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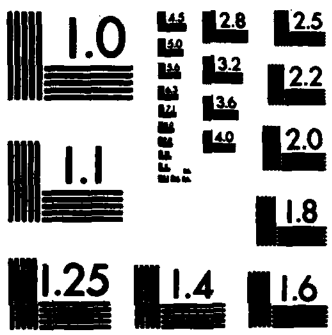
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A TWO-WAY LAYOUT PROBLEM*

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Technical Report #83-41

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Abstract

This paper deals with a nonparametric subset selection procedure for a two-way layout problem. The treatment effect with the largest unknown value is of interest to us. The block effect is a nuisance parameter in this problem. The proposed procedure is based on the Hodges-Lehmann estimators of location parameters. The asymptotic relative efficiency of the proposed procedure with the normal means procedure is evaluated. It is shown that the proposed procedure has a high efficiency.

Key words: Nonparametric procedure, subset selection, two-way layout, treatment effect, nuisance parameter, asymptotic relative efficiency.

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1. Introduction

Consider a two-factor complete block design with one observation per cell. Let the observable random variables be $X_{i\alpha}$, $i = 1,2,\ldots,k$; $\alpha = 1,2,\ldots,n$ and consider the linear model

$$(1.1)\qquad X_{i\alpha} = \mu + \theta_i + \beta_\alpha + \varepsilon_{i\alpha},\ \sum_{i=1}^{k} \theta_i = 0,$$

where $X_{i\alpha}$ is the observation under treatment i in the αth block, μ is the mean-effect, θ_i is the effect of treatment i, β_α is the block effect for the αth block (nuisance parameter), and the $\varepsilon_{i\alpha}$, $\alpha = 1,2,\ldots,n$ are error components. It is assumed that the error components are independent and identically distributed with a continuous cumulative distribution function (cdf) $F(\underline{\varepsilon})$, $\underline{\varepsilon} \in \mathbb{R}^k$ (the real k-space), where $F(\underline{\varepsilon})$ is symmetric in its arguments, that is, for any $\underline{\varepsilon} \in \mathbb{R}^k$ and any permutation $(i_1,\ldots,i_k)$ of $(1,\ldots,k)$, we have

$$(1.2)\qquad F(\varepsilon_1,\ldots,\varepsilon_k) = F(\varepsilon_{i_1},\ldots,\varepsilon_{i_k}).$$

Let $\theta_{[1]} \leq \cdots \leq \theta_{[k]}$ be the ordered θ_i's. Suppose that we are interested in the treatment with the largest unknown parameter $\theta_{[k]}$ (if more treatments than one have θ_i equal to $\theta_{[k]}$, then exactly one of these treatments is "tagged" as the best treatment). Correct selection denotes the selection of any subset containing the population with $\theta_{[k]}$ (or the "tagged" population).

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For the nonparametric approach to the one-way layout problems, most previous authors have considered procedures based on a class of rank order statistics. It was pointed out by Rizvi and Woodworth (1970) that there are difficulties associated with these procedures, mainly because the least favorable configuration is usually not known. Randles (1970) and Ghosh (1973) have considered the procedures for the one-way layout based on the Hodges-Lehmann estimators. It was shown by them that the procedures based on the Hodges-Lehmann estimators have high efficiency. In this paper we use the results of Puri and Sen (1967) to derive a subset selection procedure for the largest unknown parameter $\theta_{[k]}$.

2. Robust Compatible Estimation

In the model (1.1), for $1 \leq i < j \leq k$, let $X_{ij,\alpha} = X_{i\alpha} - X_{j\alpha}$, $e_{ij,\alpha} = \varepsilon_{i\alpha} - \varepsilon_{j\alpha}$, $\alpha = 1,2,\ldots,n$ and $\Delta_{ij} = \theta_i - \theta_j$, then for a fixed α, the αth block, we can write

(2.1) $$X_{ij,\alpha} = \Delta_{ij} + e_{ij,\alpha}.$$

From Assumption (1.2), $e_{ij,\alpha}$ have common distribution, say G, which is symmetric about zero. Hence $X_{ij,1},\ldots,X_{ij,n}$ are i.i.d. with common cdf $G(x-\Delta_{ij})$, for $1 \leq i < j \leq k$. We assume that G is continuous, but otherwise unknown. Let $R_{ij,\alpha}$ = Rank of $|X_{ij,\alpha}|$ among $|X_{ij,1}|,\ldots,|X_{ij,n}|$ and let $\underline{X}_{ij} = (X_{ij,1},\ldots,X_{ij,n})$. Consider the one-sample signed rank statistic

(2.2) $$h_{ij,n}(\underline{X}_{ij}) = n^{-1} \sum_{\alpha=1}^{n} E_{n,\alpha} Z_{n,\alpha}$$

Where $Z_{n,\alpha}$ is either one or zero as follows: if the αth smallest observation among $|X_{ij,1}|,\ldots,|X_{ij,n}|$ corresponds to $X_{ij,t}$ (for some t), then $Z_{n,\alpha} = 1$ if $X_{ij,t} > 0$ or 0 if $X_{ij,t} < 0$. $E_{n,\alpha}$ is the expected value of the

αth order statistic of a sample of size n from a distribution $\psi^*(x)$ given by

$$(2.3)\qquad \psi^*(x) = \begin{cases} \psi(x) - \psi(-x) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Throughout this paper, we shall assume that $\psi(x)$ and $G(x)$ satisfy the following assumptions (see Puri (1964) and Puri and Sen (1967)):

(1) $\psi(x)$ is symmetric about $x = 0$, that is $\psi(x) + \psi(-x) = 1$.

(2) $\frac{1}{n}\sum_{\alpha=1}^{n}[E_{n,\alpha} - \psi^{-1}(\frac{\alpha}{n+1})]Z_{n,\alpha} = O_p(n^{-\frac{1}{2}})$.

(3) $J(u) = \psi^{-1}(u)$, $0 \leq u \leq 1$ is absolutely continuous and $|J^{(i)}(u)| = |d^iJ(u)/du^i| \leq M[u(1-u)]^{-i-\frac{1}{2}+\delta}$, $i = 0,1,2$, for some M and some $\delta > 0$.

(4) G is a continuous cdf, differentiable in each of the open intervals $(-\infty,a_1)$; $(a_1,a_2),\ldots,(a_{s-1},a_s)$, (a_s,∞), for some $a_1,\ldots,a_s$ and the derivative of G is bounded in each of these intervals.

(5) The function $\frac{d}{dx}J(G(x))$ is bounded as $x \to \pm\infty$.

It is easy to see that $h_{ij,n}(x_{ij,1}+a,\ldots,x_{ij,n}+a)$ is a non-decreasing function of a for fixed $\underline{x}_{ij}$ and when $\Delta_{ij} = 0$, the distribution of $h_{ij,n}$ is symmetric about a fixed point $\mu = \frac{1}{2}E_\psi|V|$, where V has cdf ψ.

Let

$$(2.4)\qquad \begin{aligned} \Delta^*_{ij} &= \sup\{\Delta:\ h_{ij,n}(\underline{x}_{ij}-\Delta\underline{1}) > \mu\}, \\ \Delta^{**}_{ij} &= \inf\{\Delta:\ h_{ij,n}(\underline{x}_{ij}-\Delta\underline{1}) < \mu\} \end{aligned}$$

and let

$$\hat{\Delta}_{ij} = \frac{1}{2}(\Delta^*_{ij} + \Delta^{**}_{ij}).$$

Now from Hodges and Lehmann (1963), $\hat{\Delta}_{ij}$ is a translation invariant robust estimator of Δ_{ij} and has a distribution which is symmetric about Δ_{ij}. Note that the estimates $\hat{\Delta}_{ij}$ are incompatible (see Lehmann (1964)) in the sense that they do not satisfy the linear relations satisfied by the differences they estimate. This leads to certain ambiguities. To derive compatible estimators, let

$$\hat{\Delta}_i = \frac{1}{k} \sum_{\ell=1}^{k} \hat{\Delta}_{i\ell}, \ \hat{\Delta}_{ii} \equiv 0 \quad \text{for } i = 1,2,\ldots,k. \tag{2.5}$$

Then by minimizing $\sum_{i \neq j} (\hat{\Delta}_{ij}-\Delta_{ij})^2$ with respect to θ's, we obtain the compatible or adjusted estimators of Δ_{ij} as

$$Z_{ij} = \hat{\Delta}_i - \hat{\Delta}_j, \quad i \neq j. \tag{2.6}$$

Note that $E(\hat{\Delta}_i) = \theta_i$ since $\sum_{i=1}^{k} \theta_i = 0$, hence $E(Z_{ij}) = \Delta_{ij}$. Puri and Sen (1967) have proved the following theorem:

<u>Theorem 2.1</u>. The joint distribution of $\{n^{\frac{1}{2}}(Z_{ik}-\Delta_{ik});\ i = 1,2,\ldots,k-1\}$ is asymptotically normal with zero means and a covariance matrix $\Gamma = (\gamma_{ij}), i,j = 1,2,\ldots,k-1$ where

$$\gamma_{ij} = \begin{cases} 2\sigma_0^2 & \text{if } i = j \\ \sigma_0^2 & \text{if } i \neq j \end{cases} \tag{2.7}$$

and $\sigma_0^2 = [A^2 + (k-2)\lambda_J(G)]/kB^2$,

where

$$A^2 = \int_0^1 J^2(u)du, \quad B = \int_{-\infty}^{\infty} \frac{d}{dx} J(G(x))dG(x), \tag{2.8}$$

$$\text{and} \quad \lambda_J(G) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J(G(x))J(G(y))dG^*(x,y),$$

$G^*(x,y)$ is the joint cdf of $e_{ij,\alpha}$ and $e_{i\ell,\alpha} (j \neq \ell)$ whose marginal cdf's are $G(x)$ and $G(y)$, respectively.

Moreover, using the translation invariant property of $\hat{\Delta}_{ij}$, we have the following lemma:

Lemma 2.2.

(1) $\hat{\Delta}_i(x_{11}-c_1,\ldots,x_{1n}-c_1,\ldots,x_{k1}-c_k,\ldots,x_{kn}-c_k)$

$= \hat{\Delta}_i(x_{11},\ldots,x_{1n},\ldots,x_{k1},\ldots,x_{kn}) + \bar{c}-c_i,$

where $\bar{c} = \frac{1}{k}\sum_{i=1}^{k} c_i$.

(2) The distribution of $\hat{\Delta}_i-\theta_i$ is independent of $\underline{\theta}$.

3. A Nonparametric Procedure for Selecting the Best Treatment

Based on the estimators defined in (2.5), we propose a selection procedure R_1 as follows:

(3.1) R_1: Select treatment i iff $\hat{\Delta}_i \geq \max_{1\leq j\leq k} \hat{\Delta}_j - d_1$,

where $d_1 > 0$ is determined so as to satisfy the basic probability requirement.

Let $\Omega = \{\underline{\theta} = (\theta_1,\ldots,\theta_k)\}$ be the parameter space and CS stand for a correct selection which means that the selected subset contains the best treatment. For a given constant $P^*(k^{-1} < P^* < 1)$, the basic probability requirement is

$$\inf_{\underline{\theta}\in\Omega} P_{\underline{\theta}}(CS|R_1) \geq P^*.$$

Let $\hat{\Delta}_{[1]} \leq\cdots\leq \hat{\Delta}_{[k]}$ denote the ordered $\hat{\Delta}_i$'s and $\hat{\Delta}_{(i)}$ denote the unknown estimator associated with the parameter $\theta_{[i]}$, $1 \leq i \leq k$. If $P_j(\underline{\theta}|R_1)$ denotes the probability that the treatment (j) is selected (treatment (j) is associated with parameter $\theta_{[j]}$). We have the following lemma:

<u>Lemma 3.1</u>. $P_j(\underline{\theta}|R_1)$ is increasing in $\theta_{[j]}$ when all other components of $\underline{\theta}$ are fixed, and is decreasing in $\theta_{[i]}$, $i \neq j$, when all other components of $\underline{\theta}$ are fixed.

Proof. $P_j(\underline{\theta}|R_1) = P_{\underline{\theta}}(\hat{\Delta}_{(j)} \geq \max_{\underline{1 \leq i \leq k}} \hat{\Delta}_{(i)} - d_1)$

$$= P_{\underline{\theta}}(\hat{\Delta}_{(i)} - \theta_{[i]} - \hat{\Delta}_{(j)} + \theta_{[j]} \leq d_1 + \theta_{[j]} - \theta_{[i]},\ i \neq j,\ i = 1,\ldots,k).$$

By Lemma 2.2, the distribution of $\hat{\Delta}_{(i)} - \theta_{[i]} - \hat{\Delta}_{(j)} + \theta_{[j]}$, $i \neq j$, $i = 1,2,\ldots,k$ is independent of $\underline{\theta}$. Hence $P_j(\underline{\theta}|R_1)$ is increasing in $\theta_{[j]}$ and is decreasing in $\theta_{[i]}$, $i \neq j$. Note the above property is usually called strong monotonicity property.

<u>Corollary 3.2</u>. $\inf_{\underline{\theta} \in \Omega} P_{\underline{\theta}}(CS|R_1) = \inf_{\underline{\theta} \in \Omega_0} P_{\underline{\theta}}(CS|R_1)$ which is independent of $\theta_1 = \ldots = \theta_k = \theta$, where $\Omega_0 = \{\underline{\theta} \in \Omega | \theta_1 = \ldots = \theta_k\}$.

Proof. The proof follows from Lemma 3.1.

For large sample we can define d_1 as in Theorem 3.3 given below.

<u>Theorem 3.3</u>. For given $P^*(k^{-1} < P^* < 1)$. If $\sigma_0^2 < \infty$, we have

(3.2) $\quad d_1(n) = n^{-\frac{1}{2}} d\sigma_0 + o(n^{-\frac{1}{2}})$ as $n \to \infty$,

where d is the solution to the equation

(3.3) $\quad Q(d/\sqrt{2},\ldots,d/\sqrt{2}) = P^*$,

Q is the joint cdf of a normally distributed vector $(V_1,\ldots,V_{k-1})$ with

(3.4) $\quad E(V_i) = 0$, $Var(V_i) = 1$ and $Cov(V_i,V_j) = 1/2$, $i \neq j$.

Proof. $\liminf\limits_{\substack{n\to\infty \\ \underset{\sim}{\theta}\in\Omega}} P_{\underset{\sim}{\theta}}(CS|R_1) = \lim\limits_{n\to\infty} P_0(\hat{\Delta}_{(i)}-\hat{\Delta}_{(k)} \le d_1(n),\ i = 1,2,\ldots,k-1)$

$$= P(V_i \le \lim_{n\to\infty} n^{\frac{1}{2}} d_1(n)/\sqrt{2}\,\sigma_0,\ i = 1,2,\ldots,k-1), \text{ by Theorem 2.1,}$$

where P_0 denotes the probability is computed under $\theta_1 =\ldots= \theta_k$. If d is the solution of (3.3), then

$$d_1(n) = n^{-\frac{1}{2}} d\sigma_0 + o(n^{-\frac{1}{2}}) \quad \text{as } n \to \infty.$$

Remark: The solution of (3.3) is also a solution of $\int_{-\infty}^{\infty} \Phi^{k-1}(x+d)\, d\Phi(x) = P^*$, where Φ is the cdf of standard normal. This has been shown by many authors (see for example, Gupta (1963)).

Determination of the Minimum Common Sample Size

Let $E(S|R_1)$ denote the expected size of the selected subset using rule R_1, then $E(S|R_1) = \sum_{j=1}^{k} P_j(\underset{\sim}{\theta}|R_1)$. Having determined $d_1(n)$ from (3.2), one may determine the common sample size n by imposing the additional requirement that $E(S|R_1) \le 1+\varepsilon$, for some $\varepsilon > 0$, whenever $\underset{\sim}{\theta}$ lies in a given proper subset of Ω, for example, the subset defined by

$$\theta_{[1]} =\ldots= \theta_{[k-1]} = \theta_{[k]} - \delta^*,\ \delta^* > 0. \tag{3.5}$$

It will be convenient in the sequel to replace (3.5), when the sample size is n, by

$$\theta_{[1]} =\ldots= \theta_{[k-1]} = \theta_{[k]} - \delta^{(n)}. \tag{3.6}$$

(See Bartlett and Govindarajulu (1968)).

Theorem 3.4. For given $\varepsilon > 0$, with $d_1(n)$ given by (3.2) and n determined by $E(S|R_1) \le 1+\varepsilon$ for $\underset{\sim}{\theta}$ satisfying (3.6). Then as $n \to \infty$,

$$\delta^{(n} - n^{-\frac{1}{2}} c' /\sigma_0 + o(n^{-\frac{1}{2}}), \tag{3.7}$$

where $c(\varepsilon)$ is the solution to the equation

(3.8) $$Q((c+d)/\sqrt{2},\ldots,(c+d)/\sqrt{2}) + (k-1)Q(d/\sqrt{2},\ldots,d/\sqrt{2},(d-c)/\sqrt{2}) = 1+\varepsilon,$$

where Q is defined as in Theorem 3.3.

Proof. $$E(S|R_1) = \sum_{j=1}^{k-1} P_{\underline{\theta}}(n^{\frac{1}{2}}(\hat{\Delta}_{(i)}-\hat{\Delta}_{(j)})/\sqrt{2}\sigma_0 \le n^{\frac{1}{2}}d_1(n)/\sqrt{2}\sigma_0,\ i = 1,\ldots,k-1,$$
$$i \neq j,\ n^{\frac{1}{2}}(\hat{\Delta}_{(k)}-\hat{\Delta}_{(j)}-\delta^{(n)})/\sqrt{2}\sigma_0 \le n^{\frac{1}{2}}(d_1(n)-\delta^{(n)})/\sqrt{2}\sigma_0)$$
$$+ P_{\underline{\theta}}(n^{\frac{1}{2}}(\hat{\Delta}_{(i)}-\hat{\Delta}_{(k)}+\delta^{(n)})/\sqrt{2}\sigma_0 \le n^{\frac{1}{2}}(d_1(n)+\delta^{(n)})/\sqrt{2}\sigma_0,\ i=1,2,\ldots,k-1).$$

If $\underline{\theta}$ satisfies (3.6), then

$$\lim_{n\to\infty} E(S|R_1) = \sum_{j=1}^{k-1} P(V_i \le \lim_{n\to\infty} n^{\frac{1}{2}}d_1(n)/\sqrt{2}\sigma_0,\ i = 1,\ldots,k-1,\ i\neq j,$$
$$V_{k-1} \le \lim_{n\to\infty} n^{\frac{1}{2}}(d_1(n)-\delta^{(n)})/\sqrt{2}\sigma_0)+P(V_i \le \lim_{n\to\infty} n^{\frac{1}{2}}(d_1(n)+\delta^{(n)})/\sqrt{2}\sigma_0,$$
$$i = 1,2,\ldots,k-1)$$
$$= (k-1)Q(d/\sqrt{2},\ldots,d/\sqrt{2},\ (d-c)/\sqrt{2})+Q((d+c)/\sqrt{2},\ldots,(d+c)/\sqrt{2}) = 1+\varepsilon.$$

Hence $c(\varepsilon)$ is the solution of (3.8) iff

$$\delta^{(n)} = n^{-\frac{1}{2}}c(\varepsilon)\sigma_0 + o(n^{-\frac{1}{2}}) \text{ as } n \to \infty.$$

<u>Remark</u>: The common sample size n required to satisfy $\inf_{\underline{\theta}\in\Omega} P_{\underline{\theta}}(CS|R_1) = P^*$ and $E(S|R_1) \le 1+\varepsilon$ for $\underline{\theta}$ satisfying (3.5) is $(c(\varepsilon)\sigma_0/\delta^*)^2$. Note that n is a function of k, P^*, δ^*, and ε.

4. A Selection Procedure for the Normal Case

If we assume that $(\varepsilon_{1\alpha},\ldots,\varepsilon_{k\alpha})$ are jointly normally distributed with zero means and the covariance matrix $\sigma^2\begin{pmatrix} 1 & & \rho \\ & \ddots & \\ \rho & & 1 \end{pmatrix}$, $\alpha = 1,2,\ldots,n$ where $-1/(k-1) \le \rho < 1$ is known and $\sigma^2 < \infty$, may be known or unknown. The usual least square estimator of θ_i would be $\bar{X}_i-\bar{X}$ where $\bar{X}_i = \frac{1}{n}\sum_{\alpha=1}^{n} X_{i\alpha}$ and $\bar{X} = \frac{1}{k}\sum_{i=1}^{k}\bar{X}_i$. It is easy to show that the vector $(\bar{X}_1-\bar{X},\ldots,\bar{X}_k-\bar{X})$ has a joint normal distribution with mean vector $(\theta_1,\ldots,\theta_k)$ and the covariance matrix

$$(4.1) \qquad \frac{\sigma^2(1-\rho)}{n}\begin{pmatrix} 1-\frac{1}{k} & & -\frac{1}{k} \\ & \ddots & \\ -\frac{1}{k} & & 1-\frac{1}{k} \end{pmatrix}$$

and hence the vector $(\bar{X}_1 - \bar{X}_k,\ldots,\bar{X}_{k-1}-\bar{X}_k)$ has a joint normal distribution with mean vector $(\theta_1-\theta_k,\ldots,\theta_{k-1}-\theta_k)$ and the covariance matrix

$$(4.2) \qquad \frac{2\sigma^2(1-\rho)}{n}\begin{pmatrix} 1 & & \frac{1}{2} \\ & \ddots & \\ \frac{1}{2} & & 1 \end{pmatrix}.$$

We assume that σ^2 is known and propose a selection procedure R_2 by

$$(4.3) \qquad R_2\text{: Select treatment } i \text{ iff } \bar{X}_i \geq \max_{1\leq j\leq k}\bar{X}_j - d_2.$$

It is easy to see that $\inf_{\underline{\theta}\in\Omega} P_{\underline{\theta}}(CS|R_2) = \inf_{\underline{\theta}\in\Omega_0} P_{\underline{\theta}}(CS|R_2)$ and is independent of $\theta_1 = \ldots = \theta_k = \theta$. Thus, similar to Theorem 3.3 and Theorem 3.4, we have the following theorems:

<u>Theorem 4.1</u>. For given P^* $(k^{-1} < P^* < 1)$, if we let $\inf_{\underline{\theta}\in\Omega} P_{\underline{\theta}}(CS|R_2) = P^*$, then

$$(4.4) \qquad d_2(n) = n^{-\frac{1}{2}} d\sqrt{1-\rho}\,\sigma,$$

where d is the solution of (3.3).

<u>Theorem 4.2</u>. For given $\varepsilon > 0$ and $d_2(n) = n^{-\frac{1}{2}} d\sqrt{1-\rho}\,\sigma$, if n is determined by $E(S|R_2) \leq 1+\varepsilon$ for $\underline{\theta}$ satisfying (3.6), then as $n \to \infty$

(4.5) $\delta^{(n)} = n^{-\frac{1}{2}} c(\varepsilon)\sqrt{1-\rho}\ \sigma,$

where $c(\varepsilon)$ is the solution of (3.8).

Suppose that the joint distribution F of $(\varepsilon_{1\alpha},\ldots,\varepsilon_{k\alpha})$ is unknown, but the variance of $\varepsilon_{1\alpha}$ is finite. By the central limit theorem, the joint distribution of $\{n^{\frac{1}{2}}(\bar{X}_i-\bar{X}_k-\theta_i+\theta_k);\ i = 1,2,\ldots,k-1\}$ is asymptotically normal with zero means and the covariance matrix (4.2). We can still use the procedure R_2 given by (4.3). For large samples we have $d_2(n) = n^{-\frac{1}{2}}d\sqrt{1-\rho}\ \sigma + o(n^{-\frac{1}{2}})$ and $\delta^{(n)} = n^{-\frac{1}{2}}c(\varepsilon)\sqrt{1-\rho}\ \sigma + o(n^{-\frac{1}{2}})$, where d is the solution of (3.3) and $c(\varepsilon)$ is the solution of (3.8).

For any two procedures R_1 and R_2 satisfying the basic P*-condition, let us define the asymptotic relative efficiency, say $\mathrm{ARE}(R_1,R_2) = \lim_{\varepsilon\downarrow 0} n_{R_2}(\varepsilon)/n_{R_1}(\varepsilon)$ for the given parametric configuration (3.5), where $n_{R_i}(\varepsilon)$, $i = 1,2$ are the sample sizes required to achieve the same expected size, $1+\varepsilon$. Then we have the following theorem:

<u>Theorem 4.3</u>. $\mathrm{ARE}(R_1,R_2) = \{2\sigma^2(1-\rho)B^2/A^2\{[kA^2/2[A^2+(k-2)\lambda_J(G)]\}$,where A^2, B and $\lambda_J(G)$ are defined in (2.8).

Proof. For procedure R_1, putting $\delta^{(n)} = \delta^*$, from (3.7) we have

$$n_{R_1}(\varepsilon) = (c(\varepsilon)\sigma_0/\delta^*)^2,$$

where $c(\varepsilon)$ is the solution of (3.8). (Note that $\varepsilon \downarrow 0$, then $n \to \infty$).
Similarly, for procedure R_2, we have

$$n_{R_2}(\varepsilon) = (c(\varepsilon)\sqrt{1-\rho}\ \sigma/\delta^*)^2.$$

Hence

$$\mathrm{ARE}(R_1,R_2) = (1-p)\sigma^2/\sigma_0^2$$

$$= \{2\sigma^2(1-\rho)B^2/A^2\}\{kA^2/2[A^2+(k-2)\lambda_J(G)]\}.$$

Remarks:

(1) Barlow and Gupta (1969) define $ARE(R_1,R_2) = \lim_{\epsilon \downarrow 0} n_{R_2}(\epsilon)/n_{R_1}(\epsilon)$ for the given parametric configuration (3.5), where $n_{R_i}(\epsilon)$, $i = 1,2$ are the sample sizes required to achieve the same expected size (say ϵ) of non-best populations selected. If we consider the case where expected size refers only to the number of non-best populations in the selected subset, we have $n_{R_1}(\epsilon) = (c'(\epsilon)\sigma_0/\delta^*)^2$, where $c'(\epsilon)$ is the solution to the equation

$$(k-1)Q(d/\sqrt{2},\ldots,d/\sqrt{2},\ (d-c')/\sqrt{2}) = \epsilon.$$

Similarly, we have $n_{R_2}(\epsilon) = (c'(\epsilon)\sqrt{1-\rho}\ \sigma/\delta^*)^2$, and hence

$$ARE(R_1,R_2) = (1-\rho)\sigma^2/\sigma_0^2$$

which is the same as in Theorem 4.3.

(2) Puri and Sen (1967) proved that $\lambda_J(G) \leq \frac{1}{2} A^2$, hence $kA^2/2[A^2+(k-2)\lambda_J(G)] \geq 1$ and $ARE(R_1,R_2) \geq 2(1-\rho)\sigma^2B^2/A^2$. The variance of G is $2(1-\rho)\sigma^2$, hence $2(1-\rho)\sigma^2B^2/A^2$ is the ARE of the one-sample rank order tests (for location) with respect to the Student's t-test when the parent distribution is $G(x)$. If we use the normal scores estimator, we have $ARE(R_1,R_2) \geq 1$. If we use the Wilcoxon scores estimator, then for any F, we have $ARE(R_1,R_2) \geq 0.864$ and $ARE(R_1,R_2) = 3/\pi$ when F is normal. Hence the procedure given by (3.1) has "high" efficiency.

5. Relative Performance of R_1 and R_2

In Section 4, we consider the parameter points satisfying (3.6). When the condition (3.6) is not satisfied, but the ratio of sample sizes, m for R_1

and n for R_2, satisfies $\lim_{n\to\infty} \frac{n}{m} = (1-\rho)\sigma^2/\sigma_0^2$, then, for large n, the procedures R_1 and R_2 have approximately the same probability of a correct selection and expected size.

Theorem 5.1. Let n and $m = g(n)$ satisfy $\lim_{n\to\infty} \frac{n}{m} = (1-\rho)\sigma^2/\sigma_0^2$, then the procedures R_2 and R_1 have the same asymptotic probability of a correct selection and the same expected size for any parametric configuration.

Proof. For procedure R_2, consider any sequence of parameter points satisfying

$$\theta_{[k]}^{(n)} - \theta_{[i]}^{(n)} = \delta_{in} = n^{-\frac{1}{2}}\sqrt{1-\rho}\,\sigma\delta_i + o(n^{-\frac{1}{2}}),$$

$i = 1,\dots,k-1$ and for some i,j, $\delta_i \neq \delta^*$, $\delta_j \neq 0$. Then

$$\lim_{n\to\infty} P_{\underline{\theta}^{(n)}}(CS|R_2) = P(V_i \le \lim_{n\to\infty} n^{\frac{1}{2}}(d_2(n)+\delta_{in})/\sqrt{2(1-\rho)}\,\sigma,\ i = 1,\dots,k-1)$$
$$= P(V_i \le (d+\delta_i)/\sqrt{2},\ \ i = 1,2,\dots,k-1)$$
$$= Q((d+\delta_1)/\sqrt{2},\dots,(d+\delta_{k-1})/\sqrt{2})$$

and for $1 \le j \le k-1$,

$$\lim_{n\to\infty} P_j(\underline{\theta}^{(n)}|R_2) = P(V_i \le (d+\delta_i-\delta_j)/\sqrt{2},\ i = 1,2,\dots,k,\ \ i \neq j)$$
$$= Q((d+\delta_1-\delta_j)/\sqrt{2},\dots,(d+\delta_k-\delta_j)/\sqrt{2}).$$

For procedure R_1, $m^{-\frac{1}{2}}\sigma_0 \sim n^{-\frac{1}{2}}\sqrt{1-\rho}\,\sigma$, so

$$\theta_{[k]}^{(m)} - \theta_{[i]}^{(m)} = m^{-\frac{1}{2}}\sigma_0\delta_i + o(m^{-\frac{1}{2}}) = \delta_{im},\ i = 1,2,\dots,k-1.$$

Hence

$$\lim_{m\to\infty} P_{\underline{\theta}^{(m)}}(CS|R_1) = P(V_i \le (d+\delta_i)/\sqrt{2},\ \ i = 1,2,\dots,k-1)$$
$$= Q((d+\delta_1)/\sqrt{2},\dots,(d+\delta_{k-1})/\sqrt{2}).$$

and for $1 \le j \le k-1$,

$$\lim_{m\to\infty} P_j(\underline{\theta}^{(m)}|R_1) = Q((d+\delta_1-\delta_j)/\sqrt{2},\ldots,(d+\delta_k-\delta_j)/\sqrt{2}).$$

In the above parameter points, we assume that $\theta_{[k]}^{(n)} - \theta_{[i]}^{(n)}$ tend to zero at the $n^{-\frac{1}{2}}$ rate. If any difference tends to zero more rapidly, we replace δ_i by 0, and if it tends to zero more slowly, or tends to a finite limit, then we replace δ_i by ∞, and still obtain the same asymptotic behavior. This completes the proof of the above theorem.

6. Estimation of B and $\lambda_J(G)$

In practical application, for large n, the procedure R_1 can be rewritten as

(6.1) $\quad R_1$: Select treatment i iff $\hat{\Delta}_i \geq \max_{1\leq j\leq k} \hat{\Delta}_j - \frac{d\sigma_0}{\sqrt{n}}$,

where d is the solution of (3.3) or $\int_{-\infty}^{\infty} \Phi^{k-1}(x+d)d\Phi(x) = P^*$. However, σ_0 is still unknown. We need to find a consistent estimator of σ_0^2. Since $\sigma_0^2 = \{A^2+(k-2)\lambda_J(G)\}/kB^2$, where $A^2 = \int_0^1 J^2(u)du$ is known, but $B = \int_{-\infty}^{\infty} \frac{d}{dx} J(G(x))dG(x)$ and $\lambda_J(G) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(G(x))J(G(y))dG^*(x,y)$ are unknown. Our problem is to find consistent estimator of B and $\lambda_J(G)$.

The consistent estimators of B and $\lambda_J(G)$ can be found from (4.7) and Theorem 4.2 of Puri and Sen (1967); let these be $\hat{B}_n$ and $\hat{L}_n$, respectively. Then $\hat{\sigma}_0^2 = [A^2+(k-2)\hat{L}_n]/k\hat{B}_n^2$ is a consistent estimator of σ_0^2. Hence, for large n, the procedure R_1 is defined by

(6.2) $\quad R_1$: Select treatment i iff $\hat{\Delta}_i \geq \max_{1\leq j\leq k} \hat{\Delta}_j - \frac{d\hat{\sigma}_0}{\sqrt{n}}$.

<u>Remark</u>: If $\psi(x)$ is the cdf of U(-1,1), then $J(u) = 2u-1$, $0 \leq u \leq 1$, hence $A^2 = 1/3$. If $G'(x) = g(x)$ exists, then $B = 2\int g^2(x)dx$ and

$\lambda_J(G) = 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x)G(y)dG^*(x,y)-1$. It has been shown by Doksum (1967) that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x)G(y)dG^*(x,y)$ can be estimated by $\frac{1}{n(n-1)(n-2)k(k-1)(k-2)}$ {number of sixtuples $(i,j,\ell,\alpha,\beta,\gamma)$ with i,j,ℓ distinct; α,β,γ distinct, and $X_{i\alpha}-X_{j\alpha} < X_{i\beta}-X_{j\beta}$, $X_{i\alpha}-X_{\ell\alpha} < X_{i\gamma} - X_{\ell\gamma}$}.

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This paper deals with a nonparametric subset selection procedure for a two-way layout problem. The treatment effect with the largest unknown value is of interest to us. The block effect is a nuisance parameter in this problem. The proposed procedure is based on the Hodges-Lehmann estimators of location parameters. The asymptotic relative efficiency of the proposed procedure with the normal means procedure is evaluated. It is shown that the proposed procedure has a high efficiency.

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