MICROCOPY RESOLUTION TEST CHART
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MOMENT ESTIMATORS OF DEFECT PATTERN FREQUENCIES, ALLOWING FOR INSPECTION ERROR

by

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ABSTRACT

Unbiased estimators of the number of individuals in a lot possessing various patterns of types of defects are constructed. Explicit formulas are given for cases of two and three types of defect. Application of the formulas requires knowledge of the probabilities of various kinds of errors in the inspection process.

Key Words & Phrases: Inspection errors; Estimation; Quality control; Sampling inspection.
1. Introduction

In [1] [2], we introduced the term defect pattern, to reflect the incidence of defects of different types in the same individual. If there are $k$ types of defect, in all, under consideration, then an individual's defect pattern is represented by a binary $(0,1)$ sequence of $k$ digits, with the $j$-th digit equal to $0$, or $1$ according as the $j$-th type of defect is not, or is, present in the individual. There are $2^k$ possible different defect patterns.

We will denote the defect pattern $(g_1, g_2, ..., g_k)$ by $(g)$, and the number of individuals, in a lot of size $N$, possessing defect pattern $(g)$, by $D_g$.

Clearly

$$\sum (g)D_g = N$$

where $\sum (g) = \sum_{g_1=0}^{1} \sum_{g_2=0}^{1} \cdots \sum_{g_k=0}^{1}$ denotes summation over all $2^k$ patterns.

If a random sample of size $n$ is chosen (without replacement) from the lot, and $Z_g$ of these are found, on inspection, to have defect pattern $(g)$, then, if inspection is perfect a natural (and unbiased) estimator of $D_g$, obtained by equating observed and expected values of $Z_g$, is $NZ_g/n$. If inspection is imperfect, the situation is not so simple, but the same method can be used, provided the probabilities of correct and incorrect assignments of defect patterns are known. Since the resultant estimators - $D_g$, say - are obtained by equating first moments of sample and population values of the $Z_g$'s, we will call them moment estimators.

We will suppose that

(i) if a defect of type $j$ is present, the probability that it will be detected is $p_j$;
(ii) if a defect of type \( j \) is not present, the probability that it will be 
(erroneously) detected is \( p^j \); 

and 

(iii) for any defect pattern, decisions in regard to presence or absence of 
each type (or disjoint sets of types) of defect are mutually independent. 

This last assumption may not always be justified, but it serves as a useful 
point of reference. 

With assumptions (i)-(iii), the probability that an individual with defect 
pattern \( \mathbf{f} \) is classified, as a result of inspection, as having pattern \( \mathbf{g} \) is 

\[
p_{g|f} = \prod_{j=1}^{k} p_j^{f_j} \frac{g_j (1-f_j)}{(1-p_j) (1-g_j) (1-f_j)} (1-g_j) (1-f_j) 
\]

We will further assume that \( p_j > p^j \) for all \( j \). This does not affect many 
of our formulas, and it is a very reasonable assumption. It merely means that 
an individual is more likely to be classified as possessing defect type \( j \) if 
indeed it does possess this type of defect, than if it does not. 

2. Moment Estimators 

Our problem is to estimate the \( D\) 's from the observed values of the \( Z\) 's. 
The expected value of \( D\) is 

\[
E[Z_g] = nN^{-1} \sum (\mathbf{f}) D\mathbf{f} \mathbf{p} | \mathbf{f} 
\]

and so the moment estimators, \( \hat{D}\), satisfy the \( 2^k \) equations 

\[
Z_g = nN^{-1} \sum (\mathbf{f}) \hat{D} \mathbf{f} | \mathbf{f} 
\]

or, in matrix form 

\[
\hat{Z} = nN^{-1} \hat{D} \hat{P} 
\]

where \( \hat{Z} \) and \( \hat{D} \) are \( 2^k \)-rowed vectors with elements \( \{Z_g\} \) and \( \{\hat{D}_g\} \) respectively,
and \( \mathbf{P} \) is a \( 2^k \times 2^k \) matrix with \( p_{g|f} \) as the element in the \( g \)-th row and \( f \)-th column. From (3)',
\[
\mathbf{\tilde{D}} = \mathbf{N}^{-1} \mathbf{P}^{-1} \mathbf{z}
\]
where \( \mathbf{P}^{-1} \) is the left inverse of \( \mathbf{P} \). Since
\[
\mathbb{E}[\mathbf{\tilde{Z}}] = n \mathbf{N}^{-1} \mathbf{P} \mathbf{D}
\]
it follows that \( \mathbb{E}[\mathbf{\tilde{D}}] = \mathbf{D} \), so \( \mathbf{\tilde{D}} \) is an unbiased estimator of \( \mathbf{D} \) (\( \mathbb{E}[\mathbf{\tilde{D}}_g] = \mathbf{D}_g \) for all \( g \)).

However it is possible for some of the \( \mathbf{D}_g \)'s to be negative (though not all of them, because \( \sum (g) \mathbf{D}_g = \mathbf{N} \)). Similarly some (but not all) of the \( \mathbf{D}_g \)'s might exceed \( \mathbf{N} \). This latter possibility occurs much less frequently.

For \( k=1 \) (a single type of defect) we have, of course
\[
\mathbf{D}_1 = \mathbf{N} - \mathbf{D}_0, \quad \text{and} \quad \mathbb{E}[\mathbf{Z}_1] = n^{-1} \{ \mathbf{D}_1 \mathbf{p}_1 + (\mathbf{N} - \mathbf{D}) \mathbf{p}_1' \}
\]
\[
= n^{-1} \{ \mathbf{N} \mathbf{p}_1' + \mathbf{D} (\mathbf{p}_1 - \mathbf{p}_1') \}
\]
whence
\[
\mathbf{\tilde{D}}_1 = \mathbf{N} (\mathbf{p}_1 - \mathbf{p}_1')^{-1} (n^{-1} \mathbf{z}_1 - \mathbf{p}_1') \tag{5}
\]
Since \( \mathbf{p}_1 > \mathbf{p}_1' \),
\[
\mathbf{\tilde{D}}_1 < 0 \quad \text{if} \quad \mathbf{z}_1 < n \mathbf{p}_1' \tag{6.1}
\]
\[
\mathbf{\tilde{D}}_1 > \mathbf{N} \quad \text{if} \quad \mathbf{z}_1 > n \mathbf{p}_1' \tag{6.2}
\]

In practice, one would use a modified estimator such as
\[
\mathbf{\tilde{D}}_1^* = \begin{cases} 
0 & \text{if } \mathbf{\tilde{D}}_1 < 0 \\
\mathbf{\tilde{D}}_1 & \text{if } 0 \leq \mathbf{\tilde{D}}_1 \leq \mathbf{N} \\
\mathbf{N} & \text{if } \mathbf{N} < \mathbf{\tilde{D}}_1 
\end{cases} \tag{7}
\]
This estimator is not, in general, unbiased.

Table 1 shows values of
\[
\begin{align*}
(1) & \quad \Pr[\mathbf{\tilde{D}}_1 < 0] \\
(1i) & \quad \Pr[\mathbf{\tilde{D}}_1 > \mathbf{N}] \\
(1ii) & \quad \text{Bias (} \mathbf{\tilde{D}}_1^* \text{)} = \mathbb{E}[\mathbf{\tilde{D}}_1^*] - \mathbf{\tilde{D}}_1
\end{align*}
\]
for a few sets of values of the parameters.
3. **Distribution Theory**

Conditional on the numbers \( Y = \{Y_\xi\} \) of individuals with actual defect patterns \( \{(\xi)\} \) in the sample, the joint distribution of the \( 2^k \) numbers \( \{W_\xi|\xi\} \) of the \( Y_\xi \) individuals with pattern \( (\xi) \) which are assigned patterns \( \{(\xi)\} \) as a result of inspection is multinomial with parameters \( (Y_\xi;\{p_\xi|\xi\}) \). For different \( (\xi) \) and \( (\xi^*) \) and any \( (g) \), \( (g^*) \) (whether or not \( (g) \equiv (g^*) )) W_\xi|\xi \) and \( W_{\xi^*}|\xi^* \) are mutually independent binomial variables with parameters \( (Y_\xi, p_\xi|\xi) \), \( (Y_{\xi^*}, p_{\xi^*}|\xi^*) \) respectively. Since

\[
Z_g = \sum_{(\xi)} W_\xi | \xi
\]

the conditional joint distribution of \( Z \), given \( Y \), can be represented symbolically

\[
Z|Y \sim * \text{Multinomial } (Y_\xi;\{p_\xi|\xi\}) \quad (\xi)
\]

where " stands for "convolution" and " means "is distributed as".

To obtain the unconditional distribution of \( Z \), this conditional distribution has to be compounded with respect to \( Y \), which has a multivariate hypergeometric distribution with parameters \( n; D; N, \) so that

\[
\Pr[Y = y] = \frac{N!}{n!y!} \prod_{(\xi)} \left( \begin{array}{c} D_\xi \\ y_\xi \end{array} \right) \left( 0 \leq y_\xi \leq D_\xi; (\xi) \sum y_\xi = n \right)
\]

From (8) and (9), moments and product-moments of the \( Z_g \)'s can be evaluated (see [1], equations (17)).

4. **Variances of the Estimators**

From (4) the variance-covariance matrix of \( \hat{D} \) is

\[
\text{Var}(\hat{D}) = (N_0^{-1})^T \text{Var}(Z) (F^{-1})^T
\]

(where the superfix \((T)\) denotes 'transpose').
From [1] (equation (17))

\[
\text{var}(Z_g) = N^{-1} \sum (n-1) \overline{Z_g} - n \left( \overline{Z_g} \right) + \left( N-n \right) \overline{Z_g}^2 (N-1)^{-1} \tag{12.1}
\]

\[
\text{cov}(Z_g, Z_{g*}) = -n(n-1) \overline{Z_g} \overline{Z_{g*}} + (N-n) \overline{Z_g} \overline{Z_{g*}} (N-1)^{-1} \tag{12.2}
\]

where

\[
\overline{Z_g} = N^{-1} \sum (\xi) \xi D_g \xi
\]

\[
\overline{Z_{g*}} = N^{-1} \sum (\xi) \xi D_{g*} \xi
\]

and

\[
\overline{Z_g Z_{g*}} = N^{-1} \sum (\xi) \xi D_g \xi Z_{g*} \xi
\]

5. **Special Case: Uniform Accuracy of Inspection**

If \( p_j = p \) and \( p_j = p' \) for all \( j=1, \ldots, k \) - that is, the accuracy of inspection is the same for all types of defect - then

\[
p_{g|f} = \begin{pmatrix} s_{11} p_{0} & s_{10} & s_{01} & s_{00} \\ (1-p') & (1-p)(1-p') & (1-p)(1-p') & (1-p)(1-p') \end{pmatrix}
\]

where \( s_{ab} = \text{number of types (j) of defect for which} f_j = b, g_j = a \).

For \( k=2 \), we have

\[
\begin{pmatrix} g \\ f \end{pmatrix} = \begin{pmatrix} 00 & 01 & 10 & 11 \end{pmatrix} = \begin{pmatrix} 00 & 01 & 10 & 11 \end{pmatrix}
\]

\[
\begin{pmatrix} p' & (1-p') & (1-p) & (1-p) \\ (1-p') & p(1-p') & (1-p)p' & p(1-p) \\ (1-p') & p(1-p') & (1-p)p' & p(1-p) \\ p' & (1-p') & (1-p)p' & p(1-p) \\ p' & (1-p') & (1-p)p' & p(1-p) \end{pmatrix}
\]

\[
(p-p')^2 p^{-1} = \begin{pmatrix} p^2 & -p(1-p) & -p(1-p) & (1-p)^2 \\ -(1-p) & p(1-p') & (1-p)p' & -(1-p)(1-p') \\ -(1-p) & p(1-p') & (1-p)p' & -(1-p)(1-p') \\ p^2 & -p(1-p) & -p(1-p) & (1-p)^2 \end{pmatrix}
\]
Hence

\[
\tilde{D}_{00} = (Nn^{-1})(p-p')^{-2}\{p^2Z_{00}-p(1-p)(Z_{01}+Z_{10})+(1-p)^2Z_{11}\}
\]

\[
\tilde{D}_{01} = (Nn^{-1})(p-p')^{-2}\{pp'Z_{00}+p(1-p)Z_{01}+(1-p)p'Z_{10}-(1-p)(1-p')Z_{11}\}
\]

\[
\tilde{D}_{10} = (Nn^{-1})(p-p')^{-2}\{pp'Z_{00}+(1-p)p'Z_{01}+(1-p)pZ_{10}-(1-p)(1-p')Z_{11}\}
\]

\[
\tilde{D}_{11} = (Nn^{-1})(p-p')^{-2}\{p^2Z_{00}-p'(1-p')(Z_{01}+Z_{10})+(1-p')^2Z_{11}\}
\]

(14)

For example, if \( p = 0.90 \) and \( p' = 0.95 \) then

\[
\tilde{D}_{00} = Nn^{-1}(1.121Z_{00} - 0.125(Z_{01}+Z_{10}) + 0.014Z_{11})
\]

\[
\tilde{D}_{01} = Nn^{-1}(-0.062Z_{00} + 1.183Z_{01} + 0.007Z_{10} - 0.131Z_{11})
\]

\[
\tilde{D}_{10} = Nn^{-1}(-0.062Z_{00} + 0.007Z_{10} + 1.183Z_{10} - 0.131Z_{11})
\]

\[
\tilde{D}_{11} = Nn^{-1}(0.003Z_{00} - 0.066(Z_{01}+Z_{10}) + 1.249Z_{11})
\]

As is to be expected the major coefficient in the expression for \( D_{ab} \) is that of \( Z_{ab} \), but the contributions of the other terms is not negligible.
For $k = 3$ we have

$$(g) = \begin{pmatrix}
(0,0,0) & (0,0,1) & (0,1,0) & (1,0,0) & (0,1,1) & (1,0,1) & (1,1,0) & (1,1,1) \\
000 & (1-p')^3 & (1-p)(1-p')^2 & (1-p)(1-p')^2 & (1-p)(1-p')^2 & (1-p)^2(1-p') & (1-p)^2(1-p') & (1-p)^3 \\
111 & p,3 & pp'^2 & pp'^2 & pp'^2 & pp'^2 & pp'^2 & pp'^2 & p^3 
\end{pmatrix}$$

$$(f) = \begin{pmatrix}
(0,0,0) & (0,0,1) & (0,1,0) & (1,0,0) & (0,1,1) & (1,0,1) & (1,1,0) & (1,1,1) \\
000 & p^3 & -p^2(1-p) & -p^2(1-p) & -p^2(1-p) & p(1-p)^2 & p(1-p)^2 & p(1-p)^2 & -(1-p)^3 \\
\end{pmatrix}$$

$$(p-p')^3p^{-1} = \begin{pmatrix}
(0,0,0) & (0,0,1) & (0,1,0) & (1,0,0) & (0,1,1) & (1,0,1) & (1,1,0) & (1,1,1) \\
000 & p^3 & -p^2(1-p) & -p^2(1-p) & -p^2(1-p) & p(1-p)^2 & p(1-p)^2 & p(1-p)^2 & -(1-p)^3 \\
\end{pmatrix}$$
For example

\[ \bar{p}_{011} = (Nn^{-1})(p-p')^{-3}(pp'^2Z_{000} - pp'(1-p')(Z_{001} + Z_{010}) - (1-p)p'Z_{100} \]

\[ + p(1-p')^2Z_{011} + (1-p)p'(1-p')(Z_{101} + Z_{110}) - (1-p)(1-p')^2Z_{111}. \]  \hspace{1cm} (16)

In these circumstances, if also \( D_f = 2^{-k}N \) for all \( f \), then

\[ \bar{p}_g = 2^{-k}(p+p')^{m(2-p-p')} k - m \]  \hspace{1cm} (17.1)

where \( m(= m(g)) \) is the number of 1's in \( g \):

\[ \bar{p}_g = 2^{-k}(p^2+p'^2)^m(1-p)^2+(1-p')^2 \]  \hspace{1cm} (17.2)

and

\[ \bar{p}_g p^* = 2^{-k}(p^2+p'^2)^{m_{11}}(p(1-p')+(1-p)p')^{m_{01}}(1-p)^2+(1-p')^2 \]  \hspace{1cm} (17.3)

where

\[ m_{ab}(= m_{ab}(g^*, g^*)) \] is the number of types \( j \) of defect with \( g_j = a, g_j^* = b. \)

\[ m_{11} + m_{10} + m_{01} + m_{00} = k. \]

The further specialization \( p' = 1-p \), corresponding to a constant probability \( 1-p \) of error, leads to further simplification. In particular,

\[ \bar{p}_g = 2^{-k} ; \bar{p}_g = 2^{-k}(p^2+(1-p)^2)^k \] for all \( g \) \hspace{1cm} (18.1)

\[ \bar{p}_g p^* = 2^{-k}(p^2+(1-p)^2)^{m_{00}+m_{11}}(2p(1-p))^2 \]  \hspace{1cm} (18.2)

Note that \( (m_{00} + m_{11}) \) equals the number of \( j \)'s for which \( g_j = g_j^* \); and \( p^2 + (1-p)^2 = 1-2p(1-p) \). We will use the notation

\[ m_{00} + m_{11} = 6; \ p^2+(1-p)^2 = A; \ 2p(1-p) = 1-A = B \]  \hspace{1cm} (19)

From (12) and (18)

\[ \text{var}(Z_g) = n \cdot 2^{-k}(1 - \frac{N-n}{N-1} \cdot 2^{-k} - \frac{n-1}{N-1} A^k) \]  \hspace{1cm} (20.1)

\[ \text{cov}(Z_g, Z_g^*) = -n \cdot 2^{-k} \frac{N-n}{N-1} \cdot 2^{-k} + \frac{n-1}{N-1} A^k B^k - 8 \]  \hspace{1cm} (20.2)
As an example consider

\[ \overline{E}_{011} = (Nn^{-1})(2p-1)^{-6}p(1-p)^2(Z_{000} + Z_{101} + Z_{110}) - p^2(1-p)(Z_{111} + Z_{001} + Z_{010}) \]

\[ + p^3Z_{011} - (1-p)^3Z_{100} \]  

(21)

obtained from (16) by putting \( p' = 1-p \).

We obtain from (20) and (21)

\[ \text{var}(\overline{E}_{011}) = \frac{1}{n} N^2 \left\{ \frac{A^3}{(2p-1)^6} - \frac{N-9n+8}{8(N-1)} \right\} \]  

(22)

(Details are given in the Appendix. Note that

\[ A^3(2p-1)^{-6} = \frac{1}{8} \left( 1+(1-2p)^{-2} \right)^3. \]  

The proportion \( D_{011}/N \) (=2\(^{-k}\) in this case) is estimated unbiasedly by \( \overline{E}_{011}/N \) and this estimator has variance

\[ \frac{1}{64n} \left\{ (1+(1-2p)^{-2})^3 - \frac{N-9n+8}{N-1} \right\} - \frac{1}{64n} \left\{ (1+(1-2p)^{-2})^3 - 1 \right\} \]  

if \( n < N \).

REFERENCES


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APPENDIX: Derivation of the Variance of $D_{011}$

From (20) and (21)

$$(Nn^{-1})^{-2}(2p-1)^6 \text{var}(\tilde{D}_{011}) = \frac{1}{8}nS_2 - \frac{n(N-n)}{64(N-1)} S_1 - \frac{n(n-1)}{8(N-1)} [A^3S_2 + \\
2(p^2(1-p)^4(3AB^2)+p^4(1-p)^2(3AB^2) + \\
p^3(1-p)^3(B^2+2AB^2+4AB^2+2B^3)-p^3(1-p)^3B^3 \\
+ p^4(1-p)^2(3AB^2)+p^2(1-p)^4(3AB^2) + \\
= \frac{1}{8}nS_2 - \frac{n(N-n)}{64(N-1)} S_1 - \frac{n(n-1)}{8(N-1)} [A^3S_2 \\
- 6(p(1-p)^5+2p^3(1-p)^3+p^5(1-p)A^2B + \\
+12(p^2(1-p)^4+p^4(1-p)^2)AB^2-8p^3(1-p)^3B^3] \\
= \frac{1}{8}nS_2 - \frac{n(N-n)}{64(N-1)} S_1 - \frac{n(n-1)}{8(N-1)} (A^3S_2-3A^4B^2+3A^2B^4-B^6)$$

where $S_2 = \text{sum of squares of coefficients of } Z's \text{ in } (Nn^{-1})^{-1}(2p-1)^3 \tilde{D}_{011}$

$$= 3p^2(1-p)^4+3p^4(1-p)^2 + p^6(1-p)^6 = A^3$$

and $S_1 = \text{sum of coefficients of } Z's \text{ in } (Nn^{-1})^{-1}(2p-1)^3 \tilde{D}_{011}$

$$= 3p(1-p)^2-3p^2(1-p)+p^3-(1-p)^3 = (2p-1)^3$$

Noting that since $A+B = 1$,

$$A^3S_2 - 3A^4B^2 + 3A^2B^4 - B^6 = (A^2-B^2)^3 = (A-B)^3 = (2p-1)^6$$

we obtain

$$\text{var}(\tilde{D}_{011}) = \frac{1}{n} \cdot \frac{N^2}{8} \left( \frac{A^3}{(2p-1)^6} - \frac{N-9n+8}{8(n-1)} \right) = \frac{1}{n} \cdot \frac{N^2}{64} \left[ (1+(2p-1)^{-2})^{-3} \cdot 1 + \frac{8(n-1)}{N-1} \right]. \quad (22)$$

The proportion $D_{011}/N$ is estimated unbiasedly by $\tilde{D}_{011}/N$; this estimator has variance

$$\frac{1}{64n} \left[ (1+(2p-1)^{-2})^{-3} \cdot 1 + \frac{8(n-1)}{N-1} \right]. \quad (22)$$
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