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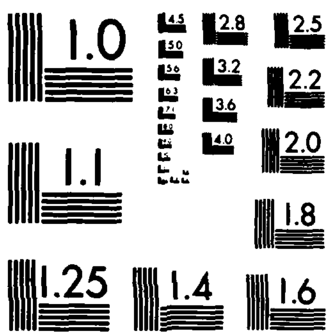

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On Mean Kullback-Leibler's Information Maximum Likelihood Principle and Uncertain Nonlinear Systems

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30 November 1983

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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FOR THE COMMANDER

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ON MEAN KULLBACK-LEIBLER'S INFORMATION MAXIMUM LIKELIHOOD PRINCIPLE AND UNCERTAIN NONLINEAR SYSTEMS

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ABSTRACT

The relationship between the (generalized) mean Kullback-Leibler's information and the (generalized) maximum likelihood principle is exploited in this report to analyze the state estimation problems of both discrete-time and continuous-time uncertain non-linear systems.

1. INTRODUCTION

In solving practical state estimation problems, we often encounter two difficult questions. The first question is related to the accuracy of the deterministic model used for fitting the dynamics of measurements. The second question concerns the accuracy of the statistical model of measurement errors. Since the exact mathematical representation of the physical measurement process is not known, a conservative but prejudiced approach to resolve the above two questions is to adjust parameters in the model until measurement residuals are acceptable. (This approach is prejudiced because almost all anomalies in the residuals can be made to disappear by carefully adjusting parameters in the model.) The resulting state estimates, therefore, differ significantly for different practitioners who depend heavily on models and personal experience in residual analysis. Moreover, the accuracy of the dynamical model and the statistical behavior of the measurement process are two compromising quantities especially well-known to those who use Kalman filters extensively. For a given measurement accuracy, it was observed in [1] and [2] that a Kalman filter might diverge due to the inaccuracy in the dynamical model. Adding so-called "process noise" to the dynamical model may prevent filter divergence [1,3,4]. A detailed analysis of filter divergence for a time-invariant linear system is documented in [1]. In regard to the selection of the covariance matrix of the process noise, people in the field often admit that it is more an art than a science.

Akaike [5] applied the mean Kullback-Leibler's information (MKLI) [6] to extend the maximum likelihood principle. Perhaps the most astonishing result of [5] in terms of the impact on time series analysis is that a computable quantity called model unreliability is introduced and applied to some practical problems. The combination of model unreliability and badness of fit was used in [5] and [7] as a measure to select parameters in a model for a stationary, ergodic process. The same idea was extended recently in [8] to determine the order of a linear time-varying auto-regressive model.

In this report, we follow the reasoning in [6] and [8] to address when to terminate adjusting parameters in a non-linear system and how to select the best non-linear state estimate among many candidates. However, only the asymptotic result is obtained. Further studies are required to extend the result reported herein to cover the finite sample cases.

The structure of this report is summarized in Figures 1 and 2. Hopefully, these figures can also be thought as the logic tree that describes the linkage of many small pieces throughout the report.

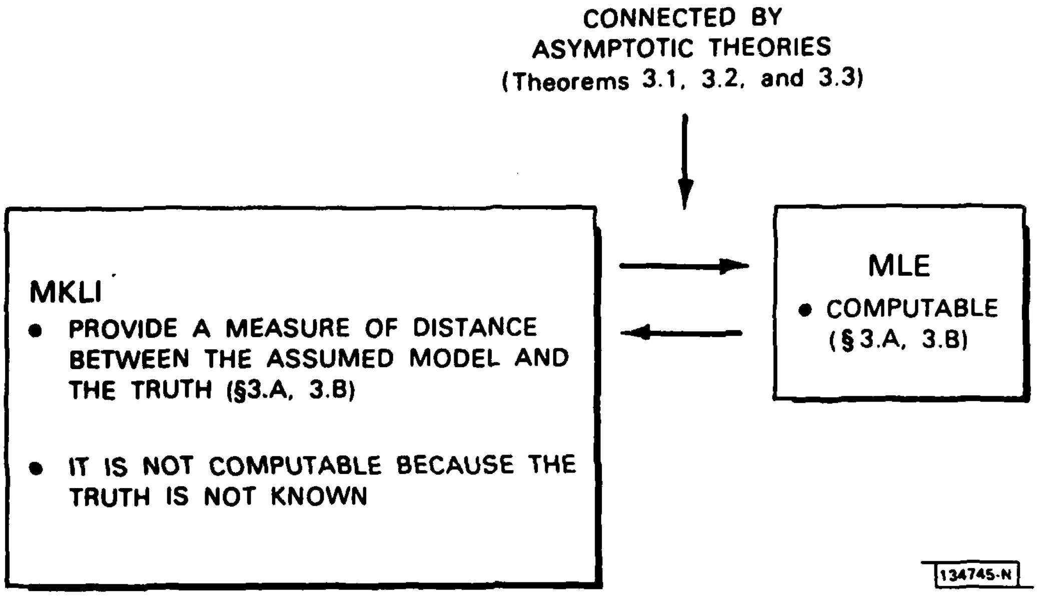

Fig. 1. The structrue of the report: fixed dynamic models.

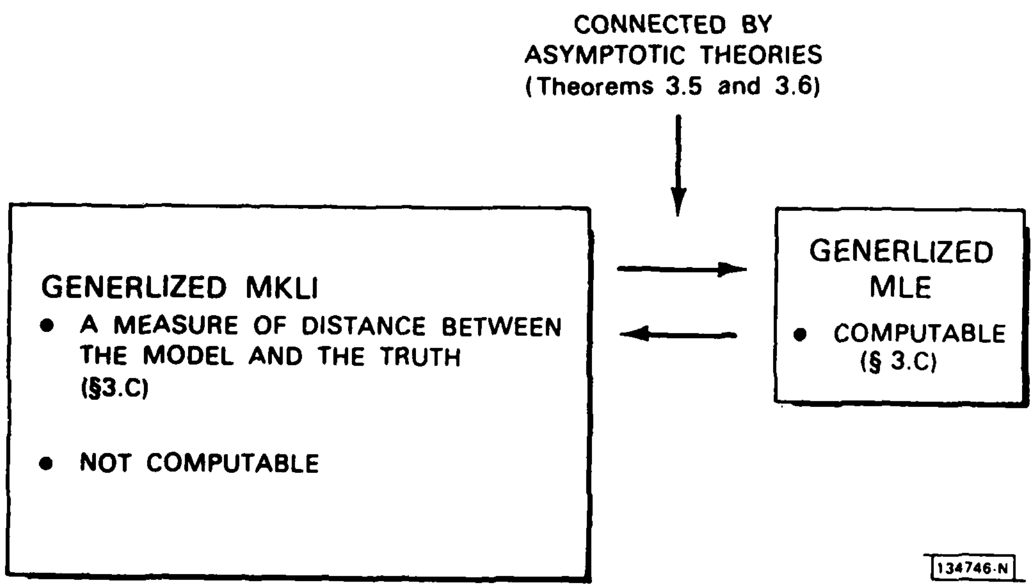

Fig. 2. The structure of the report: tunable dynamic models.

2. NOTATIONS AND PRELIMINARIES

In this section, we formulate the general problem to be addressed. Let $\underline{z}(t)$ be an m-dimensional vector measurement process. The true representation of $\underline{z}(t)$ is assumed to be given by

$$\underline{z}(t) = \underline{h}_o(\underline{x}_o(0),\underline{x}_o(1),\ldots,\underline{x}_o(t);t) + \underline{n}_o(t) \quad (2.1)$$

where $\underline{h}_o$ is an m-dimensional single-valued function differentiable with respect to the arguments, and $\underline{n}_o(t)$ is m-dimensional, zero-mean white Gaussian noise with a positive-definite covariance matrix R_o denoted by $R_o > 0$. Throughout the report, a subscript "o" refers to the true model. Note that the probability density function of $\underline{z}(0),..,\underline{z}(t_1)$, denoted by P_o is well defined and uniquely determined by System (2.1).

A mathematical model different from (2.1) is generally used. Let the mathematical model be given by

$$\underline{x}(t+1) = \underline{f}(\underline{x}(t),t) \quad , \quad \text{initial condition } \underline{x}(0) \tag{2.2a}$$

$$\underline{z}(t) = \underline{h}(\underline{x}(t),t) + \underline{n}(t) \tag{2.2b}$$

where $\underline{x}$ is a q-dimensional vector and $\underline{n}(t)$ is a zero-mean white Gaussian noise process with a positive-definite covariance matrix R. It is also assumed that $\underline{f}$ and $\underline{h}$ possess the same analytic properties as $\underline{h}_o$. The probability density function induced by (2.2) is denoted by P. Furthermore, we assume

$$\underline{h}(\underline{\phi x}_o(0,t),t) = \underline{h}(\underline{x}_o(0,t),t) \tag{2.3}$$

where $\underline{\phi}$ is a function that maps the nt-dimensional Euclidean space to the q-dimensional Euclidean space and $\underline{x}_o(0,t) = (x_o(0),\ldots,\underline{x}_o(t))$.

Many practical problems can be formulated by (2.2) for estimating the initial state $\underline{x}(0)$ from measurements. Equation (2.2a) describes the physical law governing the state vector, whereas $\underline{h}$ in (2.2b) models the measurement function. In reality, the exact physical law is either not known completely or is too complicated to be applied directly. On the other hand, the functional relationship between a given state vector and the deterministic measurements is usually known. However, exact statistical properties of measurement noise $\underline{n}(t)$ are seldom known. The trajectory estimation problem is a typical example that fits the above description exactly. The ballistic trajectory of an object is governed by Newton and Euler equations. An important part of the driving forces and torques in Newton and Euler equations is due to air pressure. In aerodynamics, pressure is best modeled by a potential equation which describes the velocity field of the air. It is impossible with current technology to incorporate a potential equation with Newton and Euler equations into the framework of the trajectory estimation problem. On the other hand, what a radar can measure about the target motion is modeled by (2.3).

The solution of the non-linear difference equation (2.2a) is unique and denoted by $\underline{x}(t;\underline{x}(0))$. The Jacobian matrices F(t) and H(t) are defined by

$$F(t) = \frac{\partial \underline{f}(\underline{x},t)}{\partial \underline{x}} \Big|_{\underline{x} = \underline{x}(t;\underline{x}(0))} \tag{2.4}$$

$$H(t) = \frac{\partial \underline{h}(\underline{x},t)}{\partial \underline{x}} \Big|_{\underline{x} = \underline{x}(t;\underline{x}(0))} \tag{2.5}$$

The transition matrix of F(t) denoted by $\phi(t,\tau)$ satisfies the following difference equation

$$\phi(t+1,\tau) = F(t)\,\phi(t,\tau) \;\; ; \;\; \phi(\tau,\tau) = I \tag{2.6}$$

where I is the q x q identity matrix. Furthermore, we define the R-observability Gramian $M(\underline{x}(0);t_1)$ by

$$M(\underline{x}(0);t_1) \triangleq \sum_{\tau=0}^{t_1-1} \phi^T(\tau,0)H^T(\tau)R^{-1}H(\tau)\phi(\tau,0) \tag{2.7}$$

where superscripts "T" and "-1" denote matrix transpose a[illegible] inverse respectively. The meaning of the observability [illegible]amian with respect to observability and unbiased estimation of [illegible]tem (2.2) is described in [10].

In the first part of this report, we address the problem of estimating $\underline{x}(t)$ from the observed sample for two different situations. When the form of $\underline{f}$ in (2.2a) is fixed (except for the initial condition $\underline{x}(0)$), we shall classify this case as a fixed dynamical model. We shall call the other case a tunable dynamical model when the functional form of $\underline{f}$ is not fixed.

3. DISCRETE-TIME UNCERTAIN NON-LINEAR SYSTEMS

A. MKLI and MLE

The mean Kullback-Leibler's information (MKLI) is a function of the likelihood ratio which gives a measure of separation between two probability distributions. The normalized MKLI of (2.1) and (2.2) is given by

$$W(t_1) \triangleq -\frac{2}{t_1} E_o\left(\ell n \frac{P(Z_0^{t_1})}{P_o(Z_0^{t_1})}\right) \tag{3.1}$$

$$= L(\underline{x}(0),R,t_1) + \frac{2}{t_1} E_o(\ell n P_o(Z_0^{t_1}))$$

where

$$L(\underline{x}(0),R,t_1) = \ln|R| + \bar{d}\ (\underline{x}(0),R,t_1) \qquad (3.2)$$

and $\bar{d}$ is defined by the following two equations:

$$d(\underline{x}(0),R,t_1) \triangleq \frac{1}{t_1} \sum_{t=0}^{t_1-1} ||\underline{z}(t)-\underline{h}(\underline{x}(t;\underline{x}(0)),t||^2_{R^{-1}} \qquad (3.3)$$

$$\bar{d}(\underline{x}(0),R,t_1) \triangleq E_o d(\underline{x}(0),R,t_1)$$

$$= \mathrm{Tr}(R_o R^{-1}) + \frac{1}{t_1} \sum_{t=0}^{t_1-1} ||\underline{h}_o(\underline{x}_o(0,t),t)-\underline{h}(\underline{x}(t;\underline{x}(0),t)||^2_{R^{-1}}. \qquad (3.4)$$

Note that E_o is the expectation operator with respect to the true probability density function P_o, $|\cdot|$ denotes the determinant of the enclosed matrix, "Tr" denotes the trace of a matrix, and $||\cdot||$ denotes the Euclidean norm. A smaller value of $W(t_1)$ means that the corresponding Model (2.2) is closer to the truth in the sense of MKLI.

The maximum likelihood estimate (MLE) of $(\underline{x}(0),R)$ denoted by $(\hat{\underline{x}}(0/t_1),\ \hat{R}(t_1))$ is defined to be the minimum point of

$$J(\underline{x}(0),R,t_1) \triangleq \ln|R| + d(\underline{x}(0),R,t_1) \qquad (3.5)$$

It is easy to verify that

$$L(\underline{x}(0),R,t_1) = E_0 J(\underline{x}(0),R,t_1) \qquad (3.6)$$

Note that the second term of (3.1) is a constant for a given observed sample and independent of the assumed mathematical model. Because of this fact and (3.6), it is not surprising to see that the MKLI and the MLE have very close relationships. Indeed, they are shown to be equivalent with probability one with respect to the true probability density function (w.p.1, P_0) asymptotically if the limit of L with respect to t_1 is unimodal [8].

B. FIXED DYNAMICAL MODELS

We shall first establish the unimodal condition of L $(\underline{x}(0),R,t_1)$ for any finite t_1 and then state the equivalent relationship between the MLE and the MKLI for the model given by (2.2).

It requires three steps to establish the unimodal condition of $L(\underline{x}(0),R,t_1)$. The first step is summarized by the following theorem.

Theorem 3.1 For a given $R > 0$, we hypothesize that

(i) $\underline{x}(0) \varepsilon S$, where S is a convex and compact subset of R^q

(ii) the observability Gramian $M(\underline{x}(0);t_1) > 0$ for all $\underline{x}(0) \varepsilon S$

(iii) for any $\underline{b}_1$, $\underline{b}_2 \varepsilon S$ that minimize $\bar{d}\,(\underline{x}(0),R,t_1)$

and $\underline{c}(t) \varepsilon R^q$ for each t, there exists $\underline{b}_3$ in S

such that

$$\sum_{t=0}^{t_1-1} \left|\left|2\underline{h}(\underline{c}(t)) - (\underline{h}(\underline{b}_1) + \underline{h}(\underline{b}_2))\right|\right|^2_{R^{-1}} \geq 4 \sum_{t=0}^{t_1-1} \left|\left|\underline{h}(\underline{c}(t)) - \underline{h}(\underline{b}_3)\right|\right|^2_{R^{-1}}$$

Under the above three hypotheses, there exists a unique minimum point of $\bar{d}(\underline{x}(0),R,t_1)$ in S.

Proof The existence of a minimum point in S is guaranteed by the hypothesis that $\bar{d}$ is a continuous function defined over a compact set S. Let $\underline{b}_1$ and $\underline{b}_2$ be two minimum points of $\bar{d}$ in S. Let

$$\underline{h}_i = \underline{h}(\underline{x}(t;\underline{b}_i),t) \quad \text{for} \quad i = 1,2.$$

By the parallelogram law, we have

$$2[||\underline{h}_o-\underline{h}_1||^2_{R^{-1}} + ||\underline{h}_o-\underline{h}_2||^2_{R^{-1}}] \tag{3.7}$$

$$= ||\underline{h}_1-\underline{h}_2||^2_{R^{-1}} + ||2\underline{h}_o-(\underline{h}_1+\underline{h}_2)||^2_{R^{-1}}$$

By Hypotheses (i), (ii) and by (2.3), we have

$$\underline{b}_0 = \frac{1}{2}(\underline{b}_1+\underline{b}_2)\varepsilon S, \text{ and}$$

$$\sum_{t=0}^{t_1-1} ||2\underline{h}(\underline{\phi x}_o(0,t)) - (\underline{h}_1+\underline{h}_2)||^2_{R^{-1}} \geq 4 \sum_{t=0}^{t_1-1} ||\underline{h}(\underline{\phi x}_o(0,t))-\underline{h}(\underline{b}_0)||^2_{R^{-1}} \tag{3.8}$$

for all $\underline{x}_O(0,t)\varepsilon R^{nt}$. Let $\underline{c}(t) = \underline{\phi}(\underline{x}_O(0,t))$. By the fact that both $\underline{b}_1$ and $\underline{b}_2$ are minimum points of $\bar{d}$ and by (3.7) as well as (3.8), we have

$$\sum_{t=0}^{t_1-1} ||\underline{h}_1-\underline{h}_2||^2_{R^{-1}} = 4 \sum_{t=0}^{t_1-1} ||\underline{h}(\underline{c}(t)) - h_1||^2_{R^{-1}} - \sum_{t=0}^{t_1-1} ||2\underline{h}(\underline{c}(t)) - (\underline{h}_1+\underline{h}_2)||^2_{R^{-1}}$$

$$\leq 0 \tag{3.9}$$

Equation (3.9) implies that

$$\sum_{t=0}^{t_1-1} ||h_1-h_2||^2_{R^{-1}} = 0 \tag{3.10}$$

By Hypothesis (ii) and Corollary 2.1.1 of [10], (3.10) implies

$$\underline{b}_1 = \underline{b}_2 \qquad \text{Q.E.D.}$$

Now let S be a set of initial states for Model (2.2). For a fixed vector $\underline{b}_1$ in S, it is known (for example, use the technique introduced in [11]) that there exists a unique covariance matrix given by

$$R_1 = R_0 + X(\underline{b}_1, t_1) \tag{3.11}$$

which minimizes $L(\underline{x}(0), R, t_1)$ where $X(\underline{b}_1, t_1)$ is defined by

$$X(\underline{b}_1,t_1) = \frac{1}{t_1} \sum_{t=0}^{t_1-1} (\underline{h}_o-\underline{h}(\underline{x}(t;\underline{b}_1),t))(\underline{h}_o-\underline{h}(\underline{x}(t;\underline{b}_1),t))^T \tag{3.12}$$

Let C be a subset of m x m positive-definite matrices containing R_0 and those generated by S through (3.11) and (3.12). Finally, the existence of a unique model is summarized by the following theorem.

<u>Theorem 3.2</u> Let S be a convex and compact set of initial states, and C be the partially-ordered set of positive-definite matrices defined above. If Hypotheses (ii) and (iii) of Theorem 3.1 hold for all R in C then there exists a unique point in S x C which minimizes $L(\underline{x}(0),R,t_1)$.

<u>Proof</u> By Theorem 3.1, there exists a unique vector $\underline{b}$ in S which minimizes $\bar{d}(\underline{x}(0),R,t_1)$ for a given R in C. We can construct a sequence in S x C as follows:

(1) Let $\underline{b}_1$ be the unique vector in S which minimizes $\bar{d}(x(0),R_o,t_1)$ and

$$R_1 = R_o + X(\underline{b}_1,t_1)$$

(2) $\underline{b}_k$ is defined to be the unique vector in S which minimizes $\bar{d}(\underline{x}(0),R_{k-1},t_1)$ and

$$R_k = R_o + X(\underline{b}_k,t_1)$$

Since $X(\underline{b},t_1)$ is continuous over a compact S, S x C is also compact. There exists a limit point of $(\underline{b}_k,R_k)$ denoted by $(\hat{b},\hat{R})$ in the compact set S x C. We shall prove that the limit point is unique.

By construction, we have

$$\bar{d}(\underline{b}_{k+1}, R_k, t_1) \leq \bar{d}(\underline{b}_k, R_k, t_1) \qquad (3.13)$$

Therefore, by definition and (3.13), we have

$$L(\underline{b}_{k+1}, R_{k+1}, t_1) \leq L(\underline{b}_{k+1}, R_k, t_1)$$

$$\leq L(\underline{b}_k, R_k, t_1)$$

Hence, $\{L(\underline{b}_k, R_k, t_1)\}$ is non-increasing and bounded below in R^1. Thus, the limit exists and is denoted by L_∞. Let $\{(\underline{b}_{k'}, R_{k'})\}$ be a subsequence such that

$$\lim_{k' \to \infty} (\underline{b}_{k'}, R_{k'}) = (\hat{\underline{b}}, \hat{R})$$

Since $L(\underline{b}, R, t_1)$ is a continuous function defined over S x C, we have

$$\lim_{k' \to \infty} L(\underline{b}_{k'}, R_{k'}, t_1) = L(\hat{\underline{b}}, \hat{R}, t_1)$$

Due to the unique property of a convergent sequence, we have

$$L(\hat{\underline{b}},\hat{R},t_1) = L_\infty$$

Suppose that $(\underline{b}_1,R_1)$ is a minimum point of L in S x C. If $\underline{b}_1=\hat{\underline{b}}$ then (3.11) implies that $\hat{R}=R_1$. If $\underline{b}_1 \neq \hat{\underline{b}}$, by Hypotheses (ii) and (iii) of Theorem 3.1 for all R in C, we should have either $X(\hat{\underline{b}},t_1) > X(\underline{b}_1,t_1)$ or $X(\hat{\underline{b}},t_1) < X(\underline{b}_1,t_1)$. Assuming $X(\hat{\underline{b}},t_1) > X(\underline{b}_1,t_1)$, we have

$$\bar{d}(\underline{b}_1,\hat{R},t_1) < \bar{d}(\hat{\underline{b}},\hat{R},t_1). \tag{3.14}$$

Inequality (3.14) is a contradiction because $\hat{\underline{b}}$ minimizes $\bar{d}(\underline{x}(0),R,t_1)$. On the other hand, if we have

$$\bar{d}(\hat{\underline{b}},R_1,t_1) < \bar{d}(\underline{b}_1,R_1,t_1)$$

then

$$L(\hat{\underline{b}},R_1,t_1) < L(\underline{b}_1,R_1,t_1) \tag{3.15}$$

Again, it contradicts the assumption that $(\underline{b}_1,R_1)$ is a minimum point of L. Therefore, $\underline{b}_1=\hat{\underline{b}}$ and $R_1=\hat{R}$.

Q.E.D.

The equivalent relationship between the MLE and the MKLI is established by two steps. For the first step, we shall assume that Theorem 3.2 holds asymptotically as t_1 approaches infinity, and let $(\hat{b},\hat{R})$ be the unique minimum point of $L_1(\underline{x}(0),R)$ in S x C, where $L_1(\underline{x}(0),R)$ is the limiting function of $L(\underline{x}(0),R,t_1)$. By Theorem 1 of [8], we have

$$\lim_{t_1 \to \infty}(\hat{\underline{x}}(0/t_1),\ \hat{R}(t_1)) = (\hat{b},\hat{R}), \quad \text{w.p.1. } P_O \qquad (3.16)$$

Note that $(\hat{\underline{x}}(0/t_1),\hat{R}(t_1))$ is the MLE of $(\underline{x}(0),R)$.

It is proved in [6] that the MKLI defined by (3.1) is a non-negative quantity and is equal to zero if and only if $P(Z_0^{t_1}) = P_O(Z_0^{t_1})$ w.p.1. P_O. This property is a basis for MKLI to provide a measure of distance between the truth (2.1) and Model (2.2). For the second step of establishing the equivalence between the MLE and the MKLI, we shall prove that this important property is preserved when $(\underline{x}(0),R)$ is replaced by the MLE $(\hat{\underline{x}}(0/t_1),\hat{R}(t_1))$ in the definition of MKLI. For this purpose, we have the following theorem.

<u>Theorem 3.3</u> Let S x C be the same set defined in Theorem 3.2 such that (3.16) holds for all $(\underline{x}(0),R) \varepsilon S \times C$ and $(\hat{\underline{b}},\hat{R})$ is the unique minimum point of $L(\underline{x}(0),R)$ in S x C. Let $(\hat{\underline{x}}(0/t_1),\hat{R}(t_1))$ be the MLE which minimizes $J(\underline{x}(0),R,t_1))$ over S x C. Then,

$$\lim_{t_1 \to \infty} E_O\, J(\hat{\underline{x}}(0/t_1),\ \hat{R}(t_1),t_1) = L(\hat{\underline{b}},\hat{R}) \qquad (3.17)$$

Proof By the definition of $\bar{d}(\underline{x}(0),R,t_1)$, we have

$$E_o J^2(\underline{x}(0),R,t_1) \leq 2[(\ln|R|)^2 + U(\underline{x}(0),R,t_1)] \tag{3.18}$$

where

$$U(\underline{x}(0),R,t_1) = \frac{1}{t_1^2} E_o\Big(\sum_{t=0}^{t_1-1} ||\underline{z}(t) - \underline{h}(\underline{x}(t;\underline{x}(0),t)||^2_{R^{-1}}\Big)^2 \tag{3.19}$$

By the hypothesis that $L(\underline{x}(0),R,t_1)$ converges for all $(\underline{x}(0),R) \in S \times C$, we have for sufficiently large t_1

$$U(\underline{x}(0),R,t_1) = (\mathrm{Tr}\, R_o R^{-1})^2 + \frac{1}{t_1^2}\Big(\sum_{t=0}^{t_1-1} ||\underline{h}_o - h||^2_{R^{-1}}\Big)^2 \tag{3.20}$$

$$+ 0\Big(\frac{1}{t_1}\Big)$$

where we have

$$\lim_{t_1 \to \infty} 0\Big(\frac{1}{t_1}\Big) = 0$$

By (3.18)-(3.20), $E_0 J_2^2(\underline{x}(0),R,t_1)$ is uniformly bounded in t_1 for all $(\underline{x}(0),R) \varepsilon S \times C$. Since $(\hat{x}(0/t_1),\hat{R}(t_1)) \varepsilon S \times C$, $\hat{J}(\underline{x}(0/t_1),\hat{R}(t_1))$ is uniformly integrable for all t_1. By the continuity assumption of J over S x C and the uniformly integrable theorem, we have

$$\lim_{t_1 \to \infty} E_0\, J(\hat{\underline{x}}(0/t_1),\hat{R}(t_1),t_1)$$

$$= E_0 \lim_{t_1 \to \infty} J(\hat{\underline{x}}(0/t_1),\hat{R}(t_1),t_1)$$

$$= L(\hat{\underline{b}},\hat{R}) \qquad \text{Q.E.D.}$$

Theorem 3.3 assures that the basis for MKLI to be an information measure is preserved asymptotically if $(\underline{x}(0),R)$ is replaced by $(\hat{\underline{x}}(0/t_1),\hat{R}(t_1))$ because the second term in (3.1) is independent of the assumed model. We shall call it the generalized mean Kullback-Leibler's information (GMKLI) if the estimate is used to replace $(\underline{x}(0),R)$ in the definition (3.1). The idea of GMKLI is applied in the next few subsections for tuning process noise.

C. TUNABLE DYNAMICAL MODELS

For a given sample function $Z_0^{t_1}$, an extended Kalman filter can be constructed based on the mathematical model given by (2.2). The predicted estimate of $\underline{x}(t)$ denoted by $\hat{\underline{x}}(t)$ is derived by

$$\hat{\underline{x}}(t) = F(t-1)\hat{\underline{x}}(t-1) + G(t-1)\,\underline{v}(t-1) \qquad (3.21a)$$

$$\underline{v}(t-1) = \underline{z}(t-1) - H(t-1)\,\hat{\underline{x}}(t-1) \qquad (3.21b)$$

where F(t-1) and H(t-1) are the Jacobian matrices of $\underline{f}(\underline{x}(t-1),t-1)$ and $\underline{h}(\underline{x}(t-1),t-1)$ evaluated at the updated estimate of $\underline{x}(t-1)$. The matrix G(t) in (3.21a) is given by

$$G(t) = F(t)\;K(t) \qquad (3.21c)$$

$$K(t) = \Sigma(t)H^T(t)[H(t)\Sigma(t)H^T(t) + R]^{-1} \qquad (3.21d)$$

$$\Sigma(t) = F(t-1)\Sigma^+(t-1)F^T(t-1) + Q \qquad (3.21e)$$

$$\Sigma^+(t) = [I-K(t)H(t)]\,\Sigma(t) \qquad (3.21f)$$

where Q is a non-negative definite matrix which is often called the covariance matrix of process noise that models the mismatch of Model (2.2). The state estimate $\hat{\underline{x}}(t)$ can be computed recursively for all t, $0 \le t \le t_1$, if $\hat{\underline{x}}(0)$, $\Sigma(0)$, R, and Q are specified. Let $\underline{\theta}$ denote the totality of all parameters for specifying $\underline{x}(0)$, $\Sigma(0)$, Q; $\hat{\underline{x}}(t,\underline{\theta},R)$ denotes the dependence of the state estimate on the parameter $\underline{\theta}$ and R. The dynamical Model (2.2a) is transformed into (3.21a) and (3.21b) which certainly are tunable. We shall address how to tune the model from the observed sample.

D. GMKLI AND GMLE

The GMKLI of (3.21) with respect to (2.1) is defined by

$$\hat{W}(t_1) = \hat{L}(\underline{\theta},R,t_1) - L_O(t_1) \qquad (3.22a)$$

where

$$\hat{L}(\underline{\theta},R,t_1) = \ln\left|R\right| + \frac{1}{t_1}\sum_{t=0}^{t_1-1} E_O\left|\left|\underline{z}(t)-\underline{h}(\underline{\hat{x}}(t,\underline{\theta},R),t)\right|\right|^2_{R^{-1}} \qquad (3.22b)$$

$$L_O(t_1) = -\frac{2}{t_1}\, E_O(\ln\, P_O(Z_0^{t_1})) \qquad (3.22c)$$

$$= \ln\left|R_O\right| + n$$

We shall first show that (3.22) indeed defines an information measure.

<u>Theorem 3.4</u> If $R > 0$ then

$$\hat{W}(t_1) \geq 0$$

and the equality holds if and only if $(\underline{\theta},R)$ minimizes $\hat{L}$ and

$$E_o\left|\left|\underline{h}_o-\underline{h}(\hat{\underline{x}}(t;\underline{\theta},R,t)\right|\right|^2_{R^{-1}} = 0$$

for all $0 \leq t \leq t_1$.

Proof By recognizing that $\underline{n}_o(t)$ is orthogonal to $\hat{\underline{x}}(t;\underline{\theta},R)$, it is not difficult to show that

$$\hat{L}(\underline{\theta},R,t_1) = \ell n\left|R\right| + \mathrm{Tr}\,(R_oR^{-1})$$

$$+\frac{1}{t_1}\sum_{t=0}^{t_1-1} E_o\left|\left|\underline{h}_o-\underline{h}(\hat{\underline{x}}(t;\underline{\theta},R),t\right|\right|^2_{R^{-1}}$$

Since $L_o(t_1)$ is the unique minimum for $\ell n\left|R\right| + \mathrm{Tr}\,R_oR^{-1}$, we complete the proof of this theorem.

Note that $L_o(t_1)$ is independent of $\underline{\theta}$ and R; therefore, we do not need to know the exact value of $L_o(t_1)$ for the purpose of estimate comparison. The state estimate which yields the smallest $\hat{L}(\underline{\theta},R,t_1)$ defined by (3.22b) is considered as the best estimate in the sense of the GMKLI. In practical applications, however, $\hat{L}(\underline{\theta},R,t_1)$ cannot be computed directly because P_o is not known. To circumvent this problem, we shall establish the equivalence relationship between the GMKLI and the generalized maximum likelihood estimate (GMLE) which is computable from the observed sample and the assumed model. The

GMLE denoted by $(\hat{\underline{\theta}}(t_1), \hat{R}(t_1))$ of $(\underline{\theta}, R)$ is defined to be the minimum point of the following function

$$\hat{d}(\underline{\theta},R,t_1) \stackrel{\Delta}{=} \ln|R| + \frac{1}{t_1} \sum_{t=0}^{t_1-1} \left|\left| \underline{z}(t) - \underline{h}\,(\hat{\underline{x}}(t;\underline{\theta},R),t)\right|\right|^2_{R^{-1}} \quad (3.23)$$

As in the case of a fixed dynamical model, we shall first study the unimodal conditions of $\hat{L}(\underline{\theta},R,t_1)$ for a finite t_1. Let $\underline{b} \stackrel{\Delta}{=} (\underline{\theta},R)$ and

$$\psi(t) \stackrel{\Delta}{=} \frac{\partial \hat{\underline{x}}(t;\underline{\theta},R)}{\partial \underline{b}} \quad (3.24)$$

The generalized R-observability Gramian is defined by

$$\hat{M}(\underline{b};t_1) \stackrel{\Delta}{=} \sum_{t=0}^{t_1-1} \psi^T(t)\ H^T(t)\ R^{-1}\ H(t)\ \psi(t) \quad (3.25)$$

We have the following theorem.

<u>Theorem 3.5</u> Let T be a compact and convex set containing elements of $\hat{\underline{x}}(0)$, $\Sigma(0)$, Q, and R defined by (3.21) such that $\Sigma(0)$ and Q are positive-semidefinite and R is positive-definite. Furthermore, we hypothesize that

(i) $E_o\hat{M}(\underline{b};t_1) > 0$ for all $\underline{b} \in T$

(ii) for any $\underline{b}_1$, $\underline{b}_2 \in T$ and $\underline{c}(t) \in R^q$ for each t, we have the same condition as (iii) of Theorem 3.1.

Under the above hypotheses, there exists a unique minimum point of $\hat{L}(\underline{\theta},T,t_1)$ in T.

The proof of this theorem can be carried out by the same way done in Theorem 3.1. The equivalent relationship between the GMLE and the GMKLI can be established similarly as introduced in Section 3.B

When hypotheses of Theorem 3.5 are too difficult to examine, the equivalence between GMLE and GMKLI can be studied as follows. First, we observe that $L_o(t_1)$ in the definition of GMKLI (Eq. 3.22a) is independent of the assumed mathematical model. The equivalent relationship will be established if $\hat{L}(\underline{\theta},R,t_1)$ can be approximated by $\hat{d}(\underline{\theta},R,t_1)$ (see Eq. 3.23). The following theorem provides conditions that the above two quantities coincide asymptotically.

<u>Theorem 3.6</u> Under the hypotheses that

(i) $\underline{h}$ is uniformly bounded in both $\underline{x}$ and t

$$\text{(ii)} \lim_{t_1 \to \infty} \frac{1}{t_1} \sum_{t=0}^{t_1-1} \left[\left|\left| \underline{h}_o - \underline{h}(\hat{\underline{x}}(t;\underline{\theta},R),t) \right|\right|^2_{R^{-1}} \right.$$

$$\left. - E_o \left|\left| \underline{h}_o - \underline{h}(\hat{\underline{x}};\underline{\theta},R),t) \right|\right|^2_{R^{-1}} \right]$$

$$= 0 \qquad\qquad \text{w.p.1} \quad P_o,$$

$$\lim_{t_1 \to \infty} \left| \hat{d}(\underline{\theta},R,t_1) - \hat{L}(\underline{\theta},R,t_1) \right| = 0 \qquad \text{w.p.1} \quad P_o$$

Proof By (2.1), we have

$$\left|\left| \underline{z}(t) - \underline{h}(\hat{x}(t;\underline{\theta},R),t \right|\right|^2_{R^{-1}} = \left|\left| \underline{h}_o - \underline{h}(\hat{\underline{x}}(t;\underline{\theta},R),t) \right|\right|^2_{R^{-1}}$$

$$+ 2\underline{n}_o^T(t)\ R^{-1}(\underline{h}_o - \underline{h}(\hat{\underline{x}}(t;\underline{\theta},R),t)) + \left|\left| \underline{n}_o(t) \right|\right|^2_{R^{-1}}$$

We claim that

$$\text{(a)} \quad \lim_{t_1 \to \infty} \frac{1}{t_1} \sum_{t=0}^{t_1-1} \left|\left|\underline{n}_O(t)\right|\right|^2_{R^{-1}} = \operatorname{Tr} R_O R^{-1} \quad \text{w.p.1} \quad P_O$$

$$\text{(b)} \quad \lim_{t_1 \to \infty} \frac{1}{t_1} \sum_{t=0}^{t_1-1} \underline{n}_O^T(t) R^{-1}(\underline{h}_O - \underline{h}(\hat{\underline{x}}(t;\underline{\theta},R),t)) = 0 \quad \text{w.p.1} \quad P_O$$

Result (a) can be proved by the law of large numbers because $\underline{n}_O(t)$ is assumed to be a zero-mean white Gaussian process with a covariance matrix R_O. To prove (b), we first recognize that $\sum_{t=0}^{k} \underline{n}_O^T(t)R^{-1}(\underline{h}_O-\underline{h}(\hat{\underline{x}}(t;\underline{\theta},R),t)$ is a martingale sequence because $\hat{\underline{x}}(t;\underline{\theta},R)$ is orthogonal to the zero-mean white process $\underline{n}_O(t)$. By Hypotheses (i), (2.3) and the discrete version of the Khazminskii lemma [12] in [13], the claim (b) can be proved. By (a), (b), and Hypothesis (ii), we complete the proof of the theorem.

The insight of Hypothesis (ii) in Theorem 3.6 is enlightened when we restrict $\underline{h}$ to be linear with constant coefficients. In this case, Hypothesis (ii) becomes

$$\lim_{t_1 \to \infty} \frac{1}{t_1} \sum_{t=0}^{t_1-1} \left[\left|\left| H\tilde{\underline{x}}(t,\underline{\theta},R)\right|\right|^2_{R^{-1}} - E_O \left|\left| H\, \tilde{\underline{x}}(t,\underline{\theta},R)\right|\right|^2_{R^{-1}} \right] = 0 \quad \text{w.p.1} \quad P_O \qquad (3.26)$$

where H is a constant matrix and

$$\tilde{\underline{x}}(t,\underline{\theta},R) = \underline{\phi}(\underline{x}_O(0,t)) - \hat{\underline{x}}(t;\underline{\theta},R)$$

If the filter error becomes stationary and ergodic then certainly (3.26) holds. For most practical applications including nonlinear state estimation problems, we find that residuals exhibit stationary sample statistics as long as filtering divergence does not occur.

4. CONTINUOUS-TIME UNCERTAIN SYSTEMS

Basically, we require two substitutions in order to extend the concepts introduced in Section 3 to cover the continuous-time systems. First, we replace the ratio P/P_0 in the definitions of MKLI and GMKLI by the Radon-Nikodym derivative (RND) of the probability measures induced by the assumed model and the true stochastic process. Secondly, we replace the likelihood functions in the definitions of MLE and GMLE by a likelihood ratio in a form of a RND with respect to a certain reference measure. These two substitutions are necessary because we are dealing with an uncountable sample space of a continuous-time stochastic process.

For example, if we model the continuous-time system by a diffusion process of an Ito differential equation [15], then the reference measure can be chosen as the Wiener measure defined over the space of continuous functions [16]. Furthermore, the RND of two Ito differential equations is well studied in the literature, e.g., [15] and [17]. After the appropriate substitutions are carried out, the analysis procedure introduced in Section 3 is directly applicable. Here, we only present two remarks that have been overlooked by researchers in this area, e.g., in [18].

To introduce these two remarks, we first look at the following simple example. Suppose that we use the model given by

$$dz(t) = a\, x(t)\, dt + \sigma\, dB(t) \tag{4.1}$$

to represent the observed scalar diffusion process z(t), where B(t) is the standard Brownian motion. However, the true representation of z(t) is given by

$$dz(t) = a_o x_o(t)\ dt + \sigma_o\ dB(t) \tag{4.2}$$

There are two problems if we want to use the maximum likelihood principle to estimate a and σ based on (4.1) and the observed sample. The first problem arises because the induced measure u and u_o of (4.1) and (4.2) respectively are singular to each other if $\sigma \neq \sigma_o$. This fact can be proved by the result reported in [19]. The second problem is explained as follows. Suppose, $\sigma = \sigma_o$, and suppose that we use the formulae provided in [20] and [21] for the RND directly. We should have

$$\ln(\mathrm{RND1}) = \frac{1}{\sigma^2}\left[\int_0^{t_1}(ax-a_o x_o)dz - \frac{1}{2}\int_0^{t_1}(a^2x^2-a_o^2x_o^2)dt\right] \tag{4.3}$$

Considering Eq. 4.3 as a function of a and σ, it is obvious that the function is not unimodal in a and σ. We arrive at the following two observations regarding the above example.

Remark 1: The representation of z(t) can be given by

$$u : \text{same as (4.1) but } z(0) \sim N(0,\sigma^2)$$

$$u_o: dz(t) = a_o x_o(t)dt + \sigma dB(t) \; ; \qquad (4.4)$$

$$z(0) \sim N(0,\sigma_o \;)$$

Remark 2: The RND of (4.4) is given by

$$\ln\left(\frac{du}{du_o}\right) = \ln(\text{RND1}) - \ln\left(\frac{\sigma}{\sigma_o}\right) + \frac{1}{2} z^2(0)\cdot[\sigma_o^{-2} - \sigma^{-2}] \qquad (4.5)$$

The above two remarks are the direct consequence of Theorem 5.3 of [15]. It is also clear that (4.5) is an unimodal function of a and σ. Inspired by Akaike's original idea, we appreciate the logarithmic term.

5. CONCLUSION

We follow Akaike's original idea to exploit the connection between the mean Kullback-Leibler's information and the maximum likelihood principle to cover the estimation problem of non-linear systems with significant model uncertainties. We introduce the concept of the generalized mean Kullback-Leibler's information and establish its relationship with the generalized maximum likelihood principle. The results of this paper have been applied to the trajectory estimation problem. Finally, we present two remarks concerning the extension of the earlier part of this paper to the diffusion process generated by the Ito differential equation.

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The relationship between the (generalized) mean Kullback-Leibler's information and the (generalized) maximum likelihood principle is exploited in this report to analyze the state estimation problems of both discrete-time and continuous-time uncertain non-linear systems.

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