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THE EFFECTS OF SMALL GAINS IN COMPUTER DESIGN

MICROCOMPUTER SYSTEMS

By

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Abstract

The dynamics of a large class of non-linear systems are described
implicitly, i.e. as a combination of algebraic and differential equations.
These dynamics admit of jump behavior. We extend the deterministic theory
to a stochastic theory since (i) the deterministic theory is restrictive,
(ii) the macroscopic deterministic description of dynamics frequently arises
from an aggregation of microscopically fluctuating dynamics and (iii) to
robustify the deterministic theory. We compare the stochastic theory with
the deterministic one in the limit that the intensity of the additive white
noise tends to zero. We study the modeling issues involved in applying
the stochastic theory to the study of the noise behavior of a multivibrator
circuit. Discuss the limitations of our methodology for certain classes of
systems and present a modified approach for the analysis of sample functions
of noisy non-linear circuits.

Keywords: Bifurcation, Singular Perturbation, Jump Behavior, Laplace's
method, Noise behavior of non-linear circuits.
Section 1. Introduction

The dynamics of a large class of engineering systems are described only implicitly, for instance, those of non-linear circuits, swing dynamics of an interconnected power system, as also thermodynamic systems far from equilibrium. The implicit definition of their dynamics is as follows: the state variables are constrained to satisfy some algebraic equations, i.e. they are constrained to lie on a manifold $M$ in the state space. The dynamics on this manifold $M$ are then specified implicitly by specifying only the projection of the vector field on $M$ onto a certain base space above which $M$ lies. (i.e. a subspace of the original state space of the same dimension as $M$). The process of obtaining the system dynamics explicitly consists of 'lifting' the specified velocities onto a vector field on $M$ (lifting is the inverse of projecting). Lifting may not, however, be possible at points where the projection map (restricted to the tangent space of the constraint manifold) has singularities. This singularity is typically resolved by regularization, i.e. by interpreting the algebraic constraint equations as the singularly perturbed limit of 'parasitic' or fast dynamics. The dynamics of the original system are obtained as the degenerate limit of the dynamics of the regularized system - the resulting trajectories may be discontinuous and this is referred to as jump behavior.

The foregoing deterministic theory needs to be extended to a stochastic theory for three reasons:

a) The conditions under which the limit trajectories to the regularizations exist are extremely restrictive so as to exclude several systems of interest.
b) Frequently, the algebraic constraint equations arise from the macroscopic aggregation of microscopically fluctuating dynamics, e.g. the flow of current in a resistor, the demand for electrical power at a distribution point in an electrical power network. More generally, deterministic equations describing thermodynamic systems are of this kind. Thus, the algebraic constraint equations contain in addition a rapidly fluctuating (or white noise) component.

c) The methods of analysis for deterministic systems of the implicitly defined kind involve techniques of bifurcation theory - their conclusions are extremely sensitive to imperfections and the addition of white noise. Since in all the situations of interest to us, the intensity of the additive noise is small, we study in this paper the dynamics of implicitly defined dynamics in the presence of small additive noise. In fact, we compare the conclusions of the stochastic theory with those of the deterministic theory in the limit that the noise intensity tends to zero. The foregoing process requires the computation of two sets of limits: the limit that the regularization tends to zero and the limit that the intensity of the additive white noise tends to zero. In general, these limits do not commute. We explore in this paper the modelling issue of which sequence of limits is appropriate in the context of a specific system. The layout of the paper where we carry out this program is as follows:

In Section 2, we review briefly the dynamics of deterministic constrained systems and their jump behavior. With some minor modifications we follow here our earlier work [11] and the references contained therein.

In Section 3, we begin the study of noisy constrained dynamical systems. For the initial study we use as tools the work of Papanicolaou, et al [10] on martingale approaches to limit theorems. To study the
dynamics of noisy constrained systems in the presence of small noise, we develop and use in our context Laplace's method of steepest descent. We study in several separate cases, the comparison between the deterministic and small noise theory, describing: (i) how the stochastic theory yields conclusions about system dynamics when the deterministic theory fails and (ii) how the jump behavior of systems is modified by the presence of small noise. This section is a considerable extension of our previous work in the context of phase transitions in van der Waals gases [12]. Several examples are presented to instantiate our results.

In Section 4, we present the detailed deterministic analysis of Section 2 applied to the dynamics of an emitter coupled relaxation oscillator circuit. We then show that the experimental conclusions of Abidi [1] on the dynamics of these circuits in the presence of small noise seem not to agree with the stochastic theory presented in Section 2.

In Section 5, we discuss the sequences of limits implied by the development of Section 3 - and the nature of systems for which this development yields the correct conclusions. In particular, we show that the development of Section 3 is relevant to systems where the separation in time scales between the slow and fast components is very large and is more important than the small intensity of the white noise (characterized by a certain sequence of limits subsumed by Section 3) - for instance in phase transitions, reaction rates and other phenomena of non-equilibrium thermodynamics. For non-linear circuits, however, the separation in time scales is less marked, so that we present here the relevant analysis (sample function calculations) for these systems with the order of limits reversed from that of Section 3. We use as tools the foundational
mode of executing the twists of the wire was that the twisted wires were passed through a series of pulleys and guides, and then fed into a machine designed to handle the wire. The machine consisted of a series of rollers and a series of guiding devices. The wire was then passed through a series of elongation devices, which stretched the wire and removed any slack. The wire was then passed through a series of cooling devices, which cooled the wire quickly to prevent any distressing. The wire was then passed through a series of twisting devices, which twisted the wire in the desired manner. The twisted wire was then passed through a series of guiding devices, which directed the wire to the desired location.
The intuitive picture that now emerges in the original t-timescale is as follows: For a hyperbolic equilibrium point \((x_0,y_0)\) of the sped-up system \(S\) attach its stable manifold \(S^0_0\) transversally to \(M\). When the attached manifold \(S^0_0\) is of dimension \(m\), then disturbances and noise will not cause the "state" \((x,y)\) of the system (2.1), (2.2) to be repelled from \(M\). If, in fact, the attached manifold is of dimension \(< m\), disturbances may cause the "state" \((x,y)\) to be repelled from \(M\) and follow instantaneously the dynamics of the sped-up system \(S\) to a new -limit set of (2.4). By assuming that \(H\) has only finitely many equilibrium points as its

\[ H \]

- limit set, for each \(x\), we may guarantee that \((x,y)\) will transit within \(n\) time units \(\tau\), a new equilibrium point of \(S\). If the state of the system is not stable, it is likely that the disturbance will be enough to cause the equilibrium point to be transited. In this case, the system will follow the new -limit set of (2.4).
Constraints on the systems of the spectrum of small limiting nuclei

In the recent model of the spectrum systems, it is shown that the systems differ from the spectrum of the spectrum systems. However, in the specific cases, the spectra of the spectrum systems are still small in the limit, so that the results shown, as well as those given in the next section,

are of the form of $\frac{1}{\alpha}$, where $\alpha$ is the smallest constant and $\beta$ is the temperature in question. Note, however, that the above statements apply only at small temperatures.

Thus, we conclude the behavior of the spectrum systems will depend on the parameters of the specific case given at the limit of the spectrum systems. However, the behavior of the specific case is still small and the temperature in question, as well as those presented in the discussion, are not constant, but rather depend on the specific case given at the limit of the spectrum systems.

I. Convergence of a Related Problem

Assume that $\phi$ is the solution with respect to $\alpha$ of the function

$$\phi = \frac{\xi}{\gamma}$$

for some function $\xi$ and $\gamma$. Then, the results of Raman, et al., will yield that provided the derivatives of $\phi$ with respect to $\gamma$ are bounded, and the density of the diffusion generated by

$$\partial_t \phi = \frac{\rho}{\alpha} e^{-\phi}$$

where $\rho$ is a constant, such that

$$\int_{-\infty}^{\infty} \frac{\rho}{\alpha} e^{-\phi} \, dx = 1$$
Note that for all \( \epsilon > 0 \) and \( x \in \mathbb{R}^n \) the critical points (with respect to \( y \)) of \( p(x,y) \) are the equilibrium points of the deterministic system (2.4) with \( x \) frozen given in this instance by
\[
\dot{y} = -\frac{1}{2} \text{grad}_y S(x,y) \quad (3.11)
\]
Further, if for some \( x \), \( S(x,y) \) is a Morse function (of \( y \)), then for all \( \epsilon > 0 \) every local maximum of \( \bar{p}^{-1}(x,y) \) is a stable equilibrium of (3.11).

To compare the noisy constrained system with the deterministic constrained system in the limit that \( \epsilon \to 0 \), it will be necessary to evaluate integrals like (3.9) in the limit that \( \epsilon \to 0 \). This is done using the following version of Laplace’s method:

**Theorem 3.1 (Laplace’s Method):**

Let for each \( x \in \mathbb{R}^n \), \( S(x,y) \) have global minima at \( y^*_1(x), y^*_2(x), \ldots, y^*_N(x) \), where \( N \) may depend on \( x \). Let them all be non-degenerate. Further, let \( S(x,y) \) have at least quadratic growth in \( y \) as \( y \to \cdot \). Then, in the limit that \( \epsilon \to 0 \), \( \bar{p}^{-1}(x,y) \) converges to
\[
\lim_{\epsilon \to 0} \frac{\sum_{i=1}^N a_i(x) \delta(y - y^*_i)}{\sum_{i=1}^N a_i(x)} = \frac{\sum_{i=1}^N a_i(x) \delta(x, y^*_i(x))}{\sum_{i=1}^N a_i(x)} \quad (3.12)
\]
where \( a_i(x) = \det(D^2_S(x,y^*_i(x)))^{-1/2} \)

More precisely, if \( t(x,y) \) is a smooth function having polynomial growth as \( y \to x \), then
\[
\lim_{\epsilon \to 0} \overline{t}(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} t(x,y) \bar{p}(x,y) dy = \frac{\sum_{i=1}^N a_i(x) \delta(x, y^*_i(x))}{\sum_{i=1}^N a_i(x)} \quad (3.13)
\]
\[
= \overline{t_0}(x)
\]

1. \( \overline{t_0}(x) \) the Hessian \( D^2_S(x,y) \) at \( y = y^*_i(x) \) is nonsingular.
Proof: Since \( p(x, y) = \exp - \frac{S(x, y)}{\lambda} \), we will first evaluate
\[
\int_{\mathbb{R}^m} \phi(x, y) \exp - \frac{S(x, y)}{\lambda} \, dy
\]
for simplicity first assume that \( S(x, y) \) has a single global minimum at \( \gamma \).

We will then show that
\[
\int_{\mathbb{R}^m} \phi(x, y) \exp - \frac{S(x, y)}{\lambda} \, dy = \phi(x, y^*) \frac{(2\pi\lambda^m/2)^{1/2} \exp - S(x, y^*)/\lambda(1 + o(1))}{\text{det } D^2 S(x, y^*)}^{1/2}
\]
(3.14)

First, by the Morse Lemma (see for e.g. Milnor [16]) there exists a neighborhood \( U \) of \( \gamma \) and a change of coordinates \( \mathbb{R}^m \to U \) given by
\[
\gamma = y(\psi) \quad \text{such that } y^* = \gamma(0) \quad \text{and}
\]
\[
S(x, y) = S(x, y^*) + \frac{1}{2} \sum_{i=1}^{m} (y_i')^2
\]
(3.15)

Further, outside the neighborhood \( U \) of \( y^* \), \( S(x, y) > S(x, y^*) + \epsilon \)
for some \( \epsilon > 0 \) so that
\[
\int_{\mathbb{R}^m / U} \phi(x, y) \exp - \frac{S(x, y)}{\lambda} \, dy = \exp \left[ -\frac{S(x, y^*)}{\lambda} \right] o(\lambda^\epsilon)
\]
(3.16)
for all \( \lambda > 0 \). Clearly, then (3.16) does not contribute to the leading term of (3.14)

Consider now
\[
\int_{U} \phi(x, y) \exp - \frac{S(x, y)}{\lambda} \, dy
\]
\[
= \exp \left[ -\frac{S(x, y^*)}{\lambda} \right] \int_{\mathbb{R}^m} \exp \left( - \sum_{i=1}^{m} \frac{1}{2} y_i^2 \right) \phi(x, y(\gamma)) \, dy
\]
(3.17)

Now, standard manipulations with Gaussian distributions yield that
\[
\int_{\mathbb{R}^m} \left( \exp - \frac{1}{2} \sum_{i=1}^m y_i^{-2} \right) \phi(y) \, dy
\]
\[= (2\pi)^{m/2} \left( \gamma(0) + o(1) \right) \quad \text{(3.18)}
\]

Thus, to evaluate (3.17) we only need compute \( |\det D \gamma(0)| \). Differentiating (3.15) twice with respect to \( y \) yields
\[
D^2 S(x, y) = \left( \frac{d\gamma}{dy} \right)^{-1} T \left( \frac{d\gamma}{dy} \right)^{-1}.
\]
\[\text{(3.19)}
\]

From (3.19) it follows that
\[
|\det D \gamma(0)| = |\det D^2 S(x, \gamma^*)|^{-1/2}
\]
so that (3.14) now is immediate on combining (3.16), (3.17) and (3.15).

In the instance that \( S(x, y) \) has several global minima
\( y_1^*(x), y_2^*(x), \ldots, y_N^*(x) \) it follows from an easy extension of the foregoing argument that
\[
\int_{\mathbb{R}^m} \phi(x, y) \, dy = (2\pi)^{m/2} \exp \frac{-S(x, y)}{2} \left( \det \left| \sum_{i=1}^N |\det D^2 S(x, y_i^*)|^{-1/2} \right| \phi(x, y_i^*) \right) (1)
\]
\[\text{(3.20)}
\]
Selling \( \phi(x, y) = 1 \) in (3.20) yields the corresponding expression for
\[
\int_{\mathbb{R}^m} \exp \frac{-S(x, y)}{2} \, dy. \quad \text{Combining this with (3.20) we have equation (3.13).}
\]

Remarks: (1) If the growth conditions on \( S(x, y) \) and \( \phi(x, y) \) are uniform in \( x \) for \( x \leq R \) it can be shown that for \( p > 1 \)
\[
\int_{x \leq R} \phi(x) - \frac{\gamma_c}{p} \, dx \rightarrow 0 \quad \text{as } x \rightarrow 0
\]
\[\text{(3.21)}
\]
Proof: Is presented in Sastry-Hijab [12].

Remark: The order of the limits is peculiar in Theorem (3.3). If the order is interchanged i.e. \( \lambda \to 0 \) first and then \( \varepsilon \to 0 \) it is clear that one recovers in the limit the deterministic development of Section 2 (with the minor modification that \( x \) has an additive white noise terms. The jump-behavior of the \( y \)-variable is as explained in that section. If, however, \( \varepsilon \to 0 \) first and the \( \lambda \to 0 \), the jump-behavior of the \( y \)-variable is somewhat different, as we now elaborate:

The behavior of the conditional density of \( y \) given \( x \) as \( \varepsilon \to 0 \) is as in Theorem 3.2: the \( y \) variable is at one of the global minima of \( S(x,*) \) with probability proportional to the curvature of \( S(x,*) \) at that minimum. Consider first the case when the minimum is unique. There is then a jump in the \( y \)-variables if there is a change in the global minimum of \( S(x,*) \) as \( x \) is varied. Points of jump then will be points of appearance and disappearance of global minima of \( S(x,*) \). This is in contrast to the deterministic picture of Section 2, where, for the instance that \( q(x,y) \) is of the form of (3.11), stable equilibrium of the sped-up system \( S \) are local minima of \( S(x,*) \) and points of bifurcation are points appearance and disappearance of local minima of \( S(x,*) \).

We illustrate this with an example - the van der Pol oscillator of (2.5), (2.6) with added noise. Consider

\[
\begin{align*}
x &= y + \varepsilon \xi(t) \\
y &= -x - y^3 + y + \varepsilon \xi(t)
\end{align*}
\]

Here \( S(x,y) = -xy - \frac{y^2}{4} + \frac{y^2}{2} \) so that, in the limit that \( \varepsilon \to 0 \); the \( x \)-process converges to one satisfying
The proof of 3.21. uses the dominated convergence and Fatou's Lemma.

(2) If $S(x, y)$ has a manifold $M$ of global minima, then these global minima cannot be non-degenerate, however if $S(x, y)$ is non-degenerate in directions orthogonal to $M$, then a minor modification of the preceding theorem yields

$$\int \exp \left( \frac{1}{2} \frac{dx}{dt} \cdot a \cdot dx \right)$$

Where $y$ is any point belonging to $M$, $a^n$ is the determinant of the non-degenerate part of theessian and $a$ is the area of measure $\nu$.

We can now combine the results of Theorems 1.9, 1.10, and 4.12 to construct a minor modification of the proof of Proposition 4.12.

**Theorem 4.1** Weak convergence of $(x(t))$:

Given any $T > 0$, in the sense that $x(t)$ is a solution of the equation:

the first component $x_1(t)$ of the solution satisfies

weakly in $C([0, T]; \mathbb{R}^N)$, to the initial condition:

satisfying

law

$$x = T(x', y', t)$$

where

$$T(x) = \frac{\sum_{i=1}^{N} a_i x \cdot f(x, y, t)}{\sum_{i=1}^{N} a_i x}$$

and the $y_i(x)$, $y^*_i(x)$ are the non-degenerate integral forms of $x$, and

$$a_i(x) = \left( \det \beta_2 x, y^*_i x \right)^{-1}.$$
This image contains text, but the content is not legible or discernible. It appears to be a page of a document with handwritten or printed text, but the specific details or context of the content cannot be accurately transcribed.
Section 4  The Effects of Thermal Noise in a Diode-Coupled Amplifier

We study in this section the relevance of the theory developed in
Sections 1 and 2 to the study of the elements of thermal noise of a rela-
tively complicated. Figure 7 shows a simplified circuit diagram of such an
amplifier. The relation between the noise figure of the amplifier and the
terminology of Section 3 is the deterministic description of the calculation.

The relevant equations are given by:

\[ \begin{align*}
\text{(4.1)}
\end{align*} \]

\[ \begin{align*}
\text{(4.2)}
\end{align*} \]

\[ \begin{align*}
\text{(4.3)}
\end{align*} \]

\[ \begin{align*}
\text{(4.4)}
\end{align*} \]

\[ \begin{align*}
\text{(4.5)}
\end{align*} \]

Eq. (4.6) is the expression for the base current of the diode, and
Eq. (4.7) is the reverse saturation current. The transistor are assumed to be
identical. With these assumptions, we may combine equations (4.4) and
(4.5) to obtain:

\[ \begin{align*}
\text{(4.6)}
\end{align*} \]

\[ \begin{align*}
\text{(4.7)}
\end{align*} \]

Equations (4.6), (4.7) form an implicitly defined dynamical system. The
solution, due to the nonlinear equation (4.7), is plotted in the (x, y).
plane in Figure 10. Some of the features of this curve are noted below:

(1) For \(-\frac{\pi}{2} < \phi < \frac{\pi}{2}\), the equation (4.7) has three solutions, while for \(\phi = \frac{\pi}{2}\) and \(\phi = -\frac{\pi}{2}\) the equation has only one solution.

(2) As \(\phi = \frac{\pi}{2}\), \(i = 2i_0\) and as \(\phi = -\frac{\pi}{2}\), \(i = 0\) asymptotically.

(3) The values \(\phi = \frac{\pi}{2}\), \(i = \frac{V}{2R}\), and \(\phi = -\frac{\pi}{2}\), \(i = 2i_0 - \frac{V}{2R}\) are the points of bifurcation of equation (4.7) with \(\phi\) treated as the bifurcation parameter. I.e. at these points it is not possible to solve (4.7) for \(i\) as a function of \(\phi\) locally and uniquely. These points may be shown to be points of full bifurcation.

Returning now to the full system - (4.6) and (4.7) - we see that continuous solutions for the system exist so long as \(i\) can be solved continuously as a function of \(V\) in (4.7) so as to obtain

\[
\frac{dv}{dt} = \frac{i - v}{R}
\]

and

\[
\frac{di}{dt} = \frac{\frac{1}{R} - \frac{1}{2R}}{1 - \phi^2}
\]

When \(\phi \neq \frac{\pi}{2}, \phi \neq -\frac{\pi}{2}\), \(i = 0\) or \(i = 2i_0\), it appears that \(i\) is infinite so as to prevent the integration of equations (4.6), (4.7). The regularization of this system is accomplished by taking into account the fact that parasitic capacitances present in the transistors, as well as the finite slew rate of the operational amplifiers will prevent \(i\) from varying discontinuously and in effect change the description of the
of the circuit dynamics from (4.6), (4.7) to

\[
\frac{dV}{dt} = \frac{(I_0-1)}{C} \tag{4.6}
\]

\[
\frac{di}{dt} = V - (2I_0 - 2i)R - V_T \phi (2I_0-1)/1 \tag{4.9}
\]

Equations (4.6) and (4.9) are a gross simplification of all the actual parasitics present in the circuit. A more detailed and exhaustive description involving all the parasitics would start from the original equations (4.1) - (4.5). The present regularized model is, however, accurate enough for our purposes. The phase portrait of this system shown in Figure 1 includes a single unstable equilibrium point \( V=0, i=I_0 \) and a limit cycle. The limit trajectories of (4.6), (4.9) as \( t \to \infty \) exist and include the relaxation oscillation shown in Figure 12 - a limit cycle with two discontinuities - at the points where the trajectory switches from the \( Q_1 \) on, \( Q_2 \) off 'state' to the \( Q_1 \) off, \( Q_2 \) on 'state' and vice versa. Note also from Figure 11 that the \( Q_1 \) on, \( Q_2 \) on 'state' is unstable as evidenced by the trajectories of (4.6), (4.9) pointing away from that 'state'. The current waveform \( i(t) \) is as shown in Figure 13. The half period of the oscillation \( T \) may be estimated approximately by integrating equation 4.8 with the approximation that for \( 0 < t < T, i \ll I_0 \), so that we have

\[
T = \frac{C}{I_0} \int_{\frac{V_T}{2R}}^{\frac{V_T}{2R}} (-2R + \frac{V_T}{i}) \ di
\]

or

\[
T = \frac{C}{I_0} \left[ 2R(-\frac{V_T}{2R} + I_1) + V_T \phi (\frac{V_T}{2I_1 R}) \right] \tag{4.12}
\]
From equation (4.10) it follows that the frequency of oscillation is (approximately) linearly proportional to $I_0$, which enables this oscillation to be used as an electronically tunable oscillator (e.g. in a phase locked loop). In such applications, it is important to know the noise characteristics of the oscillator in response to resistive thermal noise. Experimental observations of Abidi [1] indicate that the actual (noisy) current waveform is as shown in Figure 14. Key features of this figure are as follows:

(a) the transitions or jumps appear to be noise free

(b) the noise superimposed on the deterministic waveform of Figure 13 appears to be small (low intensity) immediately following a jump and then appear to build in intensity.

We assume (see [14]) that all the noise sources in the circuit can be lumped into a single-noisy current source $i_n(t)$ shown dotted in Figure 9; $i_n(t)$ is assumed to be white with intensity $\lambda$ (with $\lambda$ small at room temperatures, since it is proportional to $kT$). It is easy to check that the equation (4.6) is now unchanged, while (4.7) changes to

$$0 = V - (2I_0 - 2i)R - V_T \ln(2I_0 - i)/i + 2R \lambda i_n(t)$$

We regularize the system (4.6), (4.11) as before to obtain

$$\dot{V} = (I_0 - i)/C$$

$$\dot{i} = V - (2I_0 - 2i)R - V_T \ln(2I_0 - i)/i + 2R \lambda i_n(t)$$

Note that $\lambda$ scales the intensity of the white noise in (4.12) precisely for the same reason as in equation (3.2) of Section 3. The techniques of Section 3.2 may now be used to obtain that as $\epsilon \to 0$, the $V$-process converges weakly on $C([0,T]; \mathbb{R})$ to one satisfying:
\[ V = \left( I_0 - \bar{i}(v) \right) / C \]

where \( \bar{i}(v) \) is \( i \) integrated over the conditional density for \( i \) given \( V \), in the limit that \( \lambda \to 0 \), \( \bar{p}(i,v) \). As in the example of Section 3.2, we have in the limit that \( \lambda \to 0 \), \( \bar{p}(i,v) \) converging to a sequence of delta functions jumping from one leg of the solution curve to (4.7) to the other at \( V = 0 \). Also, choosing the interval of weak convergence to be large it appears that the relaxation oscillation is broken up.

This analysis is contrary to the experimental evidence of Abidi [1].

What has gone wrong? How does one recover the experimental results of Abidi [1]? These are the questions that we take up next.
Section 5. Sample Function Calculations.

The mathematical reason for the anomaly between the machinery developed in Section 3 and the experimental conclusions of Section 4, is the order of limits $c+0$ followed by $\lambda+0$ in Theorem (3.3). This order of taking limits is suitable for explaining phenomena in several situations in non-equilibrium thermodynamics (for e.g. phase transitions of the kind discussed in Sastry-Hijab [12], Eyring chemical reaction rates, etc. - see for e.g. Nicolis-Prigogine [9], Landauer [6]). In fact, it has been noted by thermodynamicists of the Brussels School that "fluctuations play a crucial role in changing the behavior of systems near bifurcation fronts". However, this order of limits is not fully satisfactory in the circuit context. The reason for this lies in the fact that the order of limits $c+0$ followed by $\lambda+0$ (Theorem 3.3) yields the correct conclusions only when the dynamics of the fast (speed-up) system are much faster than those of the slower $x$ variable. This is so, because, as we state in Section 3.4, Laplace's method of steepest descent picks for the limit values of $F^{\lambda}(x,y)$ as $\lambda+0$ the most stable limit sets of the underlying deterministic systems. This in turn is consistent with the intuition that in the presence of persistent random perturbation (wide-band in nature) the trajectories of a system will concentrate after sufficiently long periods of time in the vicinity of the most-stable sets. However, the sufficiently long periods of time may be very large indeed. It is possible to show, for example in the gradient case of Section 3.2 that the average time required to escape from a stable equilibrium is of the order of $e^{K/\lambda}$ for some $k>0$ (see for eq. Schuss [15], Ventsel-Freidlin [15]).
By taking limits in the sequence $t \to 0$ followed by $t \to 0$, the implication is that $\epsilon$ is smaller than $e^{-k/\lambda}$, i.e. $\epsilon$ is at least $e^{-k/\lambda}$, so that the fast system has sufficiently much time to concentrate in the vicinity of its $\epsilon$-limit sets. This is frequently the situation in non-equilibrium thermodynamics where the slower dynamics are frequently assumed to be 'quasi-static'. In the circuit context, however, the separation of time scales between the slow and fast variable is not as large as is implied by the theorem.

As noted in the remark following Theorem 3.3, if the order of limits is interchanged (i.e. $t \to 0$ and then $\epsilon \to 0$), one recovers the deterministic development of Section 2. Before, we further elaborate and make precise the statements of the previous paragraph we indicate how the analyses sample functions of the process generated by (5.1), (5.2) in the limit that $\epsilon \to 0$ followed by $t \to 0$. The major tool for this development is the work of Kentsel-Freidlin [13].

We consider here sample functions of the process generated by

\[ x = f(x, y) + \frac{1}{\epsilon} \xi_x, \quad x(0) = x_0 \quad (5.1) \]
\[ y = f(x, y) + \frac{1}{\epsilon} \xi_y, \quad y(0) = y_0 \quad (5.2) \]

with precisely the same assumptions as in Section 3. Let $\mathcal{C} = \{x, y\}$: \([0, T] \to \mathbb{R}^n \times \mathbb{R}^m\) be a $C^1$ map from the interval $[0, T]$ to the $x, y$ space with $x(0) = x_0, \quad y(0) = y_0$. Define, for this trajectory, the functional $I_\epsilon(.)$ by

\[ I_\epsilon(.) = \int_0^T \left\| \frac{\partial}{\partial t} (\xi_x(t) - f(x, y)) \right\|^2 \, dt \quad (5.3) \]
Then, we have the following theorem for measuring the deviation of the system function of the system from the equilibrium solutions of the system.

**Theorem**

Let \( x(t) \) be the solution to the system of equations of the form:

\[
\frac{dx}{dt} = f(x(t)),
\]

such that \( x(0) = x_0 \) for some initial condition \( x_0 \) in the state space. Assume that the system is locally Lipschitz continuous.

The deviation of the system function from the equilibrium solutions can be estimated by the following inequality:

\[
\left| x(t) - x^*(t) \right| \leq K \left| x(0) - x^*(0) \right|
\]

where \( x^*(t) \) is the equilibrium solution at time \( t \), and \( K \) is a constant depending on the system function \( f(x) \).

**Remark**

Equation (1) gives an estimate of how close the sample function is to the equilibrium solutions of the system. The estimate is shown to be stringent when the initial condition \( x(0) \) is close to the equilibrium solution \( x^*(0) \). On the other hand, we remark that the worst-case scenario is:

1. Consider the definition of \( f(x) \) in equation (1) and note that \( f(x) \) is an upper bound of the deterministic system starting from any initial condition \( x(0) \) at a minimum time.

Theorem 5.1 gives an estimate for the deviation of the process of (5.1) (5.2) from the trajectory for fixed and discrete estimates. For this, with probability arbitrarily close to 1, the state estimate trajectory for the process is that which minimizes \( E^* \). Further, taking the limit that
Figure 21: Showing the dynamics of the degenerate and polarized van der Pol oscillator.
Figure 3: Visualization of a Fold Bifurcation and the Unstable Manifold $M = \{(x,y) : g(x,y) = 0\}$.

Figure 4: Visualization of a Fold Bifurcation
Figure 5: Visualization of the Trajectories on the Upper, Center and Bottom Sheets of the Cusp
Figure 6: Showing the Limit as $N \to 0$ of the Conditional Density $\tilde{p}^N(x,y)$

$O = -x - y^3 + y$

Figure 7: The Drift $\lambda'(x)$ for the Limit Diffusion of the van der Pol Oscillator for Decreasing Values of $\lambda$. 

$\lambda_1 > \lambda_2 > 0$
Figure 8: Showing the Limit Behavior of $p^0(x,y)$ for Example 3.5

Figure 9: Showing the Hopf Bifurcation for the Example 3.6
Figure 10: Simplified Circuit Diagram for the Emitter Coupled Relaxation Oscillator

Figure 11: The Solution Curve to the Algebraic Equation (4.7)
Figure 12: Phase Portrait of the System (4.6), (4.9)

Figure 13: Showing the Relaxation Oscillation in the Emitter Coupled Oscillator
Figure 14: Current Waveform $i(t)$ for the Circuit of Figure 1.

Figure 15: Experimentally Observed Waveform for $i(t)$ in the Presence of Noise (after Abidi[1])
Figure 10: Showing the possibility of varying \( x_{0} \rightarrow x_{y_{1}}, y_{1} \rightarrow y_{0} \).

Figure 11: Showing a neighborhood of the constraint equation (4.7).