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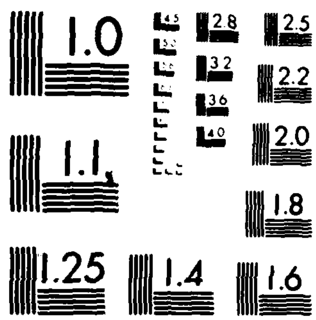

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MRC Technical Summary Report #2556

CAUCHY FLUX AND SETS OF FINITE PERIMETER

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August 1983

(Received July 21, 1983)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

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William P. Ziemer*

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ABSTRACT

A Cauchy flux Q is a real-valued, additive, area-bounded function whose domain is the class of all Borel subsets of the reduced boundary of sets of finite perimeter. If the flux Q is also volume bounded, it is shown that Q can be represented as the integral of the normal component of some vector field.

AMS (MOS) Subject Classification: 80A05, 49F20

Key Words: Cauchy flux, sets of finite perimeter, Cauchy's Theorem.

Work Unit Number 1 - Applied Analysis

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Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. Supported in part by a grant from the National Science Foundation.

SIGNIFICANCE AND EXPLANATION

Balance laws of the form

$$(*) \qquad \frac{d}{dt} E_t(B) = Q_t(B, B^e)$$

are basic to classical physics. For example, (*) represents balance of energy for a rigid heat conductor provided $E_t(B)$ is the internal energy of B and $Q_t(B,B^e)$ is the heat flow into B from its exterior B^e. Fundamental axioms of continuum physics require that (*) holds for any subbody A of B and that $Q_t(A,C)$ be well-defined for A and C in a suitably large class of sets. It is shown that Q_t can be represented as a flux over the reduced boundaries of sets of finite perimeter, thus showing that sets of finite perimeter form the suitably large class of sets in which it is possible to establish an axiomatic development of continuum physics.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

CAUCHY FLUX AND SETS OF FINITE PERIMETER

William P. Zeimer*

1. Introduction. Balance laws of the form

$$\frac{d}{dt} E_t(B) = Q_t(B,B^e) \tag{1}$$

are basic to classical physics. For example, (1) represents balance of energy for a rigid heat conductor provided $E_t(B)$ is the internal energy of B and $Q_t(B,B^e)$ is the heat flow into B from its exterior B^e. Also, (1) represents balance of momentum if $E_t(B)$ is the momentum of B and $Q_t(B,B^e)$ is the force exerted on B by its exterior B^e. Fundamental axioms of continuum physics require that (1) holds, not only for B, but for any subbody A of B and that $Q_t(A,C)$ be well-defined for any pair (A,C) where A is a body, C is a body or the exterior of a body, and A and C are separate in the sense that they intersect at most along their boundaries.

The determination of the appropriate class of sets for the family of subbodies is fundamental in the axiomatic development of continuum physics. Indeed, first considerations require that the family of subbodies be closed under intersection and union, that the concept of separate subbodies be meaningful, and that the boundary of a subbody be sufficiently regular to facilitate the basic operations of analysis. In particular, the boundary must have a general (and useful) notion of exterior normal. If domains with

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piece-wise smooth boundaries are considered as the family of subbodies, inconsistencies appear in the axiomatic structure.

In work that is currently being developed by Morton E. Gurtin, William O. Williams, and the author, it is shown that the class generated by sets of finite perimeter (see §2 below) provides the appropriate context for the axiomatic development of continuum physics. In that work it is shown that there is a function q such that

$$Q(A,C) = \int_S q(x)\,dH^{n-1}(x) \tag{2}$$

where $S = A \cap C$ is the surface of contact between A and C. In the classical context, Cauchy assumed that q in (2) depends on S only through the normal at x. Noll [N] has shown that Cauchy's assumption actually follows from the general balance law (1) under reasonable assumptions.

In this paper we will establish Noll's theorem in the more general framework of sets of finite perimeter. This result is of independent mathematical interest and is intimately related to the flat forms and cochains of Whitney, [W, Chapter IX]. The context for the work in this paper is motivated primarily by the development in the paper by Gurtin, Williams, and the author referred to above. Moreover, many of the concepts and techniques related to the Cauchy flux in this paper originate with [GW] and [GM].

The author is indebted to Morton Gurtin for suggesting this investigation and would like to thank both Morton Gurtin and William O. Williams for several helpful conversations related to the work in this paper.

2. Notation and Preliminaries.

We let R^n denote Euclidean n-space and $|A|$ will denote the Lebesgue measure of a measurable set $A \subset R^n$. We denote by H^{n-1} the (n-1)-dimensional Hausdorff measure defined on R^n. The open ball centered at x of radius r is denoted by $B(x,r)$.

If $D \subset R^n$, we will let bdry D stand for the topological boundary of D. In the development of this paper, the topological boundary of a set plays a small role and will be replaced by the notion of the measure-theoretic boundary of D, denoted by ∂D. It is defined as

$$\partial D = R^n \cap \{x : d(D,x) \neq 0 \text{ and } d(R^n - D, x) \neq 0\}$$

where

$$d(A,x) = \lim_{r \downarrow 0} \frac{|A \cap B(x,r)|}{|B(x,r)|}$$

whenever $A \subset R^n$ is a measurable set. Clearly, $\partial D \subset \text{bdry } D$. A bounded measurable set $D \subset R^n$ is called a set of finite perimeter if $H^{n-1}(\partial D) < \infty$. Notice that a set D of finite perimeter may be altered by a set of Lebesgue measure 0 and still determine the same measure-theoretic boundary ∂D. To eliminate this ambiguity, whenever a set D of finite perimeter is designated, it will be understood that D denotes the set $\{x : d(D,x) = 1\} \cup \partial D$. A complete investigation of these sets is presented in [F, Chapter 4]. We recall here some of the basic properties of sets of finite perimeter that will be used in the sequel.

Let $D \subset R^n$ be a measurable set. Then D has the unit vector $\nu(D,x)$ as <u>the measure theoretic exterior normal</u> at x if, letting

$$B^+(x,r) = B(x,r) \cap \{y : (y-x) \cdot \nu(D,x) \geq 0\}$$

$$B^-(x,r) = B(x,r) \cap \{y : (y-x) \cdot \nu(D,x) \leq 0\}$$

we have

$$\lim_{r \downarrow 0} \frac{|B^+(x,r) \cap D|}{|B^+(x,r)|} = 0$$

and

$$\lim_{r \downarrow 0} \frac{|B^-(x,r) \cap D|}{|B^-(x,r)|} = 1 .$$

If $\partial^* D$ denotes the set of points at which the exterior normal to D at x exists, then clearly $\partial^* D \subset \partial D$. However, if D is a set of finite perimeter then

$$H^{n-1}[\partial D - \partial^* D] = 0 , \tag{3}$$

and

$$\int_D \operatorname{div} V(x) = \int_{\partial^* D} V(x) \cdot \nu(D,x) dH^{n-1}(x) \tag{4}$$

whenever $V : R^n \to R^n$ is a Lipschitzian vector field with compact support, c.f. [F, Theorem 4.5.6]. We also recall from the same reference that

$$(5) \qquad \lim_{r\downarrow 0} \frac{H^{n-1}[(\partial^*D) \cap B(x,r)]}{\alpha(n-1)r^{n-1}} = 1$$

for H^{n-1} - a.e. $x \in \partial^*D$. Here $\alpha(n-1)$ denotes the volume of the unit ball in R^{n-1}. Further regularity of ∂^*D is provided by the following result: <u>If</u> D <u>is a set of finite perimeter, then there exists a countable number of</u> $(n-1)$ <u>dimensional</u> C^1 <u>manifolds</u> M_i <u>such that</u>

$$(6) \qquad H^{n-1}[\partial^*D - \bigcup_{i=1}^{\infty} M_i] = 0 ,$$

[F, Theorem 3.2.29].

A useful tool in geometric measure theory is the co-area formula. It states that if $f : R^n \to R^1$ is a Lipschitzian function, then

$$(7) \qquad \int_A |\nabla f(x)|dx = \int_{R^1} H^{n-1}[f^{-1}(t) \cap A]dt$$

whenever $A \subset R^n$ is a measurable set, [F, Theorem 3.2.11]. Later in the paper, we will apply (7) in the following form. Let $x_0 \in R^n$ and let $f(x) = |x-x_0|$. If $A \subset R^n$ is measurable, then $f^{-1}(t) \cap A = \partial B(x_0,t) \cap A$ and because $|\nabla f(x)| = 1$ for all $x \neq x_0$, it follows from (7) that

(8) $$|A| = \int_{R^1} H^{n-1}[\partial B(x_0,t) \cap A]dt \ .$$

An <u>oriented surface</u> is a pair (S,ν) where $S \subset R^n$ is a Borel set and $\nu : R^n \to R^n$ is a Borel measurable vector field that are related in the following manner. There is a (bounded) set D of finite perimeter such that

(9)
$$S \subset \partial^* D \quad \text{and}$$
$$\nu(x) = \nu(D,x)\chi_S(x)$$

where χ_S is the characteristic function of the set S . For simplicity of notation we will denote by S the pair (S,ν) , it being understood that S is oriented by the exterior normal of some set D of finite perimeter. We define

(10) $$-S = (S,-\nu)$$

Note that this is meaningful because there is a bounded set E of finite perimeter such that $S \subset \partial^* E$ and $-\nu(x) = \nu(E,x)\chi_S(x)$. Indeed, the set $R^n - D$ has the property that $\partial^*(R^n - D) = \partial^* D$. However, in our definition of finite perimeter, we require the set to be bounded. Therefore, if we let B be an open ball that contains $D \cup (\text{bdry } D)$, and define

$$E = B \cap (R^n - D) ,$$

then clearly E is of finite perimeter, $S \subset \partial^* E$, and $-\nu(x) = \nu(E,x)\chi_S(x)$.

We say that $S_1 = (S_1, \nu_1)$ and $S_2 = (S_2, \nu_2)$ are compatible if there is a set D of finite perimeter such that $S_1 \subset \partial^* D$. $S_2 \subset \partial^* D$, $\nu_1(x) = \nu(D,x)\chi_{S_1}(x)$, and $\nu_2(x) = \nu(D,x)\chi_{S_2}(x)$. We define $S_1 \cup S_2$ as $(S_1 \cup S_2, \nu)$ where $\nu(x) = \nu(D,x)\chi_{S_1 \cup S_2}(x)$ and $S_1 \cap S_2$ is defined similarly.

A <u>Cauchy flux</u> is a function Q that assigns to each oriented surface $S = (S, \nu)$ a real number and has the following properties:

(11) (i) there is a number $K > 0$ such that $|Q(S)| \leq KH^{n-1}(S)$ whenever S is an oriented surface,

(ii) $Q(S_1 \cup S_2) = Q(S_1) + Q(S_2)$ whenever S_1 and S_2 are disjoint, compatible oriented surfaces.

Observe that if D is a set of finite perimeter, it follows from (i) and (ii) that Q is countably additive on all compatible oriented surfaces $S \subset \partial^* D$.

A Cauchy flux Q is said to be <u>weakly balanced</u> if there exists a number $M > 0$ such that

$$|Q(\partial^* D)| \leq M|D| \tag{12}$$

whenever D is a set of finite perimeter. Here, in keeping with our convention, the symbol ∂^*D that appears in (13) denotes the oriented surface (∂^*D,ν) where $\nu(x) = \nu(D,x)\chi_{\partial^*D}(x)$.

Notice that (12) conceivably allows the possibility of ∂^*D being the oriented boundary of some other (bounded) set of finite perimeter, say E . That is, $(\partial^*D,\nu) = (\partial^*E,\nu_1)$ where $\nu_1(x) = \nu(E,x)\chi_{\partial^*E}(x)$. This can only happen if the symmetric difference of D and E had Lebesgue measure zero, for in the language of geometric measure theory, no non-trivial n-dimensional integral current (in our case $D - E$ or $E - D$) can have zero boundary, vide [F,§4.5.2].

3. The Existence Of A Normal-Dependent Density.

In this section it will be shown that a weakly balanced Cauchy flux can be essentially described in terms of a field.

If S is an oriented surface, a point $x \in R^n$ is called a point of density of Q on S if the following limit exists:

(13) $$\lim_{r \downarrow 0} \frac{Q[S \cap B(x,r)]}{\alpha(n-1)r^{n-1}} .$$

Because S is an oriented surface, there is a set D of finite perimeter such that $S \subset \partial^* D$. Hence, referring to (5), we see that

$$\lim_{r \downarrow 0} \frac{Q[S \cap B(x,r)]}{\alpha(n-1)r^{n-1}} = \lim_{r \downarrow 0} \frac{Q[S \cap B(x,r)]}{H^{n-1}[S \cap B(x,r)]}$$

at H^{n-1} - a.e. $x \in S$. We define $q_S(x)$ by

(14) $$q_S(x) = \lim_{r \downarrow 0} \frac{Q[S \cap B(x,r)]}{\alpha(n-1)r^{n-1}}$$

If D is a set of finite perimeter, then by virtue of (11) and the Radon-Nikodym theorem, there is Borel function $q_{\partial D} : \partial D \to R^1$ property that

(15) $$Q(S) = \int_S q_{\partial D}(x)\, dH^{n-1}(x)$$

whenever $S \subset \partial^* D$ is an oriented surface. Using again the fact that Q is countably additive on compatible elements of $\partial^* D$, it follows from the general theory of differentiation [F, §2.9] that

$$q_S(x) = q_{\partial D}(x) \tag{16}$$

for H^{n-1} - a.e. $x \in S$.

We now proceed to extend a result of Noll [N] to our context of sets of finite perimeter.

3.1 Lemma. *If* E *and* B *are sets of finite perimeter, then*

$$\text{(i)}\quad \partial(E \cap B) \supset \Big[(\partial E) \cap B\Big] \cup \Big[E \cap (\partial B)\Big] - \Big[(\partial E) \cap (\partial B)\Big]$$

$$\text{(ii)}\quad \partial(E \cap B) \subset \Big[(\partial E) \cap B\Big] \cup \Big[E \cap (\partial B)\Big] \cup \Big[(\partial E) \cap (\partial B)\Big]$$

Proof. We prove (i) first. Choose $x \in (\partial E) \cap B - (\partial E) \cap (\partial B)$. Then clearly, $d(B,x) = 1$ and $d^*(E,x) > 0$ where

$$d^*(E,x) = \limsup_{r \downarrow 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} .$$

Hence, there is a number $a > 0$ and a sequence $\{r_i\} \to 0$ such that

$$|E \cap B(x,r_i)| \geq a|B(x,r_i)| \quad , \quad i = 1,2,\ldots$$

and

$$\lim_{i\to\infty} \frac{|(R^n-B) \cap B(x,r_i))|}{|B(x,r_i)|} = 0 .$$

But

$$|E \cap B(x,r_i)| = |E \cap B \cap B(x,r_i)| + |E \cap (R^n-B) \cap B(x,r_i)|$$

and therefore

$$d^*(E \cap B , x) > 0 .$$

If also $d_*(E \cap B , x) < 1$ where

$$d_*(E \cap B , x) = \liminf_{r\downarrow 0} \frac{|E \cap B \cap B(x,r)|}{|B(x,r)|}$$

then it would follow that $x \in \partial(E \cap B)$. If it were not true that $d_*(E \cap B , x) < 1$, then $d(E \cap B , x) = 1$ or what is the same, $d[(R^n-E) \cup (R^n-B) , x] = 0$. This implies that $d(E,x) = 1$ which contradicts the fact that $x \in \partial E$. Hence $x \in \partial(E \cap B)$. The same conclusion would be reached if we had taken $x \in E \cap (\partial B) - [(\partial E) \cap (\partial B)]$ and therefore (i) is established.

In order to establish (ii), let $x \in \partial(E \cap B)$. Let

$$B^0 = \{x : d(B,x) = 1\} \text{ and}$$

$$B^e = \{x : d(B,x) = 0\} .$$

Then $R^n = B^0 \cup B^e \cup \partial B$. Note that $x \notin B^e$, for otherwise $d(E \cap B , x) = 0$ which contradicts the fact that $x \in \partial(E \cap B)$. Thus, $x \in B^0 \cup \partial B$ and similarly, $x \in E^0 \cup \partial E$. Hence

$$\partial(E \cap B) \subset (B^0 \cap E^0) \cup (B^0 \cap \partial E) \cup (E^0 \cap \partial B) \cup [(\partial E) \cap (\partial B)] .$$

Note that $[B^0 \cap E^0] \cap \partial(E \cap B) = 0$ for if $x \in B^0 \cap E^0$, then $d(E \cap B , x) = 1$ and therefore, $x \notin \partial(E \cap B)$. This completes the proof of the lemma.

3.2 Remark. If E is a set of finite perimeter and $x \in R^n$, observe that

$$H^{n-1}[(\partial E) \cap \partial B(x,r)] = 0$$

for all but countably many r , because $H^{n-1}(\partial E) < \infty$ and $\partial B(x,r) \cap \partial B(x,t) = 0$ if $r \neq t$. To see this, let

$$A_i = \{r : H^{n-1}[(\partial E) \cap \partial B(x,r)] > i^{-1}\} ,$$

$$A = \bigcup_{i=1}^{\infty} A_i .$$

If A were uncountable, then some A_i would be uncountable which would imply that $H^{n-1}(\partial E) = \infty$.

3.3 Theorem. To every weakly balanced Cauchy flux Q corresponds a density function $q : R^n \times S^{n-1} \to R^1$

<u>such that for every oriented surface</u> $S = (S,\nu)$,

$$Q(S) = \int_S q(x,\nu(x))dH^{n-1}(x) \ .$$

<u>Proof</u>. Choose $(x,\nu_0) \in R^n \times S^{n-1}$ and consider all oriented surfaces $S = (S,\nu)$ with the property that $x \in S$ and $\nu(x) = \nu_0$. If x is not a point of density for Q on any such S , define $q(x,\nu_0) = 0$. If x is a point of density of Q for some $S_1 = (S_1,\nu_1)$ with $\nu_1(x) = \nu_0$, set

$$q(x,\nu_1(x)) = q_{S_1}(x) \ . \tag{17}$$

We now show that if x is also a point of density of Q on $S_2 = (S_2,\nu_2)$ with $\nu_2(x) = \nu_0$, then $q_{S_1}(x) = q_{S_2}(x)$.

To this end, let D_1 and D_2 be sets of finite perimeter such that $S_1 \subset \partial^*D_1$, $S_2 \subset \partial^*D_2$, and $\nu(D_1,x) = \nu(D_2,x) = \nu_0$. From Lemma 3.1 we have

$$\begin{aligned} \partial(D_1 \cap B_r) &\supset \Big[(\partial D_1) \cap B_r\Big] \cup \Big[(D_1 - D_2) \cap (\partial B_r)\Big] \\ &\quad \cup \Big[(D_1 \cap D_2) \cap \partial B_r\Big] - \Big[(\partial D_1) \cap (\partial B_r)\Big] , \\ \partial(D_2 \cap B_r) &\subset \Big[(\partial D_2) \cap B_r\Big] \cup \Big[(D_2 - D_1) \cap (\partial B_r)\Big] \\ &\quad \cup \Big[(D_1 \cap D_2) \cap \partial B_r\Big] \cup \Big[(\partial D_2) \cap (\partial B_r)\Big] \end{aligned} \tag{18}$$

where, for convenience, we have set $B_r = B(x,r)$. It follows from Remark 3.2 that

$$H^{n-1}[(\partial D_1) \cap (\partial B_r)] = H^{n-1}[(\partial D_2) \cap (\partial B_r)] = 0 ,$$

and therefore that

$$Q[(\partial D_1) \cap (\partial B_r)] = Q[(\partial D_2) \cap (\partial B_r)] = 0$$

for all but countably many $r > 0$. Therefore, it follows from (18) that

$$\partial(D_1 \cap B_r) = \left[(\partial D_1) \cap B_r\right] \cup \left[(D_1 - D_2) \cap (\partial B_r)\right]$$

$$\cup \left[(D_1 \cap D_2) \cap (\partial B_r)\right] \cup N_1(r) ,$$

$$\partial(D_2 \cap B_r) = \left[(\partial D_2) \cap B_r\right] \cup \left[(D_2 - D_1) \cap (\partial B_r)\right]$$

$$\cup \left[(D_1 \cap D_2) \cap (\partial B_r)\right] \cup N_2(r) ,$$

where $Q[N_1(r)] = Q[N_2(r)] = 0$ for all but countably many $r > 0$. Therefore, for all but countably many $r > 0$,

(19) $$Q[(\partial D_1) \cap B_r] - Q[(\partial D_2) \cap B_r]$$

$$= Q[\partial(D_1 \cap B_r)] - Q[\partial(D_2 \cap B_r)]$$

$$+ Q[(D_1 - D_2) \cap \partial B_r] - Q[(D_2 - D_1) \cap \partial B_r] .$$

A similar equality holds with the subscripts 1 and 2 interchanged. In order to prove that $q_{S_1}(x) = q_{S_2}(x)$, it is sufficient to show that for every $\varepsilon > 0$, there exists a sequence $\{r_i\} \to 0$ such that

$$|Q[(\partial D_1) \cap B_{r_i}] - Q[(\partial D_2) \cap B_{r_i}]| < \varepsilon r_i^{n-1} \tag{20}$$

for $i = 1,2,\ldots$. Because of property (12) and (19), it is sufficient to show that for every $\varepsilon > 0$ there is a sequence $\{r_i\} \to 0$ such that

$$|Q[(D_1-D_2) \cap (\partial B_{r_i})]| < \varepsilon r_i^{n-1} \quad \text{and} \tag{21}$$

$$|Q[(D_2-D_1) \cap (\partial B_{r_i})]| < \varepsilon r_i^{n-1}$$

for $i = 1,2,\ldots$. Because $\nu(D_1,x) = \nu(D_2,x)$ it follows that there is $R^* > 0$ such that for $0 < r < R^*$,

$$|B_r^- \cap (R^n-D_i)| < \frac{\varepsilon}{8n} r^n \tag{22}$$

$$|B_r^+ \cap D_i| < \frac{\varepsilon}{8n} r^n \quad , \quad i = 1,2 \ .$$

Now let

$$A_{ij}^+(\varepsilon) = \{r : H^{n-1}[(\partial B_r^+) \cap (D_i-D_j)] < \varepsilon/2 \ r^{n-1}\} \ , \tag{23}$$

$$A_{ij}^-(\varepsilon) = \{r : H^{n-1}[(\partial B_r^-) \cap (D_i-D_j)] < \varepsilon/2 \ r^{n-1}\} \ .$$

Set

$$A(\varepsilon) = A_{12}^{+}(\varepsilon) \cap A_{21}^{+}(\varepsilon) \cap A_{12}^{-}(\varepsilon) \cap A_{21}^{-}(\varepsilon)$$

and suppose

(24) $$|A(\varepsilon) \cap [0,R]| = 0$$

for some $R < R^*$. Then

(25) $$R \leq |\tilde{A}_{12}^{+} \cup \tilde{A}_{21}^{+} \cup \tilde{A}_{12}^{-} \cup \tilde{A}_{21}^{-}|$$

where, for example, we set $\tilde{A}_{12}^{+} = [0,R] - A_{12}^{+}$. It follows from (8) and (23) that

(26) $$|B_R^- \cap (R^n - D_1)| \geq |B_R^- \cap (D_2 - D_1)|$$

$$\geq \int_0^R H^{n-1}[(\partial B_r^-) \cap (D_2 - D_1)]dr$$

$$\geq \int_{\tilde{A}_{12}^{-} \cap [0,R]} H^{n-1}[(\partial B_r^-) \cap (D_2 - D_1)]dr$$

$$\geq \int_{\tilde{A}_{12}^{-} \cap [0,R]} \varepsilon/2\ r^{n-1} .$$

Similar inequalities hold for each of the sets $B_R^- \cap (R^n - D_2)$, $B_R^+ \cap D_1$, and $B_R^+ \cap D_2$. Obviously, from (25),

$$(27) \quad \int_{\tilde{A}_{12}^- \cap [0,R]} \varepsilon/2 \; r^{n-1} dr + \int_{\tilde{A}_{12}^+ \cap [0,R]} \varepsilon/2 \; r^{n-1} dr$$

$$+ \int_{\tilde{A}_{21}^- \cap [0,R]} \varepsilon/2 \; r^{n-1} dr + \int_{\tilde{A}_{21}^+ \cap [0,R]} \varepsilon/2 \; r^{n-1} dr$$

$$= \int_0^R \varepsilon/2 \; r^{n-1} dr = \frac{\varepsilon}{2n} R^n .$$

Thus, one of the integrals in (27), say the first, has the property that

$$\int_{\tilde{A}_{12}^- \cap [0,R]} \varepsilon/2 \; r^{n-1} dr \geq \frac{\varepsilon}{8n} R^n .$$

Therefore, (26) implies that

$$|B_R^- \cap (R^n - D_1)| > \frac{\varepsilon}{8n} R^n$$

which contradicts (22). Therefore (24) must be false for all R such that $0 < R < R^*$. Consequently, there is a sequence $\{r_i\} \to 0$ such that $r_i \in A(\varepsilon)$ for $i = 1,2,\ldots$. This implies that

$$H^{n-1}[(\partial B_{r_i}) \cap (D_1-D_2)] < \varepsilon\, r_i^{n-1} \quad \text{and}$$

$$H^{n-1}[(\partial B_{r_i}) \cap (D_2-D_1)] < \varepsilon\, r_i^{n-1} \quad , \quad \text{for} \quad i = 1,2,\ldots .$$

In view of property (11(i)), this is sufficient to establish the validity of (21) and therefore, the proof of the theorem is complete.

Following the proof of Theorem 1 in [GM], it is easy to conclude that

$$Q(S) = -Q(-S) \tag{28}$$

whenever S is an oriented surface such that S is contained in some hyperplane and the topological boundary of S relative to the hyperplane has fintie H^{n-2} measure. To see this, assume that the hyperplane is defined by $x_n = 0$, and that $S = (S,\nu)$ where $\nu(x) = (0,0,\ldots,1)$ for $x \in S$. For every $\varepsilon > 0$, let

$$D_\varepsilon^+ = S \times [0,\varepsilon]$$

$$D_\varepsilon^- = S \times [-\varepsilon,0] .$$

Since the topological boundary of S relative to the hyperplane $x_n = 0$ has finite H^{n-2} measure, it follows that $H^{n-1}(\text{bdry } D_\varepsilon^+) < \infty$ and $H^{n-1}(\text{bdry } D_\varepsilon^-) < \infty$. Therefore, both D_ε^+ and D_ε^- are sets of finite perimeter and letting

$D_\varepsilon = D_\varepsilon^+ \cup D_\varepsilon^-$, we have

$$(29) \qquad Q(\partial D_\varepsilon^+) + Q(\partial D_\varepsilon^-) - Q(\partial D_\varepsilon) = Q(S) + Q(-S) \ .$$

In view of (12) the left side of (29) tends to 0 as $\varepsilon \to 0$ and therefore (28) is established.

It is in fact possible to prove (28) for any arbitrary oriented surface S , but this additional information is not needed in this paper.

We now can employ the results of [GM], particularly Theorems 3 and 6, to conclude that the density $q(x,\nu)$ corresponding to Q in Theorem 3.3 is linear at a.e. $x \in R^n$, We state this as

3.4 Theorem. *Let Q be a weakly balanced Cauchy flux with associated density function $q : R^n \times S^{n-1} \to R^1$. Then there exists a measurable vector field $q^* : R^n \to R^n$ such that for* a.e. $x \in R^n$

$$q(x,\nu) = q^*(x) \cdot \nu$$

whenever $\nu \in S^{n-1}$.

We now proceed to investigate the divergence (in some suitable weak sense) of the vector field q^* . It easily follows from (11(i)) that there is a constant K such that

$$|q^*(x)| \leq K \tag{30}$$

for a.e. $x \in R^n$.

We say that a set I is an n-<u>dimensional closed interval</u> if I is of the form

$$I = \{x : a_i \leq x_i \leq b_i \ , \ i = 1,2,\ldots,n\} \ .$$

The closed interval I is called <u>admissible</u> if the integral

$$\int_{\partial I} q^*(x) \cdot \nu(x) dH^{n-1}(x) \tag{31}$$

exists. Note that almost all intervals I are admissible. For each admissible interval I , set $\mu(I)$ equal to the integral in (31) and defined for $x \in R^n$,

$$\operatorname{div}^* q^*(x) = \limsup \frac{\mu(I)}{|I|} \tag{32}$$

where the lim sup is taken over a regular family of admissible intervals I containing x , [SA, p. 106]. Define $\operatorname{div}_* q^*(x)$ as the corresponding lim inf and if $\operatorname{div}_* q^*(x) = \operatorname{div}^* q^*(x)$, this common value will be called $\operatorname{div} q^*(x)$. Note from (30) that $|\operatorname{div} q^*(x)| \leq K$ when it exists.

3.5 <u>Lemma</u>. <u>For each admissible interval</u> I ,

$$\int_I \operatorname{div}^* q^* \geq \mu(I) \geq \int_I \operatorname{div}_* q^*$$

<u>Proof</u>. Suppose for some admissible I_0 and $\varepsilon > 0$, that

$$\int_{I_0} \operatorname{div}^* q^*(x)dx < \mu(I_0) - \varepsilon|I_0|$$

Let Ω be an open bounded set and let f be a lower semi-continuous function such that $f(x) \geq \operatorname{div}^* q^*(x)$ for $x \in R^n$ and

$$(33) \qquad \int_{\Omega} f(x) - \operatorname{div}^* q^*(x) < \varepsilon|I_0| \, .$$

For each admissible $I \subset \Omega$, let

$$(34) \qquad \theta(I) = \int_I f(x)dx - \mu(I)$$

and observe that, in view of the lower semicontinuity of f, $\theta_*(x) \geq f(x) - \operatorname{div}^* q^*(x)$ for every $x \in \Omega$. Here, $\theta_*(x)$ is defined in a manner similar to that in (32), with lim sup replaced by lim inf. Thus, it easily follows that $\theta(I) \geq 0$ for every admissible $I \subset \Omega$, [SA, p. 190]. Therefore from (33) and (34),

$$\mu(I_0) \leq \int_{I_0} f(x)dx \leq \int_{I_0} \operatorname{div}^* q^*(x)dx + \varepsilon|I_0| < \mu(I_0) \, ,$$

a contradiction. Thus

$$\int_I \operatorname{div}^* q^*(x)\,dx \geq \mu(I)$$

for each admissible I and similar reasoning yields the remaining inequality of the lemma.

We now show that $\operatorname{div} q^*(x)$ exists for a.e. $x \in R^n$. To this end, let A be the family of all half-open intervals $J = \{x : a_i \leq x_i < b_i \,,\ i = 1,2,\ldots,n\}$, and let F denote the field of all finite unions of intervals $J \in A$ and note that F generates the Borel sets in R^n. If we define

$$\psi(J) = \mu(I)$$

where I is the closure of J, then ψ is finitely additive and a theorem of B. Fuglede is now applicable, [FU, Theorem III]:

<u>In order that there exists an integrable function</u> f <u>such that</u>

$$\psi(J) = \int_J f(x)\,dx$$

<u>for every</u> $J \in A$, <u>the following two conditions are necessary and, when combined, sufficient</u>:

(i) <u>For every</u> $\varepsilon > 0$ <u>there is a</u> $\delta > 0$ <u>such that</u> $\sum_{k=1}^{\ell} |\psi(J_k)| < \varepsilon$ <u>for every finite number of</u>

<u>intervals</u> $J_1, J_2, \ldots, J_\ell$ <u>from</u> A <u>for which</u> $\sum_{k=1}^{\ell} |J_k| < \delta$.

(ii) <u>There is a constant</u> C <u>such that</u> $\sum_{k=1}^{\ell} |\psi(J_k)| \leq C$ <u>for every finite system of disjoint intervals</u> $J_1, J_2, \ldots, J_\ell$ <u>from</u> A .

Lemma 3.5 implies that conditions (i) and (ii) are satisfied and therefore

$$\mu(I) = \int_I f(x)dx$$

for each admissible $I \subset \Omega$. However, standard differentiation theory shows that

$$\text{div } q^*(x) = f(x)$$

for a.e. x . Thus, we have proved

3.6 <u>Lemma</u>. <u>For</u> a.e. $x \in R^n$, div $q^*(x)$ <u>exists and</u>

$$\int_I \text{div } q^*(x)dx = \int_{\partial I} q^*(x) \cdot \nu(x)dH^{n-1}(x)$$

<u>for almost every interval</u> I .

We will conclude by showing that the divergence of q^* in the sense of distributions is equal to the bounded function, div q^* .

3.7 Theorem. For every test function $\phi \in C_0^\infty(\Omega)$,

$$\int q^*(x) \cdot \nabla\phi(x)dx = \int \mathrm{div}\, q^*(x)\phi(x)dx .$$

Proof. Let Ω' be an open set whose closure is contained in Ω and choose an arbitrary interval $I \subset \Omega'$. For each $\phi \in C_0^\infty(\Omega)$ with $\int_{R^n} \phi = 1$ and $t > 0$ let

$$\phi_t(x) = t^{-n}\, \phi(x/t) ,$$

it being understood that only those $t > 0$ for which t is less than the distance from Ω' to $R^n - \Omega$ will be considered. Define

$$(\mathrm{div}\, q^*)_t = (\mathrm{div}\, q^*)*\phi_t$$

and q_t^* will denote the vector field whose coordinate functions are those of q^* convolved with ϕ_t .

With the help of Lemma 3.6 and Fubini's theorem, we have

$$\int_I (\mathrm{div}\, q^*)_t(x)dx = \int_I \int_{R^n} \mathrm{div}\, q^*(x-y)\phi_t(y)dy\, dx$$

$$= \int_{R^n} \int_{I_y} \mathrm{div}\, q^*(x)\phi_t(y)dx\, dy$$

$$= \int_{R^n} \int_{\partial I_y} q^*(x) \cdot \nu(x)\phi_t(y)\,dH^{n-1}(x)\,dy$$

where $I_y = I - y$. From (4) and Fubini's theorem, we have

$$\int_I \operatorname{div} q_t^* = \int_{\partial I} q_t^*(x) \cdot \nu(x)\,dH^{n-1}(x)$$

$$= \int_{R^n} \int_{\partial I_y} q^*(x) \cdot \nu(x)\phi_t(y)\,dH^{n-1}(x)\,dy \ .$$

Thus, for every $I \subset \Omega'$ we have shown that

$$\int_I \operatorname{div} q_t^* = \int_I (\operatorname{div} q^*)_t \tag{35}$$

from which it follows that $\operatorname{div} q_t^* = (\operatorname{div} q^*)_t$ a.e. in Ω' .

Now let $\phi \in C_0^\infty(\Omega)$ and let Ω' be as above that contains the support of ϕ . From (35) and Lebesgue's dominated convergence theorem, we have

$$\int \operatorname{div} q^*\phi = \lim_{t \downarrow 0} \int (\operatorname{div} q^*)_t\phi$$

$$= \lim_{t \downarrow 0} \int \operatorname{div} q_t^*\phi$$

$$= \lim_{t \downarrow 0} \int q_t^* \cdot \nabla\phi$$

$$= \int q^* \cdot \nabla\phi \ .$$

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2556	2. GOVT ACCESSION NO. AD-A132 834	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) CAUCHY FLUX AND SETS OF FINITE PERIMETER		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) William P. Ziemer		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS * see below		12. REPORT DATE August 1983
		13. NUMBER OF PAGES 26
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

* U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Cauchy flux, sets of finite perimeter, Cauchy's Theorem.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A Cauchy flux Q is a real-valued, additive, area-bounded function whose domain is the class of all Borel subsets of the reduced boundary of sets of finite perimeter. If the flux Q is also volume bounded, it is shown that Q can be represented as the integral of the normal component of some vector field.

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