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ASYMMETRIC ERRORS

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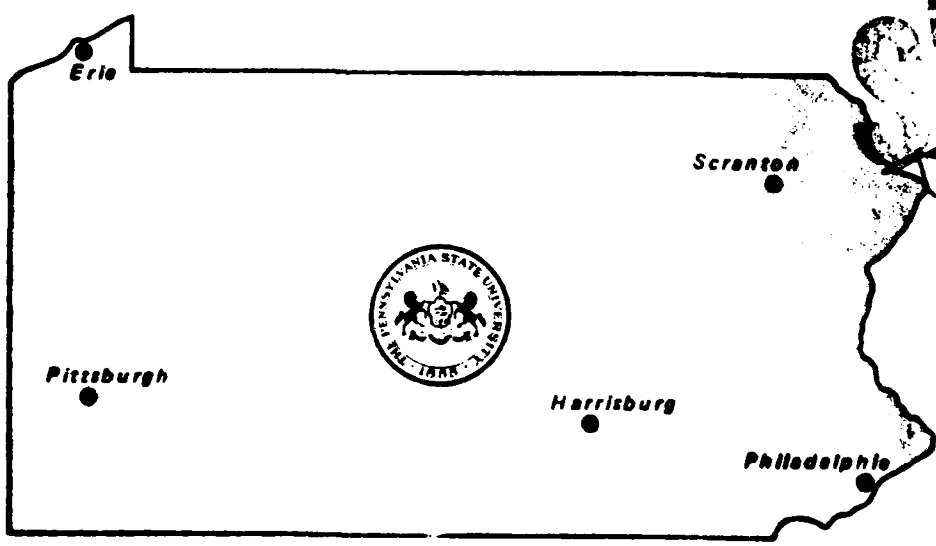

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Summary

Rank-based Inference for Linear Models: Asymmetric Errors

In this paper robust, rank-based inference procedures are considered for general linear models with (possibly) asymmetric errors. Approximating standard errors of estimates and testing hypotheses about the model parameters require estimating a scaling functional, and an approach is developed which, unlike previous work, does not require symmetry of the underlying error distribution or replicates in the design matrix. Hence, important asymmetric models such as arise in life testing can now be handled. Further, it is shown that the asymptotic properties of the inference procedures hold with simpler conditions on the design matrix than previously required. In addition an estimate of the intercept is developed without requiring the assumption of a symmetric error distribution.

1. Introduction In this paper robust, rank-based inference procedures are considered for general linear models with (possibly) asymmetric errors. Approximating standard errors of estimates and testing hypotheses about the model parameters require estimating a scaling functional, and an approach is developed which, unlike previous work, does not require symmetry of the underlying error distribution (McKean and Hettmansperger 1976) or replicates in the design matrix (Draper 1981). Hence, important asymmetric models such as arise in life testing can now be handled. Further, it is shown that the asymptotic properties of the inference procedures hold with simpler conditions on the design matrix than previously required. In addition an estimate of the intercept is developed without requiring the assumption of a symmetric error distribution.

The models to be considered and the basic assumptions are now given. The vector of observations $\underset{\sim}{Y} = (Y_1, Y_2, Y_3, \ldots, Y_n)'$ is assumed to satisfy either

(1.1) $$\underset{\sim}{Y} = \alpha \underset{\sim}{1}_n + X_n \underset{\sim}{\beta} + \underset{\sim}{e}$$

or

(1.2) $$\underset{\sim}{Y} = X_n \underset{\sim}{\beta} + \underset{\sim}{e} ,$$

where $\underset{\sim}{1}_n$ is the n x 1 vector of ones, $\underset{\sim}{\beta}$ is the p x 1 vector of unknown regression coefficients, α -if it is included- is the unknown intercept, and $\underset{\sim}{e} = (e_1, e_2, \ldots, e_n)'$ is an n x 1 vector of independent, identically distributed random errors with continuous cumulative distribution function F. We will impose the following assumptions as needed:

(1.3) F has density f having finite Fisher information; that is, f is absolutely continuous and

$$\int_{-\infty}^{\infty} (f'(e)/f(e))^2 f(e)de < \infty.$$

(1.4) X_n is a known, n x p full-rank matrix with i^{th} row $\underset{\sim}{x}_i'$. The centered matrix $C_n = X_n - n^{-1} \underset{\sim}{1}_n \underset{\sim}{1}_n' X_n$ is also full rank.

(1.5) $$n^{-1} [\underset{\sim}{1}_n, X_n]' [\underset{\sim}{1}_n, X_n] \longrightarrow \Lambda ,$$

a positive definite matrix, as $n \longrightarrow \infty$.

(1.6) $$n^{-1} C_n' C_n \longrightarrow \Sigma,$$

a positive definite matrix, as $n \longrightarrow \infty$.

We note in passing that (1.5) implies (1.6). Where possible without confusion, the dependence of various quantities on n will be suppressed.

The inference procedures for $\underset{\sim}{\beta}$ to be considered are based on a measure of dispersion proposed by Jaeckel (1972). For an n x 1 vector $\underset{\sim}{v}$, define the dispersion as

(1.7) $$D(\underset{\sim}{v}) = \sum_{i=1}^{n} a(i)\, v_{(i)},$$

where $v_{(1)} \leq v_{(2)} \leq \cdots \leq v_{(n)}$ are the ordered elements of $\underset{\sim}{v}$ and a(1),...,a(n) are a set of scores satisfying some regularity conditions. Consideration in this paper is restricted to Wilcoxon scores,

(1.8) $$a(i) = 12^{1/2} (i/[n+1] - 1/2).$$

Thus each element of $\underset{\sim}{v}$ is assigned a weight proportional to the difference between its rank among the n elements and the average rank. Procedures based on these scores generalize the Mann-Whitney-Wilcoxon two-sample procedure and inherit its asymptotic efficiency.

Jaeckel proposed estimating $\boldsymbol{\beta}$ by minimizing $D(\mathbf{Y} - X\hat{\boldsymbol{\beta}})$ and showed the estimate thus obtained is equivalent to an estimate previously suggested by Jureckova (1971) in the sense that $n^{1/2}$ times the difference converges to zero in probability. Thus the two estimates have the same limiting distribution; specifically, under conditions to be weakened in this paper in the case of Wilcoxon scores, $n^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ converges in distribution to a multivariate normal random variable with mean $\mathbf{0}$ and variance-covariance matrix $\tau^2\Sigma^{-1}$, where

$$\tau^{-1} = 12^{1/2} \int_{-\infty}^{\infty} f^2(e)\, de. \tag{1.9}$$

It is often desirable to test hypotheses about $\boldsymbol{\beta}$ of the form

$$H_0: \ H\boldsymbol{\beta} = \mathbf{0} \quad \text{versus} \quad H_A: \ H\boldsymbol{\beta} = \mathbf{0}\ , \tag{1.10}$$

where H is a full-rank $q \times p$ matrix with $q \leq p$. A consistent test for (1.10) can be based on a quadratic form in the full-model estimate of $\boldsymbol{\beta}$:

$$Q = \hat{\tau}^{-2}\, \hat{\boldsymbol{\beta}}'\, H'[H(C'C)^{-1} H']^{-1} H\, \hat{\boldsymbol{\beta}}, \tag{1.11}$$

where $\hat{\tau}$ is some consistent estimate of τ. The statistic Q has an asymptotic $X^2(q)$ distribution under H_0. If $q < p$, a consistent test for (1.10) can also be obtained by fitting both the full model and the reduced model induced by H_0. Letting $\hat{\boldsymbol{\beta}}_{FULL}$ and $\hat{\boldsymbol{\beta}}_{RED}$ be the corresponding estimates, McKean and Hettmansperger (1976) showed that

$$D^* = 2[D(\mathbf{Y} - X\hat{\boldsymbol{\beta}}_{RED}) - D(\mathbf{Y} - X\hat{\boldsymbol{\beta}}_{FULL})]/\hat{\tau} \tag{1.12}$$

has an asymptotic $X^2(q)$ distribution under H_0 and can be used as a test statistic. Both test statistics require a consistent estimate of the scaling functional τ. The estimates previously proposed necessitate either the assumption of a symmetric error distribution or replicate rows in X. In Section 3 an estimate is developed without either of these assumptions.

The asymptotic theory for $\hat{\beta}$, Q, and D* is founded on the asymptotic linearity of the gradient of the dispersion $D(\underset{\sim}{Y} - X\underset{\sim}{b})$ treated as a function of $\underset{\sim}{b}$. Thus the technical assumptions adopted by Jureckova (1971) in her proof of the asymptotic linearity have been carried over by subsequent authors. In Section 4 it is shown that this linearity property - for the important case of Wilcoxon scores - can be obtained without some of the complicated assumptions on the design matrix required by Jureckova; specifically, her assumptions 3a, 3b, and 3c are eliminated. The results of Kraft and van Eeden (1972) for linearized rank statistics should also hold under less complicated assumptions on the design.

Since $D(\underset{\sim}{Y} - a\underset{\sim}{1} - X\underset{\sim}{b}) = D(\underset{\sim}{Y} - X\underset{\sim}{b})$, Jaeckel's dispersion function provides no information concerning the intercept. McKean and Hettmansperger (1978) showed that α can be estimated by applying a one-sample signed-rank procedure to the residuals after estimating $\underset{\sim}{\beta}$, if the error distribution is symmetric. In Section 5 an estimate is proposed which does not require symmetry of F and its joint asymptotic distribution with $\hat{\underset{\sim}{\beta}}$ is stated.

2. <u>A Preliminary Lemma</u> The proofs of the results in this paper rely on a lemma which is, in essence, imbedded in the proof of Theorem 3.1 of Jureckova (1969). We state the lemma here for convenient reference.

In the sections to come, we are concerned with the asymptotic behavior of some random variable $H_n(\hat{\underset{\sim}{\gamma}}_n, \underset{\sim}{V}_n)$, where $\hat{\underset{\sim}{\gamma}}_n$ is a consistent estimate of a parameter $\underset{\sim}{\gamma}$ and $\underset{\sim}{V}_n$ is a random vector. For simplicity the dependence on n is suppressed. In many cases it is easy to determine the behavior of $H(\boldsymbol{\gamma},\boldsymbol{Y})$, and if it can be shown that $H(\boldsymbol{\gamma},\boldsymbol{Y}) - H(\hat{\boldsymbol{\gamma}},\boldsymbol{Y})$ converges to zero in probability, the behavior of $H(\hat{\boldsymbol{\gamma}},\boldsymbol{Y})$ is also determined.

For example, let W be a function of the residuals $\hat{e}_i = Y_i - \hat{\alpha} - \boldsymbol{x}_i' \hat{\boldsymbol{\beta}}$ after fitting a linear model. Then one can think of W as $H((\alpha\ \boldsymbol{\beta}')', \boldsymbol{Y})$, where $H((a\ \boldsymbol{b}')', \boldsymbol{Y})$ is that same function of the "residuals" $Y_i - a - \boldsymbol{x}_i' \boldsymbol{b}$. Since $H((\alpha\ \boldsymbol{\beta}')', \boldsymbol{Y})$ is a function of the independent, indentically distributed errors $e_i = Y_i - \alpha - \boldsymbol{x}_i' \boldsymbol{\beta}$, its behavior may be simple to determine.

The following lemma gives sufficient conditions for $H(\hat{\boldsymbol{\gamma}},\boldsymbol{Y}) - H(\boldsymbol{\gamma},\boldsymbol{Y})$ to converge to zero in probability.

<u>Lemma</u> 2.1 Suppose $H(\boldsymbol{g},\boldsymbol{Y}) = Q(\boldsymbol{g},\boldsymbol{Y}) + \boldsymbol{Z}'(\boldsymbol{g}-\boldsymbol{\gamma})$, where $Q(\boldsymbol{g},\boldsymbol{Y})$ is monotone in each of the components of $\boldsymbol{g}$, perhaps nondecreasing in some, nonincreasing in others and $||\boldsymbol{Z}||/n^t \leq K < \infty$ for some $t > 0$. Here $\boldsymbol{Z}$ may depend on n; $||\cdot||$ is the sup-norm throughout this paper. If $H(\boldsymbol{g},\boldsymbol{Y}) - H(\boldsymbol{\gamma},\boldsymbol{Y})$ converges to zero in probability for $\boldsymbol{g} = \boldsymbol{\gamma} + \boldsymbol{d}/n^t$ with $\boldsymbol{d}$ fixed, then for each B, $0 < B < \infty$,

$$\sup \left| H(\boldsymbol{g},\boldsymbol{Y}) - H(\boldsymbol{\gamma},\boldsymbol{Y}) \right|$$

converges to zero in probability, where the supremum is over $\{\boldsymbol{g}:\ n^t||\boldsymbol{g} - \boldsymbol{\gamma}|| \leq B\}$. Furthermore, if $n^t(\hat{\boldsymbol{\gamma}}-\boldsymbol{\gamma})$ is bounded in probability, then $H(\hat{\boldsymbol{\gamma}},\boldsymbol{Y}) - H(\boldsymbol{\gamma},\boldsymbol{Y})$ converges to zero in probability.

3. <u>Scaling Functional</u> In the one-sample setting, several authors (Bhattacharyya and Roussas 1969; Schuster 1974; Schweder 1975; Ahmad 1976; Cheng and Serfling 1981) have considered estimating τ^{-1} using window (kernel) density estimates, sometimes as a particular case of more general estimation

problem. However, no one has treated the problem when the estimate is computed using dependent quantities, such as the residuals in regression. Let $f_n(e)$ be the window estimate of $f(e)$ given by

$$(3.1) \qquad f_n(e) = (nh_n)^{-1} \sum_{i=1}^{n} w([e-e_i]/h_n),$$

where $e_1, \ldots, e_n$ are a random sample from a distribution having density f, w is a density, and h_n $(n=1, 2, 3, \ldots)$ is a sequence of constants converging to zero. Under appropriate regularity conditions, the estimate for $\theta = \tau^{-1}/12^{1/2}$, given by

$$(3.2) \qquad \int_{-\infty}^{\infty} f_n(e)dF_n(e),$$

where F_n is the usual empirical distribution function, is strongly consistent and asymptotically normal.

The estimate (3.2) can be written as

$$(3.3) \qquad (n^2h_n)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} w([e_i-e_j]/h_n)$$

$$= w(0)(nh_n)^{-1} + (n^2h_n)^{-1} \sum\sum_{i \neq j} w([e_i-e_j]/h_n).$$

Since the nonrandom contribution of the $i=j$ terms is of smaller order than the random portion due to the $i \neq j$ terms, for the asymptotic theory we work only with the latter. Thus we consider

$$(3.4) \qquad \hat{\theta} = [n^2h_n]^{-1} \sum\sum_{i \neq j} w([e_i-e_j]/h_n),$$

and show that $\hat{\theta}$ remains consistent for θ when the e_i are replaced by the dependent residuals $\hat{e}_i = Y_i - \hat{\alpha} - \underset{\sim}{x}_i'\hat{\underset{\sim}{\beta}}$ (or $\hat{e}_i = Y_i - \underset{\sim}{x}_i'\hat{\underset{\sim}{\beta}}$) and h_n is replaced by a random $\hat{h}_n$, under mild conditions on w, h_n, $\hat{\underset{\sim}{\beta}}$, and $\hat{h}_n$. In applications it is necessary to use the data to determine the window width, in order to obtain good performance and to make $\hat{\theta}$ scale equivariant. Note that in this section $\hat{\beta}$ is not necessarily the rank estimate.

Theorem 3.1 Suppose;

(i) w is a square-integrable, strongly unimodal density, symmetric about zero, with finite second moment;

(ii) $h_n = n^{-r}h$, where $0 < r \leq 1/2$ and h is a positive constant;

(iii) $n^{1/2}(\hat{\beta}-\beta)$ is bounded in probability; and

(iv) $n^q(\hat{h}-h)$ is bounded in probability for some $q > 0$, and $\hat{h}_n = n^{-r}\hat{h}$

Then under assumptions (1.3), (1.4), and (1.6), for model (1.1) with $\hat{e}_i = Y_i - \hat{\alpha} - x_i'\hat{\beta}$ (or for model (1.2) with $\hat{e}_i = Y_i - x_i'\hat{\beta}$),

(3.5) $$\hat{\theta} = [n^2\hat{h}_n]^{-1} \sum\sum_{i \neq j} w([\hat{e}_i - \hat{e}_j]/\hat{h}_n) \text{ converges in probability to } \theta.$$

Proof: For arbitrary $p \times 1$ b and $k > 0$ define

(3.6) $$\begin{aligned} T(b;k) &= [n^2h_n]^{-1} \sum\sum_{i \neq j} w([Y_i - x_i'b - Y_j + x_j'b]/[n^{-r}k]) \\ &= [n^2h_n]^{-1} \sum\sum_{i \neq j} w([e_i - e_j - (x_i - x_j)'(b-\beta)]/k_n), \end{aligned}$$

where $k_n = n^{-r}k$. Then $T(\beta;h)$ is the estimate of θ based on the independent, identically distributed e_i and on the nonrandom window width h_n; and $\hat{\theta} = [h_n/\hat{h}_n]T(\hat{\beta};\hat{h})$ is the estimate based on the residuals $\hat{e}_i$ and random window width $\hat{h}_n$. It is now shown that $T(\hat{\beta};\hat{h}) - T(\beta;h)$ converges to zero in probability. Since, under the conditions of the theorem, $T(\beta; h)$ is easily shown to converge to θ in probability (Aubuchon(1982) used a projection argument to show that $n^{\frac{1}{2}}(T(\beta;h) - \theta)$ converges in distribution to a normal random variable under these conditions) and $h_n/\hat{h}_n$ converges to 1 in probability, this implies that $\hat{\theta}$ also converges to θ in probability.

Although $T(b;k)$ is not of the form required by Lemma 2.1, it is possible to split $T(b;k)$ into two pieces, each of which is of the necessary form. To this end, we define the following monotone functions. Let

$$(3.7)\qquad \begin{aligned} w_1(z) &= w(z) &&, \text{ if } z \leq 0 \\ &= w(0) &&, \text{ if } z > 0; \text{ and} \\ w_2(z) &= 0 &&, \text{ if } z \leq 0 \\ &= w(z)-w(0) &&, \text{ if } z > 0. \end{aligned}$$

Then $w(z) = w_1(z) + w_2(z)$. Also, let the $2p \times 1$ vector

$$(3.8)\qquad \underset{\sim}{D}_{ij} = \begin{pmatrix} (\underset{\sim}{x}_i-\underset{\sim}{x}_j)^+ \\ (\underset{\sim}{x}_i-\underset{\sim}{x}_j)^- \end{pmatrix},$$

where $(\underset{\sim}{x}_i-\underset{\sim}{x}_j)^+$ is the vector of positive parts of $(\underset{\sim}{x}_i-\underset{\sim}{x}_j)$ and $(\underset{\sim}{x}_i-\underset{\sim}{x}_j)^-$ is the vector of negative parts, so that $(\underset{\sim}{x}_i-\underset{\sim}{x}_j) = (\underset{\sim}{x}_i-\underset{\sim}{x}_j)^+ + (\underset{\sim}{x}_i-\underset{\sim}{x}_j)^-$; recall that x_i' is the i^{th} row of the design. Letting (for $2p \times 1$ $\underset{\sim}{\delta}$)

$$(3.9)\qquad T^*(\underset{\sim}{\delta};k) = [n^2h_n]^{-1} \sum_{i\neq j}\sum w([e_i-e_j-\underset{\sim}{D}_{ij}'(\underset{\sim}{\delta}-\binom{\underset{\sim}{\beta}}{\underset{\sim}{\beta}})]/k_n),$$

we have $T^*(\underset{\sim}{\delta};k) = T(\underset{\sim}{b},k)$ when $\underset{\sim}{\delta}' = (\underset{\sim}{b}'\,\underset{\sim}{b}')$. Finally, note $T^*(\underset{\sim}{\delta};k) = T_1^*(\underset{\sim}{\delta};k) + T_2^*(\underset{\sim}{\delta};k)$, where

$$(3.10)\qquad T_m^*(\underset{\sim}{\delta};k) = [n^2h_n]^{-1} \sum_{i\neq j}\sum [w_m(e_i-e_j-\underset{\sim}{D}_{ij}'(\underset{\sim}{\delta}-\binom{\underset{\sim}{\beta}}{\underset{\sim}{\beta}}))/k_n) \\ -(-1)^m\theta w(0)\, \underset{\sim}{D}_{ij}'(\underset{\sim}{\delta}-\binom{\underset{\sim}{\beta}}{\underset{\sim}{\beta}}))]$$

for $m = 1, 2$.

The following lemma establishes conditions under which $T_m^*(\underset{\sim}{\delta};k) - T_m^*((\underset{\sim}{\beta}'\ \underset{\sim}{\beta}')';h)$ converges to zero in probability for $m = 1, 2$. The proof of the lemma involves lengthy calculations and is deferred to Appendix A. The linear term subtracted in T_1^* and added in T_2^* plays a crucial role in the proof.

<u>Lemma</u> 3.1 For $m = 1, 2$, under the conditions of Theorem 3.1,

$$E[T_m^*(\underset{\sim}{\delta};k) - T_m^*(\binom{\underset{\sim}{\beta}}{\underset{\sim}{\beta}};h)]^2 \to 0$$

as $n \to \infty$, where $\underset{\sim}{\delta} = (\underset{\sim}{\beta}' \ \underset{\sim}{\beta}')' + \underset{\sim}{d}/n^{1/2}$ and $k = h + a/n^q$ for any fixed $\underset{\sim}{d}$ and a.

Note that

(3.11) $$T_m^*(\underset{\sim}{\delta};k) - T_m^*((\begin{smallmatrix}\underset{\sim}{\beta}\\ \underset{\sim}{\beta}\end{smallmatrix});h)$$
$$= [n^2h_n]^{-1} \sum_{i\neq j}\sum [w_m([e_i-e_j-\underset{\sim}{D}_{ij}'(\underset{\sim}{\delta}-(\begin{smallmatrix}\underset{\sim}{\beta}\\ \underset{\sim}{\beta}\end{smallmatrix}))]/k_n) - w_m([e_i-e_j]/h_n)]$$
$$-(-1)^m[n^2h_n]^{-1} \sum_{i\neq j}\sum 6w(0)\, \underset{\sim}{D}_{ij}'(\underset{\sim}{\delta}-(\begin{smallmatrix}\underset{\sim}{\beta}\\ \underset{\sim}{\beta}\end{smallmatrix})).$$

The first summation is monotone in the components of $\underset{\sim}{\delta}$, while $||[n^2h_n]^{-1} \sum_{i\neq j}\sum \underset{\sim}{D}_{ij}||/n^{1/2} \leq 2(n^{\frac{1}{2}} h_n)^{-1} \max_k \ n^{-1}\sum_{i=1}^{n}[1+x_{ik}{}^2]$, which is bounded. Since $n^{\frac{1}{2}}(\hat{\underset{\sim}{\beta}}-\underset{\sim}{\beta})$ is assumed to be bounded in probability, Lemma 2.1 (with t=1/2) implies that

(3.12) $$T_m^*((\begin{smallmatrix}\hat{\underset{\sim}{\beta}}\\ \hat{\underset{\sim}{\beta}}\end{smallmatrix});k) - T_m^*((\begin{smallmatrix}\underset{\sim}{\beta}\\ \underset{\sim}{\beta}\end{smallmatrix});h)$$

converges to zero in probability for $m = 1, 2$ and $k = h + a/n^q$. Furthermore, $T_m^*((\begin{smallmatrix}\hat{\underset{\sim}{\beta}}\\ \hat{\underset{\sim}{\beta}}\end{smallmatrix});k) - T_m^*((\begin{smallmatrix}\underset{\sim}{\beta}\\ \underset{\sim}{\beta}\end{smallmatrix});h)$ is monotone in k. Since $n^q(\hat{h}-h)$ is assumed to be bounded in probability, it follows from Lemma 2.1 that

(3.13) $$T_m^*((\begin{smallmatrix}\hat{\underset{\sim}{\beta}}\\ \hat{\underset{\sim}{\beta}}\end{smallmatrix});\hat{h}) - T_m^*((\begin{smallmatrix}\underset{\sim}{\beta}\\ \underset{\sim}{\beta}\end{smallmatrix});h)$$

converges to zero in probability for $m = 1, 2$.

Recalling that

(3.14) $$T(\hat{\underset{\sim}{\beta}};\hat{h}) - T(\underset{\sim}{\beta};h)$$
$$= T^*((\begin{smallmatrix}\hat{\underset{\sim}{\beta}}\\ \hat{\underset{\sim}{\beta}}\end{smallmatrix});\hat{h}) - T^*((\begin{smallmatrix}\underset{\sim}{\beta}\\ \underset{\sim}{\beta}\end{smallmatrix});h)$$
$$= T_1^*((\begin{smallmatrix}\hat{\underset{\sim}{\beta}}\\ \hat{\underset{\sim}{\beta}}\end{smallmatrix});\hat{h}) - T_1^*((\begin{smallmatrix}\underset{\sim}{\beta}\\ \underset{\sim}{\beta}\end{smallmatrix});h)$$
$$+ T_2^*((\begin{smallmatrix}\hat{\underset{\sim}{\beta}}\\ \hat{\beta}\end{smallmatrix});\hat{h}) - T_2^*((\begin{smallmatrix}\underset{\sim}{\beta}\\ \underset{\sim}{\beta}\end{smallmatrix});h),$$

this yields the desired result.

Example For applications, we consider a modified form of (3.3):

$$\theta^* = (nh)^{-1} + [n(n-1)h_n]^{-1} \sum\sum_{i \neq j} w([e_i - e_j]/h_n), \tag{3.15}$$

where h is a constant, and we take $h_n = n^{-1/2}h$. The authors (1984, Section 4) show that, in the iid case, the bias is then of order n^{-1}. If the underlying error distribution has density $f(y,\delta) = \delta^{-1} f_1(\delta^{-1} y)$, then

$$h = [2^{-1} \int_{-\infty}^{\infty} [f_1'(y)]^2 \, dy \int_{-\infty}^{\infty} u^2 \, w(u) \, du]^{-1/3} \tag{3.16}$$

will make the first - order terms in the bias vanish.

If we take f_1 to be the normal density with interquartile range equal to 1, $\hat{\delta}$ the sample interquartile range (defining $\hat{h}$), and w the uniform density on (-1/2, 1/2), then by Theorem 3.1,

$$\theta^* = (4.11n\ \hat{\delta})^{-1} + (4.11n^{1/2}(n-1)\hat{\delta})^{-1} \sum\sum_{i \neq j} w\left(\frac{\hat{e}_i - \hat{e}_j}{4.11n^{-1/2}\ \hat{\delta}}\right) \tag{3.17}$$

is a consistent estimator of θ.

4. Linearity of the Gradient of the Dispersion In this section, it is shown that, for Wilcoxon scores, the asymptotic linearity property of the gradient of the dispersion, first proved by Jureckova (1971), holds under simpler conditions on the design matrix. Thus the work of subsequent authors in developing the asymptotic properties of $\hat{\beta}$, Q, and D*, which relied on the linearity, also is valid under simpler assumptions.

The dispersion of $Y - X b$ may be written as

$$D(Y - Xb) = \sum_{i=1}^{n} a(R(Y_i - c_i' b))(Y_i - c_i' b) \tag{4.1}$$

$$= 12^{1/2} \sum_{i=1}^{n} [R(Y_i - c_i' b)/(n+1) - 1/2](Y_i - c_i' b),$$

where c_i is the i^{th} row of the centered design matrix C (1.5) and $R(Y_i - c_i' b)$ is the rank of $Y_i - c_i' b$ among $Y_1 - c_1' b, \ldots, Y_n - c_n' b$. Then the gradient of $D(Y - X b)$, taken as a function of b, exists except on a set of Lebesque measure zero in R^p. When it exists, the negative of the k^{th} element of this gradient is

$$n^{1/2} S_k(b) = 12^{1/2} \sum_{i=1}^{n} c_{ik} [R(Y_i - c_i' b)/(n+1) - 1/2] \tag{4.2}$$

$$= 12^{1/2} (n+1)^{-1} \sum_{i=1}^{n} c_{ik}\, R(Y_i - c_i' b).$$

Since the ranks in (4.2) are translation invariant, $S_k(b)$ can be further rewritten:

$$S_k(b) = 12^{1/2}[n^{1/2}(n+1)]^{-1} \sum_{i=1}^{n} c_{ik}\, R(e_i - c_i'(b - \beta)), \tag{4.3}$$

noting that $e_i = Y_i - \alpha - x_i' \beta$. (or $e_i = Y_i - x_i' \beta$).

<u>Theorem</u> 4.1 For model (1.1) or model (1.2), under assumptions (1.3), (1.4), and (1.6), and for $S_k(b)$ defined by (4.2),

$$\sup |S_k(b) - S_k(\beta) + \tau^{-1} \sigma_{(k)}' n^{1/2} (b - \beta)| \tag{4.4}$$

converges to zero in probability as $n \to \infty$, where the supremum is over $n^{1/2} ||b - \beta|| \leq B < \infty$, and $\sigma_{(k)}$ is the k^{th} column of Σ, the limit of $n^{-1} C' C$.

<u>Proof</u> Slutsky's Theorem implies that $(n+1)^{-1} C c_{(k)}$), where $c_{(k)}$ is the k^{th} column of C, may be substituted for $\sigma_{(k)}$ since $(n+1)^{-1} C' C \to \Sigma$ as $n \to \infty$.
Define

$$T_k(b) = S_k(b) + \tau^{-1} (C c_{(k)})' n^{1/2} (b - \beta)/(n+1). \tag{4.5}$$

Again we cannot directly apply Lemma 2.1 to $T_k(b)$ but must split the quantity into two pieces. As before (3.8) let

$$(4.6) \qquad \underset{\sim}{D}_{ij} = \begin{pmatrix} (\underset{\sim}{x}_i - \underset{\sim}{x}_j)^+ \\ (\underset{\sim}{x}_i - \underset{\sim}{x}_j)^- \end{pmatrix} = \begin{pmatrix} (\underset{\sim}{c}_i - \underset{\sim}{c}_j)^+ \\ (\underset{\sim}{c}_i - \underset{\sim}{c}_j)^- \end{pmatrix} .$$

Letting

$$(4.7) \qquad T_{k1}(\underset{\sim}{\delta}) = 12^{1/2}[n^{1/2}(n+1)]^{-1} \sum_{i=1}^{n} C_{ik}^{+} \sum_{j=1}^{n} I\{e_j - e_i \le \underset{\sim}{D}_{ji}'(\underset{\sim}{\delta} - (\underset{\sim}{\beta}^{\underset{\sim}{\beta}}))\}$$

$$- \tau^{-1}[n(n+1)]^{-1} n^{1/2} (\sum_{i=1}^{n} C_{ik}^{+} \sum_{j=1}^{n} \underset{\sim}{D}_{ji})' (\underset{\sim}{\delta} - (\underset{\sim}{\beta}^{\underset{\sim}{\beta}})) \quad \text{and}$$

$$T_{k2}(\underset{\sim}{\delta}) = 12^{1/2}[n^{1/2}(n+1)]^{-1} \sum_{i=1}^{n} C_{ij}^{-} \sum_{j=1}^{n} I\{e_j - e_i \le \underset{\sim}{D}_{ji}'(\underset{\sim}{\delta} - (\underset{\sim}{\beta}^{\underset{\sim}{\beta}})]\}$$

$$- \tau^{-1}[n(n+1)]^{-1} n^{1/2} (\sum_{i=1}^{n} C_{ik}^{-} \sum_{j=1}^{n} \underset{\sim}{D}_{ji})' (\underset{\sim}{\delta} - (\underset{\sim}{\beta}^{\underset{\sim}{\beta}}))$$

for any 2p x 1 $\underset{\sim}{\delta}$, we have

$$(4.8) \qquad T_k(\underset{\sim}{b}) = T_{k1}(\underset{\sim}{\delta}) + T_{k2}(\underset{\sim}{\delta}) \text{ when } \underset{\sim}{\delta} = (\underset{\sim}{b}' \ \underset{\sim}{b}')'.$$

The following lemma establishes that $T_{km}(\underset{\sim}{\delta}) - T_{km}((\underset{\sim}{\beta}' \ \underset{\sim}{\beta}')')$ converges to zero in probability for m = 1, 2 when $\underset{\sim}{\delta} = (\underset{\sim}{\beta}' \ \underset{\sim}{\beta}')' + \underset{\sim}{d}/n^{1/2}$. The proof of the lemma involves lengthy calculations and is left to Appendix B.

<u>Lemma</u> 4.1 For m = 1, 2, under the conditions of Theorem 4.1,

$$E[T_{km}(\underset{\sim}{\delta}) - T_{km} \ ((\underset{\sim}{\beta}' \ \underset{\sim}{\beta}')')]^2 \to 0$$

as $n \to \infty$ where $\underset{\sim}{\delta} = (\underset{\sim}{\beta}' \ \underset{\sim}{\beta}')' + \underset{\sim}{d}/n^{1/2}$ for any fixed d. Now the first portions of $T_{k1}(\underset{\sim}{\delta})$ and $T_{k2}(\underset{\sim}{\delta})$ are monotone in the components of $\underset{\sim}{\delta}$. Futher,

$$||[n^{3/2}(n+1)]^{-1} n^{1/2} \sum_{i=1}^{n} C_{ik}^{+} \sum_{j=1}^{n} \underset{\sim}{D}_{ji}|| \text{ and } ||[n^{3/2}(n+1)]^{-1} n^{1/2} \sum_{i=1}^{n} C_{ik}^{-} \sum_{j=1}^{n} \underset{\sim}{D}_{ji}||$$

are bounded. Thus Lemma 2.1 implies that

$$(4.9) \qquad \sup|T_{km}(\underset{\sim}{\delta}) - T_{km}((\underset{\sim}{\beta}' \ \underset{\sim}{\beta}')')|$$

converges to zero in probability, where the supremum is over $\{\underset{\sim}{\delta}: n^{1/2} ||\underset{\sim}{\delta} - (\underset{\sim}{\beta}' \underset{\sim}{\beta}')'|| \leq B\}$. Now $A = \{\underset{\sim}{\delta}: n^{1/2} ||\underset{\sim}{\delta} - (\underset{\sim}{\beta}' \underset{\sim}{\beta}')'|| \leq B$ and $\underset{\sim}{\delta}' = (\underset{\sim}{b}' \underset{\sim}{b}')\}$ is a subset of this set. Thus (4.9) still converges to zero in probability when the supremum is taken over A. This, along with (4.8), yields the desired result, since the supremum of a sum is less than or equal to the sum of the suprema.

5. Intercept A simple estimate $\hat{\alpha}$ of the intercept for model (1.1) is given by the median of $Y_1 - \underset{\sim}{x}_1'\hat{\underset{\sim}{\beta}}, \ldots, Y_n - \underset{\sim}{x}_n'\hat{\underset{\sim}{\beta}}$. Using a proof similar to that of Theorem 4.2 (c) in McKean and Hettmansperger (1978), it is possible to show that $n^{1/2}((\hat{\alpha}-\alpha)\ (\hat{\underset{\sim}{\beta}}-\underset{\sim}{\beta})')'$ converges in distribution to a multivariate normal random varible with mean $\underset{\sim}{0}$ and variance-covariance matrix

$$V = \begin{bmatrix} (4\ f^2(0))^{-1} + \tau^2 \mu_x' \Sigma^{-1} \mu_x & -\tau^2 \mu_x' \Sigma^{-1} \\ -\tau^2 \Sigma^{-1} \mu_x & \tau^2 \Sigma^{-1} \end{bmatrix},$$

where Σ is the limit of $n^{-1} C'C$ and $\underset{\sim}{\mu}_x$ is the limit of $\bar{\underset{\sim}{x}}$, the p x 1 vector of column means of X. The conditions needed are (1.3), (1.4), and (1.5); See Aubuchon (1982) for the details.

Appendix A: Proof of Lemma 3.1

Before proceeding with the proof, some preliminary consequences of the assumptions of Theorem 3.1 are stated without proof.

Lemma A. 1 For any $\underset{\sim}{d}$ of length 2p, $\underset{\sim}{D}_{ij}' \underset{\sim}{d}/n^{1/2} \to 0$ uniformly in i, j as $n \to \infty$.

Lemma A. 2 f is bounded.

Lemma A. 3 If G is the cumulative distribution function of $e_1 - e_2$, where e_1 and e_2 are independent and have c.d.f. F, then

(i) the density of G,

$$g(z) = \int_{-\infty}^{\infty} f(e+z)f(e)de,$$

is bounded and absolutely continuous;

(ii) the derivative of g(z),

$$g'(z) = \int_{-\infty}^{\infty} f'(e+z)f(e)de,$$

is bounded and absolutely continuous; and

(iii) $g''(z)$, the derivative of $g'(z)$, is bounded.

Lemma A. 4 For any $\underset{\sim}{d}$ of length 2p, $n^{-2}\sum\sum_{i\neq j} (\underset{\sim}{D}_{ij}' d)^2$ is bounded.

Lemma 3.1 For m = 1, 2,

$$E[T_m^*(\underset{\sim}{\delta};k) - T_m^*((\underset{\sim}{\beta}^{\underset{\sim}{\beta}});h)]^2 \to 0,$$

where $\underset{\sim}{\delta} = (\underset{\sim}{\beta}' \ \underset{\sim}{\beta}')' + \underset{\sim}{d}/n^{1/2}$ and $k = h + a/n^q$, as $n \to \infty$, for any fixed $\underset{\sim}{d}$ and a.

Proof Consider the case of m = 1. Referring back to (3.10),

$$\text{(A.1)} \qquad E[T_1^*(\underset{\sim}{\delta};k) - T_1^*((\underset{\sim}{\beta}' \ \underset{\sim}{\beta}')'; h)]^2$$

$$= n^{-4}h_n^{-2} \sum\sum_{i\neq j} \sum\sum_{s\neq t} E[[w(e_i - e_j - \underset{\sim}{D}_{ij}'\underset{\sim}{d}/n^{1/2}]/k_n)I\{e_i \leq e_j + \underset{\sim}{D}_{ij}'\underset{\sim}{d}/n^{1/2}\}$$

$$- w([e_i - e_j]/h_n)I\{e_i \leq e_j\}$$

$$+\ w(0)(I\{e_i > e_j + \underset{\sim}{R}_{ij}'\underset{\sim}{d}/n^{1/2}\} - I\{e_i > e_j\} + \theta \underset{\sim}{R}_{ij}'\underset{\sim}{d}/n^{1/2})]$$

$$\times[w([e_s - e_t - \underset{\sim}{R}_{st}'\underset{\sim}{d}/n^{1/2}]/k_n) I\{e_s \le e_t + \underset{\sim}{R}_{ij}'\underset{\sim}{d}/n^{1/2}\}$$

$$-w([e_s - e_t]/h_n) I\{e_s \le e_t\}$$

$$+w(0)(I\{e_s > e_t + \underset{\sim}{R}_{st}'\underset{\sim}{d}/n^{1/2}\} - I\{e_s > e_t\} + \theta \underset{\sim}{R}_{st}'\underset{\sim}{d}/n^{1/2})]]$$

Partition the terms in this quadruple sum into three groups:

(A.2)	Group	Description	Count
	G1	Two matching pairs of subscripts	$2n(n-1)$
	G2	One matching pair	$4n(n-1)(n-2)$
	G3	No matching pair	$n(n-1)(n-2)(n-3)$
	TOTAL		$n^2(n-1)^2$

and deal with each group separately.

Consider the terms in group G1; they are uniformly bounded (using Lemma A.1), so their sum is $2n(n-1)n^{-4}h_n^{-2}O(1) = (nh_n)^{-2}O(1) \to 0$ as $n \to \infty$, where $O(1)$ is bounded.

Among the terms in group G2 consider, for example, the sum of those with $j = t$:

$$\text{(A.3)} \qquad \left| n^{-4}h_n^{-2} \sum_{\substack{G2 \\ j=t}} \int_{-\infty}^{\infty} M_{ij} \times M_{sj}\, f(e_j)\, de_j \right|,$$

$$\le\ n^{-4}h_n^{-2} \sum_{\substack{G2 \\ j=t}} \int_{-\infty}^{\infty} |M_{ij}| \times |M_{sj}|\, f(e_j)\, de_j$$

$$\le\ n^{-4}h_n^{-2} \sum_{\substack{G2 \\ j=t}} \sup|M_{ij}|\ \sup|M_{sj}|,$$

where

(A.4) $$M_{ij} = \int_{-\infty}^{e_j + D_{ij}' d/n^{1/2}} w([e_i - e_j - D_{ij} d/n^{1/2}]/k_n)\, f(e_i) de_i$$

$$- \int_{-\infty}^{e_j} w([e_i - e_j]/h_n) f(e_i) de_i$$

$$+ w(0)\int_{-\infty}^{\infty} [I\{e_i > e_j + D_{ij}' d/n^{1/2}\} - I\{e_i > e_j\} + \theta D_{ij}' d/n^{1/2}] f(e_i) de_i$$

and the suprema are over all i, j, s, and e_j. Now if $\sup|M_{ij}| \to 0$ and $\sup|M_{sj}| \to 0$ as $n \to \infty$, then the sum of terms in group G2 with j = t coverges to zero, since there are $O(n^3)$ terms in the sum and $(nh_n^2)^{-1}$ is bounded.

But letting $u = [e_i - e_j - D_{ij}' d/n^{1/2}]/k_n$ in the first integral and $u = [e_i - e_j]/h_n$ in the second,

(A.5) $$|M_{ij}| = |k_n \int_{-\infty}^{0} w(u) f(e_j + k_n u + D_{ij}' d/n^{1/2}) du$$

$$- h_n \int_{-\infty}^{0} w(u) f(e_j + h_n u) du$$

$$+ w(0)[F(e_j) - F(e_j + D_{ij}' d/n^{1/2}) + \theta D_{ij}' d/n^{1/2}]|$$

$$\leq (K_n + h_n) \sup(f)/2$$

$$+ w(0)[\theta + \sup(f)]/|D_{ij}' d/n^{1/2}| \to 0$$

uniformly in i and j.

Similar arguments hold for the other cases: i = s, i = t, and j = s.

Now turn to the sum of terms in group G3.

(A.6) $$n^{-4}h_n^{-2} \sum_{G3} E[(\cdot)(\cdot)]$$

$$= n^{-4}h_n^{-2} \sum_{G3} E[\cdot]\, E[\cdot],$$

since the multiplicands are independent when no subscripts match. Consider one of the expectations:

(A.7) $$E[w([e_i-e_j-\underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2}]/k_n)I\{e_i-e_j \le \underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2}\}$$

$$-w([e_i-e_j]/h_n)I\{e_i-e_j \le 0\}$$

$$+w(0)(I\{e_i-e_j > \underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2}\}-I\{e_i-e_j > 0\}$$

$$+ \theta\underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2})]$$

$$= \int_{-\infty}^{\underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2}} w([z-\underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2}]/k_n)g(z)dz$$

$$- \int_{-\infty}^{0} w(z/h_n)g(z)dz$$

$$+w(0)[G(0)-G(\underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2}) + \theta\underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2}],$$

recalling that G is the cdf of e_i-e_j with density g, when $i \ne j$. Letting $u = [z-\underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2}]/k_n$ in the first integral and $u = z/h_n$ in the second, this expectation is

(A.8) $$k_n \int_{-\infty}^{0} w(u)g(k_n u + \underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2})du$$

$$-h_n \int_{-\infty}^{0} w(u)g(h_n\, u)du$$

$$-w(0)(\underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2})^2\, G''(\xi_{ij})/2,$$

where $|\xi_{ij}| \leq |\underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2}|$, recalling that $\theta = G'(0)$. Making a Taylor expansion of $g(\cdot)$ about zero in each integral, further reexpress the expectation:

$$(A.9)\qquad k_n\int_{-\infty}^{0} w(u)[g(0)+(k_nu+\underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2})g'(0)+(k_nu+\underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2})^2g''(\zeta_{ij})/2]du$$

$$-h_n\int_{-\infty}^{0} w(u)[g(0)+h_nug'(0)+h_n{}^2u^2g''(\zeta)/2]du-w(0)(\underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2})^2G''(\xi_{ij})/2,$$

where $|\zeta_{ij}| \leq |k_nu+\underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2}|$ and $|\zeta| \leq |k_nu|$. Now $g'(0) = 0$ and $g''(\cdot)$ is bounded (Lemma A.3). Further, $\int_{-\infty}^{0} w(u)du = 1/2$, $\int_{-\infty}^{0} |u|w(u)du<\infty$, and $\int_{\infty}^{0} u^2w(u)du<\infty$. Thus we can write

$$(A.10)\qquad n^{-2}h_n{}^{-1}\sum_{i\neq j}\sum |E[\cdot]|$$

$$\leq n^{-2}h_n{}^{-1}\sum_{i\neq j}\sum [|k_n-h_n|g(0)/2+|k_n|^3O(1)+k_n{}^2|\underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2}|O(1)$$

$$+ |k_n(\underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2})^2|O(1)+h_n{}^3O(1)+(\underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2})^2O(1)] \to 0$$

as $n\to\infty$, since $|k_n-h_n|/h_n \to 0$, $|k_n|^3/h_n \to 0$, $k_n{}^2/h_n \to 0$, $|\underset{\sim}{D}'_{ij}\underset{\sim}{d}/n^{1/2}| \to 0$ uniformly in i, j, $n^{-2}\sum_{i\neq j}\sum (\underset{\sim}{D}'_{ij}\underset{\sim}{d})^2$ is bounded (Lemma A.4) and $(nh_n)^{-1}\to 0$.

But $|n^{-4}h_n{}^{-2}\sum_{G3} E[\cdot]E[\cdot]| \leq [n^{-2}h_n^{-1}\sum_{i\neq j}\sum |E[\cdot]|]^2 \to 0$ as $n\to\infty$. Therefore the sum of terms in group G3 converges to zero. A similar argument holds for $T_2^*(\underset{\sim}{\delta};k) - T_2^*((\underset{\sim}{\beta});h)$, and the proof of the lemma is complete.

Appendix B: Proof of Lemma 4.1

Before proceeding with the proof, a preliminary lemma which follows from the conditions of Theorem 4.1 is stated without proof.

Lemma B.1 $n^{-1/2} \max_{1 \leq i \leq n} |C_{ik}| \to 0$ as $n \to \infty$, for $k = 1, 2, \ldots, p$.

Lemma 4.1 For $m = 1, 2$, under the conditions of Theorem 4.1,

$$E[T_{km}(\underset{\sim}{\delta}) - T_{km}((\underset{\sim}{\beta}' \; \underset{\sim}{\beta}')')]^2 \to 0$$

as $n \to \infty$, where $\underset{\sim}{\delta} = (\underset{\sim}{\beta}' \; \underset{\sim}{\beta}')' + \underset{\sim}{d}/n^{1/2}$ for any fixed $\underset{\sim}{d}$.

Proof For simpler notation, the factor $(n+1)$ in $T_{km}(\underset{\sim}{\delta})$ is replaced by n; see (4.7).

Proof: Calculating the expectation for $m = 1$,

(B.1)
$$E[T_{k1}(\underset{\sim}{\delta}) - T_{k1}((\underset{\sim}{\beta}' \; \underset{\sim}{\beta}')')]^2$$

$$= 12n^{-3} E[\sum_{i=1}^{n} c_{ik}^{+} \sum_{j=1}^{n} [I\{e_j - e_i \leq \underset{\sim}{D}_{ji}' \underset{\sim}{d}/n^{1/2}\}$$

$$-I\{e_j - e_i \leq 0\} - \theta \underset{\sim}{D}_{ji}' \underset{\sim}{d}/n^{1/2}$$

$$\times \sum_{s=1}^{n} c_{sk}^{+} \sum_{t=1}^{n} [I\{e_t - e_s \leq \underset{\sim}{D}_{ts}' \underset{\sim}{d}/n^{1/2}\}$$

$$-I\{e_t - e_s \leq 0\} - \theta \underset{\sim}{D}_{ts}' \underset{\sim}{d}/n^{1/2}]]$$

$$= 12n^{-3} \sum\sum_{i \neq j} \sum\sum_{s \neq t} c_{ik}^{+} c_{sk}^{+} E[(\cdot)(\cdot)],$$

since $D_{ii} = 0$. The terms in this quadruple summation can be partitioned into two groups:

(B.2)

Group	Description	Count
G1	Some matching subscripts	$4n(n-1)(n-2) + 2n(n-1)$
G2	No matches	$n(n-1)(n-2)(n-3)$
TOTAL		$n^2(n-1)^2$

Consider the sum of terms in the first group:

(B.3) $$12n^{-3} \sum_{G1} c_{ik}^{+} c_{sk}^{+} E[(\cdot)(\cdot)]$$

$$= 12n^{-3} \sum_{G1} c_{ik}^{+} c_{sk}^{+} E[[I\{e_j - e_i \leq \underset{\sim}{D}_{ji}' \underset{\sim}{d}/n^{1/2}\}$$

$$- I\{e_j - e_i \leq 0\}] \times [I\{e_t - e_s \leq \underset{\sim}{D}_{ts}' \underset{\sim}{d}/n^{1/2}\} - I\{e_t - e_s \leq 0\}]]$$

$$+ 12n^{-3} \sum_{G1} c_{ik}^{+} c_{sk}^{+} E[-2\theta \underset{\sim}{D}_{ji}' \underset{\sim}{d}[I\{e_t - e_s \leq \underset{\sim}{D}_{ts}' \underset{\sim}{d}/n^{1/2}\}$$

$$- I\{e_t - e_s \leq 0\} - \theta \underset{\sim}{D}_{ts}' \underset{\sim}{d}/n^{1/2}]/n^{1/2}$$

$$- \theta^2 (\underset{\sim}{D}_{ji}' \underset{\sim}{d})(\underset{\sim}{D}_{ts}' \underset{\sim}{d})/n]$$

$$= Q_1 + Q_2.$$

Now

(B.4) $$|Q_1| \leq 12n^{-3} \sum_{G1} c_{ik}^{+} c_{sk}^{+} E[|\cdot| \times |\cdot|]$$

$$\leq 12n^{-3} \sum_{G2} c_{ik}^{+} c_{sk}^{+} [E|\cdot| \times E|\cdot|]^{1/2},$$

by the Cauchy - Schwarz inequality, noting that $|I\{\cdot\} - I\{\cdot\}|$ is equal to its own square. Furthermore,

(B.5) $$E|I\{e_j - e_i \leq \underset{\sim}{D}_{ji}' \underset{\sim}{d}/n^{1/2}\} - I\{e_j - e_i \leq 0\}|$$

$$= |G(\underset{\sim}{D}_{ji}' \underset{\sim}{d}/n^{1/2}) - G(0)|$$

$$= |\underset{\sim}{D}_{ji}' \underset{\sim}{d}/n^{1/2}|\, G'(\xi_{ji}),$$

where $|\xi_{ji}| \leq |\underset{\sim}{D}_{ji}' \underset{\sim}{d}/n^{1/2}|$. But G' is bounded (Lemma A.3), and $|\underset{\sim}{D}_{ji}' \underset{\sim}{d}/n^{1/2}|$ converges to zero uniformly in (i,j) as $n \to \infty$ (Lemma A.1). Further,

$$(B.6) \qquad n^{-3} \sum_{G1} c_{ik}^{+} c_{sk}^{+}$$

$$\leq n^{-1} \sum_{i=1}^{n} (c_{ik}^{+})^2 + n^{-2} \sum_{i=1}^{n} \sum_{s=1}^{n} c_{ik}^{+} c_{sk}^{+}$$

$$\leq n^{-1} \sum_{i=1}^{n} c_{ik}^{2} + [n^{-1} \sum_{i=1}^{n} |c_{ik}|]^2,$$

$$\leq n^{-1} \sum_{i=1}^{n} c_{ik}^{2} + [1 + n^{-1} \sum_{i=1}^{n} c_{ik}^{2}]^2$$

which is bounded since $n^{-1} C'C$ converges. Thus the right-hand side of (B.4) converges to zero as $n \to \infty$, and Q_1 does also.

Now consider:

$$(B.7) \qquad Q_2 = 12n^{-3} \sum_{G1} c_{ik}^{+} c_{sk}^{+} [-2\underset{\sim}{D}_{ji}'\underset{\sim}{d}[G(\underset{\sim}{D}_{ts}'\underset{\sim}{d}/n^{1/2})$$

$$- G(0) - \theta\underset{\sim}{D}_{ts}'\underset{\sim}{d}/n^{1/2}]/n^{1/2} - \theta^2(\underset{\sim}{D}_{ji}'\underset{\sim}{d})(\underset{\sim}{D}_{ts}'\underset{\sim}{d})/n]$$

$$= 12n^{-3} \sum_{G1} c_{ik}^{+} c_{sk}^{+} [-2\underset{\sim}{D}_{ji}'\underset{\sim}{d}(\underset{\sim}{D}_{ts}'\underset{\sim}{d}/n^{1/2})^2 \, G''(\xi_{ts})/(2n^{1/2})$$

$$- \theta^2(\underset{\sim}{D}_{ji}'\underset{\sim}{d})(\underset{\sim}{D}_{ts}'\underset{\sim}{d})/n].$$

where $|\xi_{ts}| \leq |\underset{\sim}{D}_{ts}'\underset{\sim}{d}/n^{1/2}|$, recalling that $G'(0) = \int_{-\infty}^{\infty} f^2(e)de = \theta$.

Since G'' is bounded (Lemma A.3), $n^{-3} \sum_{G1} c_{ik}^{+} c_{sk}^{+}$ is bounded (B.6), and $|\underset{\sim}{D}_{ji}'\underset{\sim}{d}/n^{1/2}| \to 0$ uniformly in (i,j), we have $Q_2 \to 0$ also. Referring back to (B.3), the contribution of terms in group 1 converges to zero as $n \to \infty$.

Finally, turn to the second group. These terms must go to zero faster than n^{-1}, since there are $0(n^4)$ of them and the divisor in front is only n^3. Since no subscripts match, the multiplicands are independent and the expectation factors:

$$(B.8) \qquad 12_n^{-3} \sum_{G2} c_{ik}^{+} c_{sk}^{+} \; E[(\cdot)(\cdot)]$$

$$= 12_n^{-3} \sum_{G2} c_{ik}^{+} c_{sk}^{+} \; E[\cdot]\, E[\cdot].$$

Consider one of these expectations:

$$(B.9) \qquad E[I\{e_j - e_i \le \underset{\sim}{D}_{ji}' \underset{\sim}{d}/n^{1/2}\} - I\{e_j - e_i \le 0\}$$

$$- \theta\, \underset{\sim}{D}_{ji}' \underset{\sim}{d}/n^{1/2}]$$

$$= G(\underset{\sim}{D}_{ji}' \underset{\sim}{d}/n^{1/2}) - G(0) - G'(0)\, \underset{\sim}{D}_{ji}' \underset{\sim}{d}/n^{1/2}$$

$$= (\underset{\sim}{D}_{ji}' \underset{\sim}{d})^2\, G''(\xi_{ji})/(2n),$$

where $|\xi_{ji}| \le |\underset{\sim}{D}_{ji}' \underset{\sim}{d}/n^{1/2}|$. Thus the absolute value of the summation in (B.8) is:

$$(B.10) \qquad |12n^{-3} \sum_{G2} c_{ik}^{+} c_{sk}^{+} [(\underset{\sim}{D}_{ji}' \underset{\sim}{d})^2\, G''\,(\xi_{ji})/(2n)]$$

$$\times [(\underset{\sim}{D}_{ts}' \underset{\sim}{d})^2\, G''\,(\xi_{ts})/(2n)]|$$

$$\le 3[\sup_z |G''(z)|]^2\, [n^{-1/2} \max_i |c_{ik}|]^2\, n^{-4} \sum_{G2} (\underset{\sim}{D}_{ji}' \underset{\sim}{d})^2 (\underset{\sim}{D}_{ts}' \underset{\sim}{d})^2$$

But G" is bounded (Lemma A.3), and $n^{-1/2} \max_i |c_{ik}|$ converges to zero (Lemma B.1). And

$$(B.11) \qquad n^{-4} \sum_{G2} (\underset{\sim}{D}_{ji}' \underset{\sim}{d})^2 (\underset{\sim}{D}_{ts}' \underset{\sim}{d})^2$$

$$\le [n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\underset{\sim}{D}_{ji}' \underset{\sim}{d})^2]^2,$$

which is bounded (Lemma A.4). Taking this into account, the right-hand side of (B.10) converges to zero as $n \to \infty$.

Thus by showing that the sum over terms in each of the two groups converges to zero, we have shown that the expectation on the left-hand side of (B.1) converges to zero. This establishes the lemma for $m = 1$. The proof for $m = 2$ is analogous.

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Inference based on ranks, linear models, asymmetric errors, density estimation.

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In this paper robust, rank-based inference procedures are considered for general linear models with (possibly) asymmetric errors. Approximating standard errors of estimates and testing hypotheses about the model parameters require estimating a scaling functional, and an approach is developed which, unlike previous work, does not require symmetry of the underlying error distribution or replicates in the design matrix. Hence, important asymmetric models such as arise in life testing can now be

handled. Further, it is shown that the asymptotic properties of the inference procedures hold with simpler conditions on the design matrix than previously required. In addition an estimate of the intercept is developed without requiring the assumption of a symmetric error distribution.