the hemicylinders. Comparison of this result with first-order perturbation theory showed a discrepancy that persisted even when the Rayleigh limit was taken. Krill and Farrell[8] were able to obtain the exact solution for the simplest multiple scattering case for this perfectly conducting surface, consisting of only two randomly separated, nonoverlapping, Rayleigh hemicylinders. They showed from the exact solution that the variational result accounted for multiple scattering whereas the first-order perturbation approximation did not, thus offering an explanation for the discrepancy.

The present study differs from the earlier one[7,8] as follows. First, we are testing the recent extension of the stochastic variational principle to (vector) electromagnetic wave scattering from objects or surfaces with generally nonvanishing boundary conditions. The scattering model to be considered involves dielectric cylinders. Because there are differences between the exact transverse electric (TE) and TM fields (including interactive effects), we will investigate the extent to which the vector variational principle accounts for these polarization phenomena. The model consists of an ensemble of systems, each of which contains two dielectric, parallel, Rayleigh cylinders with infinite lengths and equal radii. A plane wave is incident perpendicular to the cylinder axes and makes a fixed angle relative to the plane in which the axes lie. The cylinders are randomly separated but hard; i.e., they cannot merge or overlap, and their refractive index goes discontinuously to the value 1 at their surface.

For normal incidence and arbitrary incident polarization, the scattered field can be written as the superposition of the (decoupled) TM and TE solutions.[10,11] The exact TE and TM solutions for the differential scattering cross section are obtained in Chapter 2 for two identical dielectric Rayleigh cylinders with axis separation greater than their diameter. In Chapter 3, these solutions are ensemble averaged over a separation distance subject to the no-overlap constraint. Chapter 4 presents calculations using the vector stochastic variational principle to improve the Born approximation for random cylinder scattering and arbitrary polarization. For this simple random scattering problem, the average of the deterministic expression could also be calculated and is presented. In Chapter 5, the Born and variational results are compared to the exact TE and TM solutions, with particular attention to the accuracy with which the approximations account for polarization effects and multiple scattering. It is concluded that the variational approximation accurately describes both the polarization dependence and the multiple scattering found in the Rayleigh limit of the exact solution.

[10] H. C. Van de Hulst, *Light Scattering by Small Particles*, Wiley, New York, pp. 297-301, 306-307 (1957).

[11] G. Olaofe, "Scattering by Two Cylinders," *Radio Sci.* 5, No. 11, 1351-1360 (1970).

2.0 EXACT TE AND TM SCATTERING CROSS SECTIONS

Figure 1 presents the scattering configuration in which two parallel, infinitely long cylinders, labeled cylinder $+1$ and -1, are illuminated by a plane wave incident normal to their axes, which are located at $\zeta^{(+1)}$ and $\zeta^{(-1)}$, respectively. The cylinders have relative refractive index m and radius a, and their axes are along $\hat{z}$. The incident plane wave has electric field $\mathbf{E}_i = A\hat{e}_i \exp(i\mathbf{k}_i \cdot \mathbf{r})$, where $\mathbf{k}_i$ is the wave vector, A is the amplitude, and $\hat{e}_i$ is the polarization vector. The wave vector is in the direction $\hat{x}$, i.e., $\mathbf{k}_i \equiv k\hat{x}$, and forms a fixed angle β relative to the plane in which the cylinder axes lie. We are interested in the

differential scattering cross section $|T|^2$, where T $\equiv \hat{e}_s \cdot \mathbf{T}$, for arbitrary polarization $\hat{e}_s$. Because the plane wave is incident normal to the cylinder axes, arbitrary polarization can be treated as the sum of decoupled TE and TM field contributions. The exact TE and TM solutions will be obtained for the Rayleigh limit[10] mka and $ka \ll 1$.

The exact solution for scattering from two dielectric cylinders with arbitrary, but fixed, axis separation $\zeta \equiv \zeta^{(+1)} - \zeta^{(-1)}$ has appeared in several sources.[9,11-14] As in our two-hemicylinder solution,[8] we again follow the procedures described by Olaofe,[11,12] which are based on scalar wave functions related to the TE and TM fields.[10] First, the wave function ψ_{ex}, used to describe the total field external to the scatterers, is written as the sum of an incident plane wave and scattered wave contributions from each cylinder, i.e., $\psi_{ex} = \psi_{inc} + \psi_{sc}$. The incident plane wave is expanded in a Bessel series in terms of coordinates centered at the j-th cylinder axis, where j $= +1$ or -1 (Fig. 1). The scattered field from each individual cylinder is expressed as a sum of cylindrically outgoing waves centered at that cylinder with as-yet undetermined coefficients. Finally, the addition theorem is applied that expresses the outgoing waves from one cylinder in terms of coordinates centered at the other cylinder. The resulting equation for the external wave function, expressed in the coordinates centered on the j-th cylinder, is[11]

$$\psi_{ex}^{(j)} = A_\downarrow \sum_{n=-\infty}^{\infty} e^{in\theta^{(j)}} \left\{ i^n w_j e^{in\beta} J_n(kr^{(j)}) \right.$$

$$+ {}_j b_n H_n^{(1)}(kr^{(j)}) + J_n(kr^{(j)})(-1)^n$$

$$\left. \times \sum_{\ell=-\infty}^{\infty} {}_{j'} b_\ell H_{\ell-n}^{(1)}(k\zeta)(-1)^\ell \right\}, \qquad (1)$$
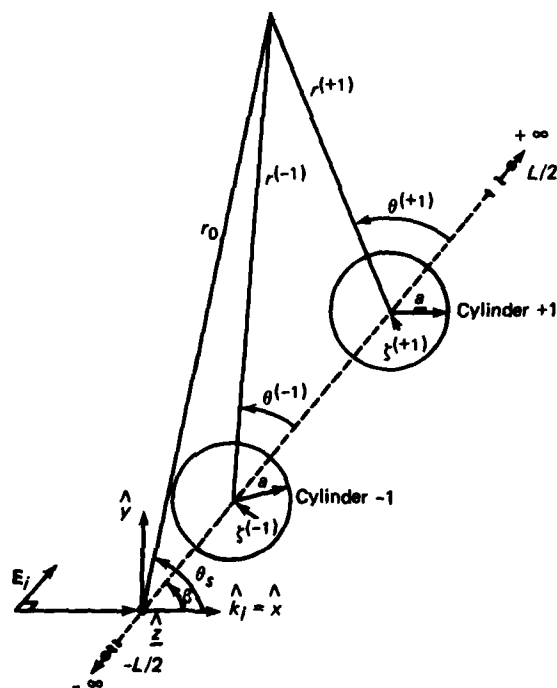

Figure 1 — The scattering problem consists of two nonoverlapping, parallel dielectric cylinders illuminated by a plane wave normal to their axes and forming a fixed angle β with the plane in which their axes lie. The infinite-length cylinders' separation $\zeta = \zeta^{(+1)} - \zeta^{(-1)}$ is a random variable, and TE and TM scattering cross sections are calculated in the Rayleigh limit.

[12] G. Olaofe, "Scattering Cross Section for Two Cylinders," *IEEE Trans. Antennas Propag.* **AP-18**, No. 6, 823-825 (1970).

[13] V. Twersky, "Scattering of Waves by Two Objects," *Electromagnetic Waves*, R. E. Langer, ed., The University of Wisconsin Press, Madison, pp. 361-389 (1962).

[14] N. Zitron and S. N. Karp, "Higher-Order Approximations in Multiple Scattering, 1, Two-Dimensional Scalar Case," *J. Math. Phys.* **2**, No. 3, 394-402 (1961).

where j' is -1 when j is $+1$ and vice versa; ξ equals ℓ or n, respectively, when j equals -1 or $+1$; and $w_j = \exp[ik\zeta^{(j)}\cos\beta]$. The factor w_j accounts for the phase shift of the incident wave that occurs when the coordinate origin is translated to the axis of cylinder j. The sum over the first term in the brackets is the incident plane wave, the sum over the second term is the scattered wave from the j-th cylinder, and the double sum represents that from the j'-th cylinder. Here the incident scalar wave function is assumed to be a plane wave of amplitude A_ψ. This amplitude can be related[10] to the corresponding amplitude A of the incident plane wave electric field. Because the amplitude of the incident wave cancels in the final expression for T (c.f. Eqs. 6a, 6b, 40, and 41), it will not be discussed further.

An expression for the coefficients in Eq. 1 can be obtained by applying the appropriate boundary conditions for the fields at the cylinders' surfaces,[10,11]

$$_jb_n = -c_n\left\{i^n w_j e^{in\beta} + (-1)^n\right.$$

$$\left. \times \sum_{\ell=-\infty}^{\infty} {}_{j'}b_\ell H^{(1)}_{\ell-n}(k\zeta)(-1)^\xi\right\}, \qquad (2)$$

$$n = 0, \pm 1, \pm 2, ..., \pm\infty,$$

where c_n is the n-th order scattering coefficient for a single cylinder.[10,11] These latter coefficients will be designated by c_n^{TE} and c_n^{TM} for the TE and TM solutions, respectively.

For nonoverlapping Rayleigh cylinders ($\zeta > 2a$, $mka << 1$, and $ka << 1$), it can be shown[8,12,13,15] that the infinite set of coupled equations in Eq. 2 may be truncated. This conclusion holds even in the limit that the cylinders touch, i.e., $\zeta \to 2a(1 + \delta_a)$, where $\delta_a \to 0$. Series convergence considerations and examination of c_n reveal[8,14,15] that, in the Rayleigh limit, the $_jb_n$ coefficients that are lowest order in ka correspond to $n = 0$ for TM fields and $n = \pm 1$ for TE fields. Moreover, the series in ℓ may be correspondingly truncated so that

[15]Ye. A. Ivanov, *Diffraction of Electromagnetic Waves on Two Bodies*, Nauka i Tekhnika Press, Minsk, pp. 66-69 (1968) (English translation).

$$_jb_0^{TM} = -c_0^{TM} w_j, \qquad (3)$$

where $c_0^{TM} = i\pi(m^2 - 1)(ka/2)^2$, and

$$_jb_{\pm 1}^{TE} = \frac{\mp ic_1^{TE}}{1 - \rho^2}[w_j e^{\pm i\beta} + \rho w_{j'} e^{\mp i\beta}], \qquad (4)$$

where $c_1^{TE} = -i\pi(ka/2)^2(m^2 - 1)/(m^2 + 1)$, and the interaction parameter $\rho = c_1^{TE}H_2(k\zeta)$ appears in both the exact and the variational solutions for the TE case. We note that in the Rayleigh limit of the two-cylinder problem, multiple scattering is expected to be significant only when the cylinders are in proximity. Observe in Eqs. 3 and 4 that the only variable that has such a characteristic is ρ, which appears only for the TE case.

Using the standard relationship[10] between the fields and the wave function, one can show that the scattering amplitude definition,

$$\psi_{sc} = A_\psi\sqrt{\frac{2}{i\pi k}}\frac{e^{ikr_0}}{\sqrt{r_0}} T, \qquad (5)$$

results in the following TE and TM amplitudes:

$$T^{TM} = \left\{_{-1}b_0^{TM} + {}_{+1}b_0^{TM} e^{-ik\zeta\cos(\theta_s - \beta)}\right\}e^{-ikr^{(-1)}\cos(\theta_s - \beta)}, \qquad (6a)$$

where it is recognized that

$$\theta^{(+1)} \cong \theta^{(-1)} \cong \theta_s - \beta \equiv \theta,$$

$$r^{(+1)} \cong r^{(-1)}, \text{ and}$$

$$r^{(+1)} - r^{(-1)} = -\zeta\cos(\theta_s - \beta),$$

and

$$T^{TE} = \left\{_{-1}B + {}_{+1}B e^{-ik\zeta\cos(\theta_s - \beta)}\right\}e^{-ikr^{(-1)}\cos(\theta_s - \beta)}, \qquad (6b)$$

where $_jB = i(_jb_{-1}^{TE}\exp(-i(\theta_s - \beta)) - {}_jb_1^{TE}\exp(+i(\theta_s - \beta))$, and we have used a coordinate system centered on the origin in Fig. 1. The cross sections $|T^{TM}|^2$ and $|T^{TE}|^2$ are readily obtained by taking the absolute square of Eqs. 5 and 6 and inserting Eqs. 3 and 4, respectively.

3.0 ENSEMBLE AVERAGES OF EXACT SOLUTIONS

Ensemble averages of the exact cross sections over random cylinder separation ζ will be evaluated in the manner of Krill and Farrell.[8] The cylinder axes point in the z-direction and lie in the plane that forms an angle β relative to the incident wave. The axes are located within the limits $-L/2 \leq \zeta^{(j)} \leq +L/2$, $j = \pm 1$, and are constrained not to overlap, i.e., $\zeta > 2a$. The two-center probability density function is

$$p\left[\zeta^{(+1)}, \zeta^{(-1)}\right] = \begin{cases} \dfrac{1}{(L - 2a)^2} & \text{if } \zeta > 2a \text{ and} \\ & -L/2 \leq \zeta^{(j)} \leq +L/2, \\ & j = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

The average $\langle \ \rangle$ of an arbitrary function $g(\zeta)$ that depends only on $\zeta \equiv \zeta^{(+1)} - \zeta^{(-1)}$ can be shown to be[8]

$$\langle g(\zeta) \rangle = \frac{1}{(L - 2a)^2} \int_{2a}^{L} (L - \zeta)[g(\zeta) + g(-\zeta)]d\zeta, \tag{7a}$$

which, in the limit $L \to \infty$, goes to[8]

$$\langle g(\zeta) \rangle = \frac{1}{L} \int_{2a}^{L} (1 - \zeta/L) [g(\zeta) + g(-\zeta)] \, d\zeta, \tag{7b}$$

provided this limit exists. Applying Eq. 7 to average the TM cross section $\langle |T|^2 \rangle^{\text{TM}}$ using Eqs. 3 and 5, one obtains

$$\langle |T_E|^2 \rangle^{\text{TM}} = \frac{\pi^2}{8} (ka)^4(m^2 - 1)^2(1 - \nu), \tag{8}$$

where the subscript E denotes the exact result, and $\nu = 4a/L$ is the packing density. The term linear in ν comes from the factor $\langle w_j w_{j'}^* \rangle$ in the absolute square of Eq. 5 and represents mean destructive interference between the cylinders caused by their nonoverlap condition.

The TE case is treated by noting that performing the absolute square of Eq. 6b and applying Eq. 4 leads to terms that are of the form

$$f(\zeta) = \frac{\rho^n \rho^* \cos(\gamma\zeta)}{|1 - \rho^2|^2}, \tag{9}$$

where $(n, \ell) = 0, 1,$ and 2, $\gamma = [k \cos \beta, k \cos \theta, k(\cos \beta \pm \cos \theta),$ or $0]$, and the interaction parameter is, again, $\rho \equiv c_1 H_2^{(1)}(k\zeta)$, which approaches (Ma^2/ζ^2) for small $k\zeta$ where $M \equiv (m^2 - 1)/(m^2 + 1)$. These averages are evaluated analytically in the same manner as in Ref. 8 (see Eqs. 18 and 20a of that reference) to give

$$\langle f(\zeta) \rangle = (-1)^{n+\ell} \frac{\nu}{2} \sqrt{M} \int_0^{\sqrt{M}/2} \frac{y^{2(n+\ell+1)}}{(1 - y^4)^2} \, dy, \tag{10}$$

where $y = \sqrt{M} \, a/\zeta$. The validity of Eq. 10 can be verified by demonstrating[8] that it differs from Eq. 9 by terms that are higher order in ka. Equation 10 can be evaluated analytically[16] to give (to first order in ν and second order in M)

$$\langle f(\zeta) \rangle = \begin{cases} -\nu M/4, & n + \ell = 1 \\ \nu M^2/48, & n + \ell = 2, \\ 0, & n + \ell = 4 \end{cases} \tag{11}$$

where we have assumed that $m = 1 + \delta$ with $\delta \ll 1$, so that $M \ll 1$. The terms that are second order in M are retained because the inaccuracies of the stochastic variational results first appear at that order.

The exact result for $\langle |T_E|^2 \rangle^{\text{TE}}$ in the limits $(mka, ka,$ and $M) \to 0$ may now be expressed by applying

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[16] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products,* Academic Press, New York (1965).

Eqs. 11, 9, and 4 to the average absolute square of Eq. 6, giving (to the fourth order in M)

$$\langle |T_E|^2 \rangle^{TE} = \frac{\pi^2}{2} (ka)^4 M^2 \left\{ \cos^2\theta_s (1 - \nu) \right.$$

$$- M\nu \cos\theta_s \cos(2\beta - \theta_s)$$

$$\left. + M^2 \frac{\nu}{24} \left[2\cos^2\theta_s + \cos^2(2\beta - \theta_s) \right] \right\}.$$

$$(12)$$

As in the TM case, the $(1 - \nu)$ term describes scattering by two independent cylinders, modified by interference effects (the $-\nu$ term). The remaining terms in Eq. 12 are due to multiple scattering.

4.0 VARIATIONAL AND BORN APPROXIMATIONS

Vector stochastic variational expressions were recently derived for general conducting, dielectric, and magnetic scatterers.[5,6] In order to investigate the extent to which these expressions can account for polarization and interactive scattering phenomena, a stochastic variational approximation to the two-Rayleigh-cylinder random scattering problem addressed in the previous sections will now be calculated. The Born approximation for this problem is computed as

an intermediate result that the stochastic variational calculation improves. We are also able to evaluate the direct average of the deterministic variational expression for this simple random scattering problem in the limit of nearly transparent Rayleigh cylinders, where $m \to 1$. The various approximate solutions are compared with the exact solutions in the Summary and Conclusions.

4.1 Variational Formulation for Infinite Cylinders

The variational expressions derived in Refs. 5 and 6 are for scatterers localized in three dimensions in which the scattered field is a spherical wave in the far-field limit. For the infinite-length, homogeneous cylinders of current interest, the scattered field E_{sc} is a cylindrical wave in this limit and is expressed as $E_{sc} \to AT e^{ikr_0}/\sqrt{r_0}$ for $r_0 >> \zeta$ (see Fig. 1). An integral wave equation for two-dimensional scatterers may be obtained from the three-dimensional results[5,6] by recognizing that E, $\bar{E}$, and $U \equiv k^2(m^2 - 1)$ are all independent of the z coordinate in Fig. 1 for a normally incident plane wave. Using the relationship[17]

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{i}{8\pi} \int_{-\infty}^{\infty} dh\, e^{ih(z-z')} H_0^{(1)}(\kappa R_2)$$

(13)

with $k^2 = \kappa^2 + h^2$ and

$$R_2 = \sqrt{((x-x')^2 + (y-y')^2)},$$

one can integrate the three-dimensional results[5,6] over the z coordinate to obtain

$$E(\zeta) = E_i + \int_{cs} \bar{G}_0^{(2)}(\zeta, \zeta') \cdot [UE(\zeta')]\, dS',$$

(14)

where cs denotes integration over the scatterer cross section, E_i is the incident plane wave with polarization $\hat{e}_i$ and propagation direction $\hat{k}_i$, the two-dimensional vector $\zeta = x\hat{x} + y\hat{y}$, and the two-dimensional dyadic Green's function is

$$\bar{G}_0^{(2)}(\zeta, \zeta') = PV[\bar{I} + \nabla\nabla/k^2]$$

$$\times \frac{i}{4} H_0^{(1)}(kR_2) - \bar{L}\delta(\zeta - \zeta')/k^2 .$$

(15)

Here $\bar{I}$ is the three-dimensional unit dyadic, $H_0^{(1)}(kR_2)$ is the 0-th order Hankel function of the first kind, PV indicates principal value integration when the first term in Eq. 15 is in the integrand, and $\bar{L}$ is the depolarization dyadic that depends on the shape of the infinitesimal excluded region used in computing the PV integral. (We note that the total integral involving $\bar{G}_0^{(2)}$ is independent of this shape.[18]) In the far field, because ζ and ζ' are never coincident, Eq. 14 can be used to express the scattering amplitude, $T = \hat{e}_i \cdot T$, as

$$T = \frac{i}{4A} N_1 ,$$

(16)

[17]L. B. Felsen and N. Marcuvitz, *Radiation and Scattering of Waves*, Prentice Hall, Inc., Englewood Cliffs, New Jersey, p. 635 (1973).

[18]A. D. Yaghjian, "Electric Dyadic Green's Functions in the Source Region," *Proc. IEEE* **68**, 248-263 (1980).

with

$$N_1 = \int_{cs} dS' \, \hat{e}_s \cdot [\bar{I} - \hat{k}_s \hat{k}_s] \cdot [U\mathbb{E}(\zeta')] \, e^{-i\hat{k}_s \cdot \Gamma'} \,, \tag{17}$$

where $\hat{e}_s$ specifies the scattered wave polarization of interest, $\hat{k}_s$ is the unit vector in the scattered direction that makes an angle θ_B relative to the incident direction in Fig. 1, and integration is over the cross-sectional area of the scatterer.

The procedures outlined in Refs. 5 and 6 may be applied to Eq. 14 to derive a stochastic variational principle for T. As for the three-dimensional case, this results in an expression of the form

$$\langle |T_V|^2 \rangle = \frac{1}{16} \frac{\langle |N_1|^2 \rangle \langle |N_2|^2 \rangle}{\langle |D|^2 \rangle} \,, \tag{18}$$

where

$$N_2(\hat{e}_i, \hat{k}_i; \hat{e}_s, \hat{k}_s) = N_1(\hat{e}_s, -\hat{k}_s; \hat{e}_i, -\hat{k}_i) \tag{19}$$

and

$$D = \int_{cs} dS' \, U\tilde{\mathbb{E}}(\zeta') \cdot \mathbb{E}(\zeta') - \int_{cs} dS \int_{cs} dS' \, [U\tilde{\mathbb{E}}(\zeta)]$$
$$\cdot \, \bar{G}_0{}^{(2)}(\zeta, \zeta') \cdot [U\mathbb{E}(\zeta')] \,. \tag{20}$$

4.2 Evaluation of the Integrals N_1, N_2, and D

We encountered calculational difficulties in our attempts to evaluate the double surface integral in Eq. 20 for two dielectric cylinders. Consequently, we employed[19] methods analogous to those used by Yaghjian[18] to obtain an alternative expression in which D is reexpressed as

$$D = D_1 - D_2 + D_3 \,, \tag{21}$$

where

$$D_1 = \int_{cs} dS' \, U\tilde{\mathbb{E}}(\zeta') \cdot [(1 + U/k^2)\mathbb{E}(\zeta')] \,, \tag{22}$$

$$D_2 = \frac{i}{4k^2} \int_{cs} dS \int_{cs} dS' \, [U\tilde{\mathbb{E}}(\zeta)] \cdot [\nabla' \times \nabla' \times (U\mathbb{E}(\zeta'))] H_0^{(1)}(kR_2) \,, \tag{23}$$

and

$$D_3 = \frac{i}{4k^2} \int_{cs} dS \, U\tilde{\mathbb{E}}(\zeta) \cdot \oint_{\Gamma} d\ell \, \hat{n}' \cdot \left\{ U\mathbb{E}(\zeta') \right.$$
$$\times \, [\nabla' H_0^{(1)}(kR_2) \times \bar{I}]$$
$$\left. + \, [\nabla' \times U\mathbb{E}(\zeta')] \times \bar{I} H_0^{(1)}(kR_2) \right\} \,, \tag{24}$$

where the contour integral in D_3 is over the boundary Γ of the scatterer cross section (cs), and $\hat{n}$ is the outward normal at the boundary. Although Eq. 21 is not as compact as Eq. 20, its evaluation is straightforward because all of the singularities are integrable.

For Rayleigh dielectric cylinders in which $ka \ll 1$ and $mka \ll 1$, it is appropriate to approximate the total original and adjoint fields appearing in the integrals of the variational expression by the corresponding fields that would occur in the absence of the cylinders, i.e., by the original and adjoint plane waves[5,6] $\mathbb{E} \approx A\hat{e}_i \exp(i\mathbf{k}_i \cdot \zeta)$ and $\tilde{\mathbb{E}} \approx A\hat{e}_s \exp(-i\mathbf{k}_s \cdot \zeta)$, respectively, where $\zeta = \mathbf{r} - z\hat{z}$. Using these trial approximations, one finds

$$N_1 = N_2 = A(\hat{e}_s \cdot \hat{e}_i) k^2 (m^2 - 1) \sum_j \int_{cs_j} e^{i\alpha \cdot \zeta_j} dS' \,, \tag{25}$$

[19] J. A. Krill, R. H. Andreo, and R. A. Farrell, "A Computational Alternative for Variational Expressions that Involve Dyadic Green Functions," *IEEE Trans. Antennas Propag.* AP-30, 1003-1005 (1982).

where the sum is over cylinders $j = \pm 1$ and the integral is over the cross section of the j-th cylinder. Further, we have used the definitions $\alpha \equiv \mathbf{k}_i - \mathbf{k}_s$ and $U = k^2(m^2 - 1)$ for $\zeta' \epsilon cs_j$, and $U = 0$ for $\zeta' \not\epsilon cs_j$. The integral is evaluated by first expressing the coordinates for points interior to the cylinders as $\zeta_j = \zeta^{(j)} + \mathbf{r}$, where $\zeta^{(j)} = [\hat{x}\zeta^{(j)} \cos \beta + \hat{y}\zeta^{(j)} \sin \beta]$ is the position of the j-th cylinder axis (see Fig. 1) and $\mathbf{r}$ is the field point within the j-th cylinder relative to its axis. Applying this transformation to Eq. 25 results in

$$N_1 = A(\hat{e}_s \cdot \hat{e}_i)k^2(m^2 - 1)\pi a^2 S_1 , \quad (26a)$$

where

$$S_1 \equiv \sum_j e^{i\alpha \cdot \zeta^{(j)}} = e^{i\alpha \cdot \zeta^{(1)}}(1 + e^{i\alpha \cdot \zeta}) ,$$
$$(26b)$$

$$\alpha \equiv |\alpha| ,$$

and we have used the fact that

$$\int_{cs_j} e^{i\alpha \cdot \mathbf{r}} dS = \pi a^2 \left[\frac{2J_1(\alpha a)}{\alpha a} \right] \xrightarrow[a \to 0]{} \pi a^2 .$$

Applying these trial approximations to the denominator integrals, Eqs. 22 through 24, results in

$$D_1 = A^2(\hat{e}_s \cdot \hat{e}_i)k^2 (m^2 - 1)m^2$$

$$\times \sum_j \int_{cs_j} e^{i\alpha \cdot \zeta_j} dS' = m^2 N_1 A , \quad (27)$$

$$D_2 = \frac{i}{4} A^2 k^2 (m^2 - 1)^2 \sum_i \sum_j \int_{cs_i} dS$$

$$\times \int_{cs_j} dS' e^{-i\mathbf{k}_s \cdot \zeta_i} \hat{e}_s$$

$$\cdot \left[\nabla' \times \nabla' \times \left(\hat{e}_i e^{i\mathbf{k}_i \cdot \zeta_j} \right) \right] \times H_0^{(1)}(kR_2) , \quad (28)$$

and

$$D_3 = \frac{i}{4} A^2 k^2 (m^2 - 1)^2 \sum_i \sum_j \int_{cs_j} dS \, e^{-i\mathbf{k}_s \cdot \zeta_i} \hat{e}_s$$

$$\cdot \oint_{\Gamma_j} d\ell' \hat{n}' \cdot \left\{ \hat{e}_i e^{i\mathbf{k}_i \cdot \zeta_j} \times \left| \nabla' H_0^{(1)}(kR_2) \times \vec{I} \right| \right.$$

$$\left. + \left[\nabla' \times \left(\hat{e}_i e^{i\mathbf{k}_i \cdot \zeta_j} \right) \right] \times H_0^{(1)}(kR_2)\vec{I} \right\} , \quad (29)$$

where Γ_j denotes the circumference of the j-th cylinder. By applying the coordinate transformation $\zeta_j = \zeta^{(j)} + \mathbf{r}$ and evaluating the double curl in Eq. 28, D_2 reduces to

$$D_2 = \frac{i}{4} A^2 k^2 (m^2 - 1)^2 (\hat{e}_s \cdot \hat{e}_i)$$

$$\times \sum_i \sum_j e^{-i\mathbf{k}_s \cdot \zeta^{(i)}} e^{-i\mathbf{k}_i \cdot \zeta^{(j)}}$$

$$\cdot \int_{cs_i} dS \int_{cs_j} dS' e^{-i\mathbf{k}_s \cdot \mathbf{r}} e^{i\mathbf{k}_i \cdot \mathbf{r}'} H_0^{(1)}(kR_2)$$
$$(30)$$

since $\hat{e}_i \cdot \mathbf{k}_i = 0$. By applying the addition theorem (e.g., see Watson[20]) for the cases $i = j$ and $i \neq j$ in Eq. 30 and evaluating the resulting integrals in the Rayleigh limit, Eq. 30 is found to be proportional to $(ka)^4$. This should be contrasted with the other integrals N_1, N_2, D_1, and (as will be shown) D_3, which are of the order $(ka)^2$. Thus, D_2 does not contribute to the variational calculation in the Rayleigh limit.

Evaluation of the D_3 integral, Eq. 29, remains. This integral arises from the discontinuity in refractive index at the surface of the scatterer and vanishes in the case of soft scatterers.[19] (Of course, D_2 would be more complicated for soft scatterers.) We will show that, for hard scatterers, D_3 accounts for all the multiple scattering and contributes to the polarization dependence in the variational approximation. We rewrite Eq. 29 in a form more convenient for calculation by using a vector identity and by translating the cylindrical coordinate systems. This gives

[20]G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, New York, Chap. XI (1962).

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$$D_3 = \frac{i}{4} A^2 k^2 (m^2 - 1)^2 \sum_i \sum_j e^{-\hat{n}_s \cdot \mathbf{r}^{(j)}} e^{i\hat{n}_i \cdot \mathbf{r}^{(i)}}$$

$$\cdot \int_0^a \int_0^{2\pi} \iota' d\iota' d\phi' e^{-ik\iota' \cos(\theta_s - \phi')} \hat{e}_s$$

$$\cdot \int_0^{2\pi} a d\phi \, e^{-ika\cos\phi} \left\{ \hat{\mathbf{r}} \cdot \left[\nabla H_0^{(1)}(kR_2) \right] \hat{e}_i \right.$$

$$- \hat{\mathbf{r}} \left(\hat{e}_i \cdot \nabla H_0^{(1)}(kR_2) \right) - i(\hat{\mathbf{r}} \doteq \mathbf{k}_i) \hat{e}_i H_0^{(1)}(kR_2)$$

$$\left. + i(\hat{\mathbf{r}} \cdot \hat{e}_i) \mathbf{k}_i H_0^{(1)}(kR_2) \right\} \Big|_{\iota = a}, \qquad (31)$$

where $\hat{\rho}$ is the radial cylindrical unit vector. When $i = j$, $R_2 = [a^2 + \iota'^2 - 2a\iota' \cos(\phi - \phi')]^{1/2}$, and when $i \neq j$, $R_2 = [P^2 + a^2 - 2Pa \cos(\phi - p)]^{1/2}$, where $P = [\mathbf{r}^2 + \iota'^2 - 2\mathbf{r}\iota' \cos(\pi - \phi' - \beta)]^{1/2}$. These geometrical parameters are illustrated in Fig. 2. The addition theorem is applied once for $i = j$ and twice for $i \neq j$, and the gradients on $H_0(kR_2)$ are then taken. The resulting series of integrals is straightforward to evaluate, and upon retaining terms that are the lowest order in ka, we find

$$D_3 = A^2 \pi (ka)^2 (m^2 - 1)$$

$$\times \left\{ (m^2 - 1) [(\hat{y} \cdot \hat{e}_s)(\hat{y} \cdot \hat{e}_i)/2 - \hat{e}_s \cdot \hat{e}_i] S_1 \right.$$

$$- \frac{i\pi}{8} (m^2 - 1)(ka)^2 H_2^{(1)}(k\mathbf{r})(\hat{y} \cdot \hat{e}_i) \hat{e}_s$$

$$\left. \cdot [\hat{x} \sin(2\beta) - \hat{y} \cos(2\beta)] S_2 \right\}, \qquad (32a)$$
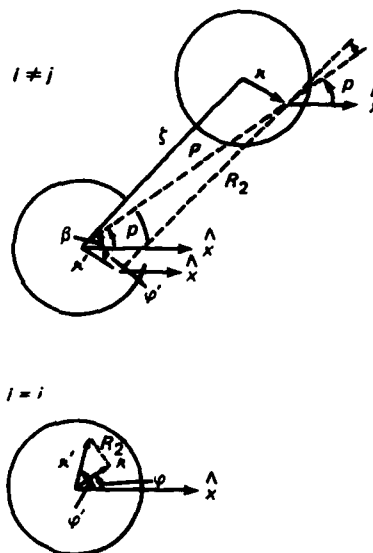

Figure 2 — Illustration of parameters that relate to R_2 using the cosine law. When $i = j$, ι and ι' are within the same cylinder; however, when $i \neq j$, ι and ι' lie in different cylinders.

where

$$S_2 = \sum_j e^{-\hat{n}_s \cdot \mathbf{r}^{(j)}} e^{i\hat{n}_i \cdot \mathbf{r}^{(j)}}$$

$$= e^{i\alpha \cdot \mathbf{r}^{(1)}} (e^{-\hat{n}_s \cdot \mathbf{r}} + e^{i\hat{n}_i \cdot \mathbf{r}}), \qquad (32b)$$

and S_1 is defined in Eq. 26b. Observe that $(ka)^4 H_2^{(1)}(k\mathbf{r})$ in Eq. 32 is of the same order as $c_i^{TE}\rho$ in the exact TE solution Eqs. 4 and 6. For $\mathbf{r} \to 2a$, this coefficient is of the order $(ka)^2$.

4.3 Evaluation of the Averages

The integrals $|N_1|^2$, $|N_2|^2$, and $|D_1|^2$ depend on the cylinder separation $\mathbf{r}$ through the phase term

Using Eq. 7 to average these integrals over random (but nonoverlapping) separations, we find

$$|S_1|^2 = 2 + 2 \, \text{Real} \, (e^{i\alpha \cdot \mathbf{r}}). \qquad (33)$$

$$\langle |N_1|^2 \rangle = |A|^2 2\pi^2 (ka)^4 (m^2 - 1)^2 (\hat{e}_s \cdot \hat{e}_i)^2 (1 - \nu) \qquad (34)$$

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and, from Eqs. 25 and 27,

$$\langle |N_1|^2 \rangle = \langle |N_2|^2 \rangle = \langle |D_1|^2 \rangle / (m^4 A^2). \quad (35)$$

The final step needed to obtain $\langle |T_1|\rangle^2$ from Eq. 18 is to complete the evaluation of $\langle |D|^2 \rangle$. From Eq. 21 and the fact that D_2 does not contribute in the Rayleigh limit,

$$\langle |D|^2 \rangle = \langle |D_1|^2 \rangle + 2 \operatorname{Real} \langle D_1^* D_3 \rangle + \langle |D_1|^2 \rangle. \quad (36)$$

The averages appearing in the latter two terms of this equation can be expressed in the general form

$$\langle f(\zeta) \rangle = \langle (ka)^{4+2(\ell+n)} H_2^{(1)n}(k\zeta) H_2^{(2)\ell}(k\zeta) \cos(\gamma\zeta) \rangle$$

$$= \frac{2(ka)^{4+2(n+\ell)}}{L}$$

$$\times \int_{2a}^{\ell} H_2^{(1)n}(k\zeta) H_2^{(2)\ell}(k\zeta) \cos \gamma \zeta \, d\zeta. \quad (37)$$

as can be shown by direct substitution of Eqs. 31 and 27 (with Eq. 26) into Eq. 36. In particular, Eq. 37 must be evaluated for the cases $n + \ell = (0,1,2)$ and $\gamma = [0, k \cos \beta, k \cos(\theta_i - \beta),$ or $k (\cos \beta \pm \cos(\theta_i - \beta))]$. In the Rayleigh limit, the evaluation of Eq. 37 can be accomplished using procedures analogous to those used to obtain the average of Eq. 9,[*] with the result

$$\langle f(\zeta) \rangle = \begin{cases} (ka)^4 & n + \ell = 0, \ \gamma = 0 \\ -\nu(ka)^4 & n + \ell = 0, \ \gamma \neq 0 \\ (-1)^n (i\nu/\pi)(ka)^4 & n + \ell = 1 \\ (i)^{n-\ell}(\nu/3\pi^2)(ka)^4 & n + \ell = 2 \end{cases} \quad (38)$$

---

[*]An alternate evaluation procedure ⋯⋯ pital's rule, is given in Ref. 21. This alternative c⋯ ⋯⋯d to evaluate the average of Eq. 9.

[21] E. P. Gray, R. W. Hart, and R. A. Farrell, *A Variational Approximation for the Scattering of Scalar Waves by Stochastic Surfaces*, JHU/APL TG 1322 (1979).

to first order in ν. These results lead to

$$\langle |D|^2 \rangle = A^4 \pi^2 (m^2 - 1)^2 (ka)^4 \Big\{ 2(1 - \nu)\Big[(m^2 - 1)^2$$

$$\times \Big(\frac{(\hat{y}\cdot\hat{e}_i)(\hat{y}\cdot\hat{e}_i)}{2} \Big)^2 + (m^2 - 1)(\hat{y}\cdot\hat{e}_i)$$

$$\times (\hat{y}\cdot\hat{e}_i)(\hat{e}_i\cdot\hat{e}_i) + (\hat{e}_i\cdot\hat{e}_i)^2 \Big]$$

$$+ \frac{\nu}{48} (m^2 - 1)^2 \Big[(\hat{y}\cdot\hat{e}_i)\hat{e}_i \cdot [\hat{x}\sin(2\beta) - \hat{y}\cos(2\beta)] \Big]^2$$

$$- \nu(\hat{y}\cdot\hat{e}_i)\hat{e}_i \cdot [\hat{x}\sin(2\beta) - \hat{y}\cos(2\beta)]$$

$$\Big[(m^2 - 1)^2 \frac{(\hat{y}\cdot\hat{e}_i)(\hat{y}\cdot\hat{e}_i)}{2} + (m^2 - 1)(\hat{e}_i\cdot\hat{e}_i) \Big] \Big\}. \quad (39)$$

The final step is to insert Eqs. 34, 35, and 39 into the variational expression Eq. 18. Expanding the result in powers of ν and retaining the first-order term yields

$$\langle |T_1|^2 \rangle^{TM} = \frac{\pi^2}{8} (ka)^4 (m^2 - 1)^2 (1 - \nu) \quad (40)$$

and

$$\langle |T_1|^2 \rangle^{TE} = \frac{\pi^2}{2} (ka)^4 M^2 \Big\{ \cos^2 \theta_i (1 - \nu)$$

$$- M\nu \cos \theta_i \cos(2\beta - \theta_i)$$

$$- M^2 \frac{\nu}{24} \cos^2 (2\beta - \theta_i) \Big\}, \quad (41)$$

where $M \equiv (m^2 - 1)/(m^2 + 1)$; and we have used the facts that

$$\hat{e}_i = \begin{cases} \hat{z} & \text{for TM waves} \\ \hat{y} & \text{for TE waves} \end{cases} \quad (42a)$$

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and

$$\hat{\epsilon}_i = \begin{cases} \hat{z} & \text{for TM waves} \\ -\hat{x}\sin\theta_i + \hat{y}\cos\theta_i & \text{for TE waves} \end{cases}$$
(42b)

This completes the evaluation of our vector stochastic variational principle. Next, we obtain the noninvariant Born approximation and the direct average of the Levine-Schwinger deterministic variational result.

The noninvariant Born approximation $\langle |T_B|^2 \rangle$ is obtained by using the plane wave trial function in the expression[8]

$$\langle |T_B|^2 \rangle = \langle |N_1|^2 \rangle / (16 |A|^2).$$
(43)

From Eqs. 34 and 42, the TE and TM solutions are

$$\langle |T_B|^2 \rangle^{TM} = \frac{\pi^2}{8} (ka)^4 (m^2 - 1)^2 (1 - \nu)$$
(44)

and

$$\langle |T_B|^2 \rangle^{TE} = \frac{\pi^2}{8} (ka)^4 (m^2 - 1)^2 \cos^2\theta_i (1 - \nu).$$
(45)

The standard Levine-Schwinger form of the variational principle is

$$|T_{LS}|^2 = \frac{|N_1|^2 |N_2|^2}{16 |D|^2}.$$
(46)

For nearly transparent Rayleigh cylinders, the denominator integral, D, can be expanded in powers of M. In particular, recognizing that $D_2 = 0$ in this limit and combining Eqs. 27 and 32,

$$D = D_1 + D_3$$

$$= \begin{cases} N_1 A & \text{for the TM case} \\ N_1 A \left(\frac{m^2 + 1}{2}\right)[1 + M\mathfrak{D}] & \text{for the TE case} \end{cases}$$
(47)

where

$$\mathfrak{D} = \frac{i\pi(ka)^2}{4} H_2^{(1)}(k\zeta) \left(\frac{\cos(2\beta - \theta_i)}{\cos\theta_i}\right) S_2/S_1.$$
(48)

It should be noted that $\mathfrak{D}$ does not depend on M and is of order $(ka)^2$ except when the cylinders are nearly touching where $\mathfrak{D}$ is of order unity. Using Eq. 47, we can write the direct average of Eq. 46 as

$$\langle |T_{LS}|^2 \rangle$$

$$= \begin{cases} \langle |N_1/A|^2 \rangle / 16 & \text{for the TM case} \\ \frac{1}{4} \left[\langle |N_1/(A(m^2 + 1))|^2 \rangle \right. & \\ \left. - M \langle \Gamma_1 \rangle + M^2 \langle \Gamma_2 \rangle \right] & \text{for the TE case} \end{cases}$$
(49)

where $|N_1/(A(m^2 + 1))|^2$, Γ_1, and Γ_2 are each proportional to M^2, and terms of order M^3 and higher have been neglected. In particular,

$$\Gamma_1 = i\frac{\pi^3}{4} (ka)^6 M^2 \cos\theta_i \cos(2\beta - \theta_i)$$
$$\times [H_2^{(1)}(k\zeta) S_1^* S_2 - H_2^{(2)}(k\zeta) S_1 S_2^*]$$
(50)

and

$$\Gamma_2 = \pi^4 M^2 \left(\frac{ka}{2}\right)^4 \cos^2(2\beta - \theta)$$
$$\times \left[\left| (ka)^2 H_2^{(1)}(k\zeta) S_2 \right|^2 \right.$$

$$- (ka)^4 \left(H_2^{(1)}(k\zeta) \right)^2 S_1^* S_2^2 / S_1$$

$$\left. - (ka)^4 \left(H_2^{(2)}(k\zeta) \right)^2 S_1 S_2^{*2} / S_1^* \right].$$
(51)

We note that $S_1^* S_2$ and S_2/S_1 depend on the cylinders' positions only through the axis separation ζ. Therefore, the averages appearing in Eq. 49 can be evaluated using Eqs. 37 and 38. The result is

$$\langle |T_{LS}|^2 \rangle = \begin{cases} \dfrac{\pi^2}{8} (ka)^4 (m^2 - 1)^2 (1 - \nu) & \text{for the TM case} \\ \\ \dfrac{\pi^2}{2} (ka)^4 M^2 \left\{ \cos^2 \theta_s (1 - \nu) \right. \\ \quad - M\nu \cos \theta_s \cos (2\beta - \theta_s) \\ \quad \left. + M^2 \dfrac{\nu}{8} \cos^2 (2\beta - \theta_s) \right] \right\} & \text{for the TE case} \end{cases} \quad (52)$$

5.0 SUMMARY AND CONCLUSIONS

The recently developed vector stochastic variational principle for electromagnetic wave scattering has been reexpressed in terms of the scalar Helmholtz-Green's function and evaluated for a test case of two randomly separated dielectric cylinders. This study examines, for the first time, the extent to which this principle can account for polarization and interactive phenomena even though the approximation that it improves does not. This has been accomplished by calculating the exact solution and comparing the result with the variational approximation. The two randomly separated cylinders have parallel axes and are infinitely long. The cylinders are not allowed to overlap, i.e., they are hard, and their diameter is small relative to a wavelength. The variational and exact results may also be compared with the Born approximation, which is obtained as an intermediate step in calculating the variational result when simple plane wave trial functions are used.

Comparison of Eqs. 8, 40, and 44 indicates that both the Born and the stochastic variational approximations give the exact result for the average TM differential scattering cross section. As the exact TM solution contains no multiple scattering, this conclusion is not surprising.

Comparison of Eqs. 12 and 45 indicates significant discrepancies between the Born approximation and the exact result for the mean TE cross section. First, the Born approximation accounts for the angular dependence of the TE solution but does not give the correct refractive index dependence in its amplitude, i.e., $(m^2 - 1)^2/8$ for the Born versus $M^2/2 \equiv [(m^2 - 1)/(m^2 + 1)]^2/2$ for the exact solution. This discrepancy arises because the Born approximation neg-

lects the geometric polarizability (resulting from the cylindrical shape) in approximating the fields inside the cylinders. Second, as expected, the Born approximation includes interference but does not account for TE multiple scattering.

The stochastic variational improvement (Eq. 41) of the Born approximation for this TE scattering corrects for the geometric polarizability. In addition, it contains a multiple scattering contribution that is correct through the lowest-order term in the variable M, i.e., through M^3. Thus the stochastic variational correction factor $|A|^2 \langle |N_2|^2 \rangle / \langle |D|^2 \rangle$ significantly improves the Born approximation to give the polarization and multiple scattering dependences accurately (even though these effects are missing from the simple Born trial field).

The direct average of the Levine-Schwinger deterministic variational expression was obtained in the limit of nearly transparent Rayleigh cylinders. That approximation also agrees with the exact result through the terms of order M^3.

As with the stochastic variational principle, the M^4 term is incorrect. (This discrepancy was uncovered subsequent to the publication of Ref. 22.) Neither variational result reproduces the $\cos^2 \theta_s$ dependence found in the M^4 term of the exact result, and both give an incorrect value for the coefficient of the $\cos^2(2\beta - \theta_s)$ term.

[22] J. A. Krill, R. H Andreo, and R. A. Farrell, "Variational Calculations of Electromagnetic Scattering from Two Randomly Separated Rayleigh Dielectric Cylinders," *J. Opt. Soc. Am.* 73, 408-410 (1983).

ACKNOWLEDGMENT

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