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New Results for Transition Probabilities in Two-Level Systems:
The Large Detuning Regime

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23. BE STUDIED IN THREE-LEVEL ATOMIC SYSTEMS. NEITHER THE PHYSICAL PROCESSES
24. NOR THE ROLE OF INTERFERENCE EFFECTS IS UNDERSTOOD IN THESE SYSTEMS. THE
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Abstract

The problem of calculating transition probabilities in two-level systems is studied in the limit where the detuning is large compared to the inverse duration of the interaction. Coupling potentials whose Fourier transforms $\tilde{V}(\omega)$ are of the form $\tilde{V}(\omega)e^{-\beta|\Omega|}$ for large frequencies give rise to solutions which may be classified into families according to the form of $\tilde{V}(\omega)$. Within each family, transition probabilities may be calculated from formulas that differ only in the numerical value of a scaling parameter. In cases where the coupling function has a pole in the complex time plane, the families are identified with the order of this singularity. In particular, for poles of first order, a connection with the Rosen-Zener solution can be made.

The analysis is performed via high-order perturbation expansions, which are shown to always converge for two-level systems driven by coupling potentials of finite pulse area.
1. Introduction

In many areas of physics, one encounters problems involving two states of a quantum-mechanical system coupled by a time-dependent potential. In the interaction representation, the equations of motion for $a_1$ and $a_2$, the probability amplitudes of levels 1 and 2, are of the form

$$i\dot{a}_1 = \sqrt{V(t)} e^{i\omega t} a_2,$$

$$i\dot{a}_2 = \sqrt{V(t)} e^{-i\omega t} a_1,$$

where $\omega$ is the frequency separation of the states and $V(t)$ is the coupling potential. Decay effects are neglected in Eqs. (1) throughout this paper and we work in a system of units in which $\hbar = 1$.

Equations of this type arise in many semiclassical problems. A problem of current interest to which they apply is the coupling of two levels of an atom by a laser pulse that has a temporal width which is small compared to the natural lifetimes of the levels. The pulse, $V(t)$, is of the form

$$V(t) = 2A(t) \cos \Omega t,$$

where $\Omega$ is the central frequency of the pulse, and $A(t)$ is the envelope function of its amplitude. Assuming that $\frac{\Omega}{\hbar \omega} \ll 1$, one can recast Eqs. (1) in terms of $\lambda$, the detuning of the pulse from resonance (rotating wave approximation) as
\[
\dot{\alpha}_1 = A(t) e^{-\alpha_1}, \quad (3)
\]
\[
\dot{\alpha}_2 = A(t) e^{\alpha_2}. \quad (4)
\]

Eqs. (3) or (4) are exceptionally simple in form, and at first glance, believe that the system must be completely understood, so that nothing remains to be investigated about the equations or their solution. Actually, there is very little known about the overall qualitative nature of the solutions to these (3) for arbitrary \(A(t)\). Apart from any intrinsic interest one might have in the dynamics of two-level systems, such information could be useful, for example, in applications where one wishes to choose the pulse shape to maximize the excitation probability for a given detuning \(\Delta\).

To appreciate that our assertion concerning the lack of knowledge about the behavior of systems described by Eqs. (3) is valid, we need only recognize that the answer to the following question is not known in general. "Starting with initial conditions \(\alpha_1(0) = 1, \alpha_2(0) = 0\) how does the probability amplitude \(\alpha_2(t)\) depend qualitatively on the pulse area \(S\), defined by
\[
S = \int_{t_0}^{t_f} A(t) \, dt,
\]
on the detuning, and on the shape of the envelope function \(A(t)\)?"}

A response to this query can be made for a limited number of cases. Analytic solutions are available if \(A(t)\) belongs to a class of functions, (including the hyperbolic secant of Rosen and Ceperley) mappable into the
hyperbolic equation, or if \( A(t) = (\text{constant}) \exp (-\alpha |t|)^{1/2} \) or if \( A(t) \) is a step function (both cases), or if the duration is zero. However, there are approximate solutions available in adiabatic or perturbative limits. Yet, there remains a wide range of parameters and pulse shapes for which an answer to the basic question cannot be provided.

In this paper, we shall examine the solutions to Eqs. (2) in the limit where the pulse \( A(t) \) and the characteristic pulse duration \( T \) are a constant, greatly in excess of unity. In other words, we are assuming that the pulse does not force the appropriate Fourier components to significantly resonate for the duration. In consequence, the transition probability \( |U_{21}(t)|^2 \) will always be very small (but will be great enough to be extremely measurable in intense states or conditions \( \lambda \approx 1 \)). We note that numerical solutions of Eqs. (2) in this scaling range may be possible but are very costly in computer time and plagued with technical difficulties.

For the case \(|A_0| = 1 \), we shall outline the following results:

1. Low-order perturbative approximations for \( a_2(t) \) are not valid for arbitrary pulse area \( A_0 \), despite the fact that \( |U_{21}(t)|^2 = 1 \) for all times.
2. An iterative solution to Eqs. (2) always converges for well-chosen envelope functions. (1) Asymptotic solutions for \( a_2(t) \), a finite, but easily found, but expressions for \( a_2(t) \) are difficult to obtain. (1)

Asymptotic solutions for \( a_2(t) \) can be obtained for a limited class of

\[ \text{functions of the applitude, one obtains closed-form expressions.} \]

\[ \text{Kaplan}^7 \text{ has also considered cases where the amplitude varies as prescribed functions of the applitude, one obtains closed-form expressions.} \]
pulse envelope functions using contour integration techniques. This is a broader set than that for which exact solutions are known. (5) The asymptotic behavior of \( a_y(x) \) depends critically on the nature of the singularities of the pulse envelope function \( a(t) \), analytically continued into the complex plane. (6) If the pulse functions have the same Fourier transforms in the limit of large frequencies and if the almost exponential of the transforms in an exponential decay in the frequency, then the asymptotic forms of the solutions \( a_y(x) \) for these functions in the limit of large \( |x| \) are simply related. In this paper, we address points (5), (6), (4), and (3); formulae for actually obtaining asymptotic solutions (points (3) and (4)) will be discussed in a future article.

II. Asymptotic solutions.

As we have indicated, the Rosenblith \(^{5,6} \) (asymmetric nonlinear coupling pulse) problem is one of the few for which exact solutions are known. In this case, a simple expression gives the transition amplitude as a function of exciting and area for all values of these parameters. Naturally, since this formula

\[
A_y(\infty) = -\frac{i}{\sqrt{2\pi}} \hat{\mathcal{V}}(A) \frac{S/\alpha S}{S},
\]

where \( \hat{\mathcal{V}} \) is the Fourier transform of \( a(t) \), is exact, it is valid in the special case of the asymptotic limit.

We shall show that there is an entire class of pulses for which the asymptotic transition amplitude, as a function of \( \alpha \) and \( S \), may be written
down by it, and if the von-Neumann problem has been solved, we
shall also demonstrate that there are other classes of points whose
solutions to the von-Neumann form are connected to those of
von-Neumann, but are connected to each other in the sense that one
or the other has been solved; the solutions for the entire class may
be obtained by inspection.

The existence of these related solutions will be established
via term-by-term comparison of the order perturbation equations with
under very general conditions, are convergent in the limit problem
(a.e. Appendix). With suitable scaling of the resulting integrals, the
series for different choices of particular values will be seen to be
identical in the limit of large arguments.

The particular potential analyzed in this paper are \( \phi(x) \) which

Fourier transforms for large values the form \( p(x) \) or \( -|p(x)| \), the

\( p \) is a slowly varying function of \( x \), and \( \alpha \) constant. It is convenient
to make a variable choice, such that \( \nu = \|x\| \) are \( \nu = \alpha/|x| \). Consequently,
the exponential decay factor in the Fourier transform becomes \( \exp(-\|x\|) \)
and the equations of motion transform to
\[
\dot{a}_1 = \beta f(x) e^{i\alpha x} a_2, \tag{3a1}
\]
\[
\dot{a}_2 = \beta f(x) e^{-i\alpha x} a_1, \tag{3a2}
\]
where \( \alpha = \|x\| \) and where the dot denotes differentiation with
respect to \( x \), which is previously designated as \( \nu \) in the planar area. The
reduced potential function \( f(x) \) is defined such that \( \int f(x) \, dx = 1 \). The
\[ \ddot{a}_1 - \left( \frac{\omega^2}{c^2} + i \omega \right) \dot{a}_1 + \beta^2 \sigma^2 a_1 = 0, \quad (a) \]
\[ \ddot{a}_2 - \left( \frac{\omega^2}{c^2} - i \omega \right) \dot{a}_2 + \beta^2 \sigma^2 a_2 = 0. \quad (b) \]

These are the results to the solutions of Eqs. (a) or (b). These are the calculation of the solutions at finite and infinite times, respectively. Any form of solution, if the transients are to be used as input to other problems, such as multiphase, including, will require a similar amount of calculation. All the results, with slight modifications, can be reproduced in the manner that most of the transients are calculated.

Apart from the real problem, the approach which is attractive, the next step is to be made an algebraic, \( \zeta(x) \rightarrow (\text{real} + \text{imaginary}) \), for which the solutions are
\[ a_1 = \, \, \, _2 F_1 \left( a, b, c, z \right), \quad (a) \]
\[ a_2 = -i \, \, \, K^{\frac{1-c^\gamma}{2}} \, \, \, _2 F_1 \left( a-c^\gamma + 1, b-c^\gamma + 1, a-c^\gamma, z \right), \quad (b) \]

or
\[ a_2 = -i \, \, \, K^{\frac{1-c^\gamma}{2}} (1-z)^{\frac{c^\gamma - a - b}{2}} \, \, \, _2 F_1 \left( 1-a, 1-b, c-c^\gamma, z \right), \quad (c) \]
where \( a = -b = \beta \left( \frac{1}{1 - \frac{i \alpha}{\sqrt{11}}} \right) \), \( c = \frac{1}{\sqrt{11}} - \frac{i \alpha}{\sqrt{11}} \),

\[ z = \frac{-\tanh \left( \frac{\beta x}{1 - \frac{i \alpha}{\sqrt{11}}} \right) + 1}{2} \quad \text{and} \quad K = \frac{\beta}{\sqrt{11}} \left( 1 - \frac{i \alpha}{\sqrt{11}} \right) \]

and \( F \) is the hyperbolic sine function. The form of \( y \) is given

\[ y = e^{+ \omega x} \] is valid for all \( x \), while that given by \( y = 0 \) is only

valid for finite \( x \), where \( \omega = \sqrt{\frac{\beta}{1 - \frac{i \alpha}{\sqrt{11}}}} \) or \( \omega = \sqrt{\frac{\beta}{1 - \frac{i \alpha}{\sqrt{11}}}} \), for which \( e^{+ \omega x} \) and \( e^{- \omega x} \) are eigenfunctions. Let us denote \( \omega \) by \( \omega_{+} \) and \( \omega_{-} \) for \( e^{+ \omega x} \) and \( e^{- \omega x} \) respectively.

The eigenvalues \( \lambda_{1}, \lambda_{2}, \ldots \) are determined by the equation

\[ \frac{(2j+1)}{2} \lambda_{j}^{2} + \frac{\omega_{+}^{2} \beta}{2} \lambda_{j} + \frac{\omega_{-}^{2} \beta}{2} = 0 \]

for \( \lambda_{j} \), where \( j = 0, 1, 2, \ldots \). The solution for \( \lambda_{j} \) is

\[ \lambda_{j} = -i \sum_{k=0}^{\infty} A_{k} \beta^{k} (-1)^{k} \]

where

\[ A_{k} = \sum_{j=0}^{(2k+1)} \frac{\omega_{+}^{2} \beta}{2} \lambda_{j}^{2} + \frac{\omega_{-}^{2} \beta}{2} \lambda_{j} + \frac{\omega_{+}^{2} \beta}{2} \lambda_{j}^{2} \]

\( \alpha_{j} \) and \( \beta_{j} \) are the eigenvalues for \( e^{+ \omega x} \) and \( e^{- \omega x} \) respectively.

In the Appendix, it is shown that this series converges for all finite

eigenvalues.

For the remainder of the paper, we will restrict our selves to

two cases of behavior that are described in the following.

- 0. -
there is only one solution. The other two terms will be

appearing in $\beta$. We shall begin by expanding the $\beta$ terms in infinite
time solutions of the same type, but with only four
proportions of the others. Let the solutions be given by

$$A_2 = -\frac{i\beta}{\alpha'} \frac{e^{i\beta x}}{\cosh \frac{\beta}{\alpha'} x} \left[ 1 + \frac{(1 - \beta^2)(1 - \beta^2) (-1 + \frac{\beta^2}{\alpha^2} + 1) + \cdots}{\alpha^2} \right].$$

For large $\alpha$, $\beta$ is approximately much smaller than $\beta^2$, we may assume

$$A_2 \approx \frac{\beta}{\alpha'} e^{i\beta x} \cosh \frac{\beta}{\alpha'} x.$$

This is equivalent to finding $\beta$ or $\beta'$, or by replacing the

integrals

$$A_2^{(n)} = -i \int_{-\infty}^{\infty} V(x') e^{-i\beta x'} dx' \approx \frac{V(x) - i\alpha' x}{\alpha'}$$

where the first part of the integral can be taken, since they are

$\mathcal{O}(\frac{1}{\alpha^2})$. With this, we can now use the integration

is unsuitable for calculating $\alpha(x)$, since each term separate to conclude

with $x = 0$. Even including the third - and higher-power terms in the
perturbations of both real and imaginary parts of the integral as done
not only are contained but are. Substitution,
hence some are necessary to calculate for each.

It is clear from the above argument that for certain eigh-
where, circumstances, or situation they are sufficiently accurate approxi-
mation for certain cases, provided A is chosen. The condition that, not
only is contained in the range of the ideal, but that, the critical
condition is contained in the range of the ideal, and that the ideal
approximation is also contained in the range of the ideal. In both.

We shall not be satisfied if it were not, or if we are satisfied if it were.
We shall not be satisfied if it were not, and if we are satisfied if it were.

Since each coupling constant \( k \) is different, one might be
led to believe that separate calculations must be performed for each
individual case. Fortunately, as we have stated earlier, there have
to be obtained for other places, if we know the fractional omission of the asymptotic transition coefficient, and, for one factor of the class, can know it for all members of the class, although the actual time dependence of the potential may be markedly different. That is significant in that their behavior transforms, and the same thing in an a → x.

When k = 1 or more or 2 or more, the factors are.* A more general problem is to solve for asymptotic pulses, both in H and valid for all cubic equations. That is essential and is that a kind of control in all cases. The only time equations for pulses in the form l(a) and cubic pulses, or x = 1, etc. However, it apply to pulses with higher order poles. This is not, although equally true, although for each order.

The L (a) n, that would be contained in the coupling pulses, l(a) and l(x) have to do with the r + l(a) and l(x). The Fourier transform of the function, for large values of the argument, has the asymptotic form \( f_\infty (\nu) \). If \( f_\infty \) is of the form, \( \nu (\nu - 1) \) where \( \nu (\nu) \) is a slowly varying function of \( \nu \), then the asymptotic transition matrix is generated by the two pulses will be the same, provided that the time interval are such that. A sufficient condition for the indicated asymptotic behavior of the Fourier transforms is that they be equal, for large \( \nu \), to a constant integration where value is given by the product of the constant at \( x = 1 \) and the usual density factor 2.1. If two such pulses

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are to have the same \( \alpha \), they must possess poles of the same order at \( z = i \).

The contribution of order \((2^3 + 1)\) to the transition amplitude may be rewritten slightly

\[
\alpha_2 = \frac{1}{(2\pi \hbar)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(x_1) e^{-i\omega x_1} \frac{\partial}{\partial x_1} \left[ \lim_{\lambda_0 \to 0} \int_{-\infty}^{\infty} A(x_j) e^{-i\omega x_j} \right] d\lambda_0 \]  

The factors \( e^{i\omega x_1} \) do not affect the integrals. They are used to remove

\[
\alpha_2 = \frac{1}{(2\pi \hbar)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(x_1) e^{-i\omega x_1} \frac{\partial}{\partial x_1} \left[ \lim_{\lambda_0 \to 0} \int_{-\infty}^{\infty} A(x_j) e^{-i\omega x_j} \right] d\lambda_0 .
\]

by working in the frequency domain, we shall be able to examine the

structure of the integrals for \( \alpha_2 \) and establish that the contribu-

tion from regions where the asymptotic form of \( \tilde{\gamma} \) is not valid is lower

by \( \mathcal{O}(1/\hbar) \) than the contributions from regions where it is valid.

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The integrals over the \( x_1 \) are trivial to perform. We obtain

\[
A_n = \lim_{\lambda \to 0} \frac{1}{(2\pi i)^{k-1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \sum_{j=2}^{2^n+1} \frac{\tilde{V}(\gamma_j)}{(\gamma_j - \alpha)^2} \right) \cdot \left( \sum_{j=2}^{2^n+1} (\gamma_j - (-1)^j - i \lambda \gamma_j) \right).
\]

We next proceed to determine the asymptotic form of the effective action.

The analysis is complex to follow for the character calculation \( A_3 \), but exactly the same formula and calculations apply for all \( A_\lambda \) with \( \lambda \neq \alpha \), the other terms. (The theorem is true up to terms determined only by \( \gamma_1 \), then that contribution will come from \( \alpha = \gamma_1 \).) The theorem itself still holds, if the counting is done with the appropriate number of the new asymptotic form, their truncation terms in the integral can be the same way up to terms of the leading order.
\[
\frac{1}{\sqrt{\det \Gamma}} \left\{ P \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\tilde{V}(x, y_1) \tilde{V}(x, y_2) \tilde{V}(x, [y_1 + y_2 - 1])}{(y_1-1)(y_1+y_2)} dy_1 dy_2 \\
+ i \pi \lim_{\lambda \to 0} \int_{-\infty}^{\infty} dy_2 \left[ \frac{\tilde{V}(x, y_2)\tilde{V}(x, y_2)^2}{1 + y_2 - i \lambda} + \frac{\tilde{V}(x, y_2)^2 \tilde{V}(x)}{-y_2 - 1 - i \lambda} \right] \right\},
\]

where \( i \) indicates that the integrals exclude infinitesimal regions near \( y_1 = -y_2 \) and \( y_2 = 1 \). We may formally integrate the first two terms. If, \( (-1) \) is factored from the second of the two integrals, they combine to be:

\[
i \pi \lim_{\lambda \to 0} \int_{-\infty}^{\infty} dy_2 \tilde{V}(x) \tilde{V}(x, y_2)^2 \left\{ \frac{1}{1 + y_2 - i \lambda} - \frac{1}{1 + y_2 + i \lambda} \right\}.
\]

It is immediately obvious that if these are partitioned according to the rule

\[
\lim_{\epsilon \to 0} \int \frac{\phi(x) dx}{x-x_0-i\epsilon} = P \int \frac{\phi(x) dx}{x-x_0} + i \pi \phi(x_0),
\]

the principal value contributions exactly cancel, while the latter are proportional to \(-\epsilon^2\), and exponentially small compared to \(\epsilon^2\), which is exponentially small. Terms proportional to exponentials which decay more rapidly than \(\epsilon^2\) do not contribute to the asymptotic form.

We now proceed to examine the remaining contributions to \(\phi(x)\),

where it is again asserted that the small regions in the right hand
of \( y_1 = -y_2 \) and \( y_2 = 1 \) are excluded from the integrals. For all regions
except where \(|y| < \frac{1}{n}\), where \(n\) is a number of order unity, \(\tilde{v}(x) \sim \tilde{v}_0(x)\).

Thus, for the entire \(y_1-y_2\) plane, except where \(y_1 \sim 0\), \(y_2 \sim 0\) (but not both simultaneously) and \(y_1 + y_2 \sim 1\), the integrand of the integrals is well represented by its asymptotic form. Furthermore, since at least one of the three Fourier transform factors departs from its asymptotic form in any given region of space, the area in the \(y_1 - y_2\) plane over which one of the \(V\) Kernel factors deviates from its asymptotic form and decays more rapidly than \(e^{-r}\) is \(O(\sqrt{t})\). In other words, implicitly we mean that the exact and asymptotic form of the Fourier transform remain “close” as their arguments \(\sim 0\). For the former, this is equivalent to the requirement, which we have already stated, that \(J = \sqrt{t}\).

Now consider the portion of the \(y_1-y_2\) plane where all factors in the numerator are well-approximated by their asymptotic forms, resulting in particular the exponential decay factors:

\[\alpha |y_1| - \alpha |y_2| - \alpha |y_1 + y_2| - 1\]

\[e \quad e \quad e \quad e\]

The only portion of the plane where the resultant effect of the exponential factors leads to an overall decay that is not faster than \(e^{-r}\) in the region \(0 < y_1 < 1\), \(0 < y_2 < 1 - y_1\). The integral over \(\{y_1, y_2\}\) is not changed sign in this portion, \(y_1 + y_2\) area, since one changes by area \(\sim 1/2\), compared to the area \(1/3\), which is the corresponding extent in which the nonasymptotic integrands drop or more rapidly than \(e^{-r}\). Note that there is no portion of the plane in which the integrand drops more slowly than \(e^{-r}\). Thus the nonasymptotic integrand contribution
is \( O\left(\frac{1}{t}\right) \) compared to that of the asymptotic integrand.

Similar considerations enable one to deduce that one may also replace the Fourier transforms in the higher-order integrals by their asymptotic forms.

We thus conclude that if the time-dependences of two coupling functions are such that the asymptotic forms of their Fourier transforms are identical out of the indicated form, the large detuning transition amplitudes are the same.

As we have indicated, a sufficient condition that two poles have the form \( \omega_n(t) \) for \( \nu > 0 \) is that both asymptotic Fourier transforms be equal to contour integrations given by (49) \( \Delta(t,\pm\nu) \). To ensure the hyperbolic tangent of zeros and zeros, \( \nu = \frac{1}{\nu} \) such \( \frac{1}{\nu} \) with the Lorentzian \( \nu = \frac{1}{\nu} (1+\omega^2)^{-1} \). The corresponding \( \alpha(x) \) are

\[
A_L(x) = \frac{\beta}{\pi} \left(1 + x^2\right)^{-1},
\]

\[
A_{H}(x) = \frac{\beta}{\alpha^2} \tanh \frac{\alpha x}{\alpha^2}.
\]

The transforms for both may be calculated via contour integrations. The Lorentzian case is trivial and applies to all \( \nu \), not just large frequencies. We choose a contour that runs along the real axis from \(-R\) to \(+R\) and is closed by a semicircle in the upper half plane. The contribution to the contour integral from the arc vanishes as \( x \to \infty \), so that the Fourier transform is identical to the contour integral, whose value is determined by the residue at the simple pole at \( x = 1 \).
The result is
\[ \tilde{\eta}_L = \frac{\beta}{\sqrt{2\pi}} e^{-|\eta|} \]  

(7a)

For the hyperbolic account we choose a rectangular contour which runs from \(-i\hbar\) to \(i\hbar\) along the real axis, that is continued by rectangular
and cuts parallel to the imaginary axis at the points \((\hbar, \pm \epsilon)\) to the points
\((\hbar, \pm i\hbar)\), and is closed by a line parallel to the real axis which runs
for \(\epsilon \rightarrow 0\) to \((-i\hbar, \hbar)\). The two vertical segments give variable con-
tributions to \(\tilde{\eta}_L\), and the horizontal segment off the real axis goes
asymptotically to zero as \(|\eta|\) to the current axis, the real axis as
\(\eta \rightarrow \hbar\). For the hyperbolic account, the Fourier transform is identical
to that of the Lorentzian in the asymptotic region. For large \(\nu\) it is
given by
\[ \tilde{\eta}_H \sim \frac{2\beta}{\sqrt{2\pi}} e^{-|\eta|} \]  

(7b)

Since the Lorentzian contour gives the transition amplitude for
all potentials, according to eq. (b), \(\nu = \sqrt{\hbar^2 - \nu^2}\), this formula
must be valid asymptotically also. As we have shown, that the asymptotic
Fourier transforms of the Lorentzian and hyperbolic account are proportional
for large potentials, the Lorentzian must induce a transition amplitude that
obeys a formula similar to eq. (b). From eq. (7), we see that to con-
struct the Lorentzian and hyperbolic account Fourier transforms so that
they are asymptotically identical, it is necessary to choose the Lorentzian
pulse area $A_L$ to be twice that of $A_L$. This immediately gives the
large detuning scaling law for the Lorentzian

$$a_{2L} = -i \sqrt{2\pi} \frac{\sqrt{2}}{a} \frac{\gamma}{L} (x) \sin \frac{\beta}{a}. \quad (8a)$$

This result has been independently obtained by carrying out an asymptotic
solution of Eqs. (3). \cite{12} One can also show that for the pulse $A_c =
\beta_c \cosh \chi$, the appropriate scaling law is

$$a_{2c} = -i \sqrt{2\pi} \frac{\sqrt{2}}{a} \frac{\gamma}{c} (x) \sin 2 \beta. \quad (8a)$$

For the hyperbolic secant pulse, the transition amplitude vanishes
for pulse areas $\beta = n\pi$, $n$ integral for all detunings. The Lorentzian,
on the other hand, has eigenvalues $\beta = n\pi$ for zero detuning, while these
for large detuning are $\beta = 2n\pi$. The eigenvalues of $A_c$ go from $n\pi$ at
$\nu = 0$ to $\frac{n\pi}{\nu}$ as $a \to \infty$.

The existence of a pole at $x = i$ is a sufficient, but not a necessary
condition that the asymptotic Fourier transform of a coupling pulse
$\sim p(x)e^{-|\omega|}$. For example, the function $(1 + x^2)^{-3/2}$ has an asymptotic
Fourier transform proportional to $\nu^{3/2} e^{-\nu}$. The factor $\nu^{1/2}$ produces
decaying the asymptotic transition amplitude from the denominator formula.
Similarly, the squares of the hyperbolic sech and of the Lorentzian each
have poles of second order at $x = i$, with the consequence that, for both
of these, $\gamma_a \sim \nu^1 e^{-|\nu|}$, so that while these will have asymptotic
transition amplitudes that are related to each other, they cannot be
obtained by scaling from Eq. (1). In our next paper, we shall show
how to calculate asymptotic transition amplitudes when the coupling pulse has second- and higher-order poles at \( z = 1 \). For now, we merely present the formulae for the transition amplitudes generated by the squares of the hyperbolic secant and Lorentzian

\[
A_a(H2) = -i \frac{2\pi}{C^2} e^{-\alpha} \sin \left[ C \sqrt{\frac{\alpha}{\pi}} \right] \sinh \left[ C \sqrt{\frac{\alpha}{\pi}} \right] \\
A_a(L2) = -i \frac{2\pi}{C^2} e^{-\alpha} \sin \left[ C \sqrt{\frac{\alpha}{\pi}} \right] \sinh \left[ C \sqrt{\frac{\alpha}{\pi}} \right]
\]

where \( C = 1 + \frac{1}{6} + \frac{1}{56} + \frac{1}{162} = 1.194 \). Equation (2a) can be obtained from eq. (3b) by scaling techniques derived in this paper.

III. Summary and Conclusion

In this paper, we have demonstrated that pulse shapes \( \Lambda(t) \) whose Fourier transforms asymptotically approach the form \( \phi(\nu)e^{-|\nu|} \), where \( \phi \) is slowly varying, may be categorized into families which differ according to the function \( \phi \). Within each family, the transition amplitudes \( a_2(\nu) \) are related by simple scaling laws, so that if one is able to derive an expression for the transition amplitude generated by one member of the family, corresponding formulae for all other members of the family may be written down by inspection.

A sufficient condition that the Fourier transform be of the required form is that it be obtainable in the asymptotic region as a contour integral evaluated from the residue at a single pole on the imaginary time axis. For the case where \( \Lambda(t) \) has single poles, \( a_2(\nu) \)
may be inferred from the solution of the non-linear problem \(^{13}\), known
for fifty years, by a trivial scaling operation.

Our results were obtained by examining the structure of the
term in perturbation expansions for transition amplitudes. (We have
demonstrated that these sequences always converge in one-level problems
provided that the pulse areas are finite. Low-order approximations,
however, are frequently not useful for times even when the exact result
at finite times.) With suitable choices of ratios of pulse areas,
corresponding terms in the series for different values of the same
family will be identical.

In a future paper\(^{15}\), we shall present solutions for explicitly
calculating transition amplitudes that apply to higher orders, as well
as single pulse. Thus, we are not restricted in practice to writing
scaling laws for pulses which may be compared in the asymptotic limit
to the hyperbolic secant.

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Appendix - Convergence of perturbation theory for the Transition

Amplitude

In this appendix we shall demonstrate how the perturbation series for \( a_0 \) converges for all finite pulse area \( A \). The contribution of order \((2n+1)\) is

\[
\begin{align*}
\mathbf{b}_1 &= -i \beta \quad a_2 = \\
2n+1 & \mathbf{f} \int_0^{\infty} -i \lambda x_i \quad 2n+1 \quad X_j \quad i \lambda x_j \quad (2n+1)
\end{align*}
\]

For example that \( A(\lambda) \) is of a single algebraic sign. Without loss of

Theorem we may take this to be positive. We compare the series with

the corresponding expansion for \( n = 0 \).

\[
\begin{align*}
\mathbf{b}_{10} &= -i \beta \quad (-1) \int_0^{\infty} \mathbf{f}(x_i)dx_i \quad \mathbf{f}(x_j)dx_j = (n-1) \\
2n+1 & \mathbf{f} \int_0^{\infty} -i \lambda x_i \quad 2n+1 \quad X_j \quad i \lambda x_j \quad (2n+1)
\end{align*}
\]

Invoking the theorem on repeated integrals of the same function

\[
\begin{align*}
\mathbf{b}_{10} &= -i \beta \frac{2n+1}{(2n+1)!} \quad (-1) \quad (\int_0^{\infty} \mathbf{f}(x)dx)^{2n+1} \\
\end{align*}
\]

and the terms are recognized as identical to those for the series \(-i \beta \).

Now consider the series
\[
F(\beta) = \sum_{k=0}^{\infty} b_{10}^{(k)} = \sum_{k=0}^{\infty} \frac{\beta^{2k+1}}{(2^{2k+1})!} \left( \int_{-\infty}^{\infty} f(x) \, dx \right)^{2k+1}
\]

This is evidently the series for some, which converges as long as \( \beta \) is finite. Hence, the series of \( b_{10}^{(k)} \) is absolutely convergent, i.e.,

\[
|b_{10}^{(k)}| = \left| \frac{\beta^{2k+1}}{(2^{2k+1})!} \right| \left| \int_{-\infty}^{\infty} f(x_1) e^{-ix_1} \, dx_1 \right| \left| \int_{-\infty}^{\infty} f(x_j) e^{-ix_j} \, dx_j \right|
\]

\[
\leq \left| \frac{\beta^{2k+1}}{(2^{2k+1})!} \right| \left( \int_{-\infty}^{\infty} |f(x_1)| \, dx_1 \right)^{2k+1} \left( \int_{-\infty}^{\infty} |f(x_j)| \, dx_j \right)^{2k+1}
\]

so that the series, \( b_{10}^{(k)} \), is also absolutely convergent, and the result is established.

We note that the same arguments will apply to perturbation series at finite times, provided purely that \( \int_{-\infty}^{\infty} f(x') \, dx' = f(x) \) is of one sign and finite. If \( f(x) \) changes sign, the results will still be valid provided the generalized area \( \int_{-\infty}^{\infty} |f(x')| \, dx' \), is finite.

A simple case where the convergence theorem does not apply is the coupling function \( k(x) = (\text{const}) \frac{\sin \alpha x}{x^3} \), since \( k(x) \) is logarithmically divergent. In addition, since the pulse area is proportional to the
Fourier transform at zero frequency, the multiple integrals in the frequency domain for the third- and higher-order contributions to the perturbation series contain regions where the integrands blow up, so that the individual terms beyond first order may not even exist. (The first-order contribution will be finite, since the Fourier transform for this pulse exists for \( v \neq 0 \). In this case, we note that the infinite area does not imply a pulse of infinite energy, so that it theoretically could exist. One obviously cannot use the methods developed here to describe the dynamics. At the very least, decay would have to be included in the analysis, and a completely non-perturbative treatment utilized.)
References


