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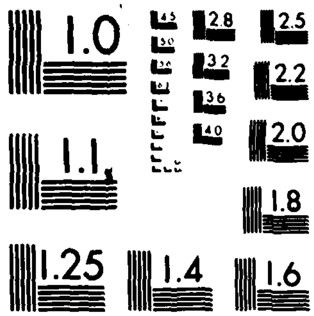

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OF THE THREE PARAMETER LOGNORMAL
DISTRIBUTION

THESIS

Jim H. Keffer
Second Lieutenant, USAF

AFIT/GOR/MA/83D-3

DEPARTMENT OF THE AIR FORCE
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THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
Air University
In Partial Fulfillment of the
Requirements for the Degree of
Master of Science in Operations Research

Jim H. Keffer, B.S.
Second Lieutenant, USAF

December 1983

Preface

Ever since my first undergraduate course in statistics, the application of statistical methods has been of great interest to me. In obtaining a Masters of Science degree in Operations Research at the Air Force Institute of Technology (AFIT), I have had the opportunity to take many statistic courses. These courses have expanded my knowledge and interest in the field. This thesis is a direct result of my interest in statistics and my education at AFIT.

No research of this magnitude is ever accomplished without the involvement of several people. I now take this opportunity to thank these participants. I am indebted to Dr Albert H. Moore, my thesis advisor, for his valuable assistance and guidance. I am also indebted to Dr Joseph P. Cain for his aid and advice as my thesis reader. And for doing an excellent job of typing the final manuscript, I thank Ms Connie Pavliga.

Lastly, I thank my wife, Donna, for her constant support and understanding during the past eighteen months.

Jim H. Keffer

Table of Contents

List of Figures

List of Tables

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Abstract

This thesis compares two modified maximum likelihood (ML) estimation techniques against three minimum distance (MD) estimation techniques in application to the three parameter lognormal distribution. The three parameter lognormal distribution has a location parameter (ξ) and two other parameters associated with the mean (μ) and standard deviation (σ) of its parent normal population. The first modified ML technique uses linear interpolation on order statistics to estimate location while the second ML technique uses the first order statistic as the location estimate. The remaining two parameters are calculated by using the location estimate in their respective censored or uncensored ML equations and solving for the parameters. Three MD techniques are used: Kolmogrov Distance, Cramer-von Mises Statistic, and the Anderson-Darling Statistic. The MD techniques refine the location estimates which are then used in the ML equations of the other two parameters to obtain their refined estimates.

Monte Carlo analysis is used to accomplish the comparison of estimation techniques. Sample sizes of 6, 8, 10, 12, and 16 are generated using three parameter sets $(\mu,\sigma,\xi) \doteq (0.0, 1.0, 10.0)$, $(1.0, 1.0, 10.0)$, and $(1.0, 2.0,$

over

10.0) Each estimation technique is applied to one-thousand replications for every combination of sample size and parameter set. Three measures of effectiveness are used to facilitate comparisons: mean square error, relative efficiency, and the Cramer-von Mises Statistic. Comparisons of these effectiveness measures across all cases reveal a clear superiority of the MD techniques over the modified ML techniques.

ROBUST MINIMUM DISTANCE ESTIMATION OF THE THREE PARAMETER LOGNORMAL DISTRIBUTION

I. Introduction

Statistical analysis is continuously playing an ever-increasing role in all branches of the present day military services. Being of such diverse application, statistical analysis is being used for optimally operating under constrained budgets, determining manpower and equipment requirements, and increasing the level of operational readiness and availability. Procurement of multi-million dollar weapon systems, desired increases in the prediction of system and component reliabilities, and logistics planning all usually involve statistical analysis before the decision-making process begins. The application of more efficient and accurate statistics to these and other areas leads to better informed decision makers and more effective decisions being made, thereby strengthening our military forces while simultaneously operating under the imposed constraints.

One very important aspect of statistical analysis is that of parameter estimation of distributions. Often times an analyst can look at a histogram of some data and determine which density function belongs to the data set. However, in most cases, simply identifying the data's density function will not yield vital information such as the mean, variance, and standard deviation. If an analyst had some reliability data on a piece of operational equipment and was asked to determine some statistics such as the mean time between failures (MTBF) or the mean time to repair (MTTR), simply fitting a density function by graphical methods and estimating its parameters would most surely yield inaccurate estimates at best. The military services recognize the importance of accurately determining these parameter estimates and have devoted much time and money to pursue this end. For example, a commander of an F-16 squadron would certainly like to know, as accurately as possible, how often the aircraft breaks down and how much time is needed to repair or replace the faulty component(s). These and other factors bear directly on the survivability and operational capability of the squadron, not to mention national defense. In order to obtain these vital statistics, the parameters of an assumed distribution must be estimated from sample data. Of course, better estimation procedures result in more accurate parameter estimates and yield more accurate information about the data. With the importance of statistical analysis and parameter estimation in mind, this thesis effort focuses on parameter estimation as applied to the three parameter lognormal (3-LN) distribution.

The method of moments and maximum likelihood are two classical parameter estimation techniques with the method of maximum likelihood usually outperforming the method of moments. However, a recent technique called the minimum distance method developed by J. Wolfowitz in the 1950s, seems to yield better parameter estimates than those from the method of maximum likelihood (3; 18; 25). This thesis applies both the minimum distance method and the method of maximum likelihood to the three parameter lognormal distribution and compares the estimates from both methods. A Monte Carlo analysis conducts the investigation. Both methods are discussed extensively in Chapters IV and V respectively.

Dr Albert H. Moore, professor of statistics at the Air Force Institute of Technology, and his past thesis students have devoted much time and study to the area of parameter estimation. Over the past several years, he and his thesis students have been extensively involved with the application of robust minimum distance estimation techniques to a variety of distributions. Some of the distributions considered so far are the three parameter weibull, the three and four parameter gamma, the four parameter beta, the generalized t, and the generalized exponential power distribution. Parameter estimates from the robust minimum distance estimation technique usually outperform the more classical techniques.

The objective of this thesis effort is to estimate and compare the parameters of the 3-LN distribution via the method of maximum likelihood and the minimum distance method. As the name implies, the 3-LN distribution has three parameters: a location parameter and two other parameters associated with the mean and standard deviation of its parent normal distribution. Two "modified" maximum likelihood estimation techniques are developed and compared against three minimum distance estimation techniques. The procedure is straightforward and briefly stated here. Estimates obtained by the "modified" method of maximum likelihood are used by the minimum distance method to obtain a new estimate of the location parameter. This new location parameter estimate is then used by the method of maximum likelihood to obtain new estimates for the remaining two parameters. Lastly, three measures of effectiveness are applied to the parameter estimates to determine which estimation method gives the most accurate estimates. Emphasis is placed on small sample sizes since moderately large sample sizes ($n > 20$) are easier to handle due to the asymptotic properties. It is anticipated that the minimum distance estimates will be more accurate than those of the "modified" maximum likelihood method for the 3-LN distribution.

The next chapter introduces the 3-LN distribution by reporting its genesis and history, derivation and development, and some important statistics. Chapter III is a short literature review consisting of some classical and newer estimation techniques that have been applied to the 3-LN distribution. Chapter IV discusses the method of maximum likelihood revealing its history, properties, and some methods used by statisticians and mathematicians in solving the maximum likelihood equations. Chapter V presents the minimum distance theory and method. Chapter VI describes the Monte Carlo analysis procedure and computerization of the estimation methods. The final chapter, Chapter VII, reports the results and conclusions obtained from the Monte Carlo analysis.

II. Three-Parameter Lognormal Distribution

History

The normal distribution has played a dominant role in the field of theoretical and applied statistics ever since its development by Guass in 1809. However, the normal curve could not provide an adequate representation of the many different distributions found in statistical practice so, by the end of the nineteenth century, some statisticians attempted to construct systems of frequency curves representing a variety of distributions. These were commonly referred to as "skew frequency curves" (13:149). Among the most successful of these systems were those developed by K. Pearson in 1895, F. Y. Edgeworth in 1898, and C. V. L. Charlier in 1905 (13:149). Pearson's system of frequency curves was defined as the solution to a differential equation involving four parameters and Charlier's system was defined by coefficients from the expansion of derivatives from the normal distribution (13:152). Edgeworth called his system the "method of translation" in which a function of the observed random variable was sought which closely approximated the normal random variable. Normal theory was then applied to the transformed variables. Although the "method of translation" was not widely accepted due to the limited variety of possible shapes, it was this system that greatly aided in the advancement of the lognormal distribution.

The genesis of the lognormal distribution arises from a theory of elementary errors combined by a multiplicative process instead of an additive process as in the case of the normal distribution (2:2). In 1879, Galton pointed out that many situations in nature such as growth, change, or even death follow this multiplicative process and laid the foundation for the development of the lognormal distribution. Galton stated that if $x_1, x_2, \ldots, x_n$ are positive and independent random variables and

$$T_n = \prod_{i=1}^{n} x_i \tag{2.1}$$

then

$$\log(T_n) = \sum_{i=1}^{n} \log(x_i) \tag{2.2}$$

and application of the central limit theorem to the random variables, $\log(x_i)$ would result in the distribution of $\log(T_n)$ being approximated by the unit normal distribution as the sample size went to infinity. Thus, the distribution of T_n was called lognormal (12:Ch 14, 113). D. McAlister was the first to explicity develop some theory on the lognormal distribution in 1879 by deriving its mean, median, mode, and second moment (2:3). In 1903, Kapteyn further established the genesis of the distribution and also developed a crude machine for generating lognormal samples from a lognormal population. Wicksel was first to estimate the parameters of the distribution by employing the method of moments and was also the first to consider the case where a simple displacement of the variate rather than the variate itself was lognormally distributed. The parameter which assigned a value to the displacement of the variate from the origin was referred to as the threshold parameter and established the 3-LN distribution.

Since the 1930s applications of the lognormal distribution have steadily increased. It has been applied in such fields as agriculture, entomology, metallurgy, biology, economics, and reliability with much success. An extensive list of applications can be found at the end of Chapter Thirteen in Johnson (14).

Probability Density Function

By using Edgeworth's "method of translation" in which a function of the observed random variable is sought which closely approximates the normal random variable, Johnson states that the transformation of a variable, say x, to normality can be accomplished by a function, say f(x), which has a specialized form and is made to depend only on a certain number of parameters. His transformation is (13:152):

$$z = \gamma + \delta \cdot f\left[\frac{x-\xi}{\lambda}\right] \tag{2.3}$$

where:

f is a monotonic function of x and does not depend on any parameter

δ and γ are shape parameters

λ is a scale parameter

ξ is a location parameter

z is unit normal

The 3-LN probability density function (pdf) is explicity derived by allowing the logarithmic function to represent the function of the transformation in Eq (2.3). When natural logarithms are used, as in this thesis, the scale parameter can be removed (2:6). Therefore, Eq (2.3) becomes

$$z = \gamma+\delta \cdot \ln(x-\xi) \tag{2.4}$$

Application of Eq (2.4) to the general form of a normal pdf yields the pdf of the displaced lognormal variates.

$$f(x) = \frac{\delta}{\sqrt{2\pi}(x-\xi)} \exp\left[\frac{-(\gamma+\delta \ln(x-\xi))^2}{2}\right] \text{ if } x > \xi \tag{2.5}$$

$$= 0 \text{ if } x < \xi$$

Another, more common, expression of lognormal pdf uses the mean (μ) and the standard deviation (σ) of its parent normal distribution and is obtained by allowing $\mu = -\gamma/\delta$ and $\sigma = 1/\delta$. The pdf becomes:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}(x-\xi)} \exp\left[\frac{-(\ln(x-\xi)-\mu)^2}{2\sigma^2}\right] \text{ if } x > \xi \qquad (2.6)$$

$$= 0 \text{ if } x < \xi$$

Figure 1 contains two 3-LN plots with different values of μ and σ .

Distribution Function

The distribution function, commonly referred to as the cumulative distribution function (cdf), of a continuous random variable is found by integrating its respective pdf over a given range. Unfortunately, the cdf for the 3-LN distribution does not have a closed form solution so it must be integrated by numerical means. The cdf, F(x), for the 3-LN distribution is defined as (32:47):

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{\xi}^{x} \frac{1}{u} \exp\left[\frac{-(\ln(u)-\mu)^2}{2\sigma^2}\right] du \qquad (2.7)$$

where $u = x-\xi$.

Important Statistics

Using the 3-LN pdf defined by Eq (2.6), a list of some important statistics are given below. The measures of central tendency are the mean, the median, and the mode defined respectively as:

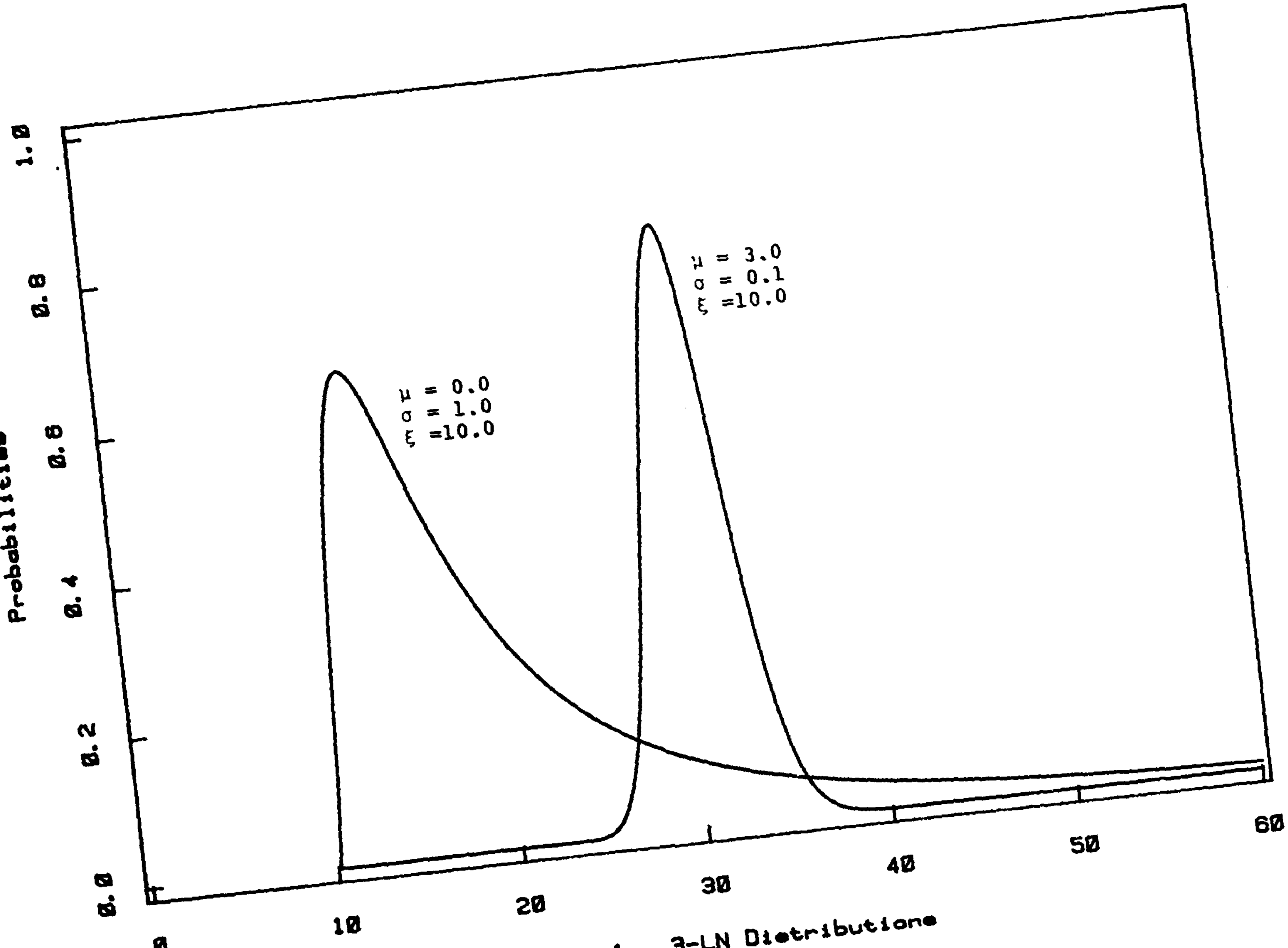

Figure 1. 3-LN Distributions

$$\text{mean} = \exp\left[\mu+1/2(\sigma^2)\right]+\xi \tag{2.8}$$

$$\text{median} = \exp[\mu]+\xi \tag{2.9}$$

$$\text{mode} = \exp\left[\mu-\sigma^2\right]+\xi \tag{2.10}$$

Note that mean > median > mode.

The r^{th} moment about ξ is:

$$E\left[(x-\xi)^r\right] = \exp\left[\mu\cdot r+1/2(r^2\cdot\sigma^2)\right] \tag{2.11}$$

The measures of dispersion are the variance ($\emptyset^2$), skewness (K_3) and kurtosis (K_4) defined respectively by:

$$\emptyset^2 = \exp\left[2\mu+\sigma^2\right]\cdot\left[\exp(\sigma^2)-1\right] \tag{2.12}$$

$$K_3 = (w+2)\cdot(w-1)^{\frac{1}{2}} \tag{2.13}$$

$$K_4 = w^4+2w^3+3w^2-3 \tag{2.14}$$

where $w = \exp\left[\sigma^2\right]$.

The standardized 100 a% deviate is

$$X_a = \frac{X_a-E[X]}{\sigma(X)} \tag{2.15}$$

where

$$X_a = \exp\left[U_a\sigma+\mu\right]+\xi$$

$$E[X] = \text{mean}$$

$\sigma(X) = \emptyset$

U_a = a- percentile from unit normal distribution

An interesting and easily verified fact is that as σ goes to zero, the standardized lognormal distribution approaches the unit normal distribution (see Figure 1).

III. Literature Review

In searching through the literature for different estimation techniques that have been applied to the 3-LN distribution, several estimation techniques applicable only to the two parameter lognormal distribution were encountered. It was interesting to note that the estimation techniques for the two parameter lognormal distribution were usually straightforward and the estimates computed with relative ease. However, with the introduction of the single location parameter, resulting in the 3-LN distribution, the estimation techniques become quite complicated and inaccurate. This finding is due to the fact that estimation of the location parameter is extremely difficult.

Classical Estimation Techniques

The most popular classical estimation techniques, as applied to the 3-LN distribution, are described in the following paragraphs.

The method of moments was formulated by K. Pearson in 1894 and is extremely simple in concept. Suppose that the pdf of a random variable, say x, is a function of h unknown parameters, say θ_1, θ_2, ..., θ_h, then by equating as many sample moments to population moments as there are unknown parameters, the equations can be solved for θ_1, θ_2, ..., θ_h, thereby yielding the parameter estimates (7:130). To avoid any ambiguity, the definition of a moment is pursued. There are two types of moments related to both the sample and population: a moment about the origin called a moment and a moment about the mean called the central moment. The k^{th} sample moment, m'_k, is computed from the sample as follows:

$$m'_k = \frac{1}{n} \sum_{i=1}^{n} x_i^k \tag{3.1}$$

where n is the sample size and x_i denotes the i^{th} observation. The k^{th} population moment, μ'_k, is derived from the population as

$$\mu'_k = E\left[X^k\right] \tag{3.2}$$

The k^{th} sample central moment, m_k, is :alculated from the sample by

$$m_k = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^k \tag{3.3}$$

where $\bar{x}$ is the sample mean. Finally, the k^{th} population central moment, μ_k, is derived from the population as

$$\mu_k = E\left[(x-\mu)^k\right] \tag{3.4}$$

where $\mu = E[X]$ is the mean of the distribution. Since the population moment is a function of the population parameters setting up the equations

$$\mu'_k = m'_k \qquad k = 1,\ldots,p \tag{3.5}$$

where p is the number of parameters being estimated, establishes a system of p equations and p unknowns. These p unknowns may be solved for yielding the method of moments estimators:

$$\hat{\sigma} = \sqrt{\ln(1+u^2)} \tag{3.6}$$

$$\hat{\mu} = \frac{1}{2}\left[\ln(m_2) - \ln(u^2(1+u^2))\right] \tag{3.7}$$

$$\hat{\xi} = m_1' - e^{\hat{\mu}}(1+u^2)^{\frac{1}{2}} \tag{3.8}$$

where u is the solution to

$$u^3 + 3u = (w-1)^{\frac{1}{2}}(w+2) \tag{3.9}$$

and $w = \exp\left[\sigma^2\right]$.

From the sample and population central moments skewness and kurtosis, two very useful statistics, denoted by K_3 and K_4 respectively, may be obtained by

$$\hat{K}_3 = \frac{m_3}{(m_2)^{\frac{3}{2}}} \tag{3.10}$$

$$K_3 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} \tag{3.11}$$

$$\hat{K}_4 = \frac{m_4}{(m_2)^2} \tag{3.12}$$

$$K_4 = \frac{\mu_4}{(\mu_2)^2} \tag{3.13}$$

The 'hat' (^) is used to identify the sample statistics. In the case of the 3-LN distribution, Eqs (3.11) and (3.13) yield Eqs (2.13) and (2.14) respectively. Equations (3.10) and (3.11) are used later in the thesis to provide an initial estimate for the σ parameter. Wicksell, Gumble, Yuan, and Aithcison and Brown have all applied the method of moments to the 3-LN distribution. The latter found that the method of moments is very inefficient when compared to other estimation methods (14:Ch 14, 124). This may be due to the fact that the moment sequence, $\{\mu'_k\}$, of the lognormal distribution is not unique to the distribution and, therefore, cannot be defined by its moments (14:Ch 14, 115). Thus, the method of moments is presented but not employed in this thesis.

The method of quantiles was also applied to the 3-LN distribution to estimate the parameters. This method results in three quantiles, say q, 1/2, 1-q, where $0 < q < 1/2$. It is clear from Eq (2.4) that the value of x such that $P[x < x_q] = q$ is related to the corresponding percentile, 1/q, of the unit normal distribution by $x_q = \hat{\xi}+\exp[\mu+v_q\cdot\sigma]$ and that the median of the lognormal sample (corresponding to $q = 1/2$ is $\hat{\xi}+\exp[\hat{\mu}]$. Denoting the sample quantile of order q by x_q and the quantiles of order q from the unit normal population by v_q such that $v_{(1-q)} = -v_q = v$ and after some rearranging, results in the following set of estimators:

$$\hat{\sigma} = \frac{1}{v}\left[\ln(x_{1-q}-x_{\frac{1}{2}})-\ln(x_{\frac{1}{2}}-x_q)\right] \tag{3.14}$$

$$\hat{\mu} = \ln(x_{\frac{1}{2}}-x_q)-\ln(1-\exp[-v\hat{\sigma}]) \tag{3.15}$$

$$\hat{\xi} = x_{\frac{1}{2}}-\exp[\hat{\mu}] \tag{3.16}$$

and explicit estimates for μ, σ, and ξ are obtained (2:58). Aitchison states that as a general rule, q should be set at 0.05 (2:58).

A final and very interesting method of parameter estimation was developed by W. F. F. Kemsley in 1952 which uses a mixture of the method of moments and quantiles. He equates the sample mean $\bar{x}$ to the population mean which is equivalent to replacing $x_{\frac{1}{2}}$ in Eqs (3.14), (3.15), and (3.16) by $\bar{x}$ and derives the function:

$$f(\sigma^2) = \frac{\bar{x}-x_q}{x_{1-q}-\bar{x}} \tag{3.17}$$

from which σ is determined. The estimates for μ and ξ are then easily found by inserting the value of σ in Eq (3.15), solving for $\hat{\mu}$, then solving for $\hat{\xi}$ using $\hat{\mu}$ in Eq (3.16). Again, a general rule is to let $q = 0.05$.

In 1957, Aitchison and Brown (2) compared these three estimation methods and reached the conclusion that, with $q = 0.05$, the method of quantiles was better than Kemsley's method, and that both were considerably better than the method of moments (2:62-63). Unfortunately, Aitchison and Brown did not apply the method of maximum likelihood to the 3-LN distribution, but they did apply it to the two parameter lognormal distribution and found that the maximum likelihood estimates were far more accurate than the estimates obtained by the method of moments or quantiles (2:53). With this in mind, it is anticipated that the method of maximum likelihood should also perform better than the method of quantiles for the 3-LN distribution. Thus, the method of maximum likelihood is used in this thesis and is extensively discussed in Chapter IV.

New Estimation Techniques

In searching through the recent literature, several articles pertaining to parameter estimation of the 3-LN distribution were discovered. Dallas R. Wingo (28) estimated the parameters by taking the conditional log-likelihood function of an ordered random sample of n independent observations, subject to several linear constraints and applied a nonlinear program to obtain the estimates. Using these estimates in a series of penalty functions and maximizing these functions for a sequence of decreasing values of the penalty parameter, he asserts that the corresponding solution sequences converges to the solutions of the nonlinear program. Thus, estimates of μ, σ, and ξ are obtained. This estimation technique was applied to several lognormal data sets which suggested that the algorithm is quite practical and easy to implement. However, a major problem with Wingo's approach is that the control parameter is arbitrarily chosen which presents several difficulties. If the control parameter is initially too large, the penalty function will be easy to maximize, but the algorithm may converge to a point that is not the solution to the constrained problem (28:58). On the other hand, if the control parameter is too small, the penalty function becomes very difficult to maximize (28:59).

In 1970, Munro and Wixley (21) estimated the parameters based on order statistics. In their approach, they take the expectations of the order statistics and express them as functions which are linear in a location and scale parameter but nonlinear in a shape parameter of the distribution. Regressing the order statistics on their expectations and implementing an iterative technique results in weighted least squares estimates of all three parameters (21:212). A Monte Carlo analysis of this methodology revealed that the iterative procedure always converges, nearly unbiased estimates are obtained, and the variances for the location and scale parameter compare favorably with those of the order statistic estimates of the standard normal distribution (21:222).

A variant on Wingo's method was developed by Gibbons and McDonald (9) in 1975. Using the expected values and covariance matrix of the standardized 2-LN distribution, they obtain the best linear unbiased estimators (BLUEs) for ξ and σ. The BLUEs fair well against the maximum likelihood estimates for sample sizes of five and under. However, their method is severely limited as it can only be applied when the shape parameter is fixed at unity (9:290).

In addition to these parameter estimation techniques, several variations of the method of maximum likelihood have been applied to the 3-LN distribution. As such, these variations are discussed in the chapter describing the method of maximum likelihood.

IV. Method of Maximum Likelihood

By far the most popular method of parameter estimation has been and still is the method of maximum likelihood. The concept of maximum likelihood was proposed by Daniel Bernoulli in 1778 and was later used by Guass in his development of the theory of least squares in 1796 (7:135). Gauss had used the method of maximum likelihood and felt that such estimation was inferior to least squares estimation. As a result, the development of the maximum likelihood method was overlooked until R. A. Fisher reintroduced the method in 1912. Since that time, the maximum likelihood method has been applied to numerous distributions and has enjoyed widespread success. As will be discussed later, the maximum likelihood estimators (MLEs) possess several desirable properties such as asymptotic unbiasedness, asymptotic efficiency, consistency, sufficiency, and invariancy. It is mostly due to these properties that the maximum likelihood method is used in this thesis.

> The principle of maximum likelihood consists in accepting as the best estimate of the parameters, say $\theta_1, \theta_2, \ldots, \theta_k$ those values of the parameters which maximize the likelihood for a given set of observations say, $x_1, x_2, \ldots, x_n$ [24:150].

Understanding what is meant by ". . . maximize the likelihood for a given set of observations [24:150]" requires the introduction of the likelihood function and its definition.

Let $x_1, x_2, \ldots, x_n$ be a set of independent random samples taken from a population described by the density function $f_n(x_1, x_2, \ldots, x_n/\theta_1, \theta_2, \ldots, \theta_k)$ where $\theta_1, \theta_2, \ldots, \theta_k$ are unknown parameters of the density function. The likelihood function, L, is then defined by:

$$\begin{aligned} L &= L(x_1, x_2, \ldots, x_n/\theta_1, \theta_2, \ldots, \theta_k) \\ &= f_n(x_1, x_2, \ldots, x_n/\theta_1, \theta_2, \ldots, \theta_k) \\ &= f(x_1/\theta_1, \theta_2, \ldots, \theta_k)\cdot f(x_2/\theta_1, \theta_2, \ldots, \theta_k), \\ &\qquad \ldots, f(x_n/\theta_1, \theta_2, \ldots, \theta_k) \quad (4.1) \\ &= \prod_{i=1}^{n} f_i(x_i/\theta_1, \theta_2, \ldots, \theta_k) \end{aligned}$$

(20:278).

The likelihood function gives the "likelihood" that the random variables assume a particular value of $x_1, x_2, \ldots, x_n$. In other words, it is a means of determining from which density function a set of values would most likely have come from. Maximization of the likelihood function yields the estimated values $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k$ for the unknown parameters and are called the maximum likelihood estimators of $\theta_1, \theta_2, \ldots, \theta_k$ respectively. The maximization process is accomplished by taking the partial derivatives of the likelihood function with respect to each unknown parameter, setting the equations equal to zero and then solving for the parameters.

$$\frac{\partial L}{\partial \theta_i} = 0 \quad i = 1,2,3,\ldots,k \tag{4.2}$$

Since L and log(L) attain their maximum at the same value of θ_i, it is usually computationally more efficient to transform L into log(L) as this transforms the product into a sum which facilitates differentiation.

Properties

The widespread use of the method of maximum likelihood is attributed to the resulting estimators being asymptotically unbiased, asymptotically efficient, consistent, sufficient, and invariant. The estimator, $\hat{\theta}$, is said to be unbiased if

$$E[\hat{\theta}] = \theta \tag{4.3}$$

the expected value of the estimator is equal to the estimate. Or simply stated, $\hat{\theta}$ is unbiased if the mean of its distribution equals, $f(\theta)$, the function of the parameter being estimated (20:293). Efficiency, mainly a large sample concept, is linked with the smallest asymptotic variance among a class of estimators which yields rapid convergence for the estimator (20:155). An alternate definition is due to the Cramer-Roa Inequality which sets a lower bound on the variance of the estimate. If equality at the lowest bound holds, then the estimator is said to be an efficient estimator (7:139). The property of consistency implies that the accuracy of the estimate increases as the sample size increases (24:151). Sufficiency merely indicates that the maximum likelihood estimates contain all the information about the unknown parameters that is contained in the sample and that the conditional distribution of the sample given the value of the statistic does not depend on the parameters (20:301). A final and important property of the MLEs is the invariance property defined as "Let $\hat{\theta}$ be the MLE of θ in the density function $f(x_1, x_2, \ldots, x_n)$. If $\tau(\cdot)$ is a function with a single valued inverse, then the MLE of $\tau(\theta)$ is $\tau(\hat{\theta})$ [20:284]." For example, it is well known that the MLE of the variance of a normal density function is

$$\sigma^2(\hat{\theta}) = \frac{1}{n}\sum_{i=1}^{n}(x_1-\mu_0)^2 \qquad (4.4)$$

where μ_0 is the mean of the distribution. Then by the invariance property, the MLE of σ is

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2} \qquad (4.5)$$

or similarly, the MLE of $\log(\sigma^2)$ is

$$\log(\sigma^2) = \log\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2\right] \qquad (4.6)$$

This property is extremely convenient when transformations are required.

Three-Parameter Lognormal Estimation

In searching through the literature, only five articles dealing with maximum likelihood estimation of the 3-LN distribution were uncovered. The first attempt was by E. Wilson and J. Worchester in 1945, followed by A. C. Cohen in 1951, B. M. Hill in 1963, J. A. Lambert in 1964, and finally H. Harter and A. Moore in 1966. The following paragraphs present a short summary of each methodology along with the maximum likelihood equations.

Employing the method of maximum likelihood, as previously described, results in the maximum likelihood equation for each parameter being:

$$\hat{\mu} = \sum_{i=1}^{n} \frac{[\ln(x_i - \xi)]}{n} \tag{4.7}$$

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} [\ln(x_i - \xi) - \hat{\mu}]^2}{n}} \tag{4.8}$$

$$(\hat{\sigma}^2 - \hat{\mu}) \sum_{i=1}^{n} \frac{1}{x_i - \xi} + \sum_{i=1}^{n} \frac{\ln(x_i - \xi)}{x_i - \xi} = 0 \tag{4.9}$$

(6:207). It is clear that if any of these equations are solved for ξ, the remaining parameters could be easily obtained by algebraic manipulation. Unfortunately, Eq (4.9) cannot be solved explicity for ξ; thus, herein lies the difficulty of parameter estimation for the 3-LN distribution.

Wilson and Worchester attempted to solve the maximum likelihood equations by a trial and error method. This resulted in computational inefficiency and extremely poor parameter estimates as could be expected.

A. C. Cohen (6) presented a more feasible and efficient technique for solving the maximum likelihood equations. He substituted Eqs (4.7) and (4.8) into Eq (4.9) yielding a function, $f(\xi)$ of the location parameter. The solution of this function is found by inverse interpolation on a small interval say (ξ_1, ξ_2) where $f(\xi_1) < 0$. The estimated value, $\hat{\xi}$, is then substituted into Eqs (4.7) and (4.8) resulting in $\hat{\mu}$ and $\hat{\sigma}$. He also uses a technique based on the least sample value to estimate ξ. This technique summarized below requires:

$$x_0 - \xi = \exp[\mu + \sigma \cdot t_0] \tag{4.10}$$

where $x_0 = x_1 + \frac{\delta}{2}$ with x_1 being the least sample observation, δ the interval of precision, and t_0 determined from the relationship

$$\frac{k}{n} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t_0} \exp\left[-\frac{t^2}{2}\right] dt \tag{4.11}$$

in which k is the number of times the least observed value occurs in the sample (6:209). Taking the natural logarithms on both sides of Eq (4.10) and substituting Eqs (4.7) and

(4.8) into the logarithm of Eq (4.10) gives a function, $f(\xi)$ of the location parameter equivalent to the maximum likelihood equation for ξ. A Monte Carlo analysis of these techniques showed that the method based on the least observed sample values performs better than the inverse interpolation method.

A significant fact with respect to the maximum likelihood estimation of the 3-LN distribution was reported by B. M. Hill in 1963. He proved that there exists a path, henceforth referred to as the "path of no return," along which the likelihood function, $L(\xi,\hat{\mu}(\xi), \hat{\sigma}^2(\xi))$ goes to ∞ as ξ goes to x_1, the least observed sample value, and goes to a positive constant as ξ goes to $-\infty$ (11:73). This leads to the ridiculous statement that the maximum likelihood estimates of ξ, μ, and σ^2 are x_1, $-\infty$, and $+\infty$ respectively (11:75). To overcome this difficulty, he introduces a joint prior distribution for ξ, μ, and σ then applies Bayes Theorem. This approach leads to the conclusion that the solution to the formal likelihood equations should be used with $\hat{\xi}$ satisfying

$$\sum_{i=1}^{n} (x_i-\hat{\xi}) + \frac{1}{\hat{\sigma}(\hat{\xi})}\left[\sum_{i=1}^{n} \frac{Z'_j}{(x_i-\hat{\xi})}\right] = 0 \qquad (4.12)$$

where

$$Z'_j = \frac{(\ln(x_i-\hat{\xi})-\hat{\mu}(\hat{\xi}))}{\hat{\sigma}(\hat{\xi})} \qquad (4.13)$$

and $\hat{\mu}(\hat{\xi})$, $\sigma(\hat{\xi})$ satisfying Eqs (4.7) and (4.8) respectively with ξ being replaced by $\hat{\xi}$ (14:Ch 14, 132).

In 1964, Lambert (16) conducted an empirical investigation of Hill's proposed estimation methodology. Using twenty-three different samples, each with different values for the parameters (ξ was always zero), he found the estimates fairly inaccurate in a majority of the samples. Also, when σ^2 was less than 0.04 or larger than 4.0, the procedure had a tendency to diverge.

The final article found on maximum likelihood estimation of the 3-LN distribution was coauthoried by Drs H. Harter and A. Moore (10) in 1966. Recognizing that an algebraic solution to the maximum likelihood equations is impossible and that the likelihood function may take the "path of no return" which yields ridiculous estimates, as proven by Hill, they develop an iterative technique to solve the maximum likelihood equations. When the likelihood function gets on the "path of no return," a modification of the technique is employed which circumvents this problem. Their technique is very flexible in that it allows for samples censored both from above and/or below. A Monte Carlo analysis was conducted with various parameter values, sample sizes, and censorings. In cases of no censoring, and all parameters unknown, the mean of each estimator is very close to the true parameter values with the corresponding variances being very small. These values indicate that the iterative technique provides good estimates of the parameters. Since this technique has been validated, verified, and yields accurate estimates of the parameters, a modification of this method is used in this thesis as one of the bases for determining the parameter estimates by the method of maximum likelihood.

The iterative procedure is straightforward and involves estimating the three parameters, one at a time in the cyclical order μ, σ, and ξ assuming all parameters unknown. First, the observations are ordered and initial estimates chosen. The initial estimate of ξ is chosen as the first order statistic. Secondly, the iterative technique procedure begins where, at each step, the rule of false position (iterative linear interpolation) is employed to determine if the value of the parameter being currently estimated satisfies its respective maximum likelihood equation, given the latest and/or known values of the other parameters (10:848). The possibility of encountering the "path of no return" occurs when no value of ξ in the permissible interval $\xi < x_1$ satisfies Eq (4.9). In these cases, the likelihood function is monotonically increasing so that $\hat{\xi} = x_1$, $\hat{\mu} \to -\infty$, and $\hat{\sigma} \to +\infty$ (10:848). Fortunately, the modified procedure alleviates this problem by censoring the smallest observation and all those equal to it. The initial estimate of ξ is then equated to the smallest uncensored observation and the iterative procedure is reapplied. The censored observation(s) is subsequently not considered in the estimation procedure, but it does become an upper bound on $\hat{\xi}$. With the likelihood function bounded, finite estimates of the remaining two parameters can be calculated. Harter and Moore suggest the iteration continue until either the results from successive steps agree with some assigned tolerances or a specified number of steps is reached.

V. The Minimum Distance Method

Minimum distance (MD) estimation is a relatively new development in the field of statistical parameter estimation. It was first introduced in the 1950s by J. Wofowitz as a method which ". . . in a wide variety of cases, will furnish super consistent estimators even when classical methods . . . fail to give consistent estimators [30:9]." In 1957, he published a fundamental paper proving the consistency of the estimator along with several examples of its use. Basically, Wolfowitz's technique was to let $\delta(F_1,F_2) = \sup_x [F_1(x)-F_2(x)]$ be a measure of the discrepancy between two distribution functions, and emphasized the applicability of this method for a broad range of distance techniques. From that time on, the minimum distance method slowly evolved. Working with absolutely continuous distributions, Blackman (4) proved consistency and asymptotic normality for an estimator of location via the MD-method. In 1969, Knusel (15) investigated the "robustness" of MD estimators and showed that they exhibit the same properties as the class of maximum likelihood type estimators (M-estimators). The term "robustness" refers to the ability of an estimator to adopt to deviations in the underlying model while remaining efficient (23:3). Then, in 1970, Sahler contributed a major piece of work in the area of minimum distance estimation. As Parr described it:

> Sahler systematically defines D-estimators [minimum distance type estimators] and outlines conditions for their existence and consistency. In particular, he considers discrepancies of integral type, and proves an asymptotic normality result for the general (unidimensional) parameter estimation [22:9].

It has not been until recently that statisticians began applying minimum distance techniques for the estimation of distribution parameters. In particular, Parr and Schucany (23) applied the MD method to estimate the location parameters of several symmetric distributions with emphasis placed on the normal distribution. They concluded that the method yielded ". . . strongly consistent estimators with excellent robustness properties [23:5]." The MD-method has also been applied, via Monte Carlo analysis, to a variety of symmetrical and nonsymmetrical distributions at the Air Force Institute of Technology as theses research under the supervision of Dr Albert H. Moore. Most of the theses have concluded that the MD-estimators provide more accurate parameter estimates than classical estimation methods, even the method of maximum likelihood. This thesis is yet another application of the MD-method to another distribution: the 3-LN distribution. Results from the Monte Carlo analysis should prove quite interesting in this case. Most of the other applications involved symmetrical or near-symmetrical distributions, of which the 3-LN distribution is usually neither. The shape of this distribution may take on a variety of forms ranging from positively to negatively skewed and very peaked to not very peaked depending on the parameter values. It will be interesting to investigate how well the MD-estimators react under the variety of distribution shapes. Nevertheless, it is anticipated that the MD-estimators will provide more accurate parameter estimates than the maximum likelihood estimates.

The minimum distance technique is an extension of the goodness-of-fit tests used in hypothesis testing. In goodness-of-fit testing, the analyst is concerned with determining whether or not a random variable follows a particular distribution given certain values of the parameters. This testing is accomplished by constructing a distribution function, call it $F(x:\underline{\Theta})$ where $\underline{\Theta}$ represents known parameter values, and then determining how well it fits the sample distribution function referred to as the empirical distribution function or EDF. The common measure of fit is usually the distance between $F(x:\underline{\Theta})$ and the EDF. The most popular EDF is a nondecreasing step function of size 1/n, where n is the sample size, corresponding to the ordered sample points. This EDF convention is employed in this thesis. The minimum distance method is somewhat a reverse of the goodness-of-fit philosophy in that the analyst assumes the probability density function of a random variable and attempts to estimate its parameters. Minimum distance estimation merely takes as its estimates of $\underline{\Theta}$ those values which minimize the discrepancy between $F(x:\underline{\theta})$ and the EDF. See Figure 2 for a plot of an estimated cdf against its EDF.

Further discussion of MD concepts necessitates the use of some notation which will be utilized throughout this chapter. Let $\Gamma = \{F_\theta(\cdot), \theta \varepsilon \Omega\}$ represent the parametric family of three parameter lognormal distributions. $F_\theta(\cdot)$ defines the estimated 3-LN distribution using the parameter estimates found by the maximum likelihood method and Ω corresponds to the parameter space. Let $G_n(\cdot)$ denote the EDF based on a random sample from the true but unknown distribution function $G(\cdot)$. The EDF is a nondecreasing step function of size $1/n$ at each point x_1, x_2, ..., x_n as previously defined and has a value of one at infinity (30:10). Finally, $\delta(G_n, F_\theta)$ denotes the discrepancy between the two distribution functions. Thus, the "D-estimators" mentioned in Parr's quote on page 33 is the ". . . value in Ω minimizing (G_n, F_θ) over Ω [22:6]."

Since the distance between two distribution functions is used in determining the parameter estimates, the distance measure is of utmost importance. In their paper, Parr and Schucany (23) describe the most commonly applied distance measures, and it is from this paper that most of the following definitions are taken (23:4-5).

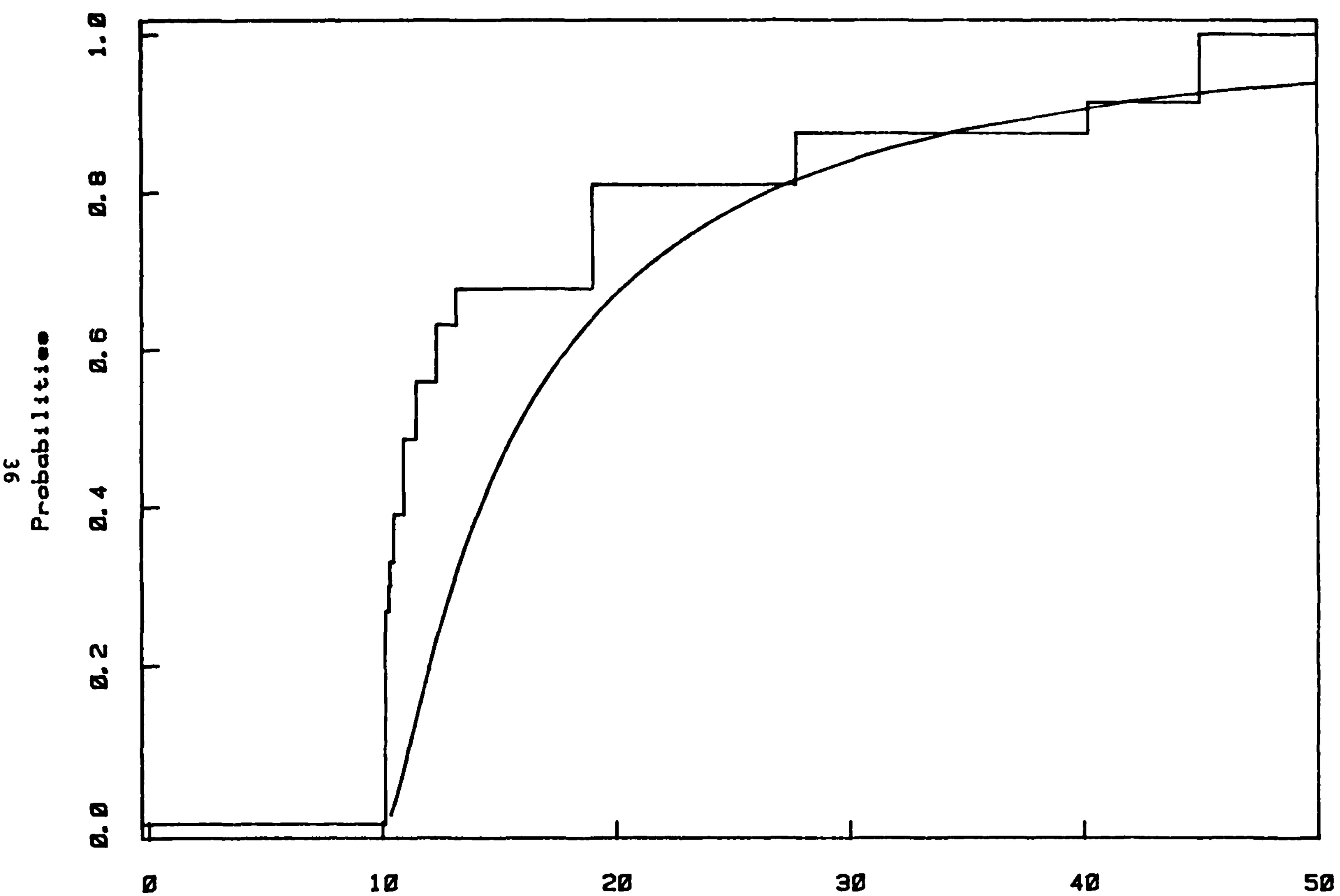

Figure 2. EDF versus Estimated CDF

Weighted Kolmogrov Distance

The weighted Kolmogrov distance is defined as:

$$D_{\psi}(G_n, F_{\theta}) = \sup_x |G_n(x) - F_{\theta}(x)| \psi(F_{\theta}(x)) \qquad (5.1)$$

This is one of the most popular distance measures as a result of its use in the Kolmogrov-Smirnov goodness-of-fit tests. The discrepancy measure minimizes the maximum distance between $G_n(x)$ and $F_{\theta}(x)$ when evaluated at each of the sample points. $\psi(F_{\theta}(x))$ is a weighting function which allows the user to assign a weight to each observation. In this thesis, uniform weights of 1.0 are assigned to each observation.

Weighted Cramer-von Mises Distance

The weighted Cramer-von Mises distance is defined as:

$$W_{\psi}^2(G_n, F_{\theta}) = \int_{-\infty}^{+\infty} (G_n(x) - F_{\theta}(x))^2 \psi(F_{\theta}(x)) dF_{\theta}(x) \qquad (5.2)$$

This technique minimizes the discrepancy between the theoretical and empirical distribution function, whereas the Kolmogrov distance only finds the absolute differences between the two. The weighting function is set equal to one resulting in the Cramer-von Mises statistic (CVM).

Anderson-Darling Statistic

The Anderson-Darling statistic is defined as

$$A_{\psi}^2(G_n, F_{\theta}) = \int_{-\infty}^{+\infty} (G_n(x) - F_{\theta}(x)) \frac{1}{u(1-u)} dF_{\theta}(x) \qquad (5.3)$$

Careful observation should reveal that this statistic is a special case of the Cramer-von Mises distance where $\psi(F_\theta(x)) = \frac{1}{u(1-u)}$ and $u = F_\theta(x)$. This weighting places more emphasis on the tails of the distribution. The process also minimizes the area between the theoretical and empirical distribution functions.

Kupier's Maximal Interval Probability Statistic

The Kupier statistic is defined as:

$$V(G_n,F_\theta) = \sup_{-\infty<a<b<+\infty}|G_n(b)-G_n(a))-(F_\theta(b) -F_\theta(a))| \qquad (5.4)$$

This statistic is closely related to the Kolmogrov distance where the parameters a and b define the maximal probability interval. The method yields more accurate estimates for a distribution scale parameter than for a location parameter.

Watson Statistic

The Watson statistic is defined as:

$$U^2(G_n,F_\theta) = \int_{-\infty}^{+\infty}(G_n(x)-F_\theta(x))^2dF_\theta(x) + \left|\int_{-\infty}^{+\infty} (G_n(x) -F_\theta(x)df_\theta(x)\right|^2 \qquad (5.5)$$

and is related to the Cramer-von Mises distance. As with the Kupier statistic, this technique is more appropriate for estimating the distribution scale parameter than the location parameter.

The discrepancy measures listed represent some of the more popular D-estimators. The popularity of these estimators is attributed to their invariance property, similar to that of the maximum likelihood estimators.

> This invariance is a result of the estimator not taking advantage of the function $g(\theta)$ of the point $\theta \varepsilon \Omega$ to be estimated, instead it simply selects a best approximating distribution [18:27].

Or as Parr stated it:

> It may well be inquired as to why an estimator obtained by minimization of a discrepancy measure which is useful in goodness-of-fit purposes (and, hence in many cases, extremely sensitive to outliers or general discrepancies from the model) should be hoped to possess any desirable "robustness" properties. It turns out that, in most cases (although no for, say, A^2 while the discrepancy measure itself may be fairly sensitive to the presence of outliers, the value $\hat{\theta}$ which minimizes the discrepancy $\delta(G_n, F_\theta)$ is much less so [23].

Since this thesis is concerned with the application of minimum distance estimation to determine the location parameter for the 3-LN distribution, the Kolmogrov distance, the Cramer-von Mises statistic, and the Anderson-Darling statistic are employed.

VI. Monte Carlo Analysis

This chapter deals extensively with the procedures used in conducting the empirical investigation and comparison of the interpolative and iterative modified maximum likelihood (ML) estimators (described later) along with each of the minimum distance (MD) estimators. Specifically, data generation, computerization of estimation techniques, and evaluation criteria are covered in detail. In this research, Monte Carlo analysis is used to evaluate the properties of the estimators. Basically, Monte Carlo analysis of estimation methods consists of the following three steps:

1. Generate independent random variables from a specific distribution (the 3-LN distribution in this case) to form random samples of a given size.

2. Apply the estimation method(s) to the random samples to obtain the parameter estimates.

3. Compare the estimates from each estimation method(s) using one or more evaluation criteria.

Since Monte Carlo analysis requires a vast amount of data and many computations, a high speed computer is a necessity. For the purposes of this analysis, the Control Data Corporation (CDC) 6600 computer located at the Aeronautical Systems Division, Wright-Patterson Air Force Base, Ohio is used. Three programs are developed to carry out this research. Each program is written in FORTRAN V and makes use of several subroutines from the International Mathematical and Statistics Library (IMSL) (12). The first program generates samples of size n from a 3-LN distribution and calculates the interpolative and iterative modified ML-estimators. The second program calculates the MD-estimators. And the third program evaluates the estimates from each estimation technique. Before proceeding further, some notation and comments are noted:

1. Let μ_0, σ_0, and ξ_0 represent initial estimates for μ, σ, and ξ.

2. Let $\hat{\mu}$, $\hat{\sigma}$, and $\hat{\xi}$ represent the ML-estimates of μ, σ, and ξ.

3. Let $\hat{\hat{\mu}}$, $\hat{\hat{\sigma}}$, and $\hat{\hat{\xi}}$ represent the MD-estimates of μ, σ, and ξ.

4. Let $X_{(i)}$ represent the i^{th} order statistic from a sample.

5. All estimates are calculated to three significant digits, except for the Kolmogrov estimates which are to two significant digits.

6. Recall that in the 3-LN pdf and cdf (Eqs (2.6) and (2.7)) there is an ln $(X-\xi)$ term; therefore, no estimate of ξ can be greater than or equal to the first order statistic $(X_{(i)})$, otherwise the term goes to minus infinity and the pdf and cdf are undefined.

7. Evaluation of the 3-LN cdf at point X' is always computed by evaluating its respective standard normal cdf

with appropriate limits. The percentile points are identical. A proof follows:

Let the transformation be

$$t = \frac{\ln(x-\xi)-\mu}{\sigma} \tag{6.1}$$

then

$$dt = \frac{1}{\sigma(x-\xi)}dx \tag{6.2}$$

Now since the 3-LN cdf is defined by

$$F(x') = \int_{\xi}^{x'} \frac{1}{(x-\xi)\sigma\sqrt{2\pi}} \exp\left[\frac{-(\ln(x-\xi)-\mu)^2}{2\sigma^2}\right]dx \quad \xi<x'<\infty \tag{6.3}$$

application of the transformation yields

$$F(t') = \int_{-\infty}^{t'} \frac{1}{\sqrt{2\pi}} \exp\left[{-t^2}/{2}\right]dt \quad -\infty<t'<\infty \tag{6.4}$$

$F(t')$ is easily recognized as a standard normal cdf and $F(x') = F(t')$.

<u>Generation of Data</u>

As previously mentioned, the objective of this research is parameter estimation of the 3-LN distribution with emphasis on small samples sizes. As such, sample sizes of 6, 8, 10, 12, and 16 are generated. For each of these sample sizes, three combinations of μ (the mean of the parent normal distribution) and σ (the standard deviation of the parent normal distribution) are used: (0.0, 1.0), (1.0, 1.0), and (1.0, 2.0). The location parameter is always equal to 10.0 since different values of ξ only result in a translation along the x-axis and does not affect the estimation of the other two parameters. The combinations of sample sizes and parameters result in a total of fifteen different cases.

For each case, 1,000 replications of sample size n are generated for use in the analysis. Caso's study of Monte Carlo validity recommended 2,000 replications; however, he also stated that 1,000 replications would be sufficient (5:37). Since runs of 2,000 iterations are very prohibitive in terms of computer CPU time, 1,000 replications are used. Generation of the 3-LN deviates is accomplished using the IMSL routine GGLNG and adding the value of the location parameter to each deviate. GGLNG generates 2-LN deviates, given the values of μ and σ as input; the addition of the location parameter forms the 3-LN random deviates. Each sample is then sorted in ascending order by the IMSL subroutine VSRTA. With the desired samples in hand, the next step is to calculate the interpolative and iterative modified ML-estimators.

Computerization of Maximum Likelihood Estimators

Interpolative Maximum Likelihood Estimators. It was mentioned in Chapter IV that if the location parameter is known, then $\hat{\mu}$ and $\hat{\sigma}$ are easily computed using Eqs (4.7) and (4.8) respectively. In this thesis, however, ξ is assumed to be unknown and, therefore, must be estimated. Unfortunately, it is very difficult to estimate ξ. As stated in Chapter IV, iterative procedures must be applied if all three parameters are to be estimated via the ML-method. It certainly would be very convenient if a simple method existed where an accurate estimate of ξ could be found; thus, with this estimate, $\hat{\mu}$ and $\hat{\sigma}$ could be easily computed. In searching through the literature, one such method was discovered. This method involves calculating the median ranks (Y_i, i = 1, 2) of the first two order statistics ($X_{(1)}$, $X_{(2)}$). Then the slope of the line connecting the first two points (i.e., ($X_{(1)}$, Y_1), ($X_{(2)}$, Y_2) is derived and interpolated down to the x-axis. The point at which the slope intersects the x-axis is taken as the value for $\hat{\xi}$. Now, with $\hat{\xi}$ known, Eqs (4.7) and (4.8) are used to obtain the estimates for $\hat{\mu}$ and $\hat{\sigma}$.

This method of linear interpolation for estimation of location parameters was originally developed in a masters thesis by Second Lieutenant D. E. Bertrand (3). He used this procedure on the four parameter Beta distribution to locate the upper and lower location parameters and obtained excellent results. This method is extremely desirable for the estimation of ξ since the interpolation will always result in a $\hat{\xi}$ being less than the first order statistic. Furthermore, it appears intuitively obvious that when the pdf is skewed to the left, as in the 3-LN pdf, the interpolation approximation of location should be very close to its true value. Although consistency of the interpolative estimate was not proven by Bertrand, it seems apparent that as the sample size increases, the first two order statistics would get closer to the true value of the location parameter, thereby making the interpolation approximation a better fit to the tail of the true cdf (3:19).

The interpolative ML-estimator methodology is given below:

1. Arrange the sample deviates in ascending order.

2. Calculate the median ranks of the first two order statistics using the formula (3:31):

$$Y_i = MR(X_{(i)}) = \frac{i-0.3}{n+0.4} \qquad i = 1,2 \tag{6.5}$$

3. Find the slope (m) of the line between ($X_{(1)}$, Y_1) and ($X_{(2)}$, Y_2) by

$$m = (Y_2-Y_1)/(X_{(2)}-X_{(1)}) \tag{6.6}$$

4. The estimate $\hat{\xi}$ is the point at which the slope (m) intersects the x-axis and is given by

$$\hat{\xi} = X_{(1)} - Y_1/m \tag{6.7}$$

The method is presented graphically in Figure 3.

Although the estimators from this method are not "strict" ML-estimators in the statistical sense, they are believed to yield extremely good estimates. Appendix B contains a computer listing of the interpolative ML-estimator algorithm.

<u>Iterative Maximum Likelihood Estimators</u>. As stated previously, the interpolative ML-estimators are not "strict" ML-estimators in the statistical sense, so an algorithm designed to calculate the "true" ML-estimators was sought. At first, the iterative ML-estimator algorithm developed by Drs Harter and Moore was used. The purpose of their algorithm was to study the asymptotic properties of ML-estimators and, therefore, used sample sizes of 50, 100, and 200. They constantly warn throughout their paper (10) that with small sample sizes: (1) the likelihood function may have no clearly defined local maximum, (2) the first order statistic might have to be used as $\hat{\xi}$ to prevent the likelihood function from going to infinity, and (3) the iteration procedure might be slow. Their suppositions were confirmed. Using sample sizes of 12 and smaller along with different combinations of the values for μ and σ, their algorithm set $\hat{\xi}$ equal to $X_{(1)}$ over 99 percent of the time and convergence was extremely slow; in some cases the estimates diverged.

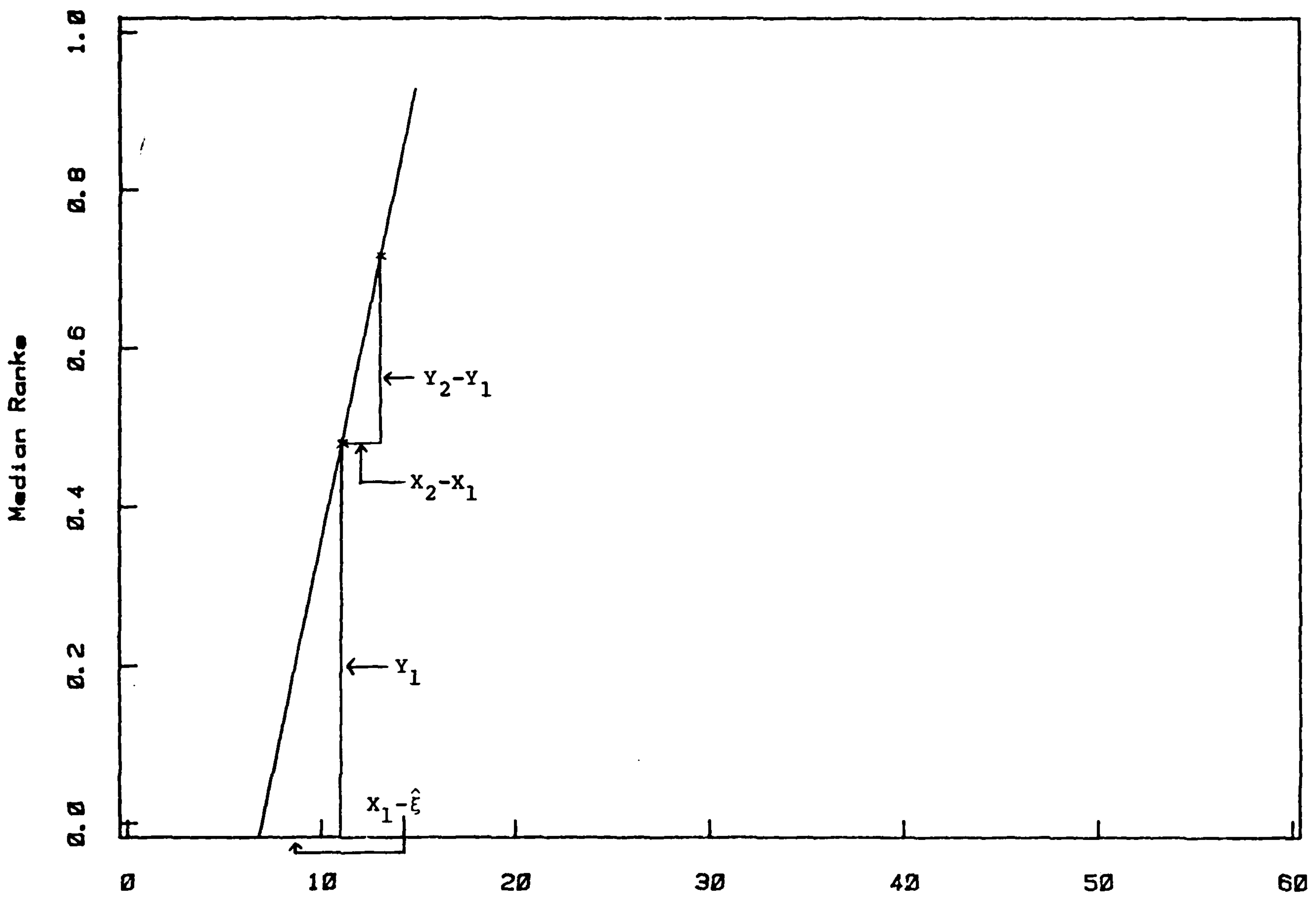

Figure 3. Interpolation for Location Estimate

These problems led to the development of a new iterative ML-estimator algorithm designed to combat the difficulties associated with the Harter and Moore algorithm. This new algorithm follows along the same lines as the Hart and Moore algorithm except that only $\hat{\mu}$ and $\hat{\sigma}$ are estimated one at a time in cyclical order while $\hat{\xi}$ is equated to $X_{(1)}$. The IMSL subroutine ZSCNT which solves a series on nonlinear equations by a variation of the secant method is used to obtain $\hat{\mu}$ and $\hat{\sigma}$. The ML-equations for μ and σ are put in separate ZSCNT subroutines. When $\hat{\mu}$ is being estimated, the latest estimate of $\hat{\sigma}$ is held constant and vice-versa until both estimates converge. Initially, the ML-equations were combined in one ZSCNT subroutine but convergence could not reached to three significant digits in 1,000 iterations so the ML-equations were put into separate subroutines. The choice of initial estimates for input into the ML-equations is crucial. Good initial values, especially for σ is vital to this algorithm if accurate estimates are desired. The following initial values were investigated (ξ always equals $X_{(1)}$): (1) μ_0 = sample mean, σ_0 = sample standard deviati (2) μ_0 = ln (sample mean), σ_0 = ln (sample standard deviat (3) μ_0 = ln (sample median), σ_0 = σ derived by equating sample and population skewness and solving for σ. Results from these runs showed that $\hat{\sigma}$ is very sensitive to σ_0 while $\hat{\mu}$ is relatively insensitive to μ_0. In this thesis, the thir set of initial values is used as input to the modified itera

ML-estimator algorithm since they give the most accurate estimates.

Although this algorithm does not produce "strict" MLEs in a statistical sense (since $\hat{\xi} = X_{(1)}$) the resulting estimators from this procedure ". . . appear to possess most of the desirable properties usually associated with maximum-likelihood estimates [10:843]." Since the constraint $\hat{\xi} = X_{(1)}$ is imposed when calculating $\hat{\mu}$ and $\hat{\sigma}$, the censored (from below) ML-equations must be used; they are (10:844):

$$\frac{\partial L}{\partial \mu} = \sum_{i=r+1}^{n} z_i - r \frac{f(z_{r+1})}{F(z_{r+1})} = 0 \tag{6.8}$$

$$\frac{\partial L}{\partial \sigma} = -(n-r) + \sum_{i=r+1}^{n} z_i^2 - r \frac{z_{r+1} f(z_{r+1})}{F(z_{r+1})} = 0$$

where

$$z_i = \frac{\ln(x_i - \xi) - \mu}{\sigma}$$

$$f(z_i) = \frac{1}{\sqrt{2\pi}} \exp\left[-z_i^2/2\right]$$

$$F(z_i) = \int_{-\infty}^{z_i} f(t)\,dt$$

with n as the sample size and r as the number of observations censored from below.

The iterative methodology is presented below:

1. Arrange sample deviates in ascending order.

2. Calculate the initial estimates: μ_0 = ln (sample median), $\sigma_0 = \sigma$ derived by equating sample and population skewness, $\hat{\xi} = X_{(1)}$

3. Given the initial estimates and $\hat{\xi} = X_{(1)}$ as input, recursively estimate μ and σ, using censored ML-equations, until the estimates agree within a certain tolerance or a maximum number of iterations is reached. Appendix B contains a computer listing of this algorithm. Attention is now turned to the computerization of the MD-estimators.

Computerization of Minimum Distance Estimators

The main thrust of this research is to estimate the location parameter of the 3-LN distribution by minimum distance (MD) estimation and then use the estimate to obtain refined estimates of μ and σ via their ML-equations. It is anticipated that these refined estimates are more accurate than the estimates from the modified methods of maximum likelihood. As described in Chapter V, the MD-estimators are obtained by finding values for the parameters which minimize the discrepancy between an empirical and an estimated distribution function. In this thesis, only the location parameter is minimized, while the other two parameter estimates, determined from the modified ML-methods, are held fixed. The empirical distribution function is the 1/n- step function and the estimated distribution is obtained by using $\hat{\mu}$ and $\hat{\sigma}$ along with the most recent MD-estimate of ξ. Once $\hat{\hat{\xi}}$ is determined, it is used in the censored or uncensored (depending on whether or not $\xi = X_{(1)}$ ML-equations of μ and σ which yields $\hat{\hat{\mu}}$ and $\hat{\hat{\sigma}}$. This procedure is done using the estimates from the interpolative and iterative modified ML-methods. Three minimum distance estimators are investigated: the Kolmogrov Distance, the Cramer-von Mises Statistic, and the Anderson-Darling Statistic.

<u>Kolmogrov</u> <u>Distance</u> <u>Estimators</u>. The computational formula which allows for calculation of the Kolmogrov Distance is defined by Stephens as (26:731):

$$D^+ = \max(1 \le i \le n) \quad [(i/n) - z_i]$$

$$D^- = \max(1 \le i \le n) \quad [z_i - (i-1)/n]$$

$$D = \max(D^+, D^-) \tag{6.9}$$

where z_i is the estimated standard normal cdf evaluated at the i^{th} sample point. To calculate the Kolmogrov Distance, $\hat{\xi}$ is shifted two units to the right and left of itself. The original shift is to the left and $\hat{\xi}$ is moved at .01 steps. With each iteration, the maximum distance between the theoretical and empirical distribution functions are recorded. Of course, location estimates are not allowed to go within .001 of $X_{(1)}$ since this implies divergence of the estimate. Also, if the final estimate is constrained by the highest or lowest incremental value (i.e., $\hat{\hat{\xi}} = \hat{\xi}+2.0$ or $\hat{\hat{\xi}} = \hat{\xi}-2.0$), a new initial estimate, which eliminates this constraint, is provided and the iterations begin anew. The final result is to choose the incremental value which minimizes the maximum distance between the two distribution functions. The actual recalculation of the new minimum distance location parameter is calculated by:

$$\hat{\hat{\xi}} = \hat{\xi} + (-2.0 + .01 \cdot \text{COUNT}) \tag{6.10}$$

where COUNT is a counter which locates the point at which the maximum distance between the estimates and empirical distribution functions is minimized. The Kolmogrov MD-estimators of μ and σ are calculated by using $\hat{\hat{\xi}}$ in the censored/uncensored ML-equations of μ and σ, thus yielding $\hat{\hat{\mu}}$ and $\hat{\hat{\sigma}}$.

<u>Cramer-von</u> <u>Mises</u> <u>Estimators</u>. Next, the program calculates the Cramer-von Mises MD-estimators. Recall that this distance measure finds the discrepancy between the estimated and empirical distribution functions. The computational formula for this measure is given by (26:731):

$$W^2 = \sum_{i=1}^{n} \left[z_i - (2_i-1)/2_n\right]^2 + (1/12n) \tag{6.11}$$

where z_i is the estimated standard normal cdf evaluated at the i^{th} sample point and n is the sample size. The minimization of the location parameter is accomplished via the IMSL subroutine ZXMIN. The subroutine uses a Quasi-Newton method to minimize a function of one or more variables and is identical to the one used by Parr in his PhD dissertation. With this method, $\hat{\mu}$ and $\hat{\sigma}$ are held constant while $\hat{\xi}$ is being minimized by ZXMIN. If the last estimate of ξ is within 0.001 of $X_{(1)}$ (implying divergence), a new initial estimate of ξ is provided. If the estimate of ξ is again within 0.001 of $X_{(1)}$, then $X_{(1)}$ is censored along with any other sample points equal to it and $\hat{\hat{\xi}}$ is set to the last censored observation. The CVM MD-estimators for μ and σ are then calculated by using $\hat{\hat{\xi}}$ in the uncensored/censored ML-equations of μ and σ thereby yielding $\hat{\hat{\mu}}$ and $\hat{\hat{\sigma}}$.

Anderson-Darling Estimators. The final distance measure investigated in this research is the Anderson-Darling Statistic. This distance measure is similar to the CVM measure in that it finds the discrepancy between two distribution functions but with more emphasis being placed on the tails of the distribution. Stephens gives the computational formula as (26:731):

$$A^2 = -\left\{ \sum_{i=1}^{n} (2i-1)\ln(z_i + \ln(1-z_{n+1-i})/n \right\} - n \tag{6.12}$$

Again, z_i is the estimated standard normal cdf evaluated at the i^{th} sample point and n is the sample size. The minimization process is identical to that of the CVM distance calculation just described. Again, once $\hat{\hat{\xi}}$ is obtained, $\hat{\hat{\mu}}$ and $\hat{\hat{\sigma}}$ are calculated via their uncensored/censored ML-equations.

Appendix C contains a listing of the computer program used to calculate each MD-estimator. The approximate CPU time required for one run of 1,000 replications varied from 250 to 300 seconds depending on the sample size.

Evaluation Criteria

The final step in this Monte Carlo analysis is to evaluate and compare all of the estimates obtained from each estimation technique. This thesis uses two modified maximum likelihood techniques along with three minimum distance techniques for each yielding a total of eight sets of parameter estimates (μ, σ, ξ) or twenty-four different parameter estimates for each sample. To make any relevant conclusions out of all the different parameters and parameter sets, certain criteria for evaluation are needed. Three approaches are used for this evaluation: mean square error (MSE), relative efficiency (REFF), and the Cramer-von Mises Statistic.

The MSE is a measure of how close each estimated parameter is to its true value and is useful for investigating the strength of each estimator. MSEs are calculated by

$$\text{MSE} = \sum_{i=1}^{1000} (\hat{\theta}_i - \theta)^2/1000$$

where θ is the true parameter value and $\hat{\theta}_i$ is the parameter estimate from the i^{th} sample and particular estimation technique. The smaller the MSE, the better the estimator. The REFF is another tool, similar to MSE, which aids in the comparisons of individual estimators. Calculation of REFFs are accomplished by:

$$REFF = MSE_m/MSE_i$$

where MSE_m is the mean square error of a reference estimator and MSE_i is the mean square error of the estimator being tested. For this analysis, MSE_m is taken as the estimators from the interpolative modified ML-method. Values of REFF greater than unity imply that the estimator is performing better than the interpolative modified maximum likelihood estimator. The third evaluator is a distance measure which provides an overall measure of how well the estimated distribution fits the true distribution. An appropriate measure of this type is the Cramer-von Mises statistic, defined by Eq (5.2). In this application, however, G_n is replaced by $\hat{F}$- the estimated cdf (see Figure 4). The integral must then be multiplied by the sample size to form the actual distance measure (3:32). Since $\partial F(x) = \frac{\partial F}{\partial x}dx = f(x)dx$, the formula used to compute the distance between the estimated cdf, $\hat{F}$, and true cdf, F, is:

$$W^2(\hat{F},F) = n \int_{\xi}^{\infty} \left[\hat{F}(x:\hat{\theta}) - F(x:\theta)\right]^2 f(x:\theta)\,dx \tag{6.13}$$

where $\hat{\theta}$ represents the parameter estimates and θ represents the true parameter values. The limits of the integral are from ξ to ∞ as the 3-LN distribution is not defined outside this range. This integral is evaluated using 15-point Laguerre integration. A function g(x) with limits ranging

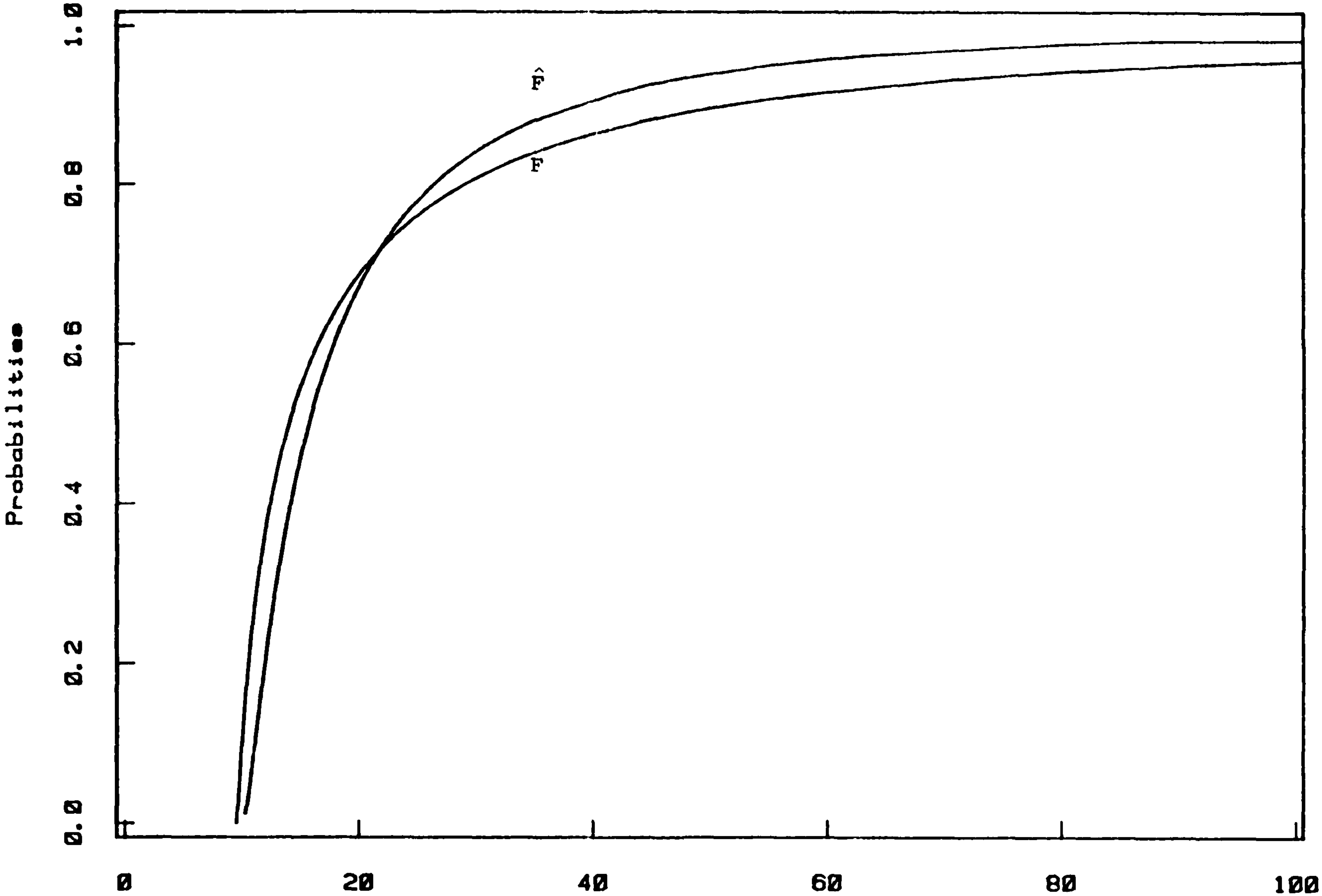

Figure 4. True CDF versus Estimated CDF

from 0 to ∞ may be numerically integrated by Laguerre integration using the relationship (1:923):

$$\int_0^\infty g(x)\,dx \approx \sum_{i=1}^{n} W_i e^{x_i} g(x_i) \tag{6.14}$$

where W_i and x_i are the i^{th} weights and abscissa of the Laguerre polynomials and n is the number of quadrature points--in this case $n = 15$. Before applying Laguerre integration, however, the lower limit is increased to ξ by adding the true value of ξ to each abcissa. The weights and abcissa are taken from the Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables (1).

The CVM statistic just described is calculated for each of the 1,000 replications of a given sample size and estimation technique. The sample mean (MCVM) and sample standard deviation (SDCVM) are calculated by:

$$\text{MCVM} = \sum_{i=1}^{1000} W_i^2/1000$$

$$\text{SDCVM} = \sqrt{\sum_{i=1}^{1000} (W_i^2 - \text{MCVM})^2/1000}$$

where W_i^2 is the CVM distance of the i^{th} replication. MCVM and SDCVM are the overall measures of how well the estimated distribution fits the true distribution. The smaller the MCVM and SDCVM the better the fit.

Much can be learned from investigating the three evaluation criteria just described. The MSEs are extremely useful in determining which estimation method worked best on a particular parameter. Unfortunately, however, the MSEs are not scale invariant; the same size MSE may be highly significant for a small valued parameter but insignificant for a larger one (2:32). The relative efficiencies yield approximately the same type of information about a particular parameter except that a base estimator is used which facilitates comparisons across all estimators. The CVM distance has the advantages over the MSE and REFF in that it is a single measure of fit for all three parameters and is also scale invariant with respect to the magnitude of the parameter estimates. Unfortunately, information concerning the individual parameters is not provided. A closing comment on the CVM distance measure is that no bias is introduced by using the CVM statistic, both for the MD-method and as an evaluation criteria. This should be apparent since during minimization the discrepancy between the estimated distribution function and 1/n step function is calculated, whereas during evaluation the discrepancy between the true and estimated cdfs is found. Investigation of these three criteria provides more than enough information to draw valid inferences about each modified maximum likelihood and minimum distance estimator. A computer listing of the program used to compute these three evaluation criteria is given in Appendix D.

VII. Results and Conclusions

Results and Comparisons

Numerical results pertaining to each of the estimation methods are listed in Appendix A. For each of the fifteen different cases (i.e., combination of parameter set (μ,σ,ξ) and sample size), there corresponds a page in Appendix A containing a table of the mean square errors (MSEs) relative efficiencies (REFFs) and the means (MCVMs) and standard deviations (SDCVMs) of the Cramer-von Mises statistics. The eight different estimation methods: modified interpolative maximum-likelihood (INT) estimation, modified iterative maximum-likelihood (ITR) estimation, and combinations of the INT and ITR estimates with Kolmogrov minimum distance (KOL) estimation, Cramer-von Mises minimum distance (CVM) estimation, and Anderson-Darling minimum distance (A-D) estimation are listed down the side of each table. The minimum distance estimators obtained by the INT estimates are listed directly under INT, and the minimum distance estimators obtained by the ITR estimates are listed directly under ITR. At the top of each table the sample size and true parameter values are given. In the following row, "MU," "SIG," and "XI" denote the parameters μ, σ, and ξ respectively. The entries in the "DIV" column are the number of times the location parameter estimate is equated to the last censored observation, usually the first order statistic. In the "CVM statistics" table, "MCVM" is the mean of the Cramer-von Mises statistic and "SDCVM" is the estimate of the standard deviation. When reference is made about a specific combination of a modified maximum likelihood estimates with a minimum distance estimator, the notation ML/MD is used where ML is either INT or ITR, and MD is either KOL, CVM, or A-D depending on the estimation technique applied. It should be noted that in some cases, the MSEs and the MCVM appear to contradict each other. In other words, one method may have larger MSEs for each parameter but a smaller overall MCVM when compared to another method. This situation arises in the case of $N = 10$, $\mu = 0.0$, $\sigma = 1.0$, $\xi = 10.0$; ITR/KOL has larger MSEs for each parameter but a smaller overall MCVM than the INT/CVM estimates. This apparent dichotomy occurs because a set of values further away from the true values in a MSE sense may be closer in a distance sense. Results obtained from the MSE, REFF, and CVM Statistics tables are summarized in the following paragraphs.

As previously stated, the MSE provides a measure of how close each estimated parameter is to its true value, while the REFF provides for comparisons across all estimators by the use of a base estimator. The MSEs from all estimation methods considered in this thesis are extremely good in that, except for cases where $N = 6$, no MSE of <u>any</u> parameter is ever greater than unity! In support of the asymptotic ML properties, the MSEs of each parameter decrease as the sample size increases for the INT and ITR estimators with the exception of the location estimator via ITR estimation. This exception seems reasonable since the location estimate is equated to the first order statistic in this method. Also, the decreasing MSEs of the interpolation estimate for the location parameter as the sample size increases lends strong support for the consistency assumption of the interpolative location estimator. When the term "best" or "better" is mentioned in the following text, it refers to best or better in the MSE sense. It should also be remembered that in every case $\xi = 10.0$. When looking at a particular sample size, the MSEs for all methods have a tendency to increase as the parameter values increase; however, this is not unexpected since the MSE is not scale invariant. In comparing the INT and ITR estimators, the REFFs reveal that, in each case, the INT estimator yields better estimates than the ITR estimators except for the location parameter when $\mu = 1$, $\sigma = 1$. The REFFs also show that the INT/CVM and INT/A-D estimators are better than their respective ITR/CVM and ITR/A-D estimators for cases of $\mu = 0$, $\sigma = 1$ and $\mu = 1$, $\sigma = 1$; in cases of $\mu = 1$, $\sigma = 2$, the reverse is true. It is also interesting to note which estimators perform the best for each parameter across all possible cases.

For cases of $\mu = 0.0$, $\sigma = 1.0$, the best estimator for μ is the ITR/KOL estimator when $N = 6,8,10$;

for $N = 12,16$ the best estimators for μ are the CVM and A-D estimators using the INT or ITR estimates. The best estimator for σ is the INT/A-D estimator except when $N = 16$ where the INT/CVM estimator is best. The best estimator for ξ is INT/A-D estimator except for $N = 6$ in which case the INT/CVM estimator is best.

For cases of $\mu = 1.0$, $\sigma = 1.0$, the best estimator of each parameter across all sample sizes is the ITR/KOL estimator.

For cases of $\mu = 1.0$, $\sigma = 2.0$, the best estimator for μ when $N = 6,8,10$ is the ITR/KOL estimator; however, when $N = 12,16$ the best estimator varies between the combinations of INT and ITR estimates with the CVM and A-D methods. The best estimator of σ is the ITR/CVM estimator followed closely by the ITR/A-D, INT/CVM, and INT/A-D estimators. The best estimator of ξ varies between the ITR/CVM and ITR/A-D estimators; the INT/CVM and INT/A-D estimators are much worse.

The MCVM measure also reveals some important information regarding the estimators. Results from this measure are usually interesting since it provides an overall measure of how well the estimated distribution function fits the true distribution function. The term "best" or "better" is now changed to mean best or better in the minimum distance sense. When the sample sizes are fixed and the parameter sets are investigated in the order ($\mu = 0.0$, $\sigma = 1.0$), ($\mu = 1.0$, $\sigma = 1.0$), and ($\mu = 1.0$, $\sigma = 2.0$), the results show that (1) the MCVM for the INT estimator gets larger, (2) the MCVM for the ITR estimator gets smaller, and (3) the MCVM for the MD-estimators are always smaller than either the INT or ITR estimators, except for one case where $N = 16$, and remain fairly constant. This indicates that in choosing between the INT and ITR estimators, the INT estimator should be used for values of μ and σ less than unity, whereas for values of μ and σ greater than unity, the ITR estimator is the correct choice.

For cases of $\mu = 0.0$, $\sigma = 1.0$, the best overall estimator is the ITR/KOL estimator except for $N = 16$ where the INT/CVM estimator is best.

For cases of $\mu = 1.0$, $\sigma = 1.0$, the best estimator is the ITR/KOL estimator except for $N = 16$ in which case the INT/A-D estimator is best.

For cases of $\mu = 1.0$, $\sigma = 2.0$, the best estimator for $N = 6,8,10$ is the ITR/CVM estimator; for $N = 12$ is the ITR/KOL estimator; and for $N = 16$ is the INT/CVM estimator. The estimators are closely followed by other combinations of INT and ITR estimates with the CVM and A-D methods.

The MCVM results clearly show that the minimum distance estimators using the modified maximum likelihood estimates are far more superior than the modified maximum likelihood estimators. The dominant estimator appears to be the ITR/KOL estimator followed closely by the ITR/CVM, ITR/A-D, INT/CVM, and INT/A-D estimators. As a final note, the SDCVM value for each estimator is extremely small indicating the stability of each estimator.

Conclusions

The objective of this thesis is the comparison of ML-estimators against MD-estimators given small samples of 3-LN deviates. It was anticipated that the MD-estimators would yield much more accurate estimates than those from the MD-estimators. Due to the small sample sizes, "strict" ML-estimators could not be obtained so two modified ML-estimation techniques are developed and compared against three MD-estimation methods. The results of the Monte Carlo analysis clearly demonstrates the superiority of the MD-estimators over both modified ML-estimators.

Considering the eight estimators tested (given the ranges of parameter values and sample sizes) in the Monte Carlo analysis, several conclusions are made. The MD-estimators are always superior to both modified ML-estimators. Of the modified ML-estimators, the interpolative method gives the best estimates. Of the MD-estimators, it appears that the ITR/KOL estimator gives the best estimates, but the INT/KOL estimator gives the worst estimates. The result may be partly due to the location parameter never being set to the last censored order statistic by the ITR/KOL estimator. The CVM and A-D estimators yield approximately equivalent estimates regardless of whether the INT or ITR estimates are used. This fact lends further support to the robustness property of the MD-estimators.

Although the estimators are rated as the "best" or "worst" estimators in the above paragraphs, it should be noted that no estimator gives extremely poor estimates. For the MD-estimators, the MSEs are rarely greater than 0.5 and the MCVM are usually less than 0.2. For the modified ML-estimators, the greatest MSE is 1.360707 while the MCVM is usually much less than 0.5. Furthermore, if great accuracy of the estimates is not required, or if a computer is not available, then the interpolative ML-method becomes a very viable estimation technique since the estimates can be easily computed using a desk or hand-held calculator; all other estimation methods require computerization due to more involved computations.

Recommendations for Further Study

Several possibilities exist for further research in parameter estimation techniques for the 3-LN distribution. First and foremost would be the development of an iterative estimation methodology allowing for the solution of the ML-equations for each parameter when working with small sample sizes. As it stands now, the MLE of the location parameter is equated to the first order statistic, while μ and σ are iteratively estimated. Perhaps using the interpolation estimate of ξ as an initial guess, an iterative procedure could be developed. In this procedure, $\hat{\xi}$ could be shifted around itself in small step lengths (i.e., 0.001) with $\hat{\mu}$ and $\hat{\sigma}$ being calculated at each step. At each step the likelihood function would also be calculated to determine the direction of the next step for the location estimate. In this way, the location estimates are independent of the steps and may yield strict ML estimators.

In this thesis, only the location parameter is refined by the MD-methods. The remaining two parameters could be similarly estimated. An iterative procedure could be developed where all three parameters are found via MD-methods. For example, σ could be estimated using Kupier's Maximal Interval Probability Statistic and/or Watson's Statistic, while μ and ξ are estimated by the Kolmogrov Distance, Cramer-von Mises Statistic and/or the Anderson-Darling Statistic. It is anticipated that this procedure would provide much more accurate estimators than those obtained in this thesis. As such, this topic would be an excellent research area for an adventurous statistician.

Appendix A

Tables of Mean Square Errors, Relative Efficiencies and CVM Statistics

Notation

MU	. . . . . . Estimate of μ
SIG	. . . . . . Estimate of σ
XI	. . . . . . Estimate of ξ
DIV	. . . . . . Number of Times $\hat{\xi} = X_{(1)}$
INT	. . . . . . Interpolative ML-Method
KOL	. . . . . . Kolmogrov MD-Method Using INT Estimates
CVM	. . . . . . CVM MD-Method Using INT Estimates
A-D	. . . . . . A-D MD-Method Using INT Estimates
ITR	. . . . . . Iterative ML-Method
KOL	. . . . . . Kolmogrov MD-Method Using ITR Estimates
CVM	. . . . . . CVM MD-Method Using ITR Estimates
A-D	. . . . . . A-D MD-Method Using ITR Estimates
MCVM	. . . . . . Mean of the CVM Distance
SDCVM	. . . . . . Standard Deviation of CVM Distance

TABLE A.1

SAMPLE SIZE 6
TRUE MU 0.00
TRUE SIGMA 1.00
TRUE XI 10.00

	MU	SIG	XI	DIV
		MEAN SQUARE ERRORS		
INT :	.655310	.286817	.105316	
KOL :	1.360707	1.103962	.143489	3
CVM :	.549699	.178744	.098908	14
A-D :	.546901	.195549	.100976	5
ITR :	.844692	.497264	.493105	
KOL :	.329784	.244997	.346987	0
CVM :	.684417	.292290	.127459	181
A-D :	.669653	.279103	.119645	280
		RELATIVE EFFICIENCIES		
INT :	1.000000	1.000000	1.000000	
KOL :	.481595	.259807	.733966	
CVM :	1.192124	1.604626	1.064791	
A-D :	1.198223	1.466725	1.042988	
ITR :	.775797	.576791	.213578	
KOL :	1.987085	1.170695	.303516	
CVM :	.957472	.981278	.826280	
A-D :	.978581	1.027640	.880239	

CVM STATISTICS

	MCVM	SDCVM
INT :	.366810	.000491
KOL :	.188285	.000189
CVM :	.055743	.000056
A-D :	.058678	.000059
ITR :	.707892	.000710
KOL :	.030761	.000031
CVM :	.062547	.000063
A-D :	.062547	.000063

TABLE A.2

SAMPLE SIZE	6
TRUE MU	1.00
TRUE SIGMA	1.00
TRUE XI	10.00

	MU	SIG	XI	DIV
		MEAN SQUARE ERRORS		
INT :	.655769	.287204	.779412	
KOL :	1.332993	1.038375	.943305	0
CVM :	.555348	.180249	.727961	0
A-D :	.555660	.198898	.745718	0
ITR :	.864145	.523615	.359428	
KOL :	.084419	.071938	.364376	0
CVM :	.753766	.385351	.926223	67
A-D :	.666059	.286989	.834891	184
		RELATIVE EFFICIENCIES		
INT :	1.000000	1.000000	1.000000	
KOL :	.491952	.276590	.826256	
CVM :	1.180826	1.593375	1.070677	
A-D :	1.180162	1.443979	1.045183	
ITR :	.758865	.546415	2.168476	
KOL :	7.768037	3.992364	2.139032	
CVM :	.869990	.745306	.841495	
A-D :	.955849	1.000751	.933549	

CVM STATISTICS

	MCVM	SDCVM
INT :	.482803	.003600
KOL :	.146933	.000032
CVM :	.056385	.000021
A-D :	.059234	.000025
ITR :	.342977	.000025
KOL :	.037457	.000041
CVM :	.062233	.000009
A-D :	.062650	.000022

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TABLE A.3

SAMPLE SIZE 6
TRUE MU 1.00
TRUE SIGMA 2.00
TRUE XI 10.00

		MU	SIG	XI	DIV
			MEAN SQUARE ERRORS		
INT	:	.943638	.521600	1.035676	
KOL	:	1.357249	.742137	.982729	2
CVM	:	.888678	.503325	1.200742	4
A-D	:	.893750	.526195	1.259358	11
ITR	:	.964483	.628586	.829922	
KOL	:	.442460	.670193	.689556	0
CVM	:	.861398	.379846	.545788	60
A-D	:	.859002	.470960	.596845	49
			RELATIVE EFFICIENCIES		
INT	:	1.000000	1.000000	1.000000	
KOL	:	.695258	.702834	1.053878	
CVM	:	1.061844	1.036308	.862530	
A-D	:	1.055818	.991267	.822384	
ITR	:	.978387	.829798	1.247920	
KOL	:	2.132709	.778283	1.501946	
CVM	:	1.095472	1.373186	1.897579	
A-D	:	1.098528	1.107523	1.735251	

CVM STATISTICS

		MCVM	SDCVM
INT	:	.617581	.021080
KOL	:	.118615	.000612
CVM	:	.059271	.000850
A-D	:	.063324	.000845
ITR	:	.272654	.000792
KOL	:	.060480	.000959
CVM	:	.057036	.000737
A-D	:	.058148	.000770

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TABLE A.4

SAMPLE SIZE	8
TRUE MU	0.00
TRUE SIGMA	1.00
TRUE XI	10.00

MEAN SQUARE ERRORS

	MU	SIG	XI	DIV
INT :	.476877	.272887	.071213	
KOL :	.922417	.939559	.100614	3
CVM :	.406065	.176728	.070241	12
A-D :	.393464	.166579	.067290	5
ITR :	.618789	.416204	.549261	
KOL :	.373430	.234557	.405017	0
CVM :	.501182	.253267	.084231	268
A-D :	.492213	.249537	.084408	290

RELATIVE EFFICIENCIES

	MU	SIG	XI
INT :	1.000000	1.000000	1.000000
KOL :	.513093	.290442	.707788
CVM :	1.174387	1.544111	1.013837
A-D :	1.211999	1.638180	1.058307
ITR :	.770662	.655657	.129653
KOL :	1.277021	1.163414	.175828
CVM :	.951505	1.077469	.845454
A-D :	.968843	1.093573	.843676

CVM STATISTICS

	MCVM	SDCVM
INT :	.488003	.000641
KOL :	.171784	.000172
CVM :	.040006	.000040
A-D :	.038185	.000038
ITR :	.952907	.000953
KOL :	.020390	.000020
CVM :	.041624	.000042
A-D :	.041624	.000042

TABLE A.5

SAMPLE SIZE 8
TRUE MU 1.00
TRUE SIGMA 1.00
TRUE XI 10.00

	MU	SIG	XI	DIV
		MEAN SQUARE ERRORS		
INT :	.476934	.273282	.526903	
KOL :	.953368	.935649	.668776	1
CVM :	.426717	.191437	.523154	3
A-D :	.398836	.172825	.497591	2
ITR :	.626356	.431749	.282776	
KOL :	.071923	.063611	.228272	0
CVM :	.539125	.352480	.635454	92
A-D :	.499798	.255878	.593109	226
		RELATIVE EFFICIENCIES		
INT :	1.000000	1.000000	1.000000	
KOL :	.500263	.292077	.787861	
CVM :	1.117684	1.427528	1.007167	
A-D :	1.195815	1.581258	1.058908	
ITR :	.761443	.632964	1.863320	
KOL :	6.631136	4.296162	2.308229	
CVM :	.884645	.775310	.829176	
A-D :	.954254	1.068013	.838374	

CVM STATISTICS

	MCVM	SDCVM
INT :	.639525	.004718
KOL :	.121675	.000079
CVM :	.039762	.000046
A-D :	.037338	.000003
ITR :	.386990	.000008
KOL :	.020569	.000016
CVM :	.169935	.000030
A-D :	.040439	.000004

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TABLE A.6

SAMPLE SIZE	8
TRUE MU	1.00
TRUE SIGMA	2.00
TRUE XI	10.00

MEAN SQUARE ERRORS

	MU	SIG	XI	DIV
INT :	.676286	.413293	.283798	
KOL :	.924357	.587918	.269736	3
CVM :	.637344	.365382	.365726	5
A-D :	.639238	.396803	.333915	20
ITR :	.692689	.480513	.660948	
KOL :	.440742	.614440	.487359	0
CVM :	.621100	.305793	.185932	62
A-D :	.623585	.366783	.176682	52

RELATIVE EFFICIENCIES

	MU	SIG	XI
INT :	1.000000	1.000000	1.000000
KOL :	.731628	.702977	1.052133
CVM :	1.061100	1.131126	.775985
A-D :	1.057955	1.041556	.849910
ITR :	.976319	.860108	.429380
KOL :	1.534424	.672634	.582318
CVM :	1.088851	1.351547	1.526353
A-D :	1.084512	1.126805	1.606268

CVM STATISTICS

	MCVM	SDCVM
INT :	.816757	.028034
KOL :	.096663	.000887
CVM :	.035378	.000723
A-D :	.036698	.000713
ITR :	.291249	.000808
KOL :	.041684	.000747
CVM :	.035354	.000707
A-D :	.035622	.000718

TABLE A.7

SAMPLE SIZE	10
TRUE MU	0.00
TRUE SIGMA	1.00
TRUE XI	10.00

MEAN SQUARE ERRORS

	MU	SIG	XI	DIV
INT :	.383317	.250480	.051575	
KOL :	.681101	.772202	.075254	3
CVM :	.335957	.185933	.050492	18
A-D :	.315620	.157262	.047732	9
ITR :	.496913	.379719	.590799	
KOL :	.402003	.227986	.461629	0
CVM :	.403179	.244989	.064706	295
A-D :	.394851	.241653	.062153	277

RELATIVE EFFICIENCIES

	MU	SIG	XI
INT :	1.000000	1.000000	1.000000
KOL :	.562790	.324371	.685348
CVM :	1.140972	1.347152	1.021448
A-D :	1.214488	1.592757	1.079382
ITR :	.771396	.659646	.087298
KOL :	.953518	1.098664	.111725
CVM :	.950736	1.022414	.797073
A-D :	.970790	1.036528	.829809

CVM STATISTICS

	MCVM	SDCVM
INT :	.612318	.000539
KOL :	.276212	.000003
CVM :	.126936	.000000
A-D :	.130615	.000000
ITR :	1.565242	.000001
KOL :	.051737	.000000
CVM :	.168352	.000001
A-D :	.168352	.000001

..

TABLE A.8

SAMPLE SIZE 10
TRUE MU 1.00
TRUE SIGMA 1.00
TRUE XI 10.00

	MU	SIG	XI	DIV
		MEAN SQUARE ERRORS		
INT :	.383927	.250430	.381625	
KOL :	.750273	.891645	.503738	0
CVM :	.363875	.220912	.381512	0
A-D :	.317965	.157387	.352664	0
ITR :	.505151	.391544	.261686	
KOL :	.061069	.057130	.178601	0
CVM :	.417473	.300833	.474617	125
A-D :	.402501	.246345	.452670	245
		RELATIVE EFFICIENCIES		
INT :	1.000000	1.000000	1.000000	
KOL :	.511716	.280863	.757586	
CVM :	1.055108	1.133620	1.000296	
A-D :	1.207451	1.591178	1.082120	
ITR :	.760024	.639597	1.458329	
KOL :	6.286779	4.383520	2.136742	
CVM :	.919647	.832456	.804068	
A-D :	.953854	1.016586	.843054	

CVM STATISTICS

	MCVM	SDCVM
INT :	.806153	.006031
KOL :	.226466	.000232
CVM :	.124742	.000030
A-D :	.127688	.000021
ITR :	.749610	.000051
KOL :	.086908	.000019
CVM :	.159530	.000035
A-D :	.159530	.000035

..

TABLE A.9

SAMPLE SIZE	10
TRUE MU	1.00
TRUE SIGMA	2.00
TRUE XI	10.00

MEAN SQUARE ERRORS

	MU	SIG	XI	DIV
INT :	.532332	.310910	.117524	
KOL :	.683867	.472942	.112787	3
CVM :	.592089	.272678	.152260	6
A-D :	.497984	.292731	.133183	29
ITR :	.547307	.374777	.684608	
KOL :	.424192	.574206	.517111	0
CVM :	.487294	.235218	.085660	71
A-D :	.505876	.277305	.082215	59

RELATIVE EFFICIENCIES

	MU	SIG	XI
INT :	1.000000	1.000000	1.000000
KOL :	.778415	.657395	1.041999
CVM :	1.060234	1.140207	.771863
A-D :	1.068974	1.062102	.882425
ITR :	.972640	.829585	.171666
KOL :	1.254932	.541461	.227270
CVM :	1.092424	1.321796	1.371976
A-D :	1.052298	1.121184	1.429475

CVM STATISTICS

	MCVM	SDCVM
INT :	1.035208	.034789
KOL :	.194634	.000954
CVM :	.112736	.000801
A-D :	.123434	.000811
ITR :	.581777	.000956
KOL :	.079166	.000870
CVM :	.112724	.000805
A-D :	.112901	.000813

..

TABLE A.10

SAMPLE SIZE	12
TRUE MU	0.00
TRUE SIGMA	1.00
TRUE XI	10.00

MEAN SQUARE ERRORS

	MU	SIG	XI	DIV
INT :	.326100	.239326	.043434	
KOL :	.546634	.655909	.060628	6
CVM :	.290527	.184550	.043426	23
A-D :	.265766	.140996	.040502	15
ITR :	.417943	.334823	.622391	
KOL :	.423729	.226359	.501794	0
CVM :	.343548	.226557	.052435	337
A-D :	.337610	.220576	.051215	309

RELATIVE EFFICIENCIES

	MU	SIG	XI
INT :	1.000000	1.000000	1.000000
KOL :	.596560	.364877	.716413
CVM :	1.122444	1.296811	1.000202
A-D :	1.227020	1.697397	1.072407
ITR :	.780250	.714785	.069786
KOL :	.769595	1.057287	.086558
CVM :	.949211	1.056364	.828350
A-D :	.965908	1.085007	.848074

CVM STATISTICS

	MCVM	SDCVM
INT :	.735408	.000703
KOL :	.210019	.000004
CVM :	.064878	.000000
A-D :	.065139	.000000
ITR :	1.624189	.000000
KOL :	.023369	.000000
CVM :	.091958	.000000
A-D :	.091958	.000000

..

TABLE A.11

SAMPLE SIZE	12
TRUE MU	1.00
TRUE SIGMA	1.00
TRUE XI	10.00

	MU	SIG	XI	DIV
MEAN SQUARE ERRORS				
INT :	.326100	.239326	.320933	
KOL :	.620869	.307474	.421894	0
CVM :	.323856	.246831	.331687	3
A-D :	.269186	.147453	.297023	2
ITR :	.427455	.358289	.272780	
KOL :	.062410	.056860	.169423	0
CVM :	.355350	.272643	.384187	169
A-D :	.347895	.236927	.374251	287
RELATIVE EFFICIENCIES				
INT :	1.000000	1.000000	1.000000	
KOL :	.525231	.296389	.760697	
CVM :	1.006730	.969595	.967577	
A-D :	1.211430	1.623070	1.080500	
ITR :	.762887	.667970	1.176527	
KOL :	5.225145	4.209080	1.894276	
CVM :	.917687	.877801	.835358	
A-D :	.937351	1.010128	.857536	

CVM STATISTICS

	MCVM	SDCVM
INT :	.975852	.007515
KOL :	.150207	.000098
CVM :	.063786	.000029
A-D :	.063633	.000022
ITR :	.657586	.000051
KOL :	.036542	.000013
CVM :	.088608	.000033
A-D :	.088597	.000033

..

TABLE A.12

SAMPLE SIZE	12
TRUE MU	1.00
TRUE SIGMA	2.00
TRUE XI	10.00

	MU	SIG	XI	DIV
		MEAN SQUARE ERRORS		
INT :	.449044	.270869	.086651	
KOL :	.539179	.379041	.082294	7
CVM :	.421967	.226249	.119563	12
A-D :	.416589	.240181	.099595	30
ITR :	.459219	.309192	.725196	
KOL :	.422344	.556705	.565643	0
CVM :	.411279	.207014	.054377	67
A-D :	.426530	.237099	.057158	67
		RELATIVE EFFICIENCIES		
INT :	1.000000	1.000000	1.000000	
KOL :	.832827	.714616	1.052942	
CVM :	1.064167	1.197219	.724734	
A-D :	1.077905	1.127772	.870033	
ITR :	.977841	.876056	.119486	
KOL :	1.063217	.486558	.153190	
CVM :	1.091821	1.308461	1.593510	
A-D :	1.052784	1.142431	1.515996	

CVM STATISTICS

	MCVM	SDCVM
INT :	1.233838	.041781
KOL :	.114867	.000514
CVM :	.049472	.000486
A-D :	.051923	.000508
ITR :	.486826	.000562
KOL :	.046033	.000585
CVM :	.051311	.000472
A-D :	.051267	.000469

..

TABLE A.13

SAMPLE SIZE	16
TRUE MU	0.00
TRUE SIGMA	1.00
TRUE XI	10.00

	MU	SIG	XI	DIV
MEAN SQUARE ERRORS				
INT :	.232383	.203052	.029973	
KOL :	.378743	.509347	.043096	4
CVM :	.222283	.179816	.030684	51
A-D :	.194003	.126010	.027965	30
ITR :	.308175	.281351	.667245	
KOL :	.456919	.221362	.569259	0
CVM :	.250650	.193066	.036964	336
A-D :	.249732	.194706	.036456	309
RELATIVE EFFICIENCIES				
INT :	1.000000	1.000000	1.000000	
KOL :	.613565	.398651	.695501	
CVM :	1.045437	1.129222	.976829	
A-D :	1.197831	1.611390	1.071822	
ITR :	.754063	.721702	.044921	
KOL :	.508587	.917282	.052653	
CVM :	.927125	1.051723	.810873	
A-D :	.930532	1.042865	.822172	

CVM STATISTICS

	MCVM	SDCVM
INT :	.058370	.000001
KOL :	.202583	.000010
CVM :	.052597	.000001
A-D :	.056141	.000001
ITR :	2.048071	.000002
KOL :	1.690798	.000008
CVM :	.083688	.000001
A-D :	.083688	.000001

..

TABLE A.14

SAMPLE SIZE	16
TRUE MU	1.00
TRUE SIGMA	1.00
TRUE XI	10.00

	MU	SIG	XI	DIV
		MEAN SQUARE ERRORS		
INT :	.232383	.203052	.221497	
KOL :	.432193	.656473	.307171	2
CVM :	.242403	.228078	.237409	3
A-D :	.190384	.119146	.202699	3
ITR :	.308562	.282413	.307384	
KOL :	.061823	.054438	.188856	0
CVM :	.256022	.220713	.270062	204
A-D :	.250442	.194406	.268905	302
		RELATIVE EFFICIENCIES		
INT :	1.000000	1.000000	1.000000	
KOL :	.537685	.309307	.721088	
CVM :	.958667	.890274	.932978	
A-D :	1.220607	1.704219	1.092739	
ITR :	.753118	.718989	.720587	
KOL :	3.758850	3.729951	1.172838	
CVM :	.907668	.919982	.820171	
A-D :	.927893	1.044473	.823699	

CVM STATISTICS

	MCVM	SDCVM
INT :	.059919	.000106
KOL :	.152448	.000483
CVM :	.076387	.000794
A-D :	.056961	.000078
ITR :	.772859	.000093
KOL :	.640381	.000413
CVM :	.080935	.000253
A-D :	.084114	.000106

..

TABLE A.15

SAMPLE SIZE	16
TRUE MU	1.00
TRUE SIGMA	2.00
TRUE XI	10.00

	MU	SIG	XI	DIV
		MEAN SQUARE ERRORS		
INT :	.303700	.200303	.030553	
KOL :	.350235	.259372	.033354	5
CVM :	.291445	.165668	.042253	11
A-D :	.287489	.177657	.032471	38
ITR :	.322491	.235817	.787416	
KOL :	.414454	.519659	.659067	0
CVM :	.293632	.162336	.024263	77
A-D :	.301267	.186689	.024930	61
		RELATIVE EFFICIENCIES		
INT :	1.000000	1.000000	1.000000	
KOL :	.867134	.772261	.916034	
CVM :	1.042050	1.209059	.723105	
A-D :	1.056387	1.127467	.940942	
ITR :	.941733	.849399	.038802	
KOL :	.732771	.385450	.046359	
CVM :	1.034290	1.233879	1.259276	
A-D :	1.008075	1.072922	1.225547	

CVM STATISTICS

	MCVM	SDCVM
INT :	.059835	.000964
KOL :	.121674	.001103
CVM :	.057521	.001007
A-D :	.072231	.001043
ITR :	.585003	.000941
KOL :	.653962	.001038
CVM :	.073151	.000925
A-D :	.059473	.000954

..

Appendix B

Computer Listing of Data Generation and Maximum Likelihood Techniques

```
      PROGRAM BEGLND
C
C     *****************************************************************
C     *                                                               *
C     *  WRITTEN BY 2LT JIM H KEFFER AFIT/GOR-83D FOR MS THESIS       *
C     *                  DECEMBER 1983                                *
C     *                                                               *
C     *  PURPOSE:     (1) GENERATE SAMPLES OF 3-LND DEVIATES          *
C     *               (2) CALCULATE THE MEAN AND STD. DEV. OF SAMPLES *
C     *               (3) CALCULATE THE MLE'S USING THE INTERPOLATION *
C     *                   METHOD                                      *
C     *               (4) CALCULATE THE MLE'S USING THE ITERATIVE     *
C     *                   METHOD                                      *
C     *                                                               *
C     *  VARIABLES:  DSEED - SEED FOR RANDOM NUMBER GENERATOR         *
C     *              TMU   - TRUE MEAN OF PARENT NORMAL               *
C     *              TSIG  - TRUE STD. DEV. OF PARENT NORMAL          *
C     *              TXI   - TRUE LOCATION VALUE                      *
C     *              BMU   - INTERPOLATIVE MLE OF TMU                 *
C     *              BSIG  - INTERPOLATIVE MLE OF TSIG                *
C     *              BXI   - INTERPOLATIVE MLE OF TXI                 *
C     *              HMU   - ITERATIVE MLE OF TMU                     *
C     *              HSIG  - ITERATIVE MLE OF TSIG                    *
C     *              HXI   - ITERATIVE MLE OF TXI                     *
C     *              N     - DESIRED SAMPLE SIZE                      *
C     *              NC    - # OF DEVIATES CENSORED FROM BELOW        *
C     *              NREPS - NUMBER OF REPLICATIONS                   *
C     *              MLE   - SUBROUTINE TO COMPUTE ITERATIVE MLE'S    *
C     *              GGLNG - IMSL ROUTINE WHICH GENERATES 2-LND DEVIATE*
C     *              VSRTA - IMSL ROUTINE WHICH ORDERS DATA           *
C     *              SUM   - DUMMY VARIABLE USED TO COMPUTE SAMPLE    *
C     *                      MEANS AND STD. DEVS.                     *
C     *              SUMR  - DUMMY VARIABLE USED FOR SUMS             *
C     *              SUMR1 - DUMMY VARIABLE USED FOR SUMS             *
C     *              X     - VECTOR OF 3-LND DEVIATES                 *
C     *              MEAN  - SAMPLE ARITHMATIC MEAN                   *
C     *              SD    - SAMPLE STANDARD DEVIATION                *
C     *              Y1    - MEDIAN RANK OF FIRST ORDER STATISTIC     *
C     *              Y2    - MEDIAN RANK OF SECOND ORDER STATISTIC    *
C     *              SLOPE - SLOPE OF INTERPOLATION LINE              *
C     *                                                               *
C     *  I/O FILES:  INPUT - UNFORMATTED INPUT OF TRUE PARAMETERS     *
C     *              TAPE5 - OUTPUT OF TRUE PARAMETERS, SAMPLES,      *
C     *                      MEANS, SD, AND ALL MLE ESTIMATES         *
C     *                                                               *
C     *  IMPORTANT:  IMSL LIBRARY MUST BE ATTACHED BEFORE THE PROGRAM *
C     *              IS RUN. REVIEW IMSL MANUAL ON GGNLG AND VSRTA    *
C     *                                                               *
C     *****************************************************************
C
```

```
      COMMON N,NC,X(50),HMU,HSIG,HXI,J,SIG,NN
      EXTERNAL GGNLG,VSTRA,MLE
      DOUBLE PRECISION DSEED
      REAL TMU,TSIG,TXI,BMU,BSIG,BXI
      REAL MEAN,SUM,SD,SLOPE,Y1,Y2
      INTEGER N
      DSEED=185921752.D0
C     *** READ PARAMETERS AND WRITE THEM TO FILE ***
      PRINT*,'ENTER TMU,TSIG,TXI,N,NREPS'
      READ*,TMU,TSIG,TXI,N,NREPS
      WRITE(5,100) NREPS,N,TMU,TSIG,TXI
 100  FORMAT(I4/I3/3(F15.6/))
C     ****** BEGIN DO-LOOP FOR GENERATION OF SAMPLES ******
      DO 999 J=1,NREPS
 103     FORMAT(I4)
C        *** GENERATE AND SORT SAMPLES FROM 2-LN DISTRIBUTION **
         CALL GGNLG(DSEED,N,TMU,TSIG,X)
         WRITE(5,103) J
         CALL VSRTA(X,N)
C        *** ADD LOCATION PARAMETER TO 2-LN DEVIATES ***
C        *** WRITE THE 3-LN DEVIATES TO FILE ***
C        *** CALCULATE SAMPLE MEAN ***
            SUM=0.0
            DO 10 I=1,N
            X(I)=X(I)+TXI
            WRITE(5,101) X(I)
 101        FORMAT(F15.6)
            SUM=SUM + X(I)
  10     CONTINUE
         MEAN=SUM/N
C        *** CALCULATE SAMPLE STANDARD DEVIATIONS ***
         SUM=0.0
         DO 20 I=1,N
            SUM=SUM+(X(I)-MEAN)*(X(I)-MEAN)
  20     CONTINUE
         SD=(SUM/N)**0.5
C        *** CALCULATE MEDIAN RANKS ***
C        *** INTERPOLATE TO ESTIMATE BXI ***
         Y1=(1.0-0.3)/(N+0.4)
         Y2=(2.0-0.3)/(N+0.4)
         SLOPE=(Y2-Y1)/(X(2)-X(1))
         BXI=X(1)-Y1/SLOPE
C        *** CALCULATE MLE OF TMU AND TSIG USING THE INTERPOLATE
C        *** VALUE FOR THE LOCATION PARAMETER, XI ***
C        *** USE FIRST ORDER STATISTIC IF THE INTERPOLATED VALUE
C        *** OF THE LOCATION PARAMETER IS CLOSE TO X(1) ***
         IF (BXI .GT. X(1)) BXI=X(1)
         SUM=0.0
         DO 30 I=1,N
           SUM=SUM+LOG(X(I)-BXI)
  30     CONTINUE
         BMU=SUM/N
         SUM=0.0
```

```
      DO 40 I=1,N
         SUM = SUM + (LOG(X(I)-BXI) - BMU)**2
 40   CONTINUE
      BSIG=SQRT(SUM/N)
C     *** CALCULATE THE MLE'S USING THE CENSORED MAXIMUM LIKELIHOOD *
C     ***EQUATIONS AND THE ITERATIVE PROCEDURE
      CALL MLE(SD)
C     ***WRITE THE MLE'S, SAMPLE MEAN AND STANDARD DEVIATION***
C     *** TO FILE FOR EACH SAMPLE ***
      WRITE(5,102) BMU,BSIG,BXI,HMU,HSIG,HXI,MEAN,SD
102   FORMAT(F15.3/F15.3/F15.3/F15.3/F15.3/F15.3/F15.6/F15.6)
C    ******* END DO-LOOP *******
999  CONTINUE
     STOP
     END
C
C
     SUBROUTINE MLE(SD)
C    ****************************************************************
C    *                                                              *
C    *  PURPOSE:     (1) CALCULATE THE ITERATIVE MLE'S FOR TMU AND  *
C    *                   TSIG WHILE TXI=X(1)                        *
C    *                                                              *
C    *  VARIABLES:   SUMN  - DUMMY USED FOR SUMS                    *
C    *               SUMD  - DUMMY USED FOR SUMS                    *
C    *               PAR1  - INITIAL & FINAL ESTIMATE OF TSIG VIA   *
C    *                       EQUATING SAMPLE AND POPULATION SKEWNESS *
C    *               XBAR  - SAMPLE MEDIAN                          *
C    *               SIG   - FINAL ESTIMATE OF TSIG FROM SAMPLE     *
C    *                       SKEWNESS                               *
C    *               C1    - ARRAY OF CONSTANTS FOR USE BY ZSCNT    *
C    *               FCN0  - FUNCTION USED TO CALCULATE EXP[SIG**2] *
C    *               FCN1  - FUNCTION USED TO CALCULATE MLE OF TMU  *
C    *               FCN2  - FUNCTION USED TO CALCULATE MLE OF TSIG *
C    *                                                              *
C    *  NOTE:        UNDEFINED VARIABLES IN THIS SUBROUTINE ARE     *
C    *               DEFINED IN THE MAIN PROGRAM                    *
C    *                                                              *
C    ****************************************************************
     COMMON N,NC,X(50),HMU,HSIG,HXI,J,SIG,NN
     EXTERNAL FCN0,FCN1,FCN2
     REAL WK(54),FNORM,PAR(1),C(1),SD,PAR1(1),C1(1)
     INTEGER J
     INTEGER N,NC,NSIG,NPAR,ITMAX,IER,MAXFN,K
C    *** CHECK IF ANY OF THE SAMPLE POINTS ARE CLOSE TO THE ***
C    *** FIRST ORDER STATISTIC - CENSOR THEM IF THEY ARE   ***
     NC=1
     DO 10 I=2,N
        IF ((X(I)-X(1)) .LE. 0.001) NC=NC+1
 10  CONTINUE
C    *** SET ZSCNT PARAMETERS FOR SKEWNESS AND MLE CALCULATIONS ***
     NPAR=1
     NSIG=3
```

```
      NPAR1=1
      ITMAX=1000
      NN=0
      C(1)=0.0
C     ***  CALCULATE SAMPLE SKEWNESS AND SIGMA FROM THE SKEWNESS ***
C     ***  EQUATION - RESULTS ARE USED AS INITIAL ESTIMATES       ***
      XBAR=X(INT((N-NC)/2))
      SUMN=0.0
      SUMD=0.0
      DO 8 I=NC,N
         SUMN=SUMN+(X(I)-XBAR)**3
         SUMD=SUMD+(X(I)-XBAR)**2
    8 CONTINUE
      SUMN=SUMN/(N-NC)
      SUMD=(SUMD/(N-NC))**1.5
      C1(1)=SUMN/SUMD
      PAR1(1)=LOG(SD)
      IF (PAR1(1) .LT. 0.001) PAR1(1)=1.0
      CALL ZSCNT(FCN0,NSIG,NPAR1,ITMAX,C1,PAR1,FNORM,WK,IER)
      SIG=SQRT(LOG(ABS(PAR1(1))))
C     *** SET INTIAL VALUES FOR THE ITERATIVE MLE PROCEDURE AND ***
C     *** CALCULATE THE MLE OF TMU AND TSIG                     ***
      HMU=LOG(XBAR)
      HSIG=SIG
      IF (SIG .LT. 1.001) HSIG=PAR1(1)
      PAR(1)=HMU
      CALL ZSCNT(FCN1,NSIG,NPAR,ITMAX,C,PAR,FNORM,WK,IER)
      HMU=PAR(1)
      PAR(1)=HSIG
      CALL ZSCNT(FCN2,NSIG,NPAR,ITMAX,C,PAR,FNORM,WK,IER)
      HSIG=PAR(1)
      RETURN
      END
C
      SUBROUTINE FCN1(PAR,F,NP,C)
C     *** CALCULATE THE CENSORED MLE OF TMU ***
      DIMENSION PAR(NP),F(NP),C(1)
      COMMON N,NC,X(50),HMU,HSIG,HXI,J,SIG,NN
      SUM=0.0
      HXI=X(NC)
      Y=(LOG(X(NC+1)-HXI)-PAR(1))/HSIG
      FU=.3989423*EXP(-(Y**2)/2)
      CALL MDNOR(Y,Z)
      R=NC*(FU/Z)
      DO 10 I=1+NC,N
         SUM=SUM+(LOG(X(I)-HXI)-PAR(1))/HSIG
   10 CONTINUE
      F(1)=SUM-R
      RETURN
      END
C
C
      SUBROUTINE FCN2(PAR,F,NP,C)
```

AD-A138 007 ROBUST MINIMUM DISTANCE ESTIMATION OF THE THREE PARAMETER LOGNORMAL DISTRIBUTION(U) AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OH SCHOOL OF ENGI.. 2/2

UNCLASSIFIED J H KEFFER DEC 83 AFIT/GOR/MA/83D-3 F/G 12/1 NL

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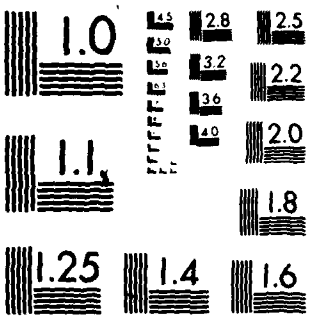

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963 A

```
C     *** CALCULATE THE CENSORED MLE OF TSIG ***
      DIMENSION PAR(NP),F(NP),C(1)
      COMMON N,NC,X(50),HMU,HSIG,HXI,J,SIG,NN
C     *** IF THE PAR(1) ESTIMATE IS INFEASIBLE, CHANGE THE INITIAL ***
C     *** ESTIMATE TO THE SIGMA OBTAINED FROM THE SKEWNESS EQN.    ***
      IF (PAR(1) .LE. SIG .AND. NN .EQ. 0) THEN
         PAR(1)=SIG
         NN=1
      ENDIF
      SUM=0.0
      HXI=X(NC)
      Y=(LOG(X(NC+1)-HXI)-HMU)/PAR(1)
      FU=.3989423*EXP(-(Y**2)/2)
      CALL MDNOR(Y,Z)
      R=NC*(Y*FU)/Z
      DO 10 I=1+NC,N
         SUM=SUM+((LOG(X(I)-HXI)-HMU)/PAR(1))**2
   10 CONTINUE
      F(1)=SUM-(N-NC)-R
      RETURN
      END
C
C
      SUBROUTINE FCNO(PAR1,F,NP,C1)
C     *** CALCULATE SIGMA FROM THE SKEWNESS EQN - PAR1(1)=W ***
      REAL F(NP),C1(NP),PAR1(NP)
      F(1)=((PAR1(1)+2)**2)*(PAR1(1)-1)- (C1(1))**2
      RETURN
      END
*EOR
..
```

Appendix C

Computer Listing of Minimum Distance Techniques

```
      PROGRAM MLEMD
C     ***********************************************************************
C     *                                                                     *
C     *  WRITTEN BY 2LT JIM H KEFFER AFIT/GOR-83D FOR MS THESIS             *
C     *                         DECEMBER 1983                               *
C     *                                                                     *
C     *  PURPOSE:    MINIMUM DISTANCE ESTIMATION OF THE LOCATION            *
C     *              PARAMETER FOR THE 3-LN DISTRIBUTION USING THE          *
C     *              INTERPOLATIVE AND ITERATIVE MLE'S.  THE MINIMUM        *
C     *              DISTANCE ESTIMATORS ARE THE KOLMOGROV DISTANCE,        *
C     *              CRAMER-VON MISES DISTANCE, AND THE ANDERSON-           *
C     *              DARLING STATISTICS                                     *
C     *                                                                     *
C     *  VARIABLES:  NREPS - NUMBER OF REPLICATIONS                         *
C     *              N     - SAMPLE SIZE (INPUT)                            *
C     *              TMU   - TRUE MEAN OF PARENT NORMAL (INPUT)             *
C     *              TSIG  - TRUE STANDARD DEVIATION OF PARENT NORMAL       *
C     *              TXI   - TRUE VALUE OF THE LOCATION PARAMETER           *
C     *              BMU   - INTERPOLATIVE MLE OF TMU                       *
C     *              BSIG  - INTERPOLATIVE MLE OF TSIG                      *
C     *              BSIG  - INTERPOLATIVE MLE OF TSIG                      *
C     *              BXI   - INTERPOLATIVE MLE OF TXI                       *
C     *              HMU   - ITERATIVE MLE OF TMU                           *
C     *              HSIG  - ITERATIVE MLE OF TSIG                          *
C     *              HXI   - ITERATIVE MLE OF TXI                           *
C     *              K     - SAMPLE INDEX                                   *
C     *              N@B   - DIVERGENCE COUNTER OF INTERPOLATIVE MLE'S      *
C     *                      WHERE @ IS INITIAL OF MD METHOD                *
C     *              N@H   - DIVERGENCE COUNTER OF ITERATIVE MLE'S          *
C     *                      WHERE @ IS INITIAL OF MD METHOD                *
C     *              NC    - # OF DEVIATES CENSORED FROM BELOW              *
C     *              NN    - CHECK FOR DIVERGENCE OF KOLMOGROV ESTIMATE*
C     *              BXIC  - NEW INITIAL ESTIMATE - USED WHEN FIRST         *
C     *                      INITIAL ESTIMATE DIVERGES                      *
C     *              X     - ARRAY OF 3-LN DEVIATES                         *
C     *              MEAN  - SAMPLE MEAN                                    *
C     *              SD    - SAMPLE STANDARD DEVIATION                      *
C     *              PAR   - ARRAY CONTAINING MLE OF XI FOR MD              *
C     *              KXI   - KOLMOGROV MD ESTIMATE OF XI                    *
C     *              CVMXI - CRAMER-VON MISES MD ESTIMATE OF XI             *
C     *              A2XI  - ANDERSON-DARLING MD ESTIMATE OF XI             *
C     *              MDNOR - IMSL ROUTINE FOR NORMAL CDF VALUES             *
C     *              ZXMIN - IMSL ROUTINE TO MINIMIZE THE DISTANCE          *
C     *              NPAR  - NUMBER OF VARIABLES INPUTTED BY ZXMIN          *
C     *              MAXFN - MAXIMUM # OF ITERATIONS BY ZXMIN               *
C     *              NSIG  - # OF SIGNIFICANT DIGITS ZXMIN SLOVES FOR       *
C     *              IOPT  - ZXMIN OPTION (SEE IMSL MANUAL)                 *
C     *              DKSXI - SUBROUTINE TO FIND DISTANCE VIA KOLMOGROV      *
C     *              DCVXI - SUBROUTINE TO FIND DISTANCE VIA CVM            *
C     *              DA2XI - SUBROUTINE TO FIND DISTANCE VIA A-D            *
```

```
C      *                 F      - DISTANCE VALUE IN SUBROUTINES              *
C      *                 H,G,W  - ARRAYS USED BY ZXMIN (SEE IMSL MANUAL)     *
C      *                 IER    - IMSL ERROR MESSAGE                         *
C      *                                                                     *
C      *  I/O FILES:     TAPE5  - INPUT,CONTAINS TRUE PARAMETER VALUES,      *
C      *                          SAMPLE DEVIATES,SAMPLE MEAN AND STD. DEV.  *
C      *                          AND MLE AND MD ESTIMATES                   *
C      *                                                                     *
C      *                 TAPE6  - OUTPUT, CONTAINS TRUE PARAMETER VALUES AND*
C      *                          ESTIMATED PARAMETER VALUES FOR EACH        *
C      *                          MINIMUM DISTANCE METHOD                    *
C      *                                                                     *
C      *  IMPORTANT:     IMSL LIBRARY MUST BE ATTACHED BEFORE RUNNING THE    *
C      *                 PROGRAM. REVIEW IMSL MANUAL ON ZXMIN.               *
C      *                                                                     *
C      ***********************************************************************
       COMMON X(50),NC,N,BMU,BSIG,BXI,BXIC,NN
       EXTERNAL ZXMIN,DCVXI,DA2XI,DKSXI,MLE,MDNOR
       DIMENSION PAR(1),H(1),G(1),W(3)
       INTEGER N,K
       REAL MEAN,KXI,CVMXI,A2XI,MU,SIG,XI,SD
C      *** INITIALIZE DIVERGENCE COUNTERS ***
       NKB=0
       NCB=0
       NAB=0
       NKH=0
       NCH=0
       NAH=0
C      *** INPUT TRUE PARAMETERS ***
       READ(5,100) NREPS,N,TMU,TSIG,TXI
  100  FORMAT(I4/I3/3(F15.6/))
       WRITE(7,106) NREPS
  106  FORMAT(I4)
       WRITE(7,101) N
  101  FORMAT(I3)
       WRITE(7,102) TMU,TSIG,TXI
  102  FORMAT(F15.6/F15.6/F15.6/)
C      ******* BEGIN DO-LOOP FOR NREPS SAMPLES *******
       DO 999 J=1,NREPS
C         *** INPUT SAMPLE INDEX ***
          READ(5,106) K
C         *** INPUT SAMPLE DEVIATES ***
          DO 10 I=1,N
             READ(5,103) X(I)
  103        FORMAT(F15.6)
   10     CONTINUE
          DO 40 I=1,N
   40     CONTINUE
C         *** INPUT ESTIMATE OF XI, SAMPLE MEAN AND STD. DEV.***
          READ(5,104) BMU,BSIG,BXI,HMU,HSIG,HXI,MEAN,SD
  104     FORMAT(F15.6/F15.6/F15.6/F15.6/F15.6/F15.6/F15.6/F15.6)
          BXIC=HXI-1.0
C         *** BEGIN MD ESTIMATION USING INTERPOLATIVE MLE'S ***
```

```
      DO 888 L=1,2
C        *** BEGIN MD ESTIMATION  USING ITERATIVE MLE'S ***
         IF ( L .EQ. 2 ) THEN
         BMU=HMU
         BSIG=HSIG
         BXIC=BXI
         BXI=HXI-1.0
         ENDIF
      WRITE(7,105) BMU,BSIG,BXI
 105  FORMAT(3(F15.6/))
C     *** SET ZXMIN PARAMETERS ***
      NPAR=1
      NSIG=3
      MAXFN=3000
      IOPT=0
C     *** MINIMIZE DISTANCE VIA KOLMOGROV DISTANCE ***
      PAR(1)=BXI
      NP=1
      NC=0
      NN=0
      CALL DKSXI(NP,PAR)
      KXI=PAR(1)
      IF (KXI .LT. X(1)) THEN
      CALL MLE(KXI)
         ELSE
            IF (L .EQ. 1) NKB=NKB+1
            IF (L .EQ. 2) NKH=NKH+1
            WRITE(7,107) HMU,HSIG,HXI
      ENDIF
C     *** MINIMIZE DISTANCE VIA CRAMER VON-MISES ***
      PAR(1)=BXI
      NC=0
      NN=0
      CALL ZXMIN(DCVXI,NPAR,NSIG,MAXFN,IOPT,PAR,H,G,F,W,IER)
      CVMXI=PAR(1)
      IF (CVMXI .LT. X(1)) THEN
         CALL MLE(CVMXI)
            ELSE
               IF (L .EQ. 2) NCH=NCH+1
               IF (L .EQ. 1) NCB=NCB+1
               WRITE(7,107) HMU,HSIG,HXI
 107           FORMAT(3(F15.6/))
      ENDIF
C     *** MINIMIZE DISTANCE VIA ANDERSON-DARLING STATISTIC ***
      PAR(1)=BXI
      NC=0
      NN=0
      CALL ZXMIN(DA2XI,NPAR,NSIG,MAXFN,IOPT,PAR,H,G,F,W,IER)
      A2XI=PAR(1)
      IF (A2XI .LT. X(1)) THEN
         CALL MLE(A2XI)
            ELSE
               IF (L .EQ. 1) NAB=NAB+1
```

```
                  IF (L .EQ. 2) NAH=NAH+1
                  WRITE(7,107) HMU,HSIG,HXI
         ENDIF
         NC=0
 888     CONTINUE
C     *******END DO-LOOP FOR NREPS *******
 999  CONTINUE
      WRITE(7,108) NKB.NCB,NAB,NKH,NCH,NAH
 108  FORMAT(I4,I4,I4,I4,I4,I4)
      STOP
      END
C
C
      SUBROUTINE MLE(XI)
C
C     ******************************************************************
C     *  PURPOSE:    CALCULATE THE MLE OF MU AND SIG GIVEN THE VALUE    *
C     *              OF THE LOCATION PARAMETER (XI)                     *
C     *                                                                 *
C     *  VARIABLES:  N   - SAMPLE SIZE                                  *
C     *              X   - ARRAY OF 3-LN DEVIATES                       *
C     *              MU  - MLE ESTIMATE OF PARENT NORMAL MEAN           *
C     *              SIG - MLE ESTIMATE OF PARENT NORMAL ST. DEV.       *
C     *              XI  - ESTIMATE OF LOCATION PARAMETER (INPUT)       *
C     *              SUM - DUMMY VARIABLE USED FOR SUMS                 *
C     *                                                                 *
C     ******************************************************************
C
      COMMON X(50),NC,N,BMU,BSIG,BXI,BXIC,NN
      REAL XI,MU,SIG,SUM,Y,Z
      INTEGER N
C     *** INITIALIZE PARAMETERS ***
      SUM=0.0
      MU=0.0
      SIG=0.0
C     *** CALCULATE THE MLE OF MU ***
      DO 10 I=1,N
         SUM=SUM+LOG(X(I)-XI)
  10  CONTINUE
      MU=SUM/N
C     *** CALCULATE THE MLE OF SIG ***
      SUM=0.0
      DO 20 I=1,N
         SUM = ( LOG(X(I)-XI) - MU )**2 + SUM
  20  CONTINUE
      SIG=SQRT(SUM/N)
      WRITE(7,102) MU,SIG,XI
 102  FORMAT(3(F15.6/))
      RETURN
      END
C
C
      SUBROUTINE DKSXI(NP,PAR)
```

```
C     ***********************************************************************
C     *                                                                     *
C     *  PURPOSE:     FIND DISTANCE BETWEEN ESTIMATED CDF AND 1/N           *
C     *               STEP FUNCTION FOR XI VIA KOLMOGROV DISTANCE           *
C     *                                                                     *
C     *  VARIABLES:   NP     - NUMBER OF PARAMETERS (ALWAYS 1)              *
C     *               PAR    - ARRAY OF PARAMETER VALUES                    *
C     *               PLUS   - DUMMY VARIABLE FOR D+                        *
C     *               MINUS  - DUMMY VARIABLE FOR D-                        *
C     *               DMAX   - VALUE OF D+ STATISTIC                        *
C     *               DMIN   - VALUE OF D- STATISTIC                        *
C     *               COUNT  - LOCATION OF POINT WHERE MIN[MAX DISTANCE] *
C     *               M      - # OF DEVIATES REMAINING AFTER NC CENSORED *
C     *                        FROM BELOW                                   *
C     *               Z      - PERCENTILE POINT FROM STD NORMAL CDF COR- *
C     *                        RESPONDING TO THE 3-LN CDF PERCENTILE PT. *
C     *                                                                     *
C     *  NOTE:        UNDEFINED VARIABLES ARE DEFINED IN THE MAIN PGM      *
C     *                                                                     *
C     ***********************************************************************
      COMMON X(50),NC,N,BMU,BSIG,BXI,BXIC,NN
      INTEGER NP,COUNT,M
      REAL PAR(NP),MAXI,PLUS,MINUS,DMAX,Y,Z
C     *** USE FIRST ORDER STATISTIC IF ESTIMATED VALUE OF XI IS ***
C     *** CLOSE TO THE FIRST ORDER STATISTIC ***
      IF ((X(1)-PAR(1)) .LT. 0.001) THEN
         NC=1
         DO 3 I=2,N
            IF((X(I)-X(1)) .LT. 0.001) NC=NC+1
  3      CONTINUE
         PAR(1)=X(NC)
      ENDIF
      MAXI=99999.
      M=N-NC
C     *** CALCULATE KOLMOGROV STATISTICS ***
  2   DO 5 J=1,400
         XII=PAR(1)+(-2.0+.01*J)
         IF (X(1+NC) .LT. XII + 0.001) GOTO 5
         DO 10 I=1,M
           Y=(LOG(X(NC+I)-XII)-BMU)/BSIG
           CALL MDNOR(Y,Z)
           PLUS=ABS( I/M - Z)
           MINUS=ABS( Z-(I-1)/M)
           DMAX=MAX(PLUS,MINUS)
           IF (DMAX .LT. MAXI) THEN
              MAXI=DMAX
              COUNT=J
           ENDIF
 10     CONTINUE
  5   CONTINUE
C     *** USE NEW INITIAL ESTIMATE FOR XI IF AT BOUNDARY ***
      IF (COUNT .EQ. 400 .AND. NN .EQ. 0) THEN
```

```
         PAR(1)=BXIC
         NN=1
         GO TO 2
      ENDIF
C     *** USE NEW INITIAL ESTIMATE FOR XI IF AT BOUNDARY ***
      IF (COUNT .EQ. 1 .AND. NN .EQ. 0) THEN
         PAR(1)=BXIC
         NN=1
         GO TO 2
      ENDIF
      PAR(1)=BXI+(-2.0+.01*COUNT)
      RETURN
      END
C
C
      SUBROUTINE DCVXI(NP,PAR,F)
C
C     ***************************************************************
C     *                                                             *
C     *  PURPOSE:     FIND DISTANCE BETWEEN ESTIMATED CDF AND 1/N   *
C     *               EDF FOR THE LOCATION PARAMETER, XI, VIA CRAMER *
C     *               VON-MISES DISTANCE                            *
C     *                                                             *
C     *  VARIABLES:   NP  - NUMBER OF PARAMETERS (ALWAYS 1)         *
C     *               PAR - ARRAY OF PARAMETER VALUES               *
C     *               SUM - DUMMY VARIABLE USED FOR SUMS            *
C     *               F   - DISTANCE VALUE AT THIS XI               *
C     *               Z   - PERCENTILE PT FROM STD NORMAL CORRESPOND- *
C     *                     ING TO THE 3-LN CDF PERCENTILE PT.      *
C     *                                                             *
C     *  NOTE:        UNDEFINED VARIABLES IN THIS SUBROUTINE ARE    *
C     *               DEFINED IN THE MAIN PROGRAM                   *
C     *                                                             *
C     ***************************************************************
C
      COMMON X(50),NC,N,BMU,BSIG,BXI,BXIC,NN
      INTEGER NP
      REAL PAR(NP),F,SUM,Y,Z
C     *** USE FIRST ORDER STATISTIC IF ESTIMATE OF XI LIES ***
C     *** CLOSE TO THE FIRST ORDER STATISTIC ***
      IF ((X(1)-PAR(1)) .LT. 0.001 .AND. NN .EQ. 0) THEN
         NN=1
         PAR(1)=BXIC
      ENDIF
C     *** IF MD ESTIMATE DIVERGED FOR XI USE A NEW INITIAL ESTIMATE ***
      IF ((X(1)-PAR(1)) .LT. 0.001 .AND. NN .EQ. 1) THEN
         NC=1
         DO 3 I=2,N
            IF((X(I)-X(1)) .LT.0.001) NC=NC+1
 3       CONTINUE
         PAR(1)=X(NC)
      ENDIF
      IF (NC .GT. 0) PAR(1)=X(NC)
```

```
C     *** CALCULATE CRAMER-VON MISES DISTANCE ***
      SUM=0.0
      M=N-NC
      DO 10 I=1,M
      Y=(LOG(X(I+NC)-PAR(1))-BMU)/BSIG
      CALL MDNOR(Y,Z)
         SUM= (Z - (2*I-1.0)/(2.0*M))**2 +1/12*M + SUM
   10 CONTINUE
C     *** SET F EQUAL TO CRAMER VON MISES DISTANCE ***
      F=SUM
      RETURN
      END
C
C
      SUBROUTINE DA2XI(NP,PAR,F)
C
C     ******************************************************************
C     *                                                                *
C     *  PURPOSE:    FIND DISTANCE BETWEEN ESTIMATED CDF AND 1/N       *
C     *              EDF FOR LOCATION PARAMETER, XI, VIA ANDERSON-     *
C     *              DARLING DISTANCE                                  *
C     *                                                                *
C     *  VARIABLES:  NP  - NUMBER OF PARAMETERS (ALWAYS 1)             *
C     *              PAR - ARRAY OF PARAMETER VALUES FOR XI            *
C     *              SUM - DUMMY VARIABLE USED FOR SUMS                *
C     *              F   - DISTANCE AT THIS XI                         *
C     *              Y1  - VALUE FOR A-D STATISTIC                     *
C     *              Y2  - VALUE FOR A-D STATISTIC                     *
C     *              Z1  - PERCENTILE PT FROM STD NORMAL CORRESPONDING *
C     *                    TO Y1                                       *
C     *              Z2  - PERCENTILE PT FROM STD NORMAL CORRESPONDING *
C     *                    TO Y2                                       *
C     *                                                                *
C     *  NOTE:       VARIABLES NOT DEFINED IN THIS SUBROUTINE ARE      *
C     *              DEFINED IN THE MAIN PROGRAM                       *
C     *                                                                *
C     ******************************************************************
C
      COMMON X(50),NC,N,BMU,BSIG,BXI,BXIC,NN
      INTEGER NP
      REAL PAR(NP),F,SUM,Y1,Y2,Z1,Z2
C     *** USE FIRST ORDER STATISTICS IF ESTIMATE OF XI LIES ***
C     *** CLOSE TO THE FIRST ORDER STATISTIC ***
      IF ((X(1)-PAR(1)) .LT. 0.001 .AND. NN .EQ. 0) THEN
         NN=1
         PAR(1)=BXIC
      ENDIF
C     *** IF FIRST INITIAL ESTIMATE DIVERGES USE A NEW INITIAL ESTIMATE
      IF((X(1)-PAR(1)) .LT. 0.001 .AND. NN .EQ. 1) THEN
         NC=1
         DO 3 I=2,N
            IF ((X(I)-X(1)) .LT. 0.001) NC=NC+1
    3    CONTINUE
```

```
         PAR(1)=X(NC)
      ENDIF
      IF (NC .GT. 0) PAR(1)=X(NC)
C     *** CALCULATE ANDERSON-DARLING STATISTICS ***
      SUM=0.0
      M=N-NC
      DO 10 I=1,M
      Y1=(LOG(X(I+NC)-PAR(1))-BMU)/BSIG
      CALL MDNOR(Y1,Z1)
      Y2=(LOG(X(N+1-I)-PAR(1))-BMU)/BSIG
      CALL MDNOR(Y2,Z2)
      SUM=(2*I-1)*(LOG(Z1)+LOG(1-Z2)) + SUM
   10 CONTINUE
      SUM=(-1*SUM)/M-M
C     *** SET F EQUAL TO ANDERSON-DARLING DISTANCE ***
      F=SUM
      RETURN
      END
*EOR
..
```

Appendix D

Computer Listing of Evaluation Criteria

```
      PROGRAM EVAL
C     ******************************************************************
C     *                                                                *
C     *  WRITTEN BY 2LT JIM H. KEFFER AFIT/GOR-83D FOR MS THESIS       *
C     *                         DECEMBER 1983                          *
C     *                                                                *
C     *  PURPOSE:    EVALUATE THE MLE AND MD PARAMETER ESTIMATES BY:   *
C     *                 (1) MEAN SQUARE ERROR (MSE)                    *
C     *                 (2) RELATIVE EFFICIENCY (REFF)                 *
C     *                 (3) CRAMER-VON MISES DISTANCE (CVM)            *
C     *                                                                *
C     *  VARIABLES:    NREPS  - NUMBER OF REPLICATIONS                 *
C     *                N      - SAMPLE SIZE                            *
C     *                SUM    - DUMMY USED FOR SUMS                    *
C     *                SUM1   - DUMMY USED FOR SUMS                    *
C     *                X      - ARRAY CONTINING LAGUERRE ABCISSA       *
C     *                W      - ARRAY CONTAINING LAGUERRE WEIGHTS      *
C     *                CVM    - MATRIX CONTAINING CVM DISTANCE FOR EVERY*
C     *                         SAMPLE AND ESTIMATOR                   *
C     *                MCVM   - ARRAY CONTAINING MEANS OF CVM DISTANCE *
C     *                         FOR EVERY ESTIMATOR TYPE               *
C     *                SDCVM  - ARRAY CONTAINING SD OF CVM DISTANCE FOR *
C     *                         EVERY ESTIMATOR TYPE                   *
C     *                F      - VALUE OF TRUE PDF                      *
C     *                TMU    - TRUE VALUE OF MU                       *
C     *                TSIG   - TRUE VALUE OF SIGMA                    *
C     *                TXI    - TRUE VALUE OF XI                       *
C     *                BM,BS,BX - INTERPOLATIVE MLE'S FOR TMU,TSIG,TXI *
C     *                HM,HS,HX - ITERATIVE MLE'S FOR TMU,TSIG,TXI     *
C     *                B@#    - MSE OF ESTIMATES FROM INTERPOLATIVE MLE *
C     *                         WHERE @= INITIAL OF MD ESTIMATOR       *
C     *                         AND # = M FOR MU; S FOR SIGMA; X FOR XI *
C     *                H@#    - MSSE OF ESTIMATES FROM ITERATIVE MLE'S *
C     *                         WHERE @ = INITIAL OF MD ESTIMATOR      *
C     *                         AND # = M FOR MU; S FOR SIGMA; X FOR XI *
C     *                RB/H@$ - RELATIVE EFFICIENCIES FOR B@# AND H@#  *
C     *                YI     - ADJUSTED LAGUERRE ABCISSA FOR I=0,8    *
C     *                ZI     - PERCENTILE PT FROM STD NORMAL CDF USING *
C     *                         A SET OF PARAMETER ESTIMATES           *
C     *                MDNOR  - IMSL ROUTINE WHICH CALCULATES THE CDF  *
C     *                         OF A STD NORMAL DISTRIBUTION AT PT. X  *
C     *                                                                *
C     *  I/O FILES:    TAPE7 - FORMATTED INPUT OF TRUE PARAMETERS AND  *
C     *                        ESTIMATES                               *
C     *                TAPE8 - FORMATTED OUTPUT OF MSE'S, REFF'S AND   *
C     *                        MCVM,SDCVM                              *
C     *                                                                *
C     *  IMPORTANT:    IMSL LIBRARY MUST BE ATTACHED BEFORE THE PROGRAM *
C     *                IS RUN. REVIEW IMSL CURROUTINE MDNOR            *
C     *                                                                *
```

```
C     ******************************************************************
C
      REAL SUM(8),SUM1(8),CVM(8,1000),MCVM(8),SDCVM(8)
      REAL X(15),W(15)
      DATA BM,BS,BX,BKM,BKS,BKX,BCM,BCS,BCX,BAM,BAS,BAX/12*0.0/
      DATA HM,HS,HX,HKM,HKS,HKX,HCM,HCS,HCX,HAM,HAS,HAX/12*0.0/
      DATA SUM,SUM1/8*0.0,8*0.0/
      DATA(X(K),K=1,15)/0.0933,0.49269,1.21559,2.26994,3.66762,5.42533,
     +7.56591,10.12022,13.13028,16.6540,20.77647,25.62389,31.40751,
     +38.53068,48.02608/
      DATA(W(K),K=1,15)/0.23957,0.5601,0.887,1.22366,1.57444 1.94475,
     +2.3415,2.77404,3.25564,3.80631,4.45847,5.27001,6.3595 8 03178,
     +11.52777/
      PI=3.1415927
C     *** INPUT TRUE PARAMETER ESTIMATES ***
      READ(7,100) NREPS,N,TMU,TSIG,TXI
  100 FORMAT(I4/I3/F15.6/F15.6/F15.6/)
C     ******* BEGIN DO-LOOP FOR NREPS SAMPLES *******
      DO 999 J=1,NREPS
      READ(7,101) BMU,BSIG,BXI,BKMU,BKSIG,BKXI,BCMU,BCSIG,BCXI,
     +BAMU,BASIG,BAXI,HMU,HSIG,HXI,HKMU,HKSIG,HKXI,HCMU,HCSIG,
     +HCXI,HAMU,HASIG,HAXI
  101 FORMAT(F15.6/F15.6/F15.6//F15.6/F15.6/F15.6//F15.6/F15.6/
     +F15.6//F15.6/F15.6/F15.6//F15.6/F15.6/F15.6//F15.6/F15.6/
     +F15.6//F15.6/F15.6/F15.6//F15.6/F15.6/F15.6/)
C     *** CALCULATE THE SQUARED ERRORS FOR ALL PARAMETERS ***
      BM=BM+(TMU-BMU)**2
      BS=BS+(TSIG-BSIG)**2
      BX=BX+(TXI-BXI)**2
      BKM=BKM+(TMU-BKMU)**2
      BKS=BKS+(TSIG-BKSIG)**2
      BKX=BKX+(TXI-BKXI)**2
      BCM=BCM+(TMU-BCMU)**2
      BCS=BCS+(TSIG-BCSIG)**2
      BCX=BCX+(TXI-BCXI)**2
      BAM=BAM+(TMU-BAMU)**2
      BAS=BAS+(TSIG-BASIG)**2
      BAX=BAX+(TXI-BAXI)**2
      HM=HM+(TMU-HMU)**2
      HS=HS+(TSIG-HSIG)**2
      HX=HX+(TXI-HXI)**2
      HKM=HKM+(TMU-HKMU)**2
      HKS=HKS+(TSIG-HKSIG)**2
      HKX=HKX+(TXI-HKXI)**2
      HCM=HCM+(TMU-HCMU)**2
      HCS=HCS+(TSIG-HCSIG)**2
      HCX=HCX+(TXI-HCXI)**2
      HAM=HAM+(TMU-HAMU)**2
      HAS=HAS+(TSIG-HASIG)**2
      HAX=HAX+(TXI-HAXI)**2
C     *** CALCULATE THE CRAMER VON-MISES DISTANCE USING NUMERICAL ***
C     *** QUADRATURE ***
      DO 10 I=1,15
```

```
C         *** ADD LOCATION TO QUADRATURE POINTS ***
          X(I)=X(I)+TXI
C         *** CALCULATE EST CDF FROM INTERPOLATION ***
          IF ( (X(I)-BXI) .LT. 0.00001) THEN
             Z1=0.0
             GOTO 11
          ENDIF
          Y1=(LOG(X(I)-BXI)-BMU)/BXI
          CALL MDNOR(Y1,Z1)
C         *** CALCULATE EST CDF FROM INTERPOLATION & KOL ***
  11      IF ( (X(I)-BKXI) .LT. 0.00001) THEN
             Z2=0.0
             GOTO 12
          ENDIF
          Y2=(LOG(X(I)-BKXI)-BKMU)/BKSIG
          CALL MDNOR(Y2,Z2)
C         *** CALCULATE EST CDF FROM INTERPOLATION & CVM ***
  12      IF ( (X(I)-BCXI) .LT. 0.00001) THEN
             Z3=0.0
             GOTO 13
          ENDIF
          Y3=(LOG(X(I)-BCXI)-BCMU)/BCSIG
          CALL MDNOR(Y3,Z3)
C         *** CALCULATE EST CDF FROM INTERPOLATION AND A-D ***
  13      IF ( (X(I)-BAXI) .LT. 0.00001) THEN
             Z4=0.0
             GOTO 14
          ENDIF
             Y4=(LOG(X(I)-BAXI)-BAMU)/BASIG
          CALL MDNOR(Y4,Z4)
C         *** CALCULATE EST CDF FROM ITERATION ***
  14      IF ( (X(I)-HXI) .LT. 0.00001) THEN
             Z5=0.0
             GOTO 16
          ENDIF
          Y5=(LOG(X(I)-HXI)-HMU)/HSIG
          CALL MDNOR(Y5,Z5)
C         *** CALCULATE EST CDF FROM ITERATION & KOL ***
  16      IF( (X(I)-HKXI) .LT. 0.00001) THEN
             Z6=0.0
             GOTO 17
          ENDIF
          Y6=(LOG(X(I)-HKXI)-HKMU)/HKSIG
          CALL MDNOR(Y6,Z6)
C         *** CALCULATE EST CDF FROM ITERATION & CVM ***
  17      IF ( (X(I)-HCXI) .LT. 0.00001) THEN
             Z7=0.0
             GOTO 18
          ENDIF
          Y7=(LOG(X(I)-HCXI)-HCMU)/HCSIG
          CALL MDNOR(Y7,Z7)
C         *** CALCULATE EST CDF FROM ITERATION & A-D ***
  18      IF ( (X(I)-HAXI) .LT. 0.00001) THEN
```

```
            Z8=0.0
            GOTO 19
         ENDIF
         Y8=(LOG(X(I)-HAXI)-HAMU)/HASIG
         CALL MDNOR(Y8,Z8)
C        *** CALCULATE TRUE CDF ***
   19    Y=(LOG(X(I)-TXI)-TMU)/TSIG
         CALL MDNOR(Y,Z)
C        *** EVALUATE TRUE PDF ***
         F=(1/((X(I)-TXI)*TSIG*SQRT(2*PI)))*EXP((-1.0/2.0)*Y**2)
C        *** ADD TO SUMS FOR EVALUATION OF INTEGRAL ***
         SUM(1)=SUM(1)+W(I)*(Z1-Z)**2*F
         SUM(2)=SUM(2)+W(I)*(Z2-Z)**2*F
         SUM(3)=SUM(3)+W(I)*(Z3-Z)**2*F
         SUM(4)=SUM(4)+W(I)*(Z4-Z)**2*F
         SUM(5)=SUM(5)+W(I)*(Z5-Z)**2*F
         SUM(6)=SUM(6)+W(I)*(Z6-Z)**2*F
         SUM(7)=SUM(7)+W(I)*(Z7-Z)**2*F
         SUM(8)=SUM(8)+W(I)*(Z8-Z)**2*F
   10 CONTINUE
C     *** CALCULATE CVM STATISTIC FOR EACH METHOD ***
      DO 8 I=1,8
         CVM(I,J)=N*SUM(I)
    8 CONTINUE
C     *** ADD TO SUMS FOR CVM ***
      DO 9 I=1,8
         SUM1(I)=SUM1(I)+CVM(I,J)
    9 CONTINUE
C     ******* END DO-LOOP FOR NREPS SAMPLES *******
  999 CONTINUE
C     *** CALCULATE MSE'S FOR EACH PARAMETER ***
      BM=BM/NREPS
      BS=BS/NREPS
      BX=BX/NREPS
      BKM=BKM/NREPS
      BKS=BKS/NREPS
      BKX=BKX/NREPS
      BCM=BCM/NREPS
      BCS=BCS/NREPS
      BCX=BCX/NREPS
      BAM=BAM/NREPS
      BAS=BAS/NREPS
      BAX=BAX/NREPS
      HM=HM/NREPS
      HS=HS/NREPS
      HX=HX/NREPS
      HKM=HKM/NREPS
      HKS=HKS/NREPS
      HKX=HKX/NREPS
      HCM=HCM/NREPS
      HCS=HCS/NREPS
      HCX=HCX/NREPS
      HAM=HAM/NREPS
```

```
      HAS=HAS/NREPS
      HAX=HAX/NREPS
C     *** CALCULATE THE RELATIVE EFFICIENCIES ***
      RBM=BM/BM
      RBS=BS/BS
      RBX=BX/BX
      RBKM=BM/BKM
      RBKS=BS/BKS
      RBKX=BX/BKX
      RBCM=BM/BCM
      RBCS=BS/BCS
      RBCX=BX/BCX
      RBAM=BM/BAM
      RBAS=BS/BAS
      RBAX=BX/BAX
      RHM=BM/HM
      RHS=BS/HS
      RHX=BX/HX
      RHKM=BM/HKM
      RHKS=BS/HKS
      RHKX=BX/HKX
      RHCM=BM/HCM
      RHCS=BS/HCS
      RHCX=BX/HCX
      RHAM=BM/HAM
      RHAS=BS/HAS
      RHAX=BX/HAX
C     *** CALCULATE MEAN CVM ***
      DO 35 I=1,8
         MCVM(I)=SUM1(I)/NREPS
   35 CONTINUE
C     *** CALCULATE THE STD DEV OF CVM STATISTICS ***
      DO 37 I=1,8
         SUM(I)=0.0
   37 CONTINUE
      DO 45 I=1,8
         DO 46 L=1,NREPS
            SUM(I)=SUM(I)+(CVM(I,L)-MCVM(I))**2
   46    CONTINUE
   45 CONTINUE
      DO 55 I=1,8
         SDCVM(I)=(SUM(I)/NREPS)**0.5
   55 CONTINUE
C     *** READ IN THE DIVERGENCE COUNTERS FOR THE MD METHODS ***
      READ(7,102) NKB,NCB,NAB,NKH,NCH,NAH
  102 FORMAT(I4,I4,I4,I4,I4,I4)
C     *** WRITE ALL RESULTS TO FILE ***
      WRITE(8,'(21X,"SAMPLE SIZE ",I5)') N
      WRITE(8,'(21X,"TRUE MU     ",F5.2)') TMU
      WRITE(8,'(21X,"TRUE SIGMA  ",F5.2)') TSIG
      WRITE(8,'(21X,"TRUE XI     ",F5.2,/)') TXI
      WRITE(8,'(12X,"MU",13X,"SIG",14X,"XI",5X,"DIV",/)')
      WRITE(8,'(20X,"MEAN SQUARE ERRORS",/)')
```

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      WRITE(8,'("INT :",F11.6,F16.6,F16.6)') BM,BS,BX
      WRITE(8,'("KOL :",F11.6,F16.6,F16.6,I6)') BKM,BKS,BKX,NKB
      WRITE(8,'("CVM :",F11.6,F16.6,F16.6,I6)') BCM,BCS,BCX,NCB
      WRITE(8,'("A-D :",F11.6,F16.6,F16.6,I6,/)') BAM,BAS,BAX,NAB
      WRITE(8,'("ITR :",F11.6,F16.6,F16.6)') HM,HS,HX
      WRITE(8,'("KOL :",F11.6,F16.6,F16.6,I6)') HKM,HKS,HKX,NKH
      WRITE(8,'("CVM :",F11.6,F16.6,F16.6,I6)') HCM,HCS,HCX,NCH
      WRITE(8,'("A-D :",F11.6,F16.6,F16.6,I6,/)') HAM,HAS,HAX,NAH
      WRITE(8,'(18X,"RELATIVE EFFICIENCIES",/)')
      WRITE(8,'("INT :",F11.6,F16.6,F16.6)') RBM,RBS,RBX
      WRITE(8,'("KOL :",F11.6,F16.6,F16.6)') RBKM,RBKS,RBKX
      WRITE(8,'("CVM :",F11.6,F16.6,F16.6)') RBCM,RBCS,RBCX
      WRITE(8,'("A-D :",F11.6,F16.6,F16.6,/)') RBAM,RBAS,RBAX
      WRITE(8,'("ITR :",F11.6,F16.6,F16.6)') RHM,RHS,RHX
      WRITE(8,'("KOL :",F11.6,F16.6,F16.6)') RHKM,RHKS,RHKX
      WRITE(8,'("CVM :",F11.6,F16.6,F16.6)') RHCM,RHCS,RHCX
      WRITE(8,'("A-D :",F11.6,F16.6,F16.6,/)') RHAM,RHAS,RHAX
      WRITE(8,'(21X,"CVM STATISTICS",/)')
      WRITE(8,'(18X,"MCVM",11X,"SDCVM",/)')
      WRITE(8,'("INT :",2X,F16.6,F16.6)') MCVM(1),SDCVM(1)
      WRITE(8,'("KOL :",2X,F16.6,F16.6)') MCVM(2),SDCVM(2)
      WRITE(8,'("CVM :",2X,F16.6,F16.6)') MCVM(3),SDCVM(3)
      WRITE(8,'("A-D :",2X,F16.6,F16.6,/)') MCVM(4),SDCVM(4)
      WRITE(8,'("ITR :",2X,F16.6,F16.6)') MCVM(5),SDCVM(5)
      WRITE(8,'("KOL :",2X,F16.6,F16.6)') MCVM(6),SDCVM(6)
      WRITE(8,'("CVM :",2X,F16.6,F16.6)') MCVM(7),SDCVM(7)
      WRITE(8,'("A-D :",2X,F16.6,F16.6)') MCVM(8),SDCVM(8)
      STOP
      END
*EOR
..
```

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This thesis compares two modified maximum likelihood (ML) estimation techniques against three minimum distance (MD) estimation techniques in application to the three parameter lognormal distribution. The three parameter lognormal distribution has a location parameter (ξ) and two other parameters associated with the mean (μ) and standard deviation (σ) of its parent normal population. The first modified ML technique uses linear interpolation on order statistics to estimate location while the second ML technique uses the first order statistic as the location estimate. The remaining two parameters are calculated by using the location estimate in their respective censored or uncensored ML equations and solving for the parameters. Three MD techniques are used: Kolmogrov Distance, Cramer-von Mises Statistic, and the Anderson-Darling Statistic. The MD techniques refine the location estimates which are then used in the ML equations of the other two parameters to obtain their refined estimates.

Monte Carlo analysis is used to accomplish the comparison of estimation techniques. Sample sizes of 6, 8, 10, 12, and 16 are generated using three parameter sets $(\mu,\sigma,\xi) \doteq (0.0, 1.0, 10.0)$, $(1.0, 1.0, 10.0)$, and $(1.0, 2.0, 10.0)$. Each estimation technique is applied to one-thousand replications for every combination of sample size and parameter set. Three measures of effectiveness are used to facilitate comparisons: mean square error, relative efficiency, and the Cramer-von Mises Statistic. Comparisons of these effectiveness measures across all cases reveal a clear superiority of the MD techniques over the modified ML techniques.

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