Multisignal Minimum-Cross-Entropy Spectrum Analysis
with Weighted Priors

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This report presents a generalization of multisignal minimum-cross-entropy spectrum analysis (multisignal MCESA), a method for simultaneously estimating a number of power spectra when a prior estimate of each is available and new information is obtained in the form of values of the autocorrelation function of their sum. The generalization involves attaching to each prior spectrum estimate a frequency-dependent weighting parameter that indicates its relative reliability, or the relative "degree of belief" associated with it. Mathematical properties of the generalized method are discussed, and illustrative numerical examples are given.
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MULTISIGNAL MINIMUM-CROSS-ENTROPY SPECTRUM ANALYSIS
WITH WEIGHTED PRIORS

INTRODUCTION

Multisignal minimum-cross-entropy spectrum analysis (MCESA) is a method for estimating the power spectrum of one or more signals when a prior estimate for each is available and new information is obtained in the form of values of the autocorrelation function of their sum [1]. The resultant estimates are the solution of a constrained minimization problem: they are consistent with the autocorrelation information and otherwise as similar as possible to the respective prior estimates in a precisely defined information-theoretic sense. Multisignal MCESA [1] is a generalization of MCESA [1,2] and reduces to it in the special case when the number of signals is one. Multisignal MCESA can be derived [1,2] as an application of the principle of minimum cross entropy [3-5] or, alternatively, by minimizing a sum of Itakura-Saito distortions [6].

Multisignal MCESA applies when, for instance, one obtains autocorrelation measurements for a signal corrupted by independent additive noise, and one has some prior knowledge concerning the spectra of both the uncorrupted signal and the noise. The results are posterior signal- and noise-spectrum estimates that take both the prior estimates and the autocorrelation information into account.

The multisignal MCESA procedure presented in Ref. 1 gives the same weight to each of the prior spectrum estimates—they are all treated on the same footing. However, in situations that arise in practice, one may have more reliable information about the spectra of some of the signals than about the others. Consider a speech signal corrupted by additive background noise; suppose the background is more nearly stationary than speech. If it is possible to detect pauses in the speech reliably, then measurements of the sum signal during a speech-free interval will yield an estimate of the noise spectrum that can serve as a prior noise estimate during an interval when speech is present. The result may well be a better prior estimate for the noise spectrum than any prior estimate that can be obtained for the speech spectrum. Alternatively one might be able to obtain a good estimate of the noise power spectrum by conventional spectrum analysis of a signal from a microphone exposed to the noise but not the speech. In both cases it would be desirable to give greater weight to the noise prior than to the speech prior in deriving posterior spectrum estimates. In other situations it might be desirable to rely more heavily on the speech prior than on the noise prior.

It is furthermore possible to have prior information about the spectrum of an individual signal that is more reliable in some frequency ranges than in others. Thus in some situations, for example, one might even wish to give greater weight to the noise prior at high frequencies and to the speech prior at low frequencies.

We present in this report a generalization of multisignal MCESA that allows a frequency-dependent weight to be attached to each prior estimate. Aside from the weights, inputs to the procedure are the same as to multisignal MCESA: a prior spectrum estimate for each signal and autocorrelation values for the sum. The results again are posterior spectrum estimates that are consistent with...

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the given autocorrelation information. When all of the weights are constant and equal, the results are identical to those from multisignal MCESA. In general, increasing a weighting parameter tends to bring the corresponding posterior spectrum closer (in the sense of Itakura-Saito distortion) to the corresponding prior spectrum at the expense of increasing the distortions for other prior-posterior pairs. This is demonstrated in the fourth section of this report, where it is also shown that the posterior spectra can be obtained by constrained minimization of a weighted sum of Itakura-Saito distortions. In the remainder of this section, we describe the multisignal MCESA method of [1]. In the second section of this report, we present the generalized method and the heuristic arguments that first led us to the generalization. In the third section we give a derivation based on minimizing a weighted sum of cross entropies, and in the fifth section we rework a numerical example from Ref. 1.

Multisignal MCESA estimates the power spectra $S_i(f)$ of a number $K$ of independent, real, band-limited, stationary processes (signals) with bandwidth $W$, given a prior estimate $P_i$ for each $S_i$, and given in addition the values $R_{\text{tot}}(t) = R_{\text{tot}}(f)$ of the autocorrelation function of the sum of the processes at finitely many lags $r$, $r = 0, \ldots, M$. The prior estimates $P_i$ may be thought of as the best guesses at $S_i$ we could make in the absence of the autocorrelation data. Under the assumption of independence, we write $R_{\text{tot}}$ as a sum of autocorrelation functions $R_i$ of individual processes:

$$R_{\text{tot}}(t) = \sum_{i=1}^{K} R_i(t),$$

where

$$R_i(t) = 2\int_0^W S_i(f) \cos 2\pi ft \, df.$$

The multisignal MCESA estimator has the form [1]

$$Q_i(f) = \frac{1}{\frac{1}{P_i(f)} + \sum_{i=1}^{K} 2\beta_i \cos 2\pi ft},$$

where the $\beta_i$ are chosen so that the $Q_i$ are consistent with the given autocorrelation values:

$$R_{\text{tot}} = 2\sum_{i=1}^{K} \int_0^W Q_i(f) \cos 2\pi ft \, df.$$  \hspace{1cm} (2)

The parameters $\beta_i$ are Lagrange multipliers that arise in the solution of a minimization problem with Eq. (2) as constraints. The minimization problem can be formulated as an application of the principle of minimum cross entropy or, alternatively, as the minimization of the sum

$$\sum_{i=1}^{K} \int_0^W \left( \frac{Q_i(f)}{P_i(f)} \log \frac{Q_i(f)}{P_i(f)} - 1 \right) df$$

of Itakura-Saito distortions.

THE METHOD

To use the multisignal MCESA estimate with weighted priors, we must supply not only autocorrelation values $R_{\text{tot}}$ for the sum of the signals and prior estimates $P_i$ for the individual signals, but also a weight $w_i$ associated with each $P_i$. The $w_i$ may be simply $K$ positive constants or, more generally, $K$ functions $w_i(f)$ of frequency. The estimate has the form

$$Q_i(f) = \frac{1}{\frac{1}{P_i(f)} + \frac{1}{w_i(f)} \sum_{i=1}^{K} 2\beta_i \cos 2\pi ft},$$

where the parameters $\beta_i$ are to be chosen so that the constraints of Eq. (2) are satisfied.
We first arrived at something like Eq. (4), with frequency-independent weights, by considering a somewhat artificial situation. Suppose that each signal $s$, is the sum of $n_i$ independent signals $s_{ij}$ with spectra $S_{ij}$, and that for each $i$ we have equal prior estimates $P_i/n_i$ for all the $S_{ij}$, $j = 1, \ldots, n_i$. Then we might consider $n_i$ to be a measure of the quality or reliability of $P_i$ as an estimate of $S_i = \sum S_{ij}$, by analogy with the fact that the sum of $n$ independent random variables distributed identically with $x/n$ has a smaller variance (by a factor of $n$) than a single random variable $x$. The MCESA posterior estimates $Q_{ij}$ for $S_{ij}$ in this situation are given by

$$Q_{ij}(f) = \frac{1}{1 + \sum_{j=1}^{M} 2\beta_j \cos 2\pi f t_i},$$

where the $\beta_j$ are chosen to satisfy constraints

$$R_{ij}^{\text{tot}} = 2K \sum_{i=1}^{K} \int_0^W Q_{ij}(f) \cos 2\pi f t_i \, df. \tag{5}$$

We consequently obtain

$$Q_i(f) = \sum_{j=1}^{n_i} Q_{ij}(f) = n_i Q_0(f)$$

$$= \frac{1}{P_i(f) + \sum_{j=1}^{M} 2\beta_j \cos 2\pi f t_i}, \tag{6}$$

as a reasonable posterior estimate of $S_i(f)$. When $Q_i$ is given by Eq. (6), the constraints of Eq. (5) become equivalent to those of Eq. (2).

The posterior spectra $Q_i$ are not altered if the $n_i$ are replaced with a proportional family of numbers $w_i$, since a common factor can be absorbed into the $\beta_i$. Furthermore, there does not seem to be much point to requiring the ratios of the $w_i$ to be rational—we regard the model of $s$, as a sum of components $s_{ij}$ as being suggestive, but not necessarily to be taken literally. We therefore replace $n_i$ in Eq. (6) with arbitrary positive numbers $w_i$ and obtain

$$Q_i(f) = \frac{1}{P_i(f) + \sum_{j=1}^{M} 2\beta_j \cos 2\pi f t_i}. \tag{7}$$

Allowing the weights to be frequency-dependent can be motivated heuristically by considering another somewhat artificial procedure—converting a single-signal problem into a multisignal problem. Assume we are given autocorrelation values $R_r$ and a prior estimate $P$ for the spectrum $S$ of a single signal $s$. Suppose we arbitrarily partition the band of frequencies from 0 to $W$ into two bands $B_1$ and $B_2$ and write $s$ as a sum

$$s(t) = s_1(t) + s_2(t)$$

of two signals whose spectra $S_1$ and $S_2$ are confined to the respective bands $B_1$ and $B_2$:

$$S_i(f) = \begin{cases} S(f), & f \in B_i, \\ 0, & f \notin B_i. \end{cases} \quad r = 1, 2.$$

We solve the two-signal MCESA problem with autocorrelation values given by $R_{\text{tot}} = R_r$, prior estimates $P_1$, $P_2$ given by

$$P_i(f) = \begin{cases} P(f), & f \in B_i, \\ 0, & f \notin B_i. \end{cases}$$
and frequency-independent weights \( w_i \). Strictly speaking, Eq. (4) is ill-defined where \( P_i(f) = 0 \); however, there is no problem with the following alternative form,

\[
Q_i(f) = \frac{P_i(f)}{1 + \frac{1}{w_i} P_i(f) \sum_{i=1}^{N} 2\beta_i \cos 2\pi f t_i}
\]

The sum \( Q \) of the posteriors satisfies

\[
Q(f) = Q_1(f) + Q_2(f)
\]

\[
= \begin{cases} 
Q_1(f), & f \in B_1 \\
Q_2(f), & f \in B_2 
\end{cases}
\]

and therefore

\[
Q(f) = \frac{1}{1 + \frac{1}{w_1} \sum_{i=1}^{N} 2\beta_i \cos 2\pi f t_i}
\]

when \( f \in B_i, \ (i = 1, 2). \) With equal weights \( w_1 = w_2 = 1 \), we thus merely recover the single-signal MCESA result in a roundabout way. However, we may write this as

\[
Q(f) = \frac{1}{P(f)} + \frac{1}{w_1} \sum_{i=1}^{N} 2\beta_i \cos 2\pi f t_i
\]

where, with general weights,

\[
w(f) = \begin{cases} 
w_1, & f \in B_1 \\
w_2, & f \in B_2 
\end{cases}
\]

We could equally well have partitioned the frequencies into any number of sets \( B_i \), not merely 2; from that point it is an obvious step to consider Eq. (8) with weights \( w(f) \) not restricted to a finite number of values \( w_i \). We are thus led to a single-signal version of Eq. (4). The generalizations that lead from the unweighted multisignal MCESA estimate Eq. (1) to Eq. (7) and Eq. (8) combine to yield Eq. (4) in full generality.

**DERIVATION**

We collect here some notation and results from Ref. 1. For the \( K \) signals, we use discrete-spectrum approximations

\[
s_i(t) = \sum_{k=1}^{N} (a_{ik} \cos 2\pi f_k t + b_{ik} \sin 2\pi f_k t),
\]

\((i = 1, \ldots, K)\). The \( f_k \) are nonzero frequencies, not necessarily uniformly spaced, and the \( a_{ik} \) and \( b_{ik} \) are random variables. We define random variables

\[
x_{ik} = \frac{1}{4} (a_{ik}^2 + b_{ik}^2)
\]

representing the power of process \( s_i \) at frequency \( f_k \), and we consider their joint probability density \( q^r(x) \), where \( x = (x_1, \ldots, x_N) \) and \( x_i = (x_{i1}, \ldots, x_{iN}) \). Assuming prior estimates \( P_{ik} = P_i(f_k) \) of the power spectra of the \( s_i \), we write prior estimates \( \rho \) of \( q^r \) in the form

\[
p(x) = \prod_{i=1}^{K} \prod_{k=1}^{N} P_{ik}(x_{ik}).
\]
where

\[ p_{ik}(x_{ik}) = \frac{1}{P_{ik}} \exp\left(-\frac{x_{ik}}{P_{ik}}\right) \]

Autocorrelation values \( R_{tot} = R(t_r) \) for the sum of the \( s_i \) are assumed known at lags \( t_r \) \( (r = 0, \ldots, M) \) with \( t_0 = 0 \). (In Ref. 1, \( R \) is written in place of \( R_{tot} \).) We write these as linear constraints

\[ R_{tot}^r = \sum_{i=1}^{K} \sum_{k=1}^{N} c_{ik} x_{ik} q(x) \text{d}x \]  

(9)

on expectation values of \( q^r \), where

\[ c_{ik} = 2 \cos 2\pi f_k t_r. \]

We obtain a posterior estimate \( q \) of \( Q^r \) by minimizing the cross entropy

\[ H(q, p) = \int q(x) \log \frac{q(x)}{p(x)} \text{d}x \]

subject to the constraints (Eq. (9) with \( q \) in place of \( q^r \)); the result has the form

\[ q(x) = \prod_{i=1}^{K} \prod_{k=1}^{N} q_{ik}(x_{ik}), \]  

(10)

where the \( q_{ik} \) are related to the posterior estimates

\[ Q_{ik} = Q(f_k) = \int x_{ik} q(x) \text{d}x \]

of the power spectra of the \( s_i \) by

\[ q_{ik}(x_{ik}) = \frac{1}{Q_{ik}} \exp\left(-\frac{x_{ik}}{Q_{ik}}\right). \]  

(11)

We find a discrete-frequency version of Eq. (1),

\[ Q_{ik} = \frac{1}{P_{ik}} + \sum_{j=1}^{M} \beta_j c_{jk}, \]  

(12)

where the \( \beta_j \) must be chosen so that

\[ \sum_{i=1}^{K} \sum_{k=1}^{N} c_{ik} Q_{ik} = R_{tot}^r \]

is satisfied.

Equation (10) states the posterior independence of the \( x_{ik} \). We would have obtained the same results Eq. (11), Eq. (12) if we had assumed Eq. (10) from the start, choosing the \( q_{ik} \) to minimize \( H(q, p) \) subject to the constraints with \( q \) expressed in the form of Eq. (10). For such densities \( q \), we have

\[ H(q, p) = \sum_{i=1}^{K} \sum_{k=1}^{N} H(q_{ik}, p_{ik}), \]  

(13)

and the constraints assume the form

\[ \sum_{i=1}^{K} \sum_{k=1}^{N} c_{ik} x_{ik} q_{ik}(x_{ik}) \text{d}x_{ik} = R_{tot}^r. \]  

(14)
Thus multisignal MCESA (without weighting factors) can be obtained by minimizing Eq. (13) subject to the constraints of Eq. (14). Our generalization is to replace the right-hand side of Eq. (13) with a weighted sum

$$
\sum_{k=1}^{K} \sum_{i=1}^{N} w_{ik} H(q_{ik}, p_{ik}) = \sum_{k=1}^{K} \sum_{i=1}^{N} w_{ik} \int q_{ik}(x_{ik}) \log \frac{q_{ik}(x_{ik})}{p_{ik}(x_{ik})} \, dx_{ik}.
$$

This is to be minimized with respect to variations in the $q_{ik}$ subject to the constraints Eq. (14)—together, of course, with the normalization constraints

$$
\int q_{ik}(x_{ik}) \, dx_{ik} = 1. \quad (15)
$$

The result is

$$
q_{ik}(x_{ik}) = p_{ik}(x_{ik}) \exp \left( -1 - \frac{\lambda_{ik}}{w_{ik}} - \frac{1}{w_{ik}} \sum_{i=1}^{N} \beta_{r} c_{ik} x_{ik} \right),
$$

where the $\beta_{r}$ and the $\lambda_{ik}$ are Lagrange multipliers corresponding to the constraints Eqs. (14) and (15).

From this it follows that Eq. (11) holds with

$$
Q_{ik} = \int x_{ik} q_{ik}(x_{ik}) \, dx_{ik} = \frac{1}{P_{ik}} + \frac{1}{w_{ik}} \sum_{r=1}^{M} \beta_{r} c_{ik}
$$

the argument is the same that established (13) and (14) from Eq. (11) in Ref. 2.

Passing to the continuous-frequency case, we write the weights as $w_{i}(f)$ and replace Eq. (16) with Eq. (4).

**PROPERTIES**

We begin with two trivial observations. The first is that, as noted in the second section of this report, the weights $w_{i}$ may be scaled by a common factor without affecting the results. The second is that the present method does indeed reduce to multisignal MCESA when the $w_{i}$ are all constant and equal.

Next we show that the posterior spectra $Q_{i}$ can be obtained by minimizing the sum

$$
\sum_{i=1}^{K} \int_{0}^{\infty} w_{i}(f) \left( \frac{Q_{i}(f)}{P_{i}(f)} - \log \frac{Q_{i}(f)}{P_{i}(f)} - 1 \right) \, df
$$

of weighted Itakura-Saito distortions subject to the constraints of Eq. (2). We form the expression

$$
\sum_{i=1}^{K} \int_{0}^{\infty} w_{i}(f) \left( \frac{Q_{i}(f)}{P_{i}(f)} - \log \frac{Q_{i}(f)}{P_{i}(f)} - 1 \right) \, df + \sum_{i=1}^{K} 2\beta_{i} \int_{0}^{\infty} w_{i}(f) \cos 2\pi f_{t} \, df
$$

involving Lagrange multipliers $\beta_{i}$, and we set its variation with respect to $Q_{i}(f)$ equal to zero:

$$
w_{i}(f) \left( \frac{1}{P_{i}(f)} - \frac{1}{Q_{i}(f)} \right) + \sum_{i=1}^{K} 2\beta_{i} \cos 2\pi f_{t} = 0.
$$

This implies Eq. (4). We obtain a minimum of Eq. (17) since the second variation, $w_{i}(f)/Q_{i}(f)^{2}$, is positive.
Finally, we justify the claim that increasing one of the weights tends to decrease the distortion between the corresponding prior and posterior spectra while increasing the distortions for other prior-posterior pairs. We write $D(Q,P)$ for the Itakura-Saito distortion of $P$ with respect to $Q$ (the integral in Eq. (3)). We first consider frequency-independent weights. Let $w'$ be the result of increasing $w_i$ for a particular value $a$ of $i$ and leaving the rest of the weights the same:

$$w'_a > w_a,$$
$$w'_i = w_i \quad (i \neq a).$$

Let the use of weights $w'_i$ result in posterior spectra $Q'_i$. We will show that

$$D(Q'_a,P_a) < D(Q_a,P_a), \quad (18)$$

and that

$$D(Q'_i,P) > D(Q_i,P) \quad (19)$$

for at least one value of $i$ (necessarily different from $a$).

Now $Q_i$ minimizes

$$\sum_i w_i D(Q_i,P_i)$$

subject to the constraints (2) while $Q'_i$ minimizes

$$\sum_i w'_i D(Q'_i,P_i)$$

subject to the same constraints (with $Q'_i$ in place of $Q_i$). It follows that

$$\sum_i w_i D(Q_i,P_i) < \sum_i w'_i D(Q'_i,P_i) \quad (20)$$

and

$$\sum_i w'_i D(Q'_i,P_i) > \sum_i w'_i D(Q'_i,P_i) \quad (21)$$

Subtracting Eq. (20) from Eq. (21) yields

$$(w'_a - w_a)D(Q_a,P_a) > (w'_a - w_a)D(Q'_a,P_a).$$

Since $(w'_a - w_a)$ is positive, we have Eq. (18). But it follows from Eq. (20) that Eq. (19) holds for some $i$.

A similar argument establishes a somewhat similar result for frequency-dependent weights. For simplicity we state this for the single-signal case. Let $w'$ be the result of increasing $w$ on some band $B$ of frequencies:

$$w'(f) > w(f), \quad f \in B$$
$$w'(f) = w(f), \quad f \notin B.$$ Let the use of $w$ and $w'$ result in posterior spectra $Q$ and $Q'$, respectively. Then

$$\int_B \left[ \frac{Q'(f)}{P(f)} - \frac{Q(f)}{P(f)} - 1 \right] df < \int_{B'} \left[ \frac{Q'(f)}{P(f)} - \frac{Q(f)}{P(f)} - 1 \right] df$$

for some $B' \subseteq B$, while the reverse inequality holds for some $B''$ disjoint from $B'$. That is, increasing the weight on $B$ decreases the distortion on some subset $B'$ of $B$ while increasing the distortion elsewhere.
EXAMPLE

In this section we take up a two-signal numerical example that was used in Ref. 1 to illustrate multisignal MCESA. We use the same prior spectra and the same autocorrelation data, and we show plots of the posterior spectra that result from various choices of the weights. The autocorrelation data were derived from the sum of a pair of assumed original spectra $S_S$ and $S_B$. (In Ref. 1 the indices stood for signal and background.) These spectra each have a sharp peak at a single frequency. They are shown in Fig. 1. The priors $P_S$ and $P_B$ were taken to be respectively flat and identical to $S_B$. They are shown in Fig. 2, which corresponds to Figs. 3 and 2 of Ref. 1. Figures 3–7 show posterior spectra $Q_S$ and $Q_B$ for five choices of constant weights: the ratios $w_S/w_B$ are 90/10, 75/25, 50/50, 25/75, and 10/90. The spectra in Fig. 5, the equal-weight case, are identical to those in Figs. 6 and 7 of Ref. 1. In each case there is a sharp peak in $Q_B$, corresponding to the peak that is in $S_B$ and $P_B$; each case there is a second peak in one or both of the posteriors, corresponding to the peak that is in $S_S$ but in neither prior; its appearance in the posteriors is due entirely to information contained in the autocorrelation data. In Fig. 3, for which the prior $P_S$ is heavily weighted, the posterior $Q_S$ is approximately flat, resembling the prior, and most of the power in the second peak is attributed to $Q_B$. In Fig. 4, $P_B$ is the heavily weighted prior; most of the power in the second peak is attributed to $Q_S$, whose corresponding prior received a small weight indicating prior uncertainty. The figures between show a progression from the one extreme to the other as the weight shifts.
Fig. 2 — Prior spectral estimates
Fig. 3 — Posterior spectral estimates ($w_f/w_g = 90/10$)

Fig. 4 — Posterior spectral estimates ($w_f/w_g = 75/25$)
Fig. 5 - Posterior spectral estimates ($w_2/w_B = 50/50$)

Fig. 6 - Posterior spectral estimates ($w_2/w_B = 25/75$)
DISCUSSION

The generalization of multisignal MCESA we have presented allows one to take into account the possibility that prior spectrum estimates are not equally reliable for all signals or in all frequency ranges for the same signal. One can express greater or less confidence or "degree of belief" in the prior estimates for the various signals in various frequency ranges by assigning greater or smaller values to the corresponding weights. Roughly speaking, increasing a weight tends to bring the corresponding posterior closer to the prior. This statement is made more precise in the results proved in the fourth section of this report and is illustrated in the examples.

Computer programs for multisignal MCESA spectrum estimation with weights may be found in the appendices of Ref. 7.

For a single signal, somewhat different considerations have led Chu and Messerschmitt [8] to the idea of minimizing a weighted Itakura-Saito distortion. However, their procedure involves minimizing the integral in Eq. (17) by varying $P$ rather than $Q$. Chu and Messerschmitt view the frequency-dependent weight as a means for taking into account the varying perceptual importance of various frequency ranges of a speech signal. They consider the weighted distortion between a true spectrum and an all-pole estimate, rather than that between a posterior estimate and a prior estimate, and they thus obtain a new method for choosing all-pole estimates of the usual form, rather than an estimate in a new form such as Eq. (4).
REFERENCES


