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# On the Estimation of Spectral Parameters Using Burst Waveforms

L.M. Novak

14 December 1983

**Lincoln Laboratory**
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
*LEXINGTON, MASSACHUSETTS*

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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# ON THE ESTIMATION OF SPECTRAL PARAMETERS USING BURST WAVEFORMS

*L.M. NOVAK*

*Group 32*

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## ABSTRACT

This report addresses the problem of estimating the spectral parameters of an observed doppler velocity spectrum using a burst radar waveform of arbitrary length. Maximum-likelihood theory is applied and the exact M.L.E. algorithm for estimating the spectral mean is derived. This M.L.E. algorithm is shown to include, as a special case, the spectral mean estimator originally proposed by R. W. Miller [1] for processing of burst waveforms. Also, when the burst waveform is a simple pulse pair, the M.L.E. algorithm reduces to the spectral mean estimator originally proposed by W. D. Rummler [2]. The Cramer-Rao bound for estimating the spectral mean using burst waveforms is also derived. Simplifications to the exact maximum-likelihood algorithm are proposed and the performance of various estimators is compared to the Cramer-Rao lower bound. Some preliminary results of studies of spectral width estimators are also presented.

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# I. INTRODUCTION

Estimation of the spectral moments of observed signal waveforms is currently an important topic in many fields of research. For example, the analysis of pulse doppler radar or sonar echoes from distributed scatterer targets is an important area of interest. Early work in this area was done by Rummler [2] who proposed estimators of the spectral mean and variance from pulse pair radar waveforms. His algorithms were based on covariance argument techniques and are generally referred to as time-domain processing. Rummler's proposed algorithms were subsequently shown to be maximum-likelihood estimators of spectral mean and variance (see K. Miller and M. Rochwarger[3]). Processing of independent pulse pairs requires a long radar observation time, and thus, it was suggested by Berger and Groginsky [4] that time-domain estimation of spectral moments from adjacent pulse pairs of a radar burst waveform is the most direct and efficient technique to implement on conventional radars. Since the explicit solution to the maximum-likelihood equation was not available, processing of the adjacent pulses of the contiguous-pulse burst waveform was done simply using Rummler's proposed time-domain algorithms. Estimates from contiguous pulse pairs with a common pulse between each two pairs were analyzed in Benham [5] where the performance accuracy (i.e., standard deviation of the spectral mean and variance estimators) was determined and compared with independent pulse pair processing. More recent studies of spectral moment estimators using contiguous correlated pulse pairs are presented in Zrnic [6,7]. These references provide an excellent survey of the topic of spectral moment estimation using time-domain processing and present a fairly complete mathematical approach for determining the theoretical accuracy of covariance argument estimators.

It was recognized by R. Lee et al. [8] and by R. Srivastava et. al. [9] that useable spectral moment estimates could be obtained by processing lags other than the first-order lag utilized by Rummler and Berger and Groginsky. Also, improved estimates could be obtained by averaging the individual moment estimates obtained at various lags. Lee proposed a poly-pulse pair processing scheme for combining estimates obtained at various lags and demonstrated that by utilizing higher order lags to obtain spectral estimates, performance of the estimators could be extended to lower signal-to-noise ratio situations.

Up to this point we have discussed spectral moment estimation using strictly time-domain signal processing techniques. An alternative method of estimating the spectral mean and variance utilizing autocorrelation estimates at all available lags combined with frequency domain processing was proposed by R. W. Miller [1]. An efficient implementation of Miller's approach utilizes an FFT algorithm and an appropriately selected window function. In this alternative approach, the power spectral density of the observed random process is computed from the sample autocorrelation function of the observed data and the spectral mean is then estimated to be the location of the maximum peak of the computed power spectrum.

In Section II of the report, maximum-likelihood theory is applied to the problem of spectral moment estimation using burst waveforms. The exact M.L.E. algorithm for estimating the spectral mean is derived. This optimal estimator utilizes autocorrelation estimates at all available lags and may be implemented either as a time-domain or frequency-domain algorithm. For contiguous-pulse burst waveforms the time-domain implementation of this optimal M.L.E. estimator is shown to be a generalization of Rummler's algorithm. When the observed data is comprised of independent pulse pairs the time-domain estimator reduces to the spectral mean estimator originally proposed by Rummler [2]. The frequency-domain implementation of the M.L.E. estimator is also shown to be a generalization of Miller's algorithm [1]. In either case (time-domain or frequency-domain processing) the M.L.E. estimator utilizes an optimally weighted combination of the autocorrelation estimates at all possible lags.

In Section III of the report we present an interesting analytical solution for the Cramer-Rao lower bound for estimation of the spectral mean from pulse burst data.

Section IV of the report presents the results of a comparison of burst waveform processing versus pulse-pair processing using Rummler time-domain processing. Also, practical simplifications to the M.L.E. estimator are presented and studied, both analytically and through the use of a Monte-Carlo simulation. Finally, preliminary studies of several algorithms for estimating the spectral spread are presented.

Section V provides a brief summary of the report and suggests the need for development of a more accurate algorithm for estimating the spectral spread parameter.

## II. MAXIMUM LIKELIHOOD ESTIMATION OF THE SPECTRAL MEAN

The return signal is assumed to be a discrete-time sequence of (complex) radar video samples spaced at the radar pulse interval, T. The return signal is comprised of a time-wise correlated narrow band signal component and a white noise component, i.e.,

$$z_k = s_k e^{j\omega kT} + n_k; \quad k=0,1,\ldots L-1 \tag{1}$$

Both $s_k$ and $n_k$ are zero mean Gaussian processes centered at zero frequency. The true signal power is denoted by S, while the correlation coefficient, $\beta(kT)$, depends upon the time spacing between samples of the sequence as well as the spectral width of the signal component. Thus, the autocorrelation function of the process $z_k$ is

$$R(kT) = S\beta(kT)e^{j\omega kT} + N\delta_k \tag{2}$$

where $R(kT)=E\{z_i^* z_{i+k}\}$ and N is defined as the white noise power per sample. Also, the term $\delta_k$ is defined as follows

$$\begin{aligned} \delta_k &= 1 \qquad k = 0 \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{3}$$

The signal power spectrum is assumed to be symmetric about the spectral mean "$\omega$" and is Gaussian in shape. Thus, the correlation coefficient, $\beta(kT)$, is also Gaussian and of the form

$$\beta(T) = e^{-2\pi^2 W^2 T^2} \tag{4}$$

where the spectral width is denoted "W" in the above expression.

In this section of the report we shall focus mainly on estimation of the spectral mean of the signal component and shall derive the maximum-likelihood estimate of "$\omega$" based upon the sequence $\{z_0, z_1, z_2, \ldots, z_{L-1}\}$. A brief summary of the mathematical solution to the proposed estimation problem is given in the following paragraphs.

Given the sequence of radar measurements

$$\begin{aligned} z_0 &= s_0\, e^{j\omega(0)T} + n_0 \\ z_1 &= s_1\, e^{j\omega(1)T} + n_1 \\ &\vdots \\ z_{L-1} &= s_{L-1}\, e^{j\omega(L-1)T} + n_{L-1} \end{aligned} \tag{5}$$

we write this data set in matrix form

$$\underline{z} = D\underline{s} + \underline{n} \tag{6}$$

where the diagonal matrix, D, is given by the following

$$\begin{bmatrix} e^{j\omega(0)T} & & & & \\ & e^{j\omega(1)T} & & & \\ & & e^{j\omega(2)T} & & \\ & & & \ddots & \\ & & & & e^{j\omega(L-1)T} \end{bmatrix} \tag{7}$$

Since all the variables are assumed to be Gaussian, we consider the covariance matrix of the measurement vector, $\underline{z}$ which may be expressed as follows:

$$E\{\underline{z}\ \underline{z}^{*'}\} = D(C + NI)D^{*} \tag{8}$$

In the above expression, $C = E\{\underline{s}\ \underline{s}^{*'}\}$ is the covariance matrix of the signal vector, $\underline{s}$. Denoting the logarithm of the (complex) Gaussian probability density function as "$\ln f(\underline{z})$", we have

$$\ln f(\underline{z}) = -\underline{z}^{*'}D(C+NI)^{-1}D^{*}\ \underline{z} - \ln|C+NI| - L \ln 2\pi \tag{9}$$

The maximum likelihood estimate of parameter "$\omega$" is obtained by taking

$$\frac{\partial}{\partial\omega}\{\ln f(\underline{z})\} = 0 \tag{10}$$

Since the term $|C+NI|$ is independent of the parameter "$\omega$", we need only consider the first term in the above log-likelihood expression. For purposes of simplicity, we shall present the maximum-likelihood solution for bursts of length L=2, 3, and 4. The general solution for an arbitrary number of pulses will then be presented.

In the following derivation we shall make use of the elements of the inverse convariance matrix $(C+NI)^{-1}$ which is written

$$(C+NI)^{-1} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \cdots\cdots \gamma_{1L} \\ \gamma_{12} & \gamma_{22} \cdots\cdots \vdots \\ \gamma_{1L} \cdots\cdots & \cdots\cdots \gamma_{LL} \end{bmatrix} \tag{11}$$

For the burst of length 2, we obtain

$$\frac{\partial}{\partial\omega}\{\ln f(\underline{z})\} = -[(-jT)\gamma_{12}\, z_0^*\, z_1\, e^{-j\omega T} + (jT)\gamma_{12}\, z_0 z_1^*\, e^{j\omega T}] \tag{12}$$

Setting the above equation to zero yields the expression which the maximum-likelihood estimate $\hat{\omega}$ must satisfy. When this is done, we obtain

$$z_0^*\, z_1\, e^{-j\hat{\omega}T} = z_0\, z_1^*\, e^{j\hat{\omega}T} \tag{13}$$

Clearly the above expression is equivalent to the Rummler algorithm for estimating the mean of the power spectral density function, i.e.,

$$\hat{\omega} = \frac{1}{T}\,\mathrm{ARG}\{\hat{R}(T)\} \tag{14}$$

where $\hat{R}(T) = z_0^* z_1$ is the unbiased M.L.E. estimate of the complex autocorrelation, $R(T)$.

Similarily, for a burst of length 3, given data samples $\{z_0, z_1, z_2\}$, we obtain

$$\frac{\partial}{\partial \omega}\{\ln f(\underline{z})\} = -\Big[(-jT)\gamma_{12}z_0^*z_1 e^{-j\omega T} + (-jT)\gamma_{12}z_1^*z_2 e^{-j\omega T}$$

$$+ (-j2T)\gamma_{13}z_0^*z_2 e^{-j2\omega T} + (jT)\gamma_{12}z_0 z_1^* e^{j\omega T} \tag{15}$$

$$+ (jT)\gamma_{12}z_1 z_2^* e^{j\omega T} + (j2T)\gamma_{13}z_0 z_2^* e^{j2\omega T}\Big]$$

Setting the above equation to zero yields the following expression:

$$\Big[z_0^*z_1 + z_1^*z_2 + \frac{2\gamma_{13}}{\gamma_{12}} z_0^*z_2 \, e^{-j\widehat{\omega T}}\Big]e^{-j\widehat{\omega T}}$$

$$= \Big[z_0 z_1^* + z_1 z_2^* + \frac{2\gamma_{13}}{\gamma_{12}} z_0 z_2^* e^{+j\widehat{\omega T}}\Big]e^{j\widehat{\omega T}} \tag{16}$$

which may be written as follows

$$\Big[\widehat{R}(T) + \frac{\gamma_{13}}{\gamma_{12}} \widehat{R}(2T)e^{-j\widehat{\omega T}}\Big]e^{-j\widehat{\omega T}}$$

$$= \Big[\widehat{R}^*(T) + \frac{\gamma_{13}}{\gamma_{12}} \widehat{R}^*(2T)e^{j\widehat{\omega T}}\Big]e^{j\widehat{\omega T}} \tag{17}$$

In the above equation, we have the definitions

$$\widehat{R}(T) = \frac{1}{2}(z_0^*z_1 + z_1^*z_2) \tag{18}$$

and $$\widehat{R}(2T) = (z_0^*z_2) \tag{19}$$

From the above expressions, it is clear that the maximum-likelihood estimate of parameter "$\omega$" is obtained from the algorithm

$$\hat{\omega} = \frac{1}{T} \text{ ARG} \left\{ \hat{R}(T) + \frac{\gamma_{13}}{\gamma_{12}} \hat{R}(2T) e^{-j\hat{\omega}T} \right\} \tag{20}$$

Since the above solution for $\hat{\omega}$ is implicit, one iterative method of solving the expression is suggested as follows. Given the data samples $\{z_0, z_1, z_2\}$, obtain an initial approximation of $\hat{\omega}$ from $\hat{R}(T)$, i.e.,let

$$\hat{\omega}_1 = \frac{1}{T} \text{ ARG} \{\hat{R}(T)\} \tag{21}$$

The improved estimate of $\omega$ based upon $\hat{R}(T)$ and $\hat{R}(2T)$ is then obtained iteratively using the expression

$$\hat{\omega}_{i+1} = \frac{1}{T} \text{ ARG} \left\{ \hat{R}(T) + \frac{\gamma_{13}}{\gamma_{12}} \hat{R}(2T) e^{-j\hat{\omega}_i T} \right\} \tag{22}$$

For a burst of length 4, given data samples $\{z_0, z_1, z_2, z_3\}$ we obtain the M.L.E. estimate

$$\hat{\omega} = \frac{1}{T} \text{ ARG} \left\{ \hat{R}(T) + \frac{4\gamma_{13}}{3\gamma_{12}} \hat{R}(2T) e^{-j\hat{\omega}T} + \frac{\gamma_{14}}{\gamma_{13}} \hat{R}(3T) e^{-j2\hat{\omega}T} \right\} \tag{23}$$

where we have defined

$$\hat{R}(T) = \frac{1}{3}\left(z_0^* z_1 + \frac{\gamma_{23}}{\gamma_{12}} z_1^* z_2 + z_2^* z_3\right) \tag{24}$$

$$\hat{R}(2T) = \frac{1}{2}\left(z_0^* z_2 + z_1^* z_3\right) \tag{25}$$

$$\hat{R}(3T) = \left(z_0^* z_3\right) \tag{26}$$

We note from the above equations that the estimate $\hat{R}(T)$ is biased, since in general, $\gamma_{23} \neq \gamma_{12}$. In a real-time implementation of the M.L.E. estimator one might use an unbiased estimate of R(T) in the solution.

Finally we present the solution for the estimate, $\hat{\omega}$, for an arbitrary burst length. Omitting the details, the solution is given by the expression

$$\hat{\omega} = \frac{1}{T} \text{ ARG } \left\{ \sum_{k=1}^{L-1} C_k \hat{R}(kT) e^{-j\hat{\omega}(k-1)T} \right\} \tag{27}$$

where the weighting coefficients, $C_k$, are defined in terms of the elements, $\gamma_{ij}$, of the inverse covariance matrix, $(C+NI)^{-1}$. We note that, in general, the M.L.E. estimator utilizes estimated autocorrelations $\hat{R}(T)$, $\hat{R}(2T)$,...., $\hat{R}(L-1)T$ at all available lags to determine "$\omega$", since, in general, the coefficients $\gamma_{ij} \neq 0$. The estimator is implemented as a time-domain algorithm and it is seen (by comparing Equations (14) and 27)) to be a generalization of the spectral mean estimator proposed by Rummler[2]. This estimator, however, is implicit in structure and an iterative scheme must be utilized to find the $\hat{\omega}$ which satisfies Equation (27). Thus, we are interested in determining an explicit method for obtaining the estimate $\hat{\omega}$. In the following paragraphs, we develop a frequency-domain implementation of the estimator which is shown to provide an explicit solution for $\hat{\omega}$. This alternative implementation is a generalization of the spectral mean estimator proposed by Miller [1]. From the estimates of the autocorrelation function $\{\hat{R}(0), \hat{R}(T), \ldots, \hat{R}(L-1)T\}$ we compute an estimated power spectral density, $S(\omega)$, as follows

$$S(\omega) = \sum_{k=-(L-1)}^{(L-1)} A_k \hat{R}(kT) e^{j\omega kT} \tag{28}$$

where the coefficients $(A_k)$ comprise an arbitrary set of weights or equivalently a window function. Since $\hat{R}(-kT)=\hat{R}^*(kT)$ we may write

$$S(\omega) = \sum_{k=1}^{(L-1)} A_{-k} \hat{R}^*(kT) e^{-j\omega kT} + A_0 \hat{R}(0) + \sum_{k=1}^{(L-1)} A_k \hat{R}(kT) e^{j\omega kT} \tag{29}$$

Miller [1] took as the estimate of spectral mean, $\omega$, that frequency $\hat{\omega}$ which maximized the function $S(\omega)$. That is, $\hat{\omega}$ is estimated as that frequency corresponding to the peak of the power spectral density. Thus, $\hat{\omega}$ is found by taking

$$\frac{d}{d\omega} S(\omega) \bigg|_{\hat{\omega}} = 0 \tag{30}$$

Performing the indicated differentiation one may easily obtain the result

$$\sum_{k=1}^{L-1} C_k \hat{R}(kT) e^{-j\hat{\omega}(k-1)T} \; e^{-j\hat{\omega}T}$$
$$= \sum_{k=1}^{L-1} C_k \hat{R}(kT) e^{j\hat{\omega}(k-1)T} \; e^{j\hat{\omega}T} \tag{31}$$

where we have assumed $A_k = A_{-k}$ and substituted $C_k = kA_k$ into the above. But the above expression may be shown to be equivalent to the M.L.E. estimate

$$\hat{\omega} = \frac{1}{T} \text{ ARG} \left\{ \sum_{k=1}^{L-1} C_k \hat{R}(kT) e^{-j\hat{\omega}(k-1)T} \right\} \tag{32}$$

Therefore, instead of solving for $\hat{\omega}$ from the implicit formula of Equation (32), we may compute the power spectral density function, $S(\omega)$ of Equation (28) and locate the peak of the function to determine $\hat{\omega}$ explicitly.

## III. CRAMER-RAO BOUND FOR SPECTRAL MEAN ESTIMATES

Next, we wish to determine the Cramer-Rao lower bound on achievable estimation accuracy by determining the Fisher information matrix for the parameter "$\omega$". Again we shall present the general solution by first deriving the results for short bursts of length 2 and 3. The general solution for a burst of length L will then be presented.

For the case of L=2 pulses, we obtain

$$\frac{\partial^2}{\partial \omega^2} \{\ln f(\underline{z})\} = -[(-jT)^2 \gamma_{12} z_0{}^* z_1 e^{-j\omega T} + (jT)^2 \gamma_{12} z_0 z_1{}^* e^{j\omega T}] \quad (33)$$

Taking the "expectation" of the above yields the Fisher information matrix, which is found to be

$$E\left(\frac{-\partial^2}{\partial \omega^2} \{\ln f(z)\}\right) = -2T \quad \gamma_{12} S \ \beta(T) \quad (34)$$

Substituting $\gamma_{12}$ into the above expression, one obtains the Cramer-Rao lower bound for the variance of the estimate $\hat{\omega}$.

$$\sigma_{\hat{\omega}}^2 \geq \frac{1}{2T^2} \left[\frac{(1+N/S)^2}{\beta^2(T)} - 1\right] \quad (35)$$

Incidentally, the above expression is identical to the approximate predicted accuracy of the Rummler pulse pair method for estimating the spectral mean, $\hat{\omega}$ (see Rummler [2]). His derivation involved approximating the nonlinear estimator (Equation (14)) using perturbation theory and computing the variance of the linearized estimate, $\hat{\omega}$.



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Similarly, for a three pulse burst we have

$$\frac{\partial^2}{\partial\omega^2}\{\ln f(\underline{z})\} = -[(-jT)^2\gamma_{12}z_0^*z_1e^{-j\omega T} + (-jT)^2\gamma_{12}z_1^*z_2e^{-j\omega T}$$

$$+ (-j2T)^2\gamma_{13}z_0^*z_2e^{-j\omega T} + (jT)^2\gamma_{12}z_0z_1^*e^{j\omega T} \qquad (36)$$

$$+(jT)^2\gamma_{12}z_1z_2^*\, e^{j\omega T} + (j2T)^2\gamma_{13}z_0z_2^*e^{j2\omega T}]$$

Finally, for the three pulse burst, the Cramer-Rao lower bound becomes

$$\sigma_{\hat{\omega}}^2 \geq \frac{1}{4T^2[\gamma_{12}\ S\beta(T) + 2\gamma_{13}\ S\beta(2T)]} \qquad (37)$$

The Cramer-Rao bound for a pulse burst of length L is similarly determined. The derivation is briefly summarized as follows. For the general L-pulse burst we write the log-likelihood function as follows:

$$
\ln f(\underline{z}) = -(z_0^* z_1^* z_2^* \ldots z_{L-1}^*) \begin{pmatrix} \gamma_{11} & \gamma_{12}e^{-j\omega T} & \gamma_{13}e^{-j2\omega T} & & \gamma_{1L}e^{-j(L-1)\omega T} \\ \gamma_{21}e^{j\omega T} & \gamma_{22} & \gamma_{23}e^{-j\omega T} & & \\ \gamma_{31}e^{j2\omega T} & \gamma_{32}e^{j\omega T} & \gamma_{33} & & \\ & & & & \gamma_{L-1,1}e^{-j\omega T} \\ \gamma_{L1}e^{j(L-1)\omega T} & & & \gamma_{L,L-1}e^{j\omega T} & \gamma_{LL} \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ \\ z_{L-1} \end{pmatrix} + \text{CONST} \qquad (38)
$$

where the remaining terms which are independent of the parameter "$\omega$" are simply denoted as "CONST". Before proceeding with the derivation we make the following remarks. In the above expression, which is of the form

$$
\ln f(\underline{z}) = -\underline{z}^{*'} D(C+NI)^{-1} D^{*'} \underline{z} \quad + \text{CONST} \qquad (39)
$$

we have assumed the matrix inverse to be more generally of the form

$$
(C+NI)^{-1} = (\gamma_{ij}) \qquad (40)
$$

Also, we have combined the matrices D and $D^*$ with the elements of the matrix $(\gamma_{ij})$. Hopefully, this will lead to the desired solution in a clear manner. Computing the required second partial derivative with respect to the parameter "$\omega$" and then factoring out the complex exponentials leads to the result

$$\frac{1}{T^2}\frac{\partial^2}{\partial\omega^2}\ln f(z) = (z_0^*, z_1^* e^{j\omega T}, z_2^* e^{j2\omega T}, \ldots z_{L-1}^* e^{j(L-1)\omega T})$$

$$\begin{pmatrix} 0 & \gamma_{12} & 4\gamma_{13} & 9\gamma_{14} & & \\ \gamma_{21} & 0 & \gamma_{23} & 4\gamma_{24} & 9\gamma_{25} & \\ 4\gamma_{31} & \gamma_{32} & 0 & \gamma_{34} & 4\gamma_{35} & \\ 9\gamma_{41} & 4\gamma_{42} & \gamma_{43} & 0 & & \gamma_{L-1,L} \\ & & 9\gamma_{L,L-3} & 4\gamma_{L,L-2} & \gamma_{L,L-1} & 0 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 e^{-j\omega T} \\ z_2 e^{-j2\omega T} \\ \\ z_{L-1} e^{-j(L-1)\omega T} \end{pmatrix} \tag{41}$$

This expression is then expanded as the quadratic form

$$\frac{1}{T^2}\frac{\partial^2}{\partial\omega^2}\{\ln f(\underline{z})\} = \sum_{m=0}^{L-1}\sum_{n=0}^{L-1}(n-m)^2\gamma_{mn} z_m^* z_n e^{j(m-n)\omega T} \tag{42}$$

Finally, taking the "Expectation" we obtain the result

$$\frac{1}{T^2} E\left(\frac{\partial^2}{\partial\omega^2}\{\ln f(\underline{z})\}\right) = \sum_{m=0}^{L-1}\sum_{\substack{n=0 \\ n\neq m}}^{L-1}(n-m)^2\gamma_{nm} S\beta((n-m)T) \tag{43}$$

where we have used the fact that

$$E\{z_m^* z_n\} = S\,\beta((n-m)T)e^{j(n-m)\omega T}; \qquad n\neq m \tag{44}$$

Summarizing our derivation, we have found the following general solution for the Fisher Information matrix

$$E\left(-\frac{\partial^2}{\partial\omega^2}\{\ln f(\underline{z})\}\right) = -T^2\sum_{i=0}^{L-1}\sum_{j=0}^{L-1}(i-j)^2\gamma_{ij}A_{ij} \tag{45}$$

where we define the LxL matrices

$$(A_{ij}) = (C+NI) \tag{46}$$

$$(\gamma_{ij}) = (C+NI)^{-1} \tag{47}$$

Then the C-R bound is given by the following expression.

$$\sigma^2_{\hat{\omega}} \geq \frac{-1}{T^2\sum_{i=0}^{L-1}\sum_{j=0}^{L-1}(i-j)^2\,\gamma_{ij}\,A_{ij}} \tag{48}$$

## IV. ANALYSIS AND SIMULATION RESULTS

First the results of a comparison of burst waveform processing versus pulse-pair processing using Rummler techniques is presented. For this comparison, the total number of available pulses is considered to be fixed (specifically we take L=16 pulses) and compare estimator performance for a single burst with the estimator performance achieved using 8 independent pulse-pairs (again, processing a total of 16 radar pulses). Figures 1, 2 and 3 show the results of this study. In each of these figures three performance curves are shown. First, the fundamental accuracy achievable with a 16 pulse burst waveform is shown, based upon the C-R bound derived earlier in this report (see Equation 48). Also, the theoretically predicted accuracy achievable by processing the 16 contiguous pulses of the burst using the Rummler time-domain algorithm is shown. Thus, we evaluate the accuracy of the estimator (see Appendix B)

$$\hat{\omega} = \frac{1}{T} \text{ARG}\{\hat{R}(T)\} \tag{49}$$

where

$$\hat{R}(T) = \frac{1}{L-1} \sum_{i=0}^{L-2} z_i^* z_{i+1} \tag{50}$$

Finally, the accuracy achieved by Rummler processing of 8 independent pulse pairs (hence, a total of 16 transmitted radar pulses) is shown. This is considered to be a fair comparison



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in that both approaches utilize the same total number of radar pulses. Thus, both approaches utilize the same total transmitted radar energy. The major difference, of course, is that the 8 independent pulse pairs use a considerably longer radar observation time.

The curves of Figures 1 through 3 show normalized estimation accuracy versus normalized spacing between the pulses in the burst or pulse pair. The normalization used in these curves is described as follows. The assumed signal correlation function is given by the expression

$$\beta(T) = e^{-2\pi^2 (WT)^2} \tag{51}$$

where $W = 2\sigma_v/\lambda$. We have defined the signal process decorrelation time to be that time, $T_0$, required for the signal process to decorrelate to $e^{-1/2} \approx 0.6$ in value. Setting $\beta(T_0) = e^{-1/2}$ we find

$$T_0 = \frac{\lambda}{4\pi\sigma_v} \tag{52}$$

Therefore, in Figures 1, 2 and 3 we have evaluated the normalized estimation accuracy (ST.DEV$\{\hat{V}\}/\sigma_v$) versus normalized time between pulses ($T/T_0$). Note that the mean doppler velocity (m/s) is related to the spectral mean estimate through the formula $\hat{V} = (\lambda/4\pi)\hat{\omega}$. The pulse-pairs are assumed to be sufficiently separated in time so that statistically independent sampling of the signal process from pair to pair is achieved. The figures show performance for three selected signal-to-noise ratios (-6 dB, 0 dB, +10 dB) and are valid for any specific spectral spread. From these curves one may draw the following conclusions.

1. In each case it is clear that burst processing utilizing a total of 15 "lags of 1" available from the 16 contiguous pulses is superior to pulse-pair processing using 8 independent Rummler pulse-pairs. This is especially true at low signal-to-noise ratios. At high signal-to-noise ratios both estimators provide comparable performance, the burst waveform having a small advantage.

2. Comparing the performance of the 16 pulse burst waveform with its corresponding theoretical C-R bound shows clearly that the 16 pulse burst waveform with Rummler time-domain processing of the 15 "lags of 1" products achieves estimation accuracy comparable with the C-R bound except for low signal-to-noise ratios. Thus, it appears that some improvement in estimating the mean velocity is possible for only the low (S/N) ratio case.

3. From the curves shown in Figures 1, 2 and 3 it is also seen that the optimal pulse spacing is given by $T/T_0 \approx 1$, i.e., selecting the pulse spacing $T \approx T_0$ minimizes estimator errors for both pulse pair waveforms and burst waveforms. Thus, to specify an optimal interpulse spacing $T \approx T_0$ one must have knowledge of the true spectral spread, $\sigma_v$, as indicated in Equation (52).

In the remaining paragraphs we restrict our discussion to burst waveform processing and consider the design of optimal burst-forms. The basic objective will be that of achieving good estimator accuracy using a burst waveform having a small interpulse spacing ($T \ll T_0$). Note from the curves of Figures 1 through 3 that excellent performance can be obtained with the burst waveform provided the time between pulses, T, is properly selected. At high signal-to-noise ratios, selection of a good pulse spacing is not difficult since the performance curves are quite flat, i.e., are very insensitive to the parameter T over a wide range of values (see Figure 3). Even with the signal-to-noise ratio as low as a few dB this insensitivity to parameter T is true. Thus, for any reasonable signal-to-noise ratio the spacing between pulses could for example be selected between $0.3T_0$ to $1.5\ T_0$ and excellent performance will be achieved over the equivalent range of the spectral spread parameter. There is no need to improve upon the Rummler burst estimator performance over this range of parameters. Therefore, we next consider the problem of improving performance at low signal-to-noise ratios.

For lower signal-to-noise ratios (say 0 dB) selection of the interpulse spacing begins to become a more important consideration (see Figure 2). For example, selecting, $T=0.3T_0$ one observes that burst waveform performance begins to degrade relative to the estimation accuracy predicted by the C-R bound. For very poor signal-to-noise ratios (say -6 dB) selection of the interpulse spacing begins to become very critical (see Figure 1) and with $T = 0.3T_0$, burst estimator performance becomes very degraded relative to the CR bound. For this reason we have considered the problem of improving burst estimator performance at these very low signal-to-noise ratios.

We remark that in deriving the MLE spectral mean estimator (see Equation 27) it was tacitly assumed that the covariance matrix $(C+NI)^{-1}$ was exactly known a priori or has been accurately estimated from observed data. However, if the coefficients $(\gamma_{ij})$ must be estimated from the data then applying the maximum-liklihood approach to the simultaneous estimation of the parameter set $\{\omega, \gamma_{11}, \gamma_{12}, \ldots, \gamma_{LL}\}$ requires the solution to the following equations

$$\begin{aligned} \frac{\partial}{\partial \omega} \{\ln f(\underline{z})\} &= 0 \\ \frac{\partial}{\partial \gamma_{11}} \{\ln f(\underline{z})\} &= 0 \\ &\vdots \\ \frac{\partial}{\partial \gamma_{LL}} \{\ln f(\underline{z})\} &= 0 \end{aligned} \tag{53}$$

The solution to this set of equations yields the joint maximum liklihood estimates $\{\hat{\omega}, \hat{\gamma}_{11}, \ldots, \hat{\gamma}_{LL}\}$. This of course provides the estimator of "$\omega$" to be of the form

$$\hat{\omega} = \frac{1}{T} \text{ ARG } \sum_{k=1}^{L-1} \hat{C}_k \hat{R}(kT) e^{-j \hat{\omega}(k-1)T} \tag{54}$$

where the $\hat{C}_k$ are defined in terms of the M.L.E. estimates of the $\hat{\gamma}_{ij}$ of the inverse covariance matrix $(C+NI)^{-1}$. For an L-pulse burst this implies the simultaneous solution to 1 + 0.5 L(L+1) equations. Letting L=16 this means one must solve a total of 137 coupled nonlinear equations - a formidable task!

The structure of the M.L.E. spectral mean estimator suggests that using the information contained in the higher order lags of the burst waveform one may improve the accuracy in estimating the spectral mean at low signal-to-noise ratios for small interpulse spacings. To this end, we consider the simpler approach first suggested by Lee (see Reference [8]). In this appproach, estimates of the spectral mean are obtained using each estimated autocorrelation $\{\hat{R}(T), \hat{R}(2T), \hat{R}(3T), \ldots\}$ using Rummler's time-domain method. In these studies, we have considered the following three spectral mean estimators

$$\begin{aligned} \hat{V}_1 &= \frac{\lambda}{4\pi T} \text{ ARG}\{\hat{R}(T)\} \\ \hat{V}_2 &= \frac{\lambda}{4\pi(2T)} = \text{ARG}\{\hat{R}(2T)\} \\ \hat{V}_3 &= \frac{\lambda}{4\pi(3T)} = \text{ARG}\{\hat{R}(3T)\} \end{aligned} \tag{55}$$

where

$$\hat{R}(T) = \frac{1}{L-1} \sum_{i=0}^{L-2} z_i^* z_{i+1}$$

$$\hat{R}(2T) = \frac{1}{L-2} \sum_{i=0}^{L-3} z_i^* z_{i+2} \tag{56}$$

$$\hat{R}(3T) = \frac{1}{L-3} \sum_{i=0}^{L-4} z_i^* z_{i+3}$$

Then the estimates $\hat{V}_1$, $\hat{V}_2$ and $\hat{V}_3$ are combined in a linear fashion to obtain improved performance. To this end, we have considered spectral mean estimators of the form

$$\hat{V}_{12} = c_1\hat{V}_1 + c_2\hat{V}_2$$

$$\hat{V}_{123} = c_1\hat{V}_1 + c_2\hat{V}_2 + c_3\hat{V}_3 \tag{57}$$

where the coefficients $c_1$, $c_2$, $c_3$ are selected either to minimize the rms estimation error of $\hat{V}_{12}$, $\hat{V}_{123}$ (optimal weights) or simply to average the estimates (uniform weights).

Figure 4 shows the theoretical estimation accuracy of each of the spectral mean estimators $\hat{V}_1$, $\hat{V}_2$ and $\hat{V}_3$, computed from estimated autocorrelations $\hat{R}(T)$, $\hat{R}(2T)$ and $\hat{R}(3T)$, respectively. For the specific case considered (S/N = -6 dB, $\sigma_v$ = 150 m/s $\lambda$=.03m, L=16 pulses) the curves indicate theoretical accuracy versus spacing, T, between adjacent pulses of the burst waveform. If it is desirable to utilize a quick burst of pulses corresponding to a small spacing T, we see for the data shown in Figure 4 that the spectral mean estimator, $\hat{V}_3$, provides best performance. For pulse spacings greater than 8 μ seconds the performance of estimator $\hat{V}_3$ degrades rapidly. However, in this region of pulse spacing both estimators $\hat{V}_1$ and $\hat{V}_2$ provide good performance.

Given the three estimators $\hat{V}_1$, $\hat{V}_2$ and $\hat{V}_3$, each of which provide useable estimates of the spectral mean, it is reasonable to consider obtaining improved estimates using a linear combination of the individual estimates. Note however that the estimates $\hat{V}_1$, $\hat{V}_2$, and $\hat{V}_3$ are correlated and the amount of improvement in estimation accuracy achievable depends upon the degree of correlation.

Figure 5 shows the results of a theoretical analysis of the performance of the spectral mean estimators $\hat{V}_{12}$ and $\hat{V}_{123}$ using uniform weights as in Lee [8] (for comparison the figure also indicates the performance of estimator $\hat{V}_1$). The curves of Figure 6 indicate the theoretical performance of these same estimators using optimal minimum variance weights. Comparing the curves of Figure 5 and Figure 6 it is seen that the simpler method of averaging estimates using uniform weights provides significantly improved accuracy relative to the single-lag estimator $\hat{V}_1$. This is especially true for small pulse spacings and at large spacings for $\hat{V}_{123}$, and at still larger spacings for $\hat{V}_{12}$, and $\hat{V}_1$. For this case, the additional improvement achieved using optimal weights may not be justifiable. That is, since the computation of optimal weights requires knowledge of both the S/N ratio and spectral width parameters, implemention of optimal weights may not justify the computational expense.

Figure 7 presents the results of a Monte-Carlo simulation of spectral mean estimators $\hat{V}_1$, $\hat{V}_{12}$ and $\hat{V}_{123}$. For the case considered ($\sigma_v$ = 150 m/s, $\lambda$= 0.03 m, L=16 pulses and T=5 $\mu$ sec) we have evaluated estimator accuracy as a function of signal-to-noise ratio. We have also included both the Cramer-Rao lower bound and the theoretically determined accuracy for algorithm $\hat{V}_1$ [see Appendix B]. From the figure we observe that the simulation of estimator $\hat{V}_1$ agrees quite well with accuracy predictions at high signal-to-noise ratios. Also, almost no improvement in performance was obtained at high signal-to-noise ratios using estimator $\hat{V}_{12}$ or $\hat{V}_{123}$. At low signal-to-noise ratios the performance of estimator $\hat{V}_1$ departs from the theoretical predictions due to approximations used in the linearized error analysis. As expected, however, at low signal-to-noise ratios we observe a significant improvement in performance achieved using estimators $\hat{V}_{12}$ and $\hat{V}_{123}$. Finally we remark that uniform weights were used in this simulation.

Figure 8 shows the results of a Monte-Carlo simulation of several spectral width estimators which were evaluated. Among the algorithms simulated are Rummler's original spectral width estimator (Algorithm 1) and two logarithmic versions of the width estimator (see table 1 for definitions of these estimators). The first logarithmic version of the width estimator (Algorithm 2) was developed to reduce the bias error inherent in Rummler's original spread estimator and assumes the shape of the power spectral density function to be Gaussian (Reference [10]). The second logarithmic version of the width estimator (Algorithm 3) does not require an estimate of the noise power, N, as is required by the other algorithms by utilizing both $\hat{R}(T)$ and $\hat{R}(2T)$ in estimating the spectral width (Reference [9]).

The performance of each spectral width estimator was evaluated via Monte-Carlo simulation and the accuracy was plotted versus signal-to-noise ratio for the same parameter set considered previously ($\sigma_v$ = 150 m/s, L = 16 pulses, T = 5 μ sec, $\lambda$ = 0.03m). Indicated on the figure is the Cramer-Rao lower bound on estimating the spectral width parameter (from Zrnic [10]). Also shown on the figure is the theoretical accuracy of algorithm (1) (see Appendix C for accuracy formulas). For the data shown in Figure 8 we observe two significant results. First, the theoretical accuracy predictions are observed to be quite good at high signal-to-noise ratios, however, at low signal-to-noise ratios we observe significant departure of the simulation results from the theoretical predictions. Secondly, the Cramer-Rao lower bound is observed to be very optimistic and does not provide a tight bound on the performance of these spectral width estimators.

## V. SUMMARY

This report considered the problem of estimating the spectral mean and spectral width parameters of an observed doppler velocity spectrum using a burst waveform of arbitrary length. The observed waveform was modeled as a correlated Gaussian random process and the maximum likelihood algorithm for estimating the spectral mean was derived. Also, the corresponding Cramer-Rao lower bound for estimation of the spectral mean parameter was derived. The M.L.E. spectral-mean estimator was shown to include both the time-domain spectral-mean estimator proposed by Rummler [2] and the frequency domain spectral mean estimator proposed by Miller [1]. Simplications to the M.L.E. spectral mean estimator were proposed and studied, both theoretically and by using Monte-Carlo simulations. It was shown that by using additional order lag estimates (e.g., $\hat{R}(T)$, $\hat{R}(2T)$, $\hat{R}(3T)$) that improved spectral mean estimates could be obtained and this improvement in performance could be achieved at lower signal-to-noise ratio situations.

Finally, preliminary studies of several spectral width estimators were presented based upon Monte-Carlo simulations. From these simulation results it was observed that performance of the spectral width estimators was far from the Cramer-Rao lower bound, thus, there is need for development of a new improved algorithm for estimating the spectral width parameter.



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## APPENDIX A: COMPLEX RADAR SIGNAL SIMULATOR

This appendix describes the technique used to simulate complex radar signal samples for burst waveforms of arbitrary length. The radar signals being simulated are described earlier in the report (see Equations 1 through 6). The technique has been used to simulate burst waveforms with as many as 16 pulses and as few as 2 pulses. The technique will be described using, as an example, a three pulse burst. The extension to the general L-pulse case is then trivial.

Assume we have three complex pulse returns

$$\begin{aligned} z_0 &= s_0 e^{j0\omega T} + n_0 \\ z_1 &= s_1 e^{j1\omega T} + n_1 \\ z_2 &= s_2 e^{j2\omega T} + n_2 \end{aligned} \qquad \text{(A-1)}$$

From the assumed signal autocorrelation function, which is

$$E\{z_i^* \, z_{i+k}\} = S\beta(kT)e^{jk\omega T} + N\,\delta_k \qquad \text{(A-2)}$$

we obtain the covariance matrix for the vector $\underline{z}'=(z_0,z_1,z_2)$

$$E\{\underline{z}\,\underline{z}^{*'}\} = \begin{bmatrix} S+N & S\beta(T)e^{-j\omega T} & S\beta(2T)e^{-j2\omega T} \\ S\beta(T)e^{j\omega T} & S+N & S\beta(T)e^{-j\omega T} \\ S\beta(2T)e^{j2\omega T} & S\beta(T)e^{j\omega T} & S+N \end{bmatrix} \qquad \text{(A-3)}$$

The basic idea is to recognize that this covariance matrix can be factored into the product of three matrices, i.e.

$$E\{\underline{z}\ \underline{z}^{*'}\} = D(C+NI)D^{*'} \qquad \text{(A-4)}$$

$$\text{Where } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{j\omega T} & 0 \\ 0 & 0 & e^{j2\omega T} \end{bmatrix} \qquad \text{(A-5)}$$

$$\text{and } (C+NI) = \begin{bmatrix} S+N & S\beta(T) & S\beta(2T) \\ S\beta(T) & S+N & S\beta(T) \\ S\beta(2T) & S\beta(T) & S+N \end{bmatrix} \qquad \text{(A-6)}$$

Note that since the matrix (C+NI) is real and symmetric, it may be factored as follows

$$(C+NI) = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ 0 & a_{22} & a_{32} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \text{(A-7)}$$

where the unknown coefficients, $a_{ij}$, are all real. Defining the matrix factorization as (C+NI) = MM', then the reader may easily verify the following solution.

$$
\begin{aligned}
a_{11} &= \sqrt{S + N} \\
a_{21} &= \frac{S\beta(T)}{\sqrt{S + N}} \\
a_{31} &= \frac{S\beta(2T)}{\sqrt{S + N}} \qquad\qquad \text{(A-8)} \\
a_{22} &= \frac{\sqrt{(S + N)^2 - (S\beta(T))^2}}{\sqrt{S + N}} \\
a_{32} &= \frac{(S\beta(T)\,[(S+N) - (S\beta(2T))]}{\sqrt{S + N}} \\
a_{33} &= \frac{\sqrt{(S+N)^3 - [(S\beta(2T))^2 + 2(S\beta(T))^2](S+N) + 2(S\beta(T))^2(S\beta(2T))}}{\sqrt{(S + N)^2 - (S\beta(T))^2}}
\end{aligned}
$$

The sequence of complex data samples $\{z_0, z_1, z_2\}$ is then generated using the following matrix linear transformation

$$
\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{j\omega T} & 0 \\ 0 & 0 & e^{j2\omega T} \end{bmatrix} \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\underline{z} = D\,M\,\underline{g}} \begin{bmatrix} g_0 \\ g_1 \\ g_2 \end{bmatrix} \qquad \text{(A-9)}
$$

The input samples $g_0$, $g_1$, $g_2$ are uncorrelated complex Gaussian random variables

$$g_0 = x_0 + jy_0$$
$$g_1 = x_1 + jy_1 \qquad (A-10)$$
$$g_2 = x_2 + jy_2$$

Thus, to obtain the properly correlated sequence of complex data samples we generate 6 independent (mean=0, variance=1/2) Gaussian random variables $x_0$, $y_0$, $x_1$, $y_1$, $x_2$ and $y_2$ and obtain the corresponding simulated complex radar signal samples as indicated above in equation A-9. It is easy to show that the complex data samples $\{z_0, z_1, z_2\}$ have the desired Hermitian symmetric covariance matrix (see Equation A-3). It is easy to verify the fact that the variables $\{z_0, z_1, z_2\}$ also satisfy the required circular property (see reference [6])

$$E\{z_i, z_j\} = 0 \qquad \text{for all } i, j \qquad (A-11)$$

We remark that this property will not be satisfied if one were to take the input data vector to be real Gaussian variates (e.g., $g_0 = x_0$, $g_1 = x_1$, $g_2 = x_2$). Finally, it is clear that the method described in this appendix generalizes to the general L-pulse case.

## APPENDIX B ERROR ANALYSIS OF MEAN VELOCITY ESTIMATOR

This appendix provides a summary of the error analysis of the mean velocity estimator for burst waveforms which are processed using the Rummler time domain approach. Specifically we have considered a burst waveform comprised of L contiguous pulse returns, denoted $\{z_0, z_1, z_2, \ldots, z_{L-1}\}$. From this contiguous pulse train we may compute estimates of the mean doppler frequency using various lag products. For example, we might construct the following estimates.

$$\begin{aligned}
\hat{f}_1 &= \frac{1}{2\pi T} \text{ ARG } \{\hat{R}(T)\} \\
\hat{f}_2 &= \frac{1}{2\pi(2T)} \text{ ARG } \{\hat{R}(2T)\} \\
\hat{f}_3 &= \frac{1}{2\pi(3T)} \text{ ARG } \{\hat{R}(3T)\} \\
&\vdots \\
&\text{etc}
\end{aligned} \tag{B-1}$$

where

$$\begin{aligned}
\hat{R}(T) &= \frac{1}{L-1} \sum_{i=0}^{L-2} z_i^* \, z_{i+1} \\
\hat{R}(2T) &= \frac{1}{L-2} \sum_{i=0}^{L-3} z_i^* \, z_{i+2} \\
\hat{R}(3T) &= \frac{1}{L-3} \sum_{i=0}^{L-4} z_i^* \, z_{i+3} \\
&\vdots \\
&\text{etc.}
\end{aligned} \tag{B-2}$$

Each of the above estimators $\hat{f}_1$, $\hat{f}_2$, $\hat{f}_3$, etc yields valid estimates of the mean doppler frequency. One is interested in determining the accuracy of these estimates. A brief discussion of the accuracy analysis is given in the following paragraphs. This analysis is based upon the work of Reference [6].

An analysis of the accuracy of the Rummler time domain estimator (see appendix A, Reference [6]) yields the following error expression for the accuracy of the estimator $\hat{f}_K$

$$\operatorname{Var}\{\hat{f}_K\} = \frac{1}{8\pi^2(KT)^2} \operatorname{Re}\left[ E\left\{\left|\frac{\hat{R}(KT)}{R(KT)}\right|^2\right\} - E\left\{\left(\frac{\hat{R}(KT)}{R(KT)}\right)^2\right\}\right] \quad \text{(B-3)}$$

Omitting the lengthy derivations of the terms of the above expression (see Reference [6] for basic approach) one may obtain the final expression shown below.

$$\operatorname{Var}\{\hat{f}_K\} = [1/8\pi^2(KT)^2\beta(KT)]$$

$$\left[\frac{1}{(L-K)^2}\sum_{m=-(L-K-1)}^{m=(L-K-1)} (\beta^2(mT) - \beta((m+K)T)\beta((m-K)T)) \left[L-K-|m|\right] + \frac{1}{(L-K)}\left[2(N/S) + (N/S)^2\right] - \underbrace{\frac{2}{(L-K)}2 \quad [(N/S)\beta(2KT)(L-2K)]}_{L>2K,\ \text{otherwise}=0}\right] \quad \text{(B-4)}$$

Note that the third term in the above expression is valid only when L>2K, otherwise this term is zero.

Since $\hat{f}_1$ and $\hat{f}_2$ yield two viable estimates of the mean doppler frequency, one may attempt to improve the accuracy of the mean doppler frequency estimate by taking a linear combination of the estimates $\hat{f}_1$ and $\hat{f}_2$.

$$\hat{f}_{12} = c_1 \hat{f}_1 + c_2 \hat{f}_2 \tag{B-5}$$

The weights ($c_1$, $c_2$) could be selected to provide a better estimate of the mean doppler frequency (better in the minimum variance sense). Selecting uniform weights ($c_1=c_2=1/2$) might also be considered as a simpler improved estimator. In either case, the accuracy of estimate $\hat{f}_{12}$ is given by the expression

$$\begin{aligned} \mathrm{Var}\{\hat{f}_{12}\} &= c_1^2 \,\mathrm{Var}\{\hat{f}_1\} + c_2^2 \,\mathrm{Var}\{\hat{f}_2\} \\ &\quad + 2\, c_1\, c_2 \,\mathrm{Cov}\{\hat{f}_1, \hat{f}_2\} \end{aligned} \tag{B-6}$$

Selecting ($c_1$, $c_2$) to make $\hat{f}_{12}$minimum variance and unbiased we obtain the optimal solution

$$c_1^* = \frac{\mathrm{Var}\{\hat{f}_2\} - \mathrm{Cov}\{\hat{f}_2, \hat{f}_1\}}{\mathrm{Var}\{\hat{f}_1\} + \mathrm{Var}\{\hat{f}_2\} - 2\,\mathrm{Cov}\{\hat{f}_2, \hat{f}_1\}} \tag{B-7}$$

$$c_2^* = 1 - c_1^* \tag{B-8}$$

Similar expressions can be obtained for an improved estimator of the form

$$\hat{f}_{123} = c_1 \hat{f}_1 + c_2 \hat{f}_2 + c_3 \hat{f}_3 \tag{B-9}$$

To evaluate the theoretical accuracy of these various estimators it is clear that one must evaluate the covariance between the errors of each Rummler estimate $\hat{f}_1$, $\hat{f}_2$, $\hat{f}_3$, etc. If the covariance between the errors of these estimates is small then significant improvement in mean doppler frequency estimates can be achieved.

The covariance between estimate $\hat{f}_K$ and $\hat{f}_J$ (where K>J) is evaluated as follows. Extending the results of Appendix A of Reference [6] we have the expression

$$\mathrm{Cov}\ \{\hat{f}_K, \hat{f}_J\} = [1/8\pi^2(KT)(JT)]$$
$$\mathrm{RE}\left[ E\left\{ \left(\frac{\hat{R}(JT)}{R(JT)}\right)\left(\frac{\hat{R}(KT)}{R(KT)}\right)^* - \left(\frac{\hat{R}(JT)}{R(JT)}\right)\left(\frac{\hat{R}(KT)}{R(KT)}\right)\right\}\right] \tag{B-10}$$

where,

$$\hat{R}(JT) = \frac{1}{L-J} \sum_{i=0}^{L-J-1} z_i^* z_{i+J}$$
$$\hat{R}(KT) = \frac{1}{L-K} \sum_{i=0}^{L-K-1} z_i^* z_{i+K} \tag{B-11}$$

Omitting the lengthy derivations, the result may be shown to be given by the following expression.

$$\mathrm{Cov}\ \{\hat{f}_K,\ \hat{f}_J\} = \left[1/8\pi^2 (KT)(JT)(L-J)(L-K)\,\beta(KT)\,\beta(JT)\right]$$

$$\left[\sum_{i=0}^{L-J-1}\ \sum_{i=0}^{L-K-1}\ \beta((j-i)T)\,\beta((i-j+J-K)T) - \beta((j-i+K)T)\,\beta((i-j+J)T) + 2(L-K)\,\beta((J-K)T)(N/S) \underbrace{-\ 2(L-K-J)\,\beta((J+K)T)(N/S)}_{L-K-J\ >\ 0,\ \text{otherwise}\ =\ 0}\right] \tag{B-12}$$

Note that the last term in the above expression is valid only when $L-K-J > 0$, otherwise this term is zero.

## APPENDIX C ERROR ANALYSIS OF SPREAD ESTIMATOR

This appendix provides a summary of the error analysis equations used to evaluate the performance accuracy of the Rummler spread estimator. The basic algorithm is defined by the following expression

$$\hat{W} = \frac{1}{\sqrt{2}\,\pi T} \left| 1 - \left| \hat{\rho}(T) \right| \right|^{1/2} \tag{C-1}$$

where $\hat{\rho}(T)$ is the estimated correlation coefficient of the underlying signal process. The correlation coefficient is estimated from the sample correlation function as follows

$$\hat{\rho}(T) = \frac{\hat{R}(T)}{\hat{R}(0) - N} \tag{C-2}$$

Thus, it is assumed that the noise power, N, is known or can be accurately estimated from range gate data comprised of noise-only samples. Also, we have the terms

$$\hat{R}(0) = \frac{1}{L} \sum_{i=0}^{L-1} \left| z_i \right|^2 \tag{C-3}$$

$$\hat{R}(T) = \frac{1}{L-1} \sum_{i=0}^{L-2} z_i^* \, z_{i+1} \tag{C-4}$$

This estimator provides reasonably good estimates of the spread parameter "W" for small values of T. To see this, one simply considers for a large number of pulses ($L \to \infty$)

$$\hat{R}(0) \sim S+N \tag{C-5}$$

$$\hat{R}(T) \sim S\beta(T)e^{i2\pi fT} \tag{C-6}$$

Thus, for large L,

$$\hat{\rho}(T) \sim e^{-2\pi^2 W^2 T^2} \tag{C-7}$$

Approximating the right hand side of the above equation by the term $1-2\pi^2W^2T^2$ we obtain.

$$\hat{W} \sim \frac{1}{\sqrt{2}\,\pi T} \left|2\pi^2 W^2 T^2\right|^{1/2} \tag{C-8}$$

Thus, for small T we have

$$\hat{W} \sim W$$

This derivation was not intended to be rigorous but was presented simply to demonstrate that the algorithm W may provide reasonably good estimates of the spread parameter, W, for small T. This algorithm is however, a biased estimator of the parameter W since

$$E\{\hat{W}\} \neq W \tag{C-10}$$

and thus the analysis of the performance accuracy of this algorithm must include a determination of both its bias and variance terms. It may be shown (see Reference [10] for details of the derivation) that the variance and bias errors for the Rummler spread estimator are

## A. Random Error Equation

$$\mathrm{Var}\{\hat{W}\} = \left[ \frac{\beta^2(T)}{8\pi^2(1-\beta(T))} \{\mathrm{TERM}\ 1\} - 2\{\mathrm{TERM}\ 2\} + 0.5(\{\mathrm{TERM}\ 3\} + \{\mathrm{TERM}\ 4\}) \right] \quad \text{(C-11)}$$

where,

$$\mathrm{TERM}\ 1 = \mathrm{Var}\left\{\frac{\hat{R}(0)}{S}\right\}$$

$$= \frac{1}{L^2} \sum_{m=-(L-1)}^{(L-1)} \beta^2(mT)\,[L-|m|] + \frac{1}{L}\,[2(N/S)+(N/S)^2] \quad \text{(C-12)}$$

$$\mathrm{TERM}\ 2 = \mathrm{Cov}\left\{\frac{\hat{R}(0)}{S}, \frac{\hat{R}(T)}{R(T)}\right\}$$

$$= \frac{1}{L(L-1)}\left[ \sum_{m=-(L-2)}^{(L-2)} \frac{\beta((m+1)T)\,\beta(mT)}{\beta(T)}\,[(L-1)-|m|] \right. \quad \text{(C-13)}$$

$$\left. + \sum_{l=0}^{L-2} \frac{\beta((1-L-2)T)\,\beta((1-L-1)T)}{\beta(T)} + (2L-2)(N/S) \right]$$

$$\mathrm{TERM}\ 3 = E\left\{\left|\frac{\hat{R}(T)}{R(T)} - 1\right|^2\right\} = E\left\{\left|\frac{\hat{R}(T)}{R(T)}\right|^2\right\} - 1 \quad \text{(C-14)}$$

$$= \frac{1}{(L-1)}\left[ \sum_{m=-(L-2)}^{L-2} \frac{\beta(mT)}{\beta(T)}^2 [(L-1)-|m|] + \frac{(L-1)}{\beta^2(T)}\,[2(N/S)+(N/S)]^2 \right]$$

$$\text{TERM 4} = E\left\{\left(\frac{\hat{R}(T)}{R(T)} - 1\right)^2\right\} = E\left\{\left(\frac{\hat{R}(T)}{R(T)}\right)^2\right\} - 1 \quad \text{(C-15)}$$

$$= \frac{1}{(L-1)^2}\left[\sum_{m=-(L-2)}^{(L-2)} (\beta(mT))^2\left[(L-1)-|m|\right] + \frac{2(L-2)\beta(2T)}{(\beta(T))^2}(N/S)\right]$$

B. Bias Error Equation

$$\text{Bias}\{\hat{W}\} = \frac{\beta(T)}{4\pi^2T^2\bar{W}}\Big[\{\text{TERM 1}\} - \{\text{TERM 2}\} + 0.25\left(\{\text{TERM 3}\} - \{\text{TERM 4}\}\right)\Big] - \frac{\text{Var}\{\hat{W}\}}{2\bar{W}} \quad \text{(C-16)}$$

where

$$\bar{W} = \frac{1}{\sqrt{2}\,\pi T}\left|1-\beta(T)\right|^{1/2} \quad \text{(C-17)}$$

Another algorithm for estimating the spread parameter, W, is the ' ;arithmic version of Rummler's algorithm, defined by the following expression (see Reference [10]).

$$\hat{W}_{\log} = \frac{1}{\sqrt{2}\,\pi T}\left|-\ln\left|\hat{\rho}(T)\right|\right|^{1/2} \quad \text{(C-18)}$$

This algorithm is reported to have less bias than the Rummler version, especia:' for the case where the signal process has a power spectral density which is Gaussian in shape (and hence, a Gaussian autocorrelation function). Taking the same approach as for the previou pread estimator it is easy to determine the error equatons for the logarithmic spread estimator.

The variance and bias errors for this estimator are summarized below

$$\mathrm{Var}\ \{\hat{W}_{log}\} = \frac{1}{\left(4\pi^2 T^2 \bar{W}_{log}\right)^2} \Big[ \{\mathrm{TERM\ 1}\} - 2\ \{\mathrm{TERM\ 2}\} + 0.5 \{\mathrm{TERM\ 3} + \mathrm{TERM\ 4}\} \Big] \qquad \text{(C-19)}$$

$$\mathrm{Bias}\ \{\hat{W}_{log}\} = \frac{1}{8\pi^2 T^2 \bar{W}_{log}} \Big[ \{\mathrm{TERM\ 1}\} - \{\mathrm{TERM\ 4}\} \Big] \frac{-\mathrm{Var}\ \{W_{log}\}}{2\ \bar{W}_{log}} \qquad \text{(C-20)}$$

where $\bar{W}_{log} = \dfrac{(-\ln \rho(T))^{1/2}}{\sqrt{2}\,\pi T}$ (C-21)

TABLE 1

ALGORITHMS FOR ESTIMATING SPECTRAL SPREAD PARAMETER, W

$$\hat{W}_1 = \frac{1}{\sqrt{2\pi T}} \left| 1 - \left| \frac{\hat{R}(T)}{\hat{R}(0)-N} \right| \right|^{\frac{1}{2}}$$

$$\hat{W}_2 = \frac{1}{\sqrt{2\pi T}} \left| \ln \left| \frac{\hat{R}(0)-N}{\hat{R}(T)} \right| \right|^{\frac{1}{2}}$$

$$\hat{W}_3 = \frac{1}{\sqrt{6\pi T}} \ln \left| \frac{\hat{R}(T)}{\hat{R}(2T)} \right|^{\frac{1}{2}}$$

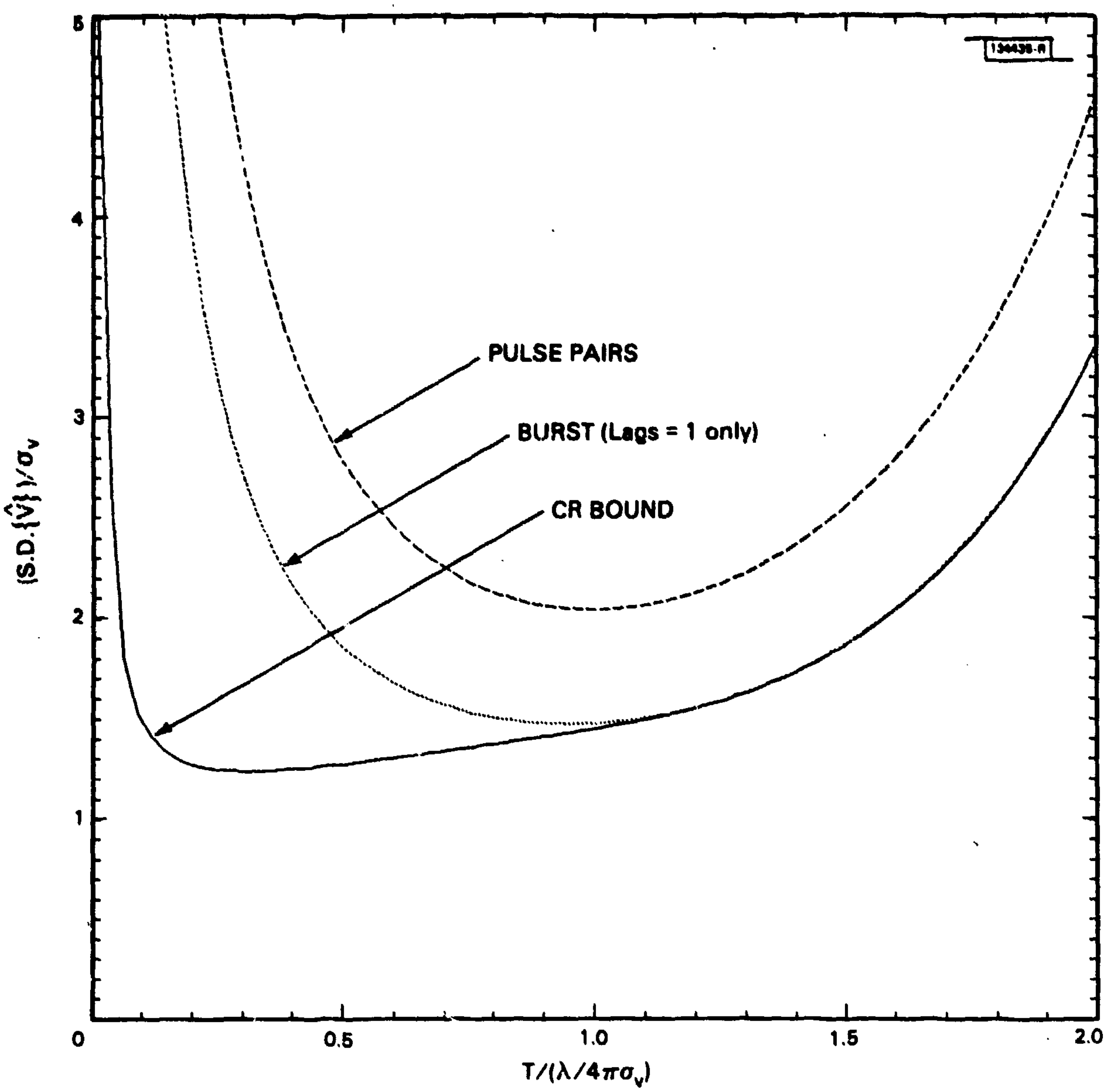


Fig. 1. Normalized standard deviation of mean velocity estimators versus normalized pulse spacing (16 pulses total, SNR = -6 dB).

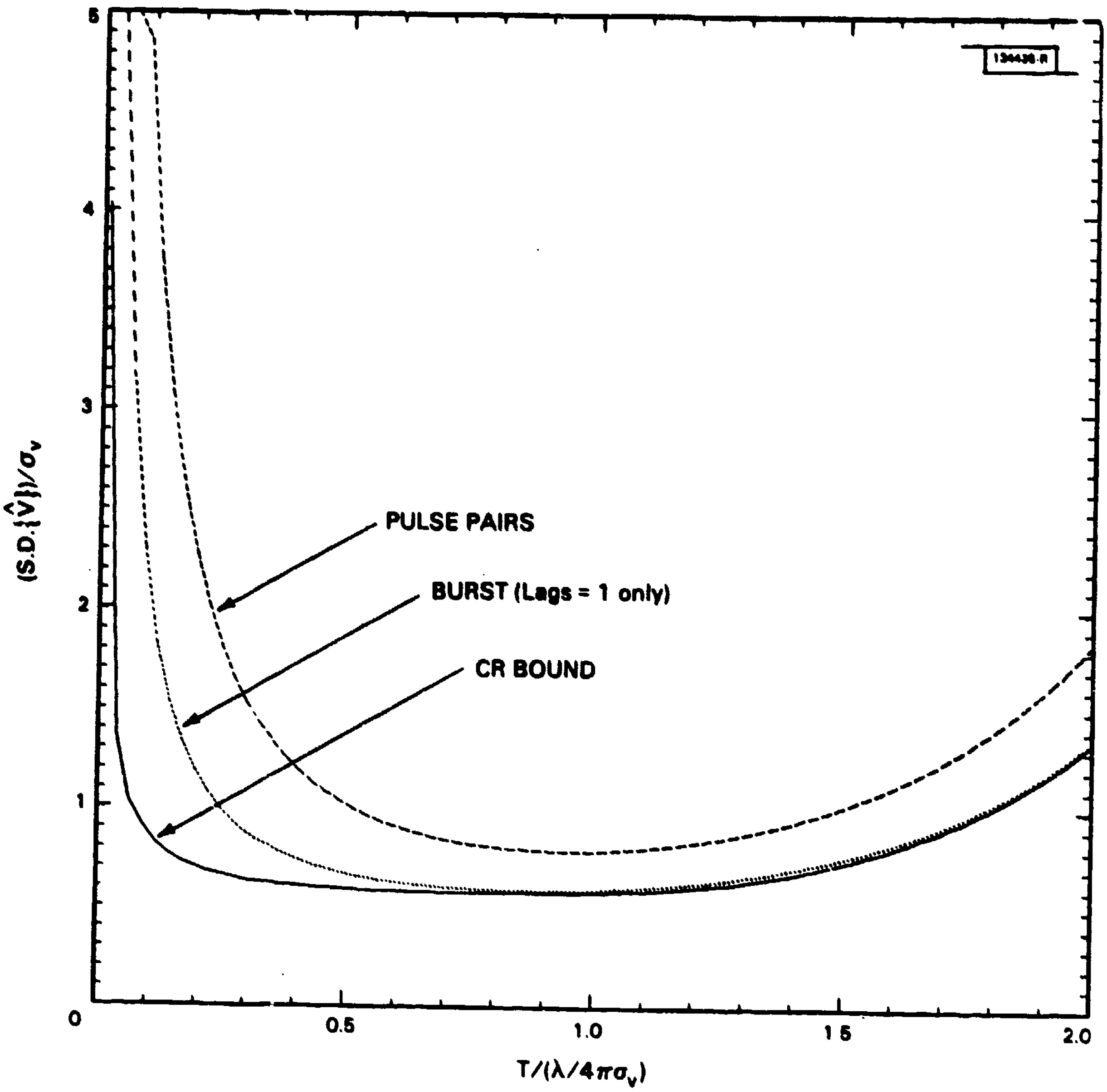


Fig. 2. Normalized standard deviation of mean velocity estimators versus normalized pulse spacing (16 pulses total, SNR = 0 dB).

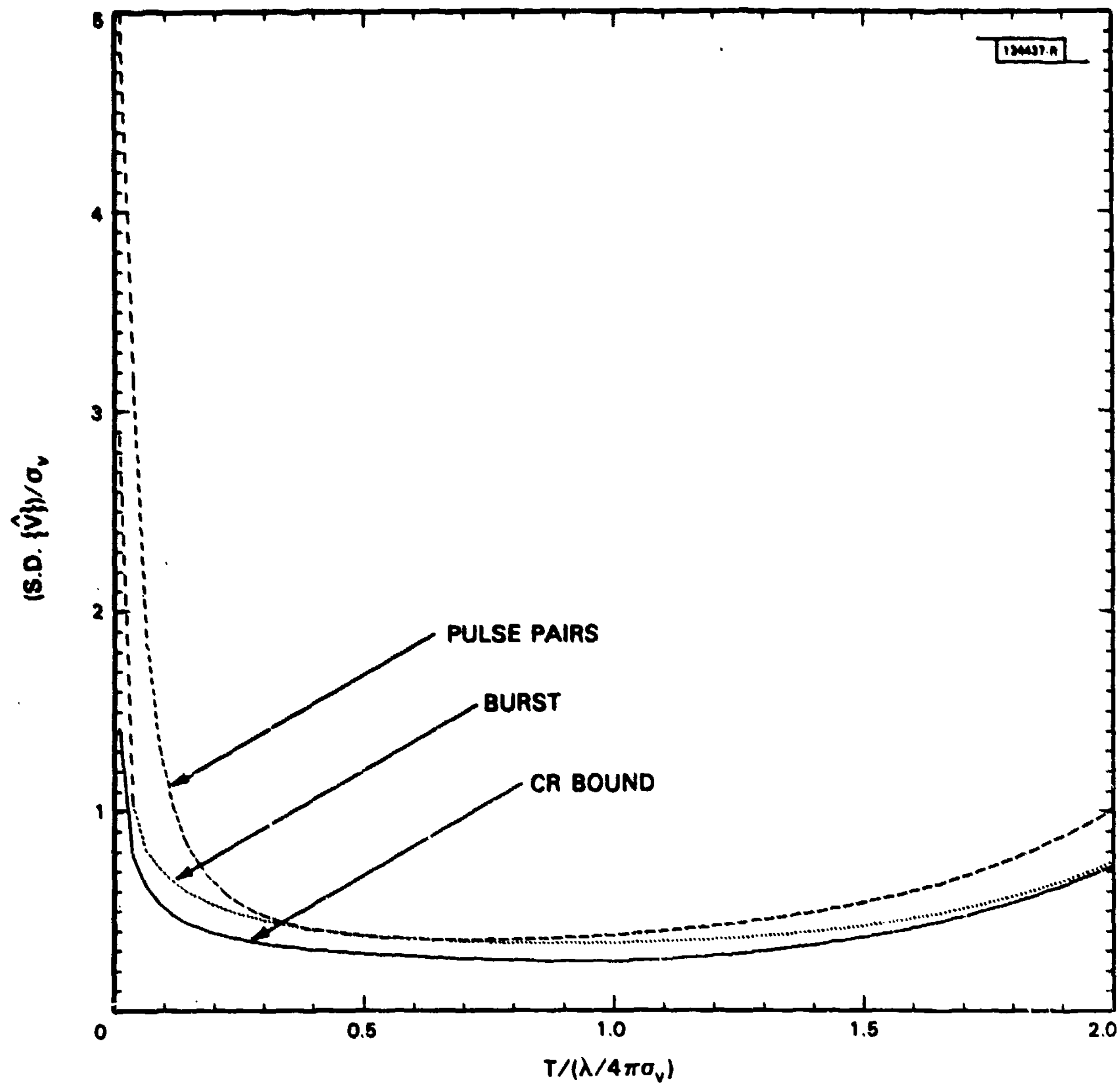


Fig. 3. Normalized standard deviation of mean velocity estimators versus normalized pulse spacing (16 pulses total, SNR = 10 dB).

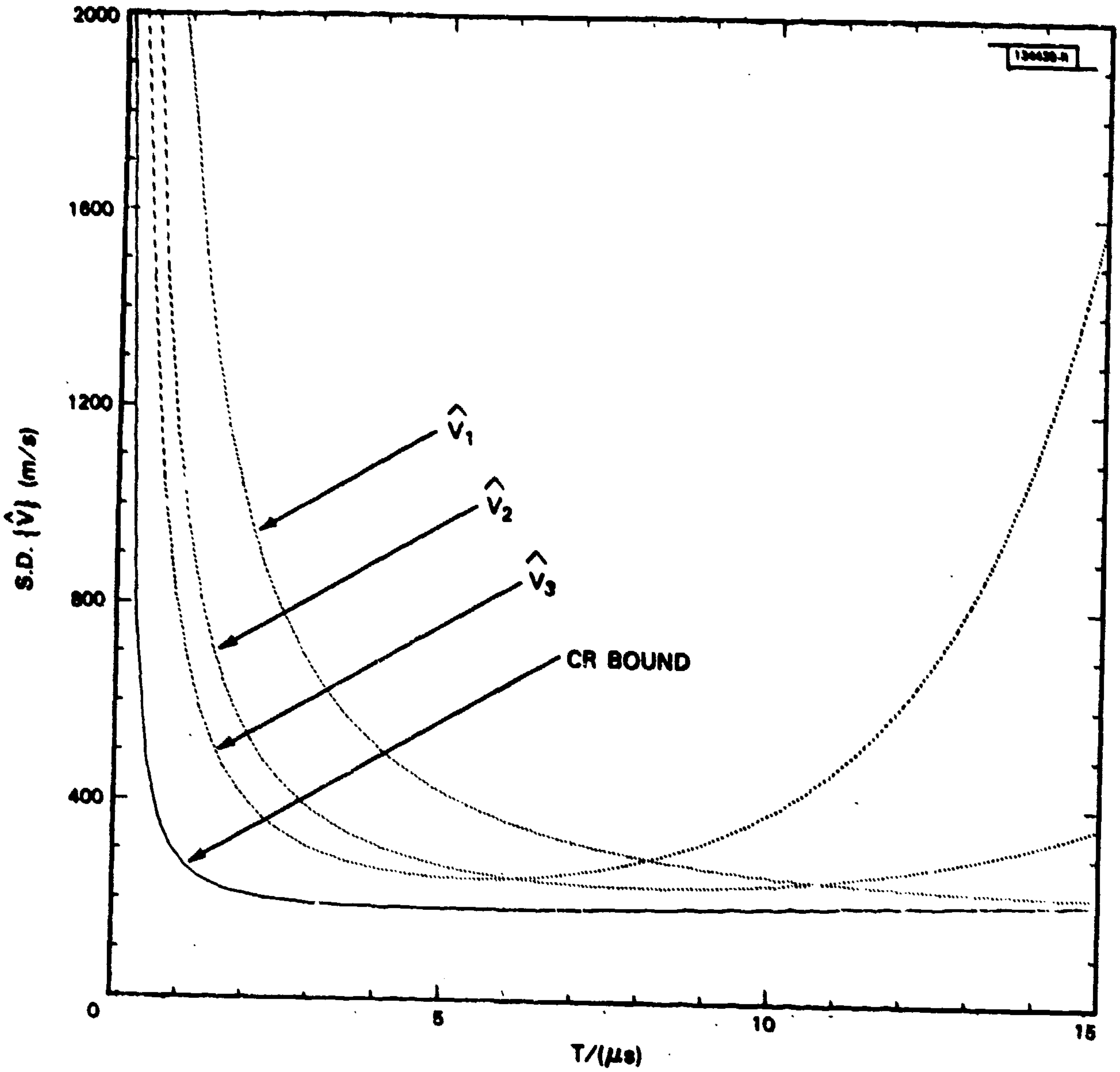


Fig. 4. Standard deviation (m/s) of mean velocity estimators $\hat{V}_1$, $\hat{V}_2$, $\hat{V}_3$ versus pulse spacing (μ sec) (16 pulse burst, SNR = - 6 dB, $\sigma_V$ = 150 m/s, f = 9.1 GHz).

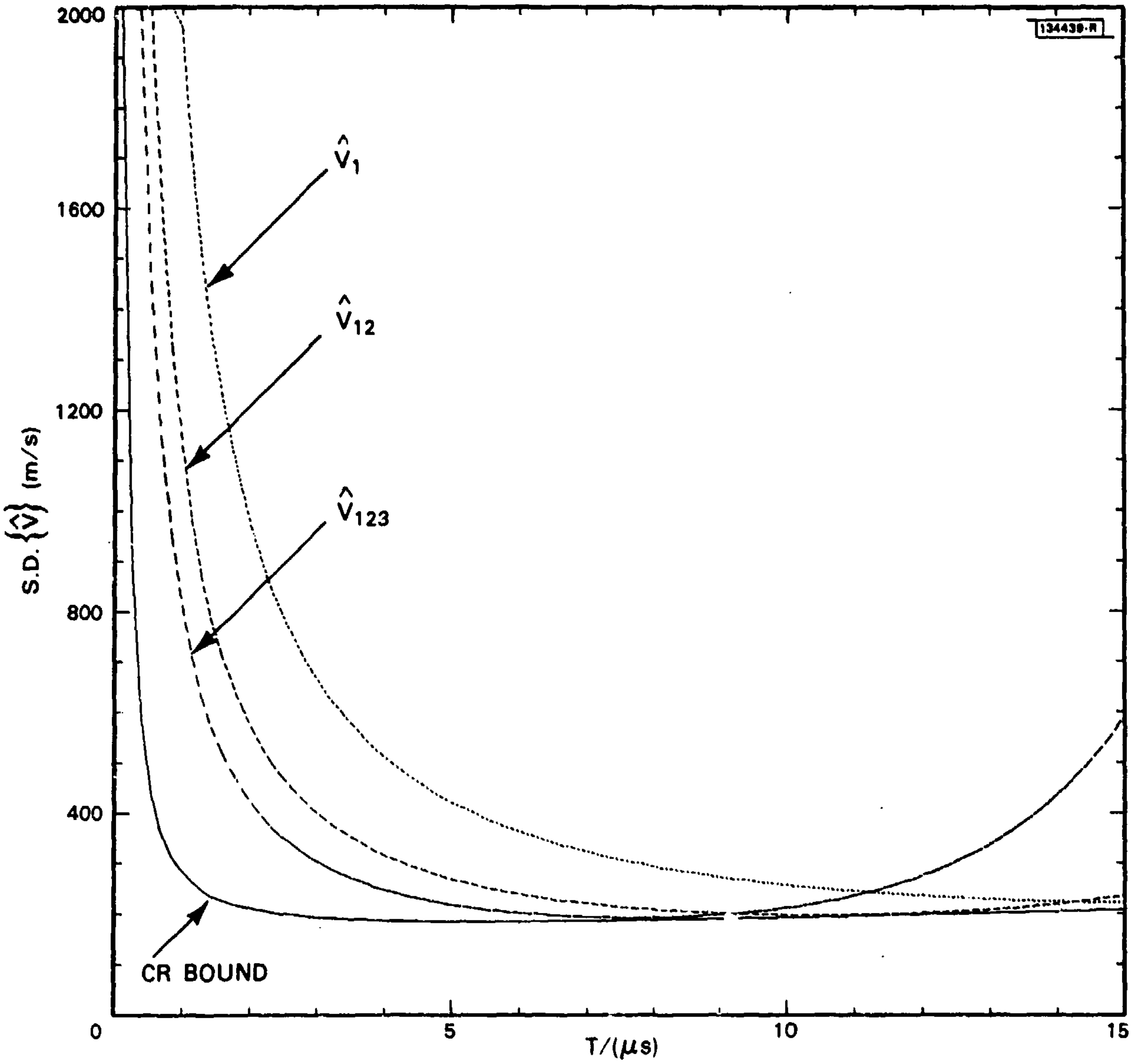


Fig. 5. Standard Deviation (m/s) of mean velocity estimators $\hat{V}_1$, $\hat{V}_{12}$, $\hat{V}_{123}$ versus pulse spacing ($\mu$ sec) with uniform weights (16 pulse burst, SNR = -6 dB, $\sigma_v$ = 150 m/s, f = 9.1 GHz).

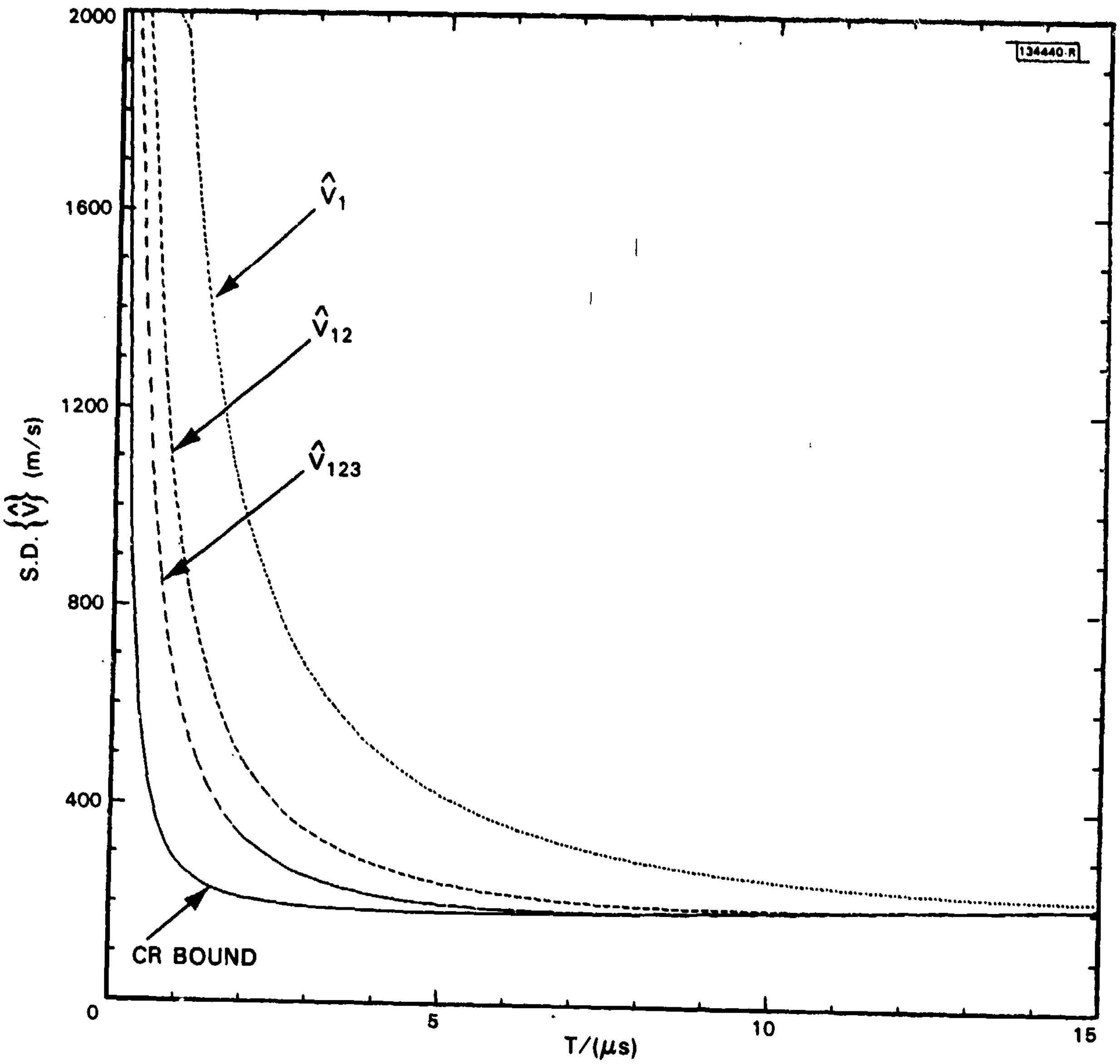


Fig. 6. Standard Deviation (m/s) of mean velocity estimators $\hat{V}_1$, $\hat{V}_{12}$, $\hat{V}_{123}$ versus pulse spacing (μ sec) with optimal weights (16 pulse burst, SNR = -6 dB, $\sigma_v$ = 150 m/s, f = 9.1 GHz).

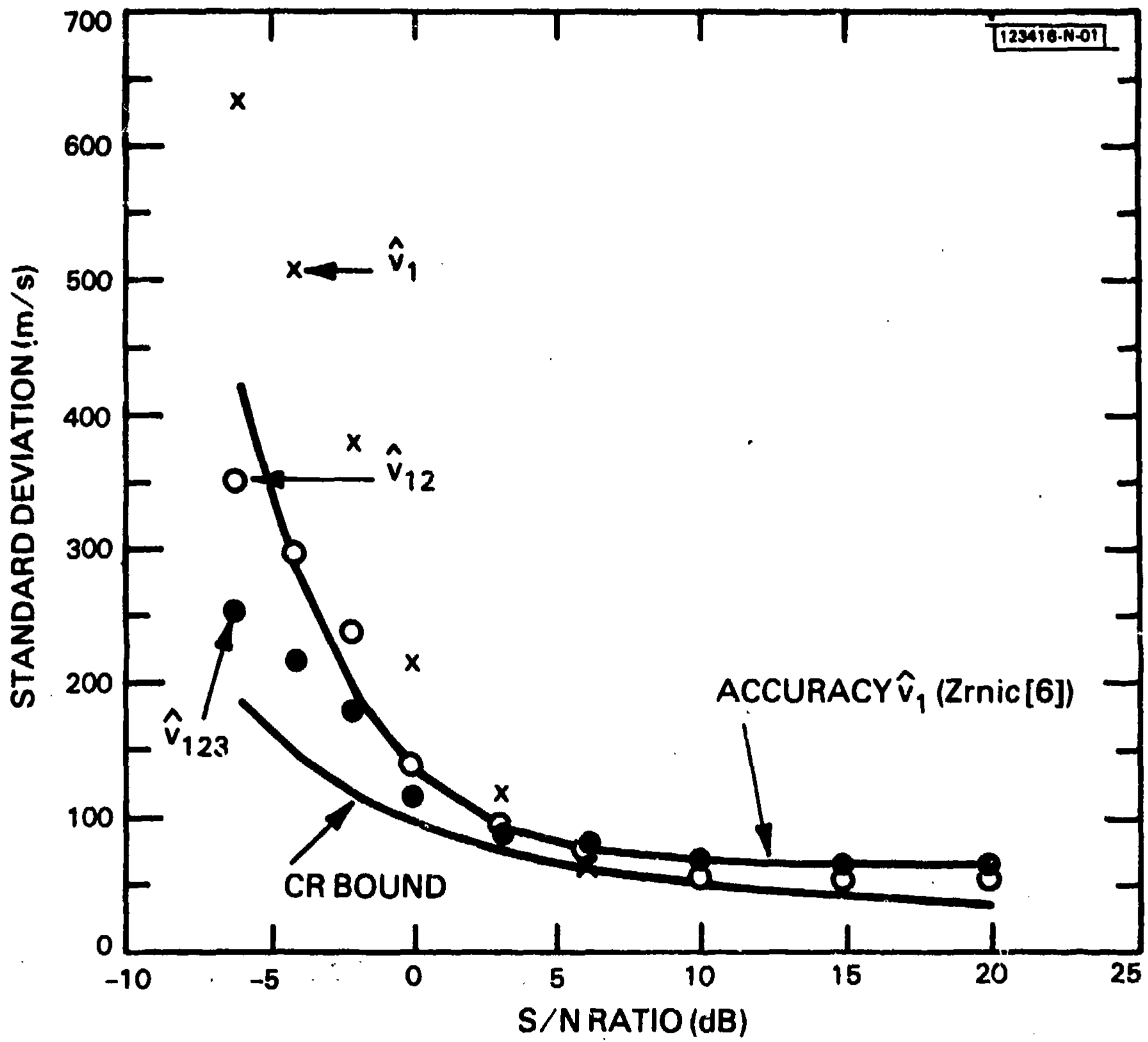


Fig. 7. Standard Deviation (m/s) of mean velocity estimators $\hat{V}_1$, $\hat{V}_{12}$, $\hat{V}_{123}$ versus signal-to-noise ratio (dB) (16 pulse burst, $\sigma_v$ = 150 m/s, f = 9.1 GHz, T = 5 μsec).

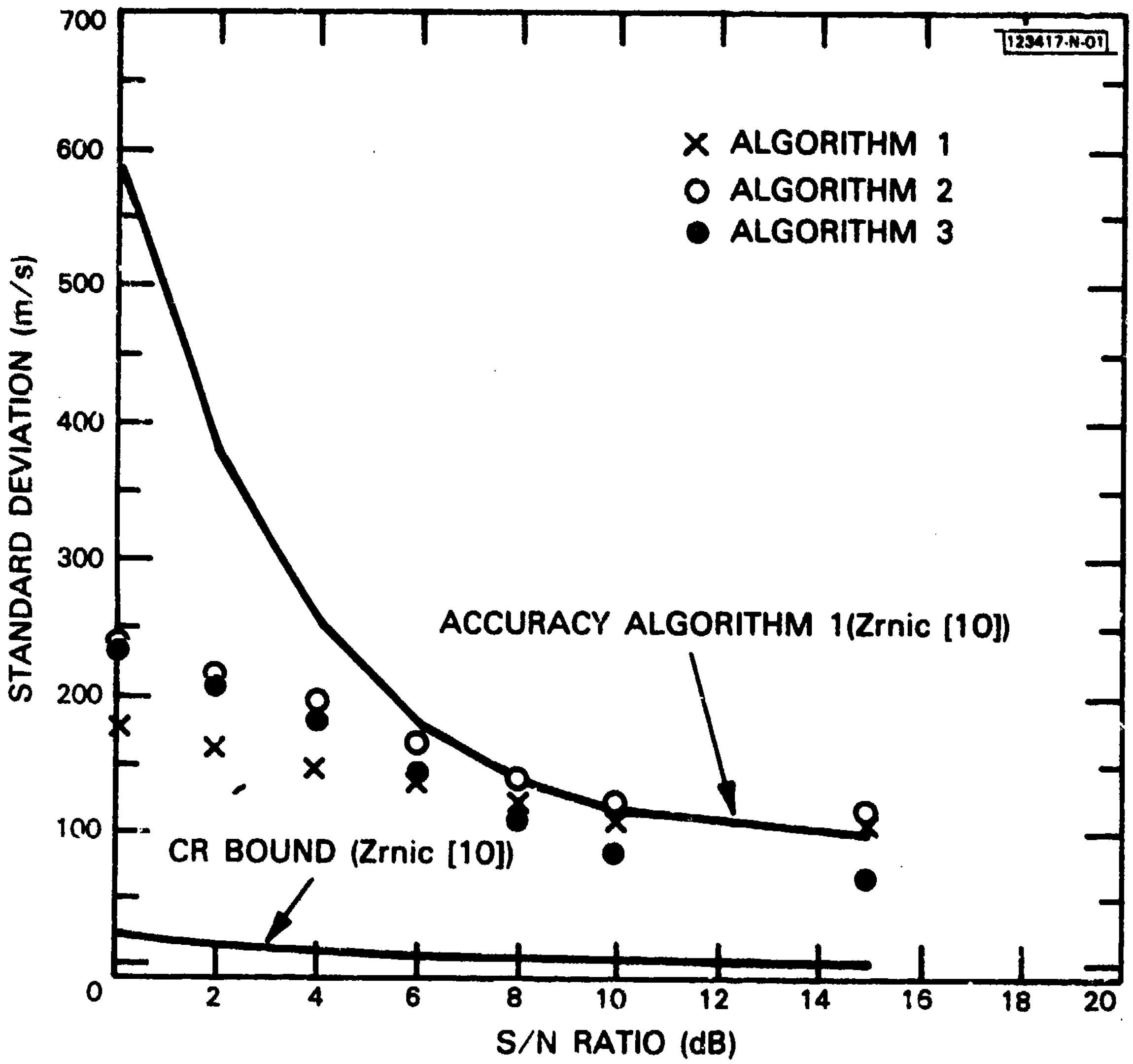


Fig. 8. Standard Deviation (m/s) of velocity spread estimators versus signal-to-noise ratio (dB) (16 pulse burst, $\sigma_v = 150$ m/s, f = 9.1 GHz, T = 5 μsec).

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This report addresses the problem of estimating the spectral parameters of an observed doppler velocity spectrum using a burst radar waveform of arbitrary length. Maximum-likelihood theory is applied and the exact M.L.E. algorithm for estimating the spectral mean is derived. This M.L.E algorithm is shown to include, as a special case, the spectral mean estimator originally proposed by R.W. Miller [1] for processing of burst waveforms. Also, when the burst waveform is a simple pulse pair, the M.L.E. algorithm reduces to the spectral mean estimator originally proposed by W.D. Rummler [2]. The Cramer-Rao bound for estimating the spectral mean using burst waveforms is also derived. Simplifications to the exact maximum-likelihood algorithm are proposed and the performance of various estimators is compared to the Cramer-Rao lower bound. Some preliminary results of studies of spectral width estimators are also presented.

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