Nonlinear Extensional Vibrations of Quartz Rods

H. F. Tiersten
A. Ballato

Rensselaer Polytechnic Institute
Troy, NY 12181

Submitted for announcement only.

UNCLASSIFIED
Nonlinear extensional vibrations of quartz rods

H. F. Tiersten
Department of Mechanical Engineering, Aeronautical Engineering and Mechanics, Rensselaer Polytechnic Institute, Troy, New York 12181

A. Ballato
U. S. Army Electronics Technology and Devices Laboratory, USAERADCOM, Fort Monmouth, New Jersey 07703

(Received 8 October 1982; accepted for publication 4 February 1983)

The one-dimensional scalar differential equation describing the extensional motion of thin piezoelectric rods is obtained from the general nonlinear three-dimensional description. Only the elastic nonlinearities are considered. The relations between the quadratic and cubic coefficients of the rod and the fundamental anisotropic elastic constants of various orders are derived. The quadratic rod coefficients are calculated for various orientations of quartz rods, but not the cubic rod coefficients because the fundamental elastic constants of fourth order, which are required for the calculation, are not presently known. The nonlinear equation and boundary conditions are applied in the analyses of both intermodulation and nonlinear resonance of quartz rods. In each instance a lumped parameter representation of the solution, which is valid in the vicinity of a resonance, is obtained and the influence of the external circuitry is included in the treatment.

PACS numbers: 43.40.Cw, 62.20.Dc, 43.40.Ga

INTRODUCTION

A one-dimensional scalar nonlinear differential equation describing the extensional motion of thin piezoelectric rods oriented in an arbitrary direction with respect to the principal axes of an arbitrarily anisotropic crystal is obtained from the general nonlinear three-dimensional equations of electroelasticity. Only the elastic nonlinearities are included in the description. The electrical behavior is taken to be linear since quartz has small piezoelectric coupling. The treatment provides the relation between the quadratic and cubic nonlinear extensional coefficients of the rod and the fundamental anisotropic elastic constants of second, third, and fourth order, along with the well-known relation between Young's modulus and the fundamental second-order elastic constants. The quadratic rod coefficients are calculated for various orientations of quartz rods with respect to the principal axes of the quartz crystal. Such calculations cannot be performed for the cubic rod coefficients because the fourth-order elastic constants of quartz, on which the cubic coefficients depend, are not presently known.

The steady-state solution of the nonlinear system is determined at an intermodulation frequency by straightforward iteration from the linear solution. The intermodulation current–voltage relation, which is valid in the vicinity of a resonance, is obtained and the crystal is incorporated in the test circuit and the relation between the intermodulation and driving voltages is determined. This relation and the calculated quadratic rod coefficients can be used along with measurements to determine the cubic rod coefficients for any desired orientation of quartz rod, including, of course, the practical 5°X cut.

In addition, the nonlinear equation and boundary conditions are applied in the analysis of nonlinear resonance in quartz rods. The steady-state solution is obtained by means of an asymptotic iterative procedure and an expansion in the linear eigensolutions. A lumped parameter representation of the nonlinear solution, which is valid in the vicinity of a resonance, is presented. The representation relates the amplitude of the rod displacement nonlinearly to the voltage across the electrodes. The external circuitry is incorporated in the analysis and an equation relating the amplitude of the rod displacement nonlinearly to the driving voltage and other circuit parameters is obtained.

It should be carefully noted that the coupling between extension and flexure, which can exist in an anisotropic rod, is not included in this description. However, since the coupling of a high overtone of flexure with a low extensional mode occurs only for certain well separated geometries, the equation for the anisotropic rod presented here is valid for all other geometric ratios. This means that the equation is valid for almost all geometries and certainly for all practical cases.

I. NONLINEAR EQUATIONS

The stress equations of motion and charge equation of electrostatics for material points of an electroelastic solid with small piezoelectric coupling may be written in the respective forms

\[ K_{L L} \ddot{u}_L = \delta_{LM} \rho \ddot{u}_M, \]

\[ \ddot{D}_{L L} = 0, \]

where we have employed the conventions that capital and lower case indices, respectively, refer to the reference Cartesian coordinates and present Cartesian coordinates of material points. We also employ the conventions that a comma followed by a capital index denotes partial differentiation with respect to the known reference coordinates, i.e., the independent variables excluding time, a dot over a variable denotes partial differentiation with respect to time, and repeated tensor indices are to be summed. The symbols \( \rho \), \( u_M \), \( K_{L L} \), and \( D_{L L} \), respectively, denote the reference mass...
density, the mechanical displacement, the Piola–Kirchhoff stress tensor, and the electric displacement vector. In the nonlinear stress equations of motion, Eq. (1), it is understood that the motion of a material point is described by the functional relation

$$y_i = y_i(X_c, t),$$

(3)

which is one-to-one and differentiable as often as required, where $y_i$ and $X_c$ respectively, denote the present and reference position of material points. The symbol $\delta_{ij}$ in Eq. (1) is a translation operator, which serves to translate a vector from the present to the reference position and vice-versa and is required for notational consistency and clarity because of the use of capital and lower case indices, respectively, to refer to the reference and present positions of material points. Clearly, the mechanical displacement vector $u_{i}$ and present and reference positions of material points $y_i$ and $X_c$ are related by

$$y_i = \delta_{ij}(X_c + u_j).$$

(4)

For purposes of this treatment, the constitutive equations for $K_{ij}$ and $D_L$ may be written in the form

$$K_{ij} = y_{ij} T_{LM},$$

(5)

$$D_L = \varepsilon_{LMK} E_{NK} + \varepsilon_{LM} \varepsilon_{KL},$$

(6)

where

$$T_{LM} = T_{LMPO} E_{PO} - \varepsilon_{KLM} \varepsilon_{KL} + \frac{1}{2} \varepsilon_{LMPO} E_{PO} E_{OBN},$$

(7)

and

$$E_{AB} = \frac{1}{2}(y_{AB} - \delta_{AB})$$

(8)

is the finite strain tensor. Substituting from Eq. (4) into Eq. (8) we obtain

$$E_{AB} = \frac{1}{2}(u_{AB} + u_{BA} + u_{RAB} u_{RAB}),$$

(9)

and, of course, since the electric field is quasistatic, we have

$$\varepsilon_{KL} = - \varphi_{KL}.$$  

(10)

In view of the small piezoelectric coupling in quartz, in (6) and (7), we have included nonlinear elastic terms only and kept the electric and electroelastic terms linear, in accordance with the discussion in the Introduction. In Eqs. (6), (7), and (10) $\varphi$ is the electric scalar potential and $c_{LMAB}^4, c_{LMKAB}^4$ are the second-order elastic, third-order elastic, fourth-order elastic, piezoelectric, and dielectric constants, respectively. The associated boundary conditions at the surface of the piezoelectric solid are on $N_L K_{ij}$ or $u_{ij}$ and $N_L D_L$, or $\varphi$, where $N_L$ denotes the unit normal to the undeformed surface.

A schematic diagram of the piezoelectric rod is given in Fig. 1, where $2l > 2w > 5h$ and the surfaces normal to $x_2$ are fully electroded. Since we are interested in solutions which vary slowly compared to $2h$, to lowest order the electrical variables may be written in the form

$$\varepsilon_{1} = \varepsilon_{2} = 0,$$

(11a)

$$\varepsilon_{3} = - \nu/2h,$$

(11b)

which is one-to-one and differentiable as often as required, where $\nu$, and $X_c$, respectively, denote the present and reference position of material points. The symbol $\delta_{ij}$ in Eq. (1) is a translation operator, which serves to translate a vector from the present to the reference position and vice-versa and is required for notational consistency and clarity because of the use of capital and lower case indices, respectively, to refer to the reference and present positions of material points. Clearly, the mechanical displacement vector $u_{i}$ and present and reference positions of material points $y_i$ and $X_c$ are related by

$$y_i = \delta_{ij}(X_c + u_j).$$

(4)

For purposes of this treatment, the constitutive equations for $K_{ij}$ and $D_L$ may be written in the form

$$K_{ij} = y_{ij} T_{LM},$$

(5)

$$D_L = \varepsilon_{LMK} E_{NK} + \varepsilon_{LM} \varepsilon_{KL},$$

(6)

where

$$T_{LM} = T_{LMPO} E_{PO} - \varepsilon_{KLM} \varepsilon_{KL} + \frac{1}{2} \varepsilon_{LMPO} E_{PO} E_{OBN},$$

(7)

and

$$E_{AB} = \frac{1}{2}(y_{AB} - \delta_{AB})$$

(8)

is the finite strain tensor. Substituting from Eq. (4) into Eq. (8) we obtain

$$E_{AB} = \frac{1}{2}(u_{AB} + u_{BA} + u_{RAB} u_{RAB}),$$

(9)

and, of course, since the electric field is quasistatic, we have

$$\varepsilon_{KL} = - \varphi_{KL}.$$  

(10)

In view of the small piezoelectric coupling in quartz, in (6) and (7), we have included nonlinear elastic terms only and kept the electric and electroelastic terms linear, in accordance with the discussion in the Introduction. In Eqs. (6), (7), and (10) $\varphi$ is the electric scalar potential and $c_{LMAB}^4, c_{LMKAB}^4$ are the second-order elastic, third-order elastic, fourth-order elastic, piezoelectric, and dielectric constants, respectively. The associated boundary conditions at the surface of the piezoelectric solid are on $N_L K_{ij}$ or $u_{ij}$ and $N_L D_L$, or $\varphi$, where $N_L$ denotes the unit normal to the undeformed surface.

A schematic diagram of the piezoelectric rod is given in Fig. 1, where $2l > 2w > 5h$ and the surfaces normal to $x_2$ are fully electroded. Since we are interested in solutions which vary slowly compared to $2h$, to lowest order the electrical variables may be written in the form

$$\varepsilon_{1} = \varepsilon_{2} = 0,$$

(11a)

$$\varepsilon_{3} = - \nu/2h,$$

(11b)

and hence in the nonlinear description being obtained here $K_{12}$ and $K_{13}$ should also vanish approximately because they vanish on the surfaces at $X_1 = \pm l$. Since $y_{ij}$ is nonsingular, from Eqs. (5) and (14) we have

$$T_{LM} = 0, \quad T_{LM} = 0,$$

(16)

which shows that in this description of long wavelength extensional vibrations of elastic rods all $T_{LM}$ vanish except $T_{11}$. At this stage the only nonzero components of the Piola-Kirchhoff stress tensor $K_{ij}$ are the $K_{ij}$, which from Eqs. (4) and (5) are given by

$$K_{11} = (1 + u_{1,1}) T_{11},$$

(17)

$$K_{12} = u_{2,1} T_{11}, \quad K_{13} = u_{3,1} T_{11},$$

(18)

In the approximation being made here $u_{1,1}$ is large and all the other $u_{i,M}$, which arise because of the vanishing of the $T_{LM}$.
\[ L_M \neq 11 \), are expressible in terms of \( u_{11} \), and are somewhat smaller. These relations are derived in the next section and in the course of the derivation it is shown that for wavelengths large compared to the cross-sectional dimensions of the rod
\[ u_{21} \approx 0, \quad u_{31} \approx 0. \]  

(19)

Thus, from Eqs. (18) and (19) we have
\[ K_{12} \approx 0, \quad K_{13} \approx 0, \]  

(20)

and from Eqs. (14) and (20) we see that all \( K_{ij} = 0 \) except \( K_{11} \). Under these circumstances all that remains of Eq. (1) is
\[ K_{11,1} = \rho \ddot{u}_{11}, \]  

(21)

which is the basic differential equation for the anisotropic rod. We must now find the nonlinear relation between \( T \), and from Eqs. (18) and (19) we have
\[ \dot{c}T_i = E_i/s_{11}, \]  

(28a)

\[ \dot{c}E_T = s_T, E_i/s_{11}, \]  

(28b)

which are the well-known results in the linear theory. Now, substituting Eqs. (28) into Eq. (26) and retaining terms no higher than quadratic, we obtain
\[ T_1 = E_i/s_{11} + (1/2J_1) s_{11} \hat{c} \]  

(29)

where \( J_1 \) is the constitutive relation for \( K_{11} \). Substituting from Eq. (29) and Eq. (28b) into Eq. (27) and retaining terms no higher than quadratic in \( E_i \), we obtain
\[ \dot{c}T_i = \frac{s_{11}}{s_{11}} E_i + \frac{1}{2s_{11}^2} \left( \frac{\hat{c}E_T}{s_{11} E_i} - \frac{s_T}{c_R} \right) \]  

(30)

The further substitution of Eq. (30) into Eq. (26) and the retention of terms no higher than cubic in \( E_i \) yields
\[ \dot{c}T_i = \beta_i E_i^3 + \gamma_i E_i^3, \]  

(31)

where
\[ \dot{c} = \frac{1}{s_{11}}, \quad \beta_i = \frac{1}{2s_{11}^2} \hat{c}, \quad \gamma_i = \frac{1}{3s_{11}^2} \hat{c}, \]  

(32)

\[ \gamma_i = \frac{1}{3s_{11}^2} \left[ s_{11} \hat{c} + \frac{s_{11}}{s_{11}} \right]. \]  

(32b)

where the caret on the subscript \( Q \) means that
\[ s_{11}^2 = 0 \text{ for } \hat{Q} = 1. \]  

(33)

The second iterate for \( T_1 \) given in Eq. (31) with Eqs. (32) is the cubic relation we need to obtain the cubic constitutive relation for \( K_{11} \), in terms of \( u_{11} \). Substituting from Eq. (31) with Eq. (9), and Eq. (19) into Eq. (17) and reintroducing the linear piezoelectric term, we obtain the result
\[ K_{11} = cu_{11} + \beta (u_{11})^2 + \gamma (u_{11})^3 - (d_{31}/s_{11})B, \]  

(34)

where
\[ \beta = \beta_i + \frac{1}{2} c, \quad \gamma = \gamma_i + 2\beta_i + \frac{1}{2} c. \]  

(35a)

(35b)

Equation (34), with Eqs. (35) and (32), is the constitutive relation we have been seeking for the arbitrarily anisotropic piezoelectric rod, which is nonlinear in the mechanical displacement gradient \( u_{11} \), but linear in the electric field.

We must now show that \( K_{12} \) and \( K_{13} \) do indeed vanish approximately as indicated in Sec. 1. To this end we note that

\[ 204 \quad J. Acoust. Soc. Am., Vol. 73, No. 6, June 1983 \]  

H. F. Tiersten and A. Ballato: Vibrations of rods

2024
at the zeroth iterate, from Eq. (28) with Eq. (9), in tensor notation, we have

\[ S_{KL} = r_{KL11}u_{11,1} \]

(36)

where

\[ S_{KL} = \frac{1}{2}(u_{KL1} + u_{LK1}) \]

(37a)

\[ r_{KL11} = s_{KL11}/s_{1111} \]

(37b)

and \( u_{11} = u_{1}(x, t) \). For \( KL = 22 \) and \( 33 \), respectively, we have

\[ u_{2,2} = r_{2211}u_{11,1}, \quad u_{3,3} = r_{3311}u_{11,1} \]

(38)

which may be integrated immediately to give

\[ u_{2} = X_{2}r_{2211}u_{11,1} + f(X_{1}, X_{3}, t) \]

\[ u_{3} = X_{3}r_{3311}u_{11,1} + g(X_{1}, X_{2}, t) \]

(39)

From the remaining three relations in Eq. (36) with Eq. (37a), we have

\[ \frac{1}{2}(u_{1,2} + u_{2,1}) = r_{1211}u_{11,1} \]

(40a)

\[ \frac{1}{2}(u_{1,3} + u_{3,1}) = r_{1311}u_{11,1} \]

(40b)

\[ \frac{1}{2}(u_{1,3} + u_{3,2}) = r_{3211}u_{11,1} \]

(40c)

Since the rod axis lies along \( X_{1} \), and we have assumed that torsion is uncoupled from extension in this thin anisotropic rod, we have

\[ w_{32} = \frac{1}{2}(w_{32} - w_{32}) = 0 \]

(41)

where \( w_{32} \) is the small (linear) local mechanical rotation about the rod axis \( X_{1} \). From Eqs. (39), (40c), and (41), we obtain

\[ f = X_{2}r_{2211}u_{11,1}, \quad g = X_{3}r_{3311}u_{11,1} \]

(42)

Now, substituting from Eqs. (39) and (42) into (40a) and (40b), respectively, we obtain

\[ u_{1,2} = 2r_{1211}u_{11,1} - (X_{2}r_{2211} + X_{3}r_{3311})u_{11,1} \]

(43)

\[ u_{1,3} = 2r_{1311}u_{11,1} - (X_{2}r_{2311} + X_{3}r_{3311})u_{11,1} \]

(44)

Since we are interested in the limit of long wavelengths (\( \lambda \)) as the rod collapses to a line, we have

\[ \max_{\lambda} \frac{w}{\lambda} = 0, \quad \max_{\lambda} \frac{h}{\lambda} = 0 \]

(45)

which with Eqs. (39), (40c), (41), and (43) yields

\[ o_{u,2} = 2r_{1211}u_{11,1} \]

(46)

\[ o_{u,3} = 2r_{1311}u_{11,1} \]

(47)

\[ o_{u,2} = 0 \]

(48)

\[ o_{u,3} = 0 \]

(49)

\[ o_{u,3,2} = r_{3211}u_{11,1} \]

(50)

\[ o_{u,1,2} = r_{2211}u_{11,1} \]

(51)

which along with (38) gives all the zero-order displacement gradients. Equations (45c) and (45e) show that at the zeroth iterate Eqs. (19) hold and since \( \kappa_{ij} \) is obtained from \( o_{u,ij} \), we have shown that

\[ \kappa_{12} = 0, \quad \kappa_{13} = 0 \]

(52)

In the case of the first iterate, from Eq. (30) with Eqs. (9) and (45), in place of Eq. (36), we obtain

\[ S_{KL} = r_{KL11}u_{11,1} + p_{KL11}(u_{11,1})^{2} \]

(53)

where in the compressed matrix notation

\[ p_{T1} = \frac{1}{2} s_{1111} + \frac{1}{2} (s_{11} s_{11} - s_{1111}) \]

(54)

and each \( \alpha_{T1} \) consists of a sum of terms, each of which contains a quadratic combination of the \( r_{T1} \) from Eqs. (45) and (9). The exact forms of the \( \alpha_{T1} \) are cumbersome to write because of the forms in Eqs. (45) and are not needed here. From Eqs. (47) and (37b) along with the procedure followed in obtaining Eqs. (45) in the case of the zeroth iterate, it is clear that we can obtain relations similar to those in (45) if we can obtain the relation equivalent to Eq. (41) in the case of the first iterate. It is shown in the Appendix that to second order in the mechanical displacement gradients \( u_{KL} \), the local orthogonal transformation tensor \( R_{KL} \) is given by

\[ R_{KL} = \delta_{KL} + \Omega_{LK} \]

(55)

where

\[ \Omega_{LK} = \omega_{LK} + \frac{1}{2} \omega_{LM} \omega_{MK} + \frac{1}{3} (\omega_{LM} \omega_{ML} - \omega_{LM} \omega_{MK}) \]

(56)

and

\[ \omega_{LK} = \frac{1}{2}(u_{KL} - u_{KL}) \]

(57)

In the linear theory the same type of procedure shows that

\[ R_{KL} = \delta_{KL} + \omega_{LK} \]

(58)

Since in the quadratic case \( \Omega_{LK} \) takes the place of \( \omega_{LK} \) in the linear, \( \Omega_{22} \) represents the torsional rotation of the rod in the quadratic case. Consequently, in order to uncouple torsion we take

\[ \Omega_{22} = 0 \]

(59)

Substituting from the zeroth order results in the nonlinear terms in Eq. (50) and using the limits in Eqs. (44), from Eq. (53) we obtain for the first iterate

\[ w_{23} = \frac{1}{2} r_{1211} r_{1311}(u_{11,1})^{2} \]

(60)

in place of Eq. (41) for the zeroth iterate. From the forms in Eq. (47) and Eq. (54), which take the place of Eqs. (36) and (41), respectively, it is clear that by following the procedure used earlier in the case of the zeroth iterate, we obtain expressions similar to those obtained with the zeroth iterate but containing additional terms depending on \( (u_{11,1})^{2} \). Furthermore, when we take the limit in Eqs. (44) we obtain

\[ u_{1,2} = 0, \quad u_{1,3} = 0 \]

(61)

and since \( \kappa_{ij} \) is obtained from \( u_{ij} \), from Eq. (18) we have shown that

\[ \kappa_{12} = 0, \quad \kappa_{13} = 0 \]

(62)

When all stress components \( T_{ij}(\kappa_{ij}) \) vanish except \( T_{11}(\kappa_{11}) \) the linear piezoelectric constitutive equation can be written in a more useful form. To this end we first write Eq. (6) in the compressed matrix notation, thus

\[ D_{L} = e_{LQ} E_{Q} + \varepsilon_{LQ} \Phi_{Q} \]

(63)

and we substitute from the linear portion of Eq. (24) into Eq.
employ Eq. (25) to obtain

\[ D_L = d_{Lx} T_P + e_{Lx} \mathbf{g}_x, \]  

where

\[ e_{Lx} = e_{Lx} + e_{Lq} d_{KQ}. \]

We now substitute from the linear portion of Eq. (24) for \( T = 1 \) into Eq. (58) and find

\[ D_L = \frac{d_{L1}}{s_{11}} S_1 + \left( e_{Lx} - \frac{d_{L1}}{s_{11}} d_{K11} \right) \mathbf{g}_x, \]

from which, with the aid of Eqs. (11), we obtain

\[ D_3 = \frac{d_{311}}{s_{1111}} u_{11} + \left( e_{313} - \frac{d_{311}^2}{s_{1111}} \right) \mathbf{g}_y, \]

in which we have reintroduced the tensor notation.

The substitution of Eq. (34) into Eq. (11b) into Eq. (21) yields

\[ c u_{11,1} + \beta \left[ \left( u_{11,1}^2 \right)_1 + \gamma \left[ \left( u_{11,1}^2 \right)_1 \right] \right]_1 = \rho \dot{u}_{11}, \]

which is the differential equation of motion of the piezoelectric (anisotropic) rod. The boundary conditions consist of the specification of \( K_{11} \) or \( u_1 \) at the ends of the rod \( X = \pm l \). After a solution has been obtained, the current \( I \) can be calculated from the relation

\[ I = 2\pi \int_{-l}^{l} \phi_3 \text{d}x, \]

and we recall that \( \phi_3 \) in Eqs. (34) and (61) is related to the driving voltage \( V \) by Eq. (11b).

The quadratic rod coefficient \( \beta \) has been calculated from Eq. (35a) with Eqs. (32a) for a number of orientations of quartz rods using the second- and third-order elastic constants of quartz and the results are plotted in Figs. 2-4. The linear rod constant \( c \) is plotted along with \( -\beta \) in each figure. The quadratic rod coefficient \( \beta \) is always negative and ranges in value from about \( -c \) to about \( -5c \). Figure 2 is for the rotated \( X \) cuts with \( \theta = 0^\circ \) corresponding to the rod axis along the crystallographic \( X \) direction. The locations of the \( +5^\circ -, MT - , \) and \( -18^\circ - X \) cuts are shown in the figure. Figure 3 is for the rotated \( Y \) cuts with \( \theta = 0^\circ \) corresponding to the rod axis along the crystallographic \( X \) direction. This figure exhibits the twofold symmetry of quartz with \( X \) the digonal axis. Figure 4 is for doubly rotated cuts with \( \theta = 0^\circ \) corresponding to the rod in the \( X-Y \) plane at \( +45^\circ \) from the \( X \) axis. In Figs. 2-4, \( \theta = \pm 90^\circ \) corresponds to the rod axis along the crystallographic \( Z \) direction.

III. INTERMODULATION IN RODS

In this section we consider the problem of intermodulation in piezoelectric rods. For clarity we reproduce certain of the pertinent equations obtained in Secs. I and II here. The
stress equation of motion and charge equation of electrostatics for the piezoelectric rod take the form

\[ K_{11} = \rho_0 \ddot{u}, \]  
\[ D_{33} = 0, \]  
where

\[ K_{11} = c_{11} + \beta (u_0 \ddot{u})^2 + \gamma (u_0 \ddot{u})^3 - \frac{d_{31}}{s_{11}} \ddot{w}_3, \]  
\[ D_3 = [d_{31}/s_{11}] u_0 + (\epsilon_r^2 - d_{31})/s_{11} \ddot{w}_3, \]

and \( c, \beta, \) and \( \gamma \) are given in Eqs. (32) and (35). From Eq. (11b) we have

\[ \ddot{w}_3 = -(V/2h) e^{i\omega t}, \]  
where \( V \) is the magnitude of the applied voltage and \( \omega \) is the steady-state driving frequency. For a rod free at both ends the boundary conditions are

\[ K_{11} = 0 \text{ at } X_1 = \pm l. \]  
(68)

Since the asymptotic solution to the nonlinear problem is obtained by iteration from the solution of the appropriate linear problem, we begin the analysis with a brief presentation of the known solution of the associated linear problem. As a solution of the linear problem, i.e., Eqs. (64a) and (68), with Eq. (65) in the absence of \( \beta \) and \( \gamma \), we write

\[ \omega u_1 = [\omega - (d_{11}/2h) V X_1] e^{i\omega t}, \]  
(69)

the substitution of which in Eqs. (64a) and (68) yields

\[ c_{11} u_{11} + \rho_0 \omega^2 u = \rho_0 \omega^2 (d_{11}/2h) V X_1, \]  
\[ \omega u_1 = 0 \text{ at } X_1 = \pm l. \]  
(71)

The eigensolutions of the homogeneous form of Eqs. (70) and (71), i.e., with \( V = 0 \), may be written in the form

\[ u_{1+} = u_{a} e^{i\omega t}, \]  
\[ u_{a} = a_{u} \sin \eta_{u} X_1, \]  
(72a)

Substituting from Eqs. (72) into the homogeneous form of Eqs. (70) and (71), we obtain

\[ \rho_0 \omega^2 = c_{11} n \eta_{u}^2, \]  
\[ \eta_{u} = n \pi/2l, \text{ } n \text{ odd.} \]  
(73b)

We write the steady-state solution of the forced vibration problem in the form

\[ \omega u = \sum_{n} A_n \sin \eta_{u} X_1, \]  
(74)

the substitution of which in Eq. (70) along with the use of the orthogonality of Eq. (72b) with Eq. (71) and Eq. (73a) yields

\[ A_n = -8l/[(1 - n \pi^2/2l) d_{11}/V] |\omega^2 - \eta_{u}^2|^2 \]  
(75)

Since Eq. (75) has a resonance denominator, in the vicinity of a resonance, say the \( N \)th, one term in the series in Eq. (74) dominates and with Eq. (69) we have

\[ \omega u_N = [A_N \sin (N \pi X_1/2l) - (d_{11}/s_{11}) V X_1] e^{i\omega t}, \]  
(76)

where we have introduced the notation \( \omega_{11}, \) \( V_{11}, \) and \( A_N \) to denote the quantities at driving frequency \( \omega_1. \) Substituting from Eqs. (67) and (68) at \( \omega_1 \) into Eq. (66), which is then substituted into Eq. (63), we obtain

\[ I_{\omega} = 4w_1 \rho_0 \omega^2 \frac{e_{13}^2}{2h} \left[ 1 + \frac{8(k_{11})^2}{\pi^2 N^2 \left[ (\omega_{11}^2/\omega_1^2 - 1) \right]} V_{11} \right], \]  
(77)

where we have made the identification \( I_{\omega} = -I \) at \( \omega_1 \) since at \( \omega_1 \) the crystal is passive\(^{13} \) and

\[ (k_{11})^2 = d_{31}^2 /s_{11} \epsilon_r^2, \]  
(78)

and we have replaced \( \omega_N \) by \( \omega_{11} \) where

\[ \omega_{11} = \omega_N + iQ_N/2Q_N, \]  
(79)

in which \( Q_N \) is the unloaded quality factor of the resonator in that mode, and serves to prevent the resonant denominator from vanishing at \( \omega = \omega_N. \)

Since we are interested in intermodulation due to the simultaneous application of test tones to the resonator at two nearby frequencies \( \omega_1 \) and \( \omega_2, \) both of which are in the neighborhood of \( \omega_N, \) we consider the steady-state linear solution to the vibration problem already presented in this section to be at the frequencies \( \omega_1 \) and \( \omega_2. \) Moreover, since we are interested only in the situation in which the intermodulation frequency \( \Omega \) is in the vicinity of \( \omega_N, \) only certain combinations of the nonlinear frequency products are significant. To obtain expressions for the nonlinear products in Eq. (65) we first write the dominant real solution in complex notation, thus

\[ u_1 = (\omega_{11} e^{i\omega_{11} t} + \omega_{N2} e^{i\omega_{N2} t} + \omega_{N2} e^{-i\omega_{N2} t} + \omega_{N1} e^{-i\omega_{N1} t}). \]  
(80)

where \( * \) denotes complex conjugate and we have ignored the terms containing \( V_{11} \) and \( V_{12} \) because they do not have resonant denominators,\(^{14} \) and then substitute into the quadratic product in Eq. (65) to obtain a linear steady-state homogeneous forced vibration problem for the first iterate solution. When the appropriate linear inhomogeneous vibration problem has been defined standard complex notation for linear systems is employed. This type of procedure, i.e., the use of the previous iterate to define the linear inhomogeneous problem for the next iterate in the above-mentioned manner, is employed at each successive stage of iteration. In view of the foregoing statements, the form of the linear solution, and previous experience\(^{15} \) only the frequencies of \( (2\omega_1 - \omega_2) \) and \( (2\omega_2 - \omega_1) \) are of interest in our intermodulation study. Furthermore, it is sufficient for our purposes to determine the response at \( (2\omega_1 - \omega_2) \) only because the response at \( (2\omega_2 - \omega_1) \) can then be obtained simply by an interchange of the subscripts 1 and 2.

Since we are interested in the response at \( (2\omega_1 - \omega_2) \) only, the form of Eq. (65) along with Eq. (80) and the surrounding discussion reveals that we need determine the first iterate solution \( \mu \), due to the quadratic nonlinearity only at \( 2\omega_1, \) and \( (\omega_1 - \omega_2). \) Furthermore, in the interest of brevity we obtain \( \mu \) at \( 2\omega_1 \) only and simply present the changes in the result at \( (\omega_1 - \omega_2). \) As already noted the first iterate equation at \( 2\omega_1 \) is obtained by substituting \( \mu \) in Eq. (76) in the quadratic term in Eq. (64a) with Eq. (65), in accordance with Eq. (80) and the associated discussion, but ignoring \( \omega_{N2} \) and the time-independent term and neglecting the term without a resonant denominator\(^{14} \) to obtain

\[ c_{11} u_{11} - \rho_0 \ddot{u}_1 = \beta, A \eta_{11} (\sin 2\eta_{11} X_1/2) e^{2i\omega_{11} t}. \]  
(81)
Since the boundary condition is given by Eq. (68), by virtue of Eqs. (73) the solution for \( i_1 \), takes the form

\[
_i_1 = \left( \sum B_m \sin \eta_m X_1 - \frac{d_{31} V_{2w_1}}{2h} X_1 \right) e^{i \omega_1 t},
\]

where the \( B_m \) are obtained from the orthogonality of the sine functions at \( i \). Including only one term, say the \( M \)th, in the series in Eq. (82), we have

\[
B_M = \frac{\beta a_A N A \eta P N M}{2p(4w_1^2 - \omega_M^2)} - \frac{8i(-1)^M - 1/2 d_{31} V_{2w_1}}{[\omega_M^2 - (4w_1^2)]} - M^2 \pi^2/2h,
\]

where we have used Eq. (73a) and

\[
P_{NM} = \frac{2l}{\pi} \left( \sin(2N - M) \pi/2 \right) \frac{\sin(2N + M) \pi/2}{2N + M}.
\]

Substituting from Eqs. (82) and (67) at \( 2w_1 \) into Eq. (66), which is then substituted into Eq. (63), we obtain

\[
I_{2w_1} = -4w_1 \omega_1 \frac{\beta A}{2h} \left( 1 + \frac{8(k_{31}^2)}{\pi^2 M^2 [\omega_M^2 - (4w_1^2)]} \right) e^{i \omega_1 t},
\]

where we have made the identification \( I_{2w_1} = I \) at \( 2w_1 \) since at \( 2w_1 \) the crystal is active. In a similar way, again including only the \( M \)th term, \( \omega_1 = \omega_2 = \alpha \), we obtain

\[
_i_1 = \left( C_M \sin \eta_m X_1 - \frac{d_{31} V_{2w_1}}{2h} X_1 \right) e^{i \omega_1 t},
\]

with

\[
C_M = \frac{\beta a_A N A \eta P N M}{2p(4w_1^2 - \omega_M^2)} - \frac{8i(-1)^M - 1/2 d_{31} V_{2w_1}}{[\omega_M^2 - (4w_1^2)]} - M^2 \pi^2/2h.
\]

The linear second iterate equation at \( \Omega = (2w_1 - \omega_2) \) is obtained by substituting the appropriate expression for \( i_1 \), i.e., a linear combination of the zeroth and first iterate solution functions at \( \omega_1, \omega_2, 2w_1 \), and \( \alpha \), which takes the form

\[
\frac{c_1 \omega_{1,11}}{2} - \rho_0 \hat{u}_1
\]

\[
= \frac{\beta}{2}/[\{2A^2 B_{NM} + i \nu A C N \eta N \eta M \}
X(\eta N \sin \eta N X_1 \cos \eta N X_1)
+ \eta M \sin \eta M X_1 \cos \eta M X_1)
- \eta N (d_{31}/2h) \left( \eta A N V_{2w_1} + i \nu A N V_{2w_1} \right) \sin \eta N X_1 \right] e^{i \omega_1 t}
+ \frac{3 \gamma A N V_{2w_1} \eta N \cos \eta N X_1 \sin \eta N X_1 e^{i \omega_1}}{2h}.
\]

As in the case of the zeroth iterate we take a series solution for \( i_1 \), in the form

\[
_i_1 = \left( \sum \eta_m \sin \eta_m X_1 - \frac{d_{31} V_{2w_1}}{2h} X_1 \right) e^{i \omega_1 t},
\]

and with the aid of the orthogonality of the sine functions at \( i \), we find for \( n = N \), which dominates the others

\[
G_N = -\frac{8l}{(2n - 1)^2 \pi^2/2h} \frac{2}{2N + M} \frac{\beta}{\pi^2/2h} \frac{\sin(2N + M) \pi/2}{N^2 \pi^2/2h} \frac{\sin(2N - M) \pi/2}{2N - M}.
\]

Substituting from Eqs. (91) and (67) at \( \Omega \) into Eq. (66), which is then substituted into Eq. (63), we obtain

\[
I_\omega = -4w_1 \omega_1 \frac{\beta A}{2h} \left( 1 + \frac{8(k_{31}^2)}{\pi^2 M^2 [\omega_M^2 - (4w_1^2)]} \right) V_{2w_1}
+ i \omega_1 \frac{d_{31}}{s_{11}} \frac{\beta a_A N A \eta P N M}{2p(4w_1^2 - \omega_M^2)}
+ \frac{i \omega_1 \frac{d_{31}}{s_{11}} \beta a_A N A \eta P N M}{2p(4w_1^2 - \omega_M^2)}
\]

where

\[
q_{NM} = \frac{\sin \eta_m l}{\eta N}
- \frac{2l}{2N + M} \frac{\sin(2N + M) l}{2N + M}.
\]

Substituting from Eqs. (91) and (67) at \( \Omega \) into Eq. (66), which is then substituted into Eq. (63), we obtain

\[
I_\omega = -4w_1 \omega_1 \frac{\beta A}{2h} \left( 1 + \frac{8(k_{31}^2)}{\pi^2 M^2 [\omega_M^2 - (4w_1^2)]} \right) V_{2w_1}
+ i \omega_1 \frac{d_{31}}{s_{11}} \frac{\beta a_A N A \eta P N M}{2p(4w_1^2 - \omega_M^2)}
+ \frac{i \omega_1 \frac{d_{31}}{s_{11}} \beta a_A N A \eta P N M}{2p(4w_1^2 - \omega_M^2)}
\]

where we have made the identification \( I_\omega = I \) at \( \Omega \) since the crystal is active at \( \omega \), and

\[
r_{NM} = \eta N q_{NM} + \eta M p_{NM}/2.
\]

A schematic diagram of the circuit, which is driven at the two-test tone frequencies \( \omega_1 \) and \( \omega_2 \), is shown in Fig. 5, where \( V_{1w} \) is the driving voltage, \( R_g \) the generator resistance, and \( R_L \) the load resistance. Application of Kirchhoff's voltage equation to the circuit shown in Fig. 5 at \( \omega_1 \) and \( \omega_2 \) yields the two equations

\[
V_{1w} + I_{\omega_1}(R_g + R_L) + V_{\omega_1} = 0,
\]

\[
V_{1w} + I_{\omega_2}(R_g + R_L) + V_{\omega_2} = 0,
\]

which with Eq. (77) for \( \omega_1 \) and the equivalent for \( \omega_2 \), enables the determination of \( V_{\omega_1} \) and \( V_{\omega_2} \) in terms of \( I_{\omega_1} \) and \( I_{\omega_2} \).
respective, from which \( \bar{A}_N \) and \( \bar{A}_N' \) are obtained in terms of \( V_g \) and \( V_{2g} \), respectively, from Eq. (75). The application of Kirchhoff's voltage equation to the circuit shown in Fig. 5 at \( 2\omega_1 \), \( \alpha \), and \( \Omega \) yields

\[
I_\omega (R_g + R_e) + V_e = 0, \quad \kappa = 2\omega_1, \alpha, \Omega,
\]

since at these frequencies \( V_e = 0 \). From Eq. (97) with \( \kappa = 2\omega_1 \) and \( \kappa = \alpha, \) respectively, and Eq. (85) and Eq. (88), \( V_{2m} \) and \( V_\alpha \) may readily be determined in terms of \( V_g \) and \( V_{2g} \). Finally, from Eq. (97) with \( \kappa = \Omega \) and Eq. (94), \( V_{2g} \) can readily be determined in terms of \( V_g \) and \( V_{2g} \) if \( \gamma \) is known. However, since actually \( \gamma \) is the only unknown quantity, a measurement of the load voltage \( I_\omega R_\omega \) enables the determination of \( I_\omega \) and \( V_\omega \) from Eqs. (97), from which \( \gamma \) can be evaluated from Eq. (94) for any desired orientation, and in particular, for the practical 5\(^\circ\) \( X \) cut.

IV. NONLINEAR RESONANCE IN RODS

In this section we consider the problem of nonlinear resonance in piezoelectric rods. The basic nonlinear equations required for the problem of interest here are Eqs. (64)--(68). We again obtain the solution by iteration from the linear solution, but the successive iterates must be corrected because we are interested in the steady-state nonlinear solution at the driving frequency \( \omega_1 \), in which there is a nonlinear dependence of the resonant frequency \( \omega_N \) on the amplitude of the extensional displacement of the rod. Clearly, the zeroth iterate, or linear solution, is given in Sec. III and Eqs. (69)--(79) hold for purposes of this section. Again, in the vicinity of the \( N \)th resonance one term dominates the series and the zeroth iterate is given by Eq. (76) with the subscript 1 stricken from the \( A_N \) and the \( \omega \).

In accordance with the discussion surrounding Eq. (80) in order to obtain the linear equation for the first iterate, the zeroth iterate solution is written in the form

\[
\varphi_1 = \frac{1}{2} \left( A_N \sin \frac{N\pi}{2l} X_1 - \frac{d_{31} V_{2w}}{2h} X_1 \right) e^{i\omega t} + \left( A_N' \sin \frac{N\pi}{2l} X_1 - \frac{d_{31} V_{2w}}{2h} X_1 \right) e^{-i\omega t},
\]

which is substituted in the quadratic term in Eq. (64a) with Eq. (65) to obtain the linear inhomogeneous equation for the first iterate

\[
c_N u_{1,11} - \mu^2 u_1 = (\beta/2) \eta^2 \sin 2\eta_N X_1 (A_N A_N' - A_N A_N') e^{i\omega t} + A_N A_N'.
\]

Since the boundary condition is given by Eq. (68) with Eq. (65) and the \( V_{2w} \) is negligible, by virtue of Eq. (73b) the solution for \( u_1 \) takes the form

\[
u_1 = \left( \sum_m B_m \sin \eta_m X_1 - \frac{d_{31} V_{2w}}{2h} \right) e^{i\omega t} + \sum_m C_m \sin \eta_m X_1,
\]

where the \( B_m \) and \( C_m \) are obtained from the orthogonality of the \( \sin \eta_m X_1 \) and we have made use of the fact that the electrodes are shorted at zero frequency. Since the case of primary interest is \( N = 1 \) and for that case \( M = 1 \) gives the largest term in each series in Eq. (100) and also results in the largest single term caused by the series in Eq. (100) at the next iterate and we are not interested here in an exhaustive treatment of all possibilities, but only in indicating the nature of the influence of the terms in the series in Eq. (100) on the nonlinear resonance equation and obtaining the expression due to the term of greatest influence for the case \( N = 1 \), we consider the term \( M = N \) only in each series in Eq. (100). Accordingly, including only the \( M \)th term in both series in Eq. (100), with the aid of the orthogonality of the \( \sin \eta_m X_1 \) and Eq. (73a), we have

\[
B_M = \frac{\beta A_N' \eta^2 N M}{2l p_0^2 (4\omega^2 - \omega_1^2)} \left( -1 \right)^M - \frac{1}{2} \frac{8d_{31} V_{2w}}{M^2 \pi^2 2h}
\]

\[
C_M = - \frac{\beta A_N \eta^2 N M}{2l p_0^2 (4\omega^2 - \omega_1^2)},
\]

where \( p_{NM} \) is given in Eq. (84). Substituting from Eq. (100) and Eq. (67) at \( 2\omega \) into Eq. (66), which is then substituted into Eq. (63), we obtain

\[
I_{2\omega} = -4wl/2\omega \left( 1 + \frac{8(k_{11}^2)^2}{M^2 \pi^2 \omega_1^2} \right) V_{2w} + \frac{\omega_1^2 V_{2w}}{2l p_0^2 (4\omega^2 - \omega_1^2)}
\]

where we have made the identification \( I_{2\omega} = I \) at \( 2\omega \), since at \( 2\omega \) the crystal is active. In accordance with earlier discussions in order to obtain the linear equation for the second iterate, the first iterate solution is written in the form

\[
u_2 = \frac{1}{2} \left[ (A_N \sin \frac{N\pi}{2l} X_1 - \frac{d_{31} V_{2w}}{2h} X_1) e^{i\omega t} + B_M \sin \frac{M\pi}{2l} X_1 - \frac{d_{31} V_{2w}}{2h} X_1) e^{i\omega t}
\]

and substituted in both the quadratic and cubic terms in Eqs. (64a) and (68), with Eq. (65), while retaining all terms of order \( A_N A_N' \) and containing the time dependence \( e^{i\omega t} \) only to obtain the homogeneous differential equation

\[
c_{2u_{1,11}} = \rho^2 \beta \mu^2,
\]

\[
= \frac{\beta}{2} \left( (A_N \sin \frac{N\pi}{2l} X_1 + A_N \cos \frac{M\pi}{2l} X_1) \sin \eta_N X_1ight) e^{i\omega t}
\]

\[
+ \eta_N A_N' \sin \frac{d_{31} V_{2w}}{2h} \sin \eta_N X_1 e^{i\omega t}
\]

and also results in the

\[
\text{FIG. 5. Schematic diagram of the reduced test circuit.}
\]
and the inhomogeneous boundary conditions

\[ c_{2M_{1,1}} - \frac{d_{31}}{s_{11}} V_{w} e^{i\omega t} \quad s_{11} \]

\[ = -\frac{\beta}{2} \left[ A_{N} \eta_{N} \cos \eta_{N} l \right] \]

\[ \times \left( B_{N} \eta_{N} \cos \eta_{N} l - \frac{d_{31}}{2h} V_{2w} \right) \]

\[ + A_{N} \eta_{N} \cos \eta_{N} l \left[ C_{M} \eta_{N} \cos \eta_{N} l \right] \]

\[ - \frac{3}{4} \gamma A_{N} \eta_{N}^{3} \cos^{3} \eta_{N} l e^{i\omega t}, \quad \text{at } x_{1} = \pm l, \]  

(105)

where we have neglected terms without resonant denominators, i.e., those containing \( \gamma \eta_{N} \) compared to terms with resonant denominators and we have not yet set \( M = N \). It should be noted that the right-hand side of Eq. (105) turns out to be negligible because \( A_{N} \) is small and \( \eta_{N} \) is nearly given by Eq. (73b). Since we regard the \( A_{N}, \eta_{N}, \) and \( \eta_{M} \) on the right-hand side of Eqs. (104) and (105) as presently unknown and ignore Eqs. (73b) and (75), the equations are homogeneous for \( V_{w} \). We say the iterate is nearly given by Eq. (73b).

Before we obtain the nonlinear eigensolutions of Eqs. (104) and (105) we first note that for the short circuit case \( V_{2w} = 0 \), and the open circuit case \( I_{2o} = 0 \) yields the greatest difference in eigenfrequency from the shorted case. Furthermore, the open circuit voltage \( V'_{2w} \) obtained from Eqs. (102) when substituted in Eq. (104) contains the product of \( (k_{31})^{2} \) and \( A_{N} A_{N}^{*} \), both of which are small. Since the product of two small quantities is negligible, we may omit \( V'_{2w} \) in Eqs. (104) and (105) in the sequel. The nonlinear eigensolutions must satisfy Eq. (104), which for \( M = N \) takes the form

\[ c_{2U_{1,11}} - \beta_{V_{2w}}^{2} u_{1} = \left( H \sin 2\eta_{N} x_{1} \right) \]

\[ + \frac{9}{16} \gamma A_{N}^{2} A_{N}^{*} \eta_{N} \left[ \sin \eta_{N} x_{1} \right] \]

\[ \sin 3\eta_{N} x_{1} \right) e^{i\omega t}, \]  

(106)

where

\[ H = -\frac{1}{3} \frac{\beta}{\gamma c} A_{N}^{2} A_{N}^{*} \frac{f_{N}}{l} \eta_{N}, \]

\[ f_{N} = (1 - 1)^{N - 1/2} - \left( -1 \right)^{N + 1/2} \left( \frac{1}{3} \right). \]  

(107)

and we have made use of the fact that we are obtaining the steady-state solution for \( \omega \) in the vicinity of \( \omega_{N} \) only and we have employed Eqs. (73), (84), and (101). The homogeneous boundary conditions are given by Eq. (105) with \( V_{w} = 0 \). As the \( N \)th eigensolution of the linear differential equation (105), we take

\[ u_{1} = u_{N} e^{i\omega_{N}}, \]  

(108)

\[ u_{N} = A_{N} \sin \eta_{N} x_{1} + G_{N} \sin 2\eta_{N} x_{1} + L_{N} \sin 3\eta_{N} x_{1}, \]  

(109)

the substitution of which in Eq. (106) yields

\[ \rho_{N}^{2} \omega_{N}^{2} = \sin^{2} \eta_{N} (1 + \mu), \]  

(110)

\[ G_{N} = \frac{h}{\pi \kappa f_{N} / N \pi \mu A_{N}}, \]

\[ L_{N} = -\mu A_{N} / 8, \]  

(111a, 111b)

where

\[ \kappa = \beta^{2} / \gamma c, \]  

(112a)

\[ \mu = \frac{1}{8} \left( \gamma c / \eta_{N}^{3} A_{N} A_{N}^{*}. \right) \]  

(112b)

Since the amplitude \( A_{N} \) is very small, Eqs. (111) with Eq. (112b) show that \( G_{N} \) and \( L_{N} \) are negligible for our purposes except insofar that they might result in a change in the boundary conditions (105). Substituting from Eqs. (108) and (109) into Eq. (105), we obtain

\[ \cos \eta_{N} l = -\mu \left( \cos \eta_{N} l + \frac{64}{27} \kappa f_{N}^{2} / N \pi \cos 2\eta_{N} l \right) \]

\[ - \frac{8}{27} \kappa f_{N} \left( 1 + \cos 2\eta_{N} l \right) \]

\[ - \frac{1}{24} \cos 3\eta_{N} l \right). \]  

(113)

The roots of Eq. (113), with Eq. (110), determine the amplitude dependent eigenfrequencies \( \eta_{N} \) of this piezoelectric rod in the nonlinear case. In the absence of the nonlinear term \( \mu \), Eq. (113) reduces to Eq. (73b). Since \( \mu \) is small, the roots \( \eta_{N} \) may be obtained iteratively from the roots in the linear case, which are given in Eq. (73b). Consequently, in the nonlinear case the roots must differ from \( N \pi / 2 \) by small quantities, say \( \Delta_{N} \), and we may write

\[ \eta_{N} l = N \pi / 2 + \Delta_{N}, \quad N \text{ odd}. \]  

(114)

Substituting from Eq. (114) into Eq. (113), expanding and retaining terms linear in \( \Delta_{N} \), we obtain

\[ \Delta_{N} = -\left( -1 \right)^{N - 1/2} \mu c f_{N} / N \pi \mu, \]  

(115)

which shows that as a consequence of the existence of the coefficient \( \beta \) of the quadratic term, the nonlinear boundary conditions have a small influence on \( \eta_{N} \). Since \( \mu < 1 \), from Eq. (110), with the aid of Eqs. (114) and (115), we have

\[ \Delta_{N} = \frac{N \pi}{2} \left( \frac{c f_{N}}{\mu} \right)^{1/2} \left( 1 + \frac{\mu}{2} \right) \]

\[ - \left( -1 \right)^{N - 1/2} \frac{128 c f_{N}^{3}}{81 N^{2} \pi^{3} H}. \]  

(116)

which gives the resonant frequency of the \( N \)th mode of extensional vibration of the thin piezoelectric rod with small piezoelectric coupling, including the dependence on the amplitude of vibration.\(^{19}\)

The solution to the nonlinear forced vibration problem at a driving frequency \( \omega \) in the vicinity of \( \omega_{N} \) may now be obtained by means of an expansion in the eigensolutions, while retaining the nonlinear correction in the dominant \( N \)th eigensolution only. Accordingly, we write

\[ z u_{1} = \left[ \mu - \left( d_{31} V_{w} / 2 h \right) x_{1} \right] e^{i\omega_{N}}, \]  

(117)

\[ z u_{1} = A_{N} \sin \eta_{N} X_{1} + G_{N} \sin 2\eta_{N} X_{1} + L_{N} \sin 3\eta_{N} X_{1} \]

\[ + \sum_{n \neq N} A_{n} \sin \eta_{N} X_{1}, \quad n \text{ odd}, \]  

(118)

and \( G_{N} \) and \( L_{N} \) are determined from Eqs. (111). In the iterative procedure we are employing the influence of the large
term in the expansion in Eq. (118), i.e., $A_N \sin \eta_N X_1$, already appears on the right-hand side of Eqs. (104) and (105) including that resulting from $\beta$ and the first iterate solution. Substituting from Eq. (117) with Eq. (118) into Eq. (104), we obtain

\[
I_v = io\omega u = \frac{\beta_t^4}{2h} \left( V_v - \frac{2h}{l} \frac{d_1}{\epsilon_{ij} s_1} \hat{A}_N \right),
\]

where we have made the identification $I_v = -I$ since the rod is passive and

\[
\hat{A}_N = A_N(1 - \eta^N - 1/2).
\]

Application of Kirchhoff's voltage equation to the circuit shown in Fig. 5 yields

\[
V_g + I_v(R_g + R_L) + V_v = 0,
\]

where $V_g$ is the generator voltage, $R_g$ and $R_L$ are the generator and load resistance, respectively, and $V_v$ is the voltage across the crystal. Substituting from Eqs. (125) and (127) into Eq. (122), we obtain

\[
N^2\pi^2 \left[ \omega^2 - \hat{\omega}_N^2 (1 + \alpha \hat{A}_N \hat{\omega}_N^2) \right] \left( 1 + i \omega \frac{2u_1 \epsilon_{ij}^s}{h} (R_g + R_L) \right)
- \frac{i \omega (R_g + R_L) 4u_1 (l/h)}{h} \epsilon_{ij}^s (k_3^s) \hat{A}_N = \frac{-d_1}{h} V_g,
\]

and, as usual, we have replaced $\omega$ by $\hat{\omega}_N$ where

\[
\hat{\omega}_N = \omega + i\omega / 2Q_N,
\]

in which $Q_N$ is the unloaded quality factor of the resonator in the $N$th mode. Equation (128) is the nonlinear equation relating the amplitude of the extensional displacement to the generator voltage. Since Eq. (128) contains $\hat{A}_N$ as well as $\hat{\omega}_N$, it is not simply a complex cubic equation. In order to facilitate the solution of Eq. (128), we multiply Eq. (128) by its complex conjugate and after some algebraic manipulation obtain

\[
\alpha \left[ d^2 + C - 2(d + C)\omega^2 \right] \hat{\omega}_N \alpha \alpha + (1 + \alpha^2) \hat{\omega}_N \alpha^2 \alpha^2
= e^2 c_s \hat{A}_N,
\]

where

\[
C = -\frac{\omega (R_g + R_L) 4u_1 (l/h)}{h} \epsilon_{ij}^s (k_3^s) \hat{A}_N / (\omega \omega / 2Q_N),
\]

\[
a = \alpha \omega (R_g + R_L) 4u_1 (l/h) / h, \quad d = \omega^2 - \hat{\omega}_N^2 + \hat{\omega}_N / (\omega / 2Q_N),
\]

\[
\epsilon = d_s (l/h) / h, \quad v = N^2 \pi^2 / 4 \omega^2, \quad b = \omega R_g + R_L / (4u_1 \epsilon_{ij}^s (l/h) / (k_3^s)^2 / v).
\]

Equation (130) is a real cubic algebraic equation in the square of the extensional vibration amplitude $A_N \hat{\omega}_N$, which may readily be solved for the $A_N \hat{\omega}_N$. For a solution $A_N \hat{\omega}_N$ to be meaningful, it must be real and positive. Consequently, there is at least one and possibly three physically meaningful $A_N \hat{\omega}_N$. When a physically meaningful $A_N \hat{\omega}_N$ has been determined from Eq. (130), it may be substituted in Eq. (128), which then becomes a linear equation in $A_N \hat{\omega}_N$ that may readily be solved for the phase if desired. When an $A_N \hat{\omega}_N$ is known, the current $I_v$ through the circuit may be evaluated from Eq.
(125). This result can be used as a check for the $y$ determined from intermodulation measurements for any desired orientation and in particular, for the practical $5^\circ X$ cut of quartz.

ACKNOWLEDGMENTS

The work of one of the authors (HFT) was supported in part by the Army Research Office under Grant No. DAAG 29-79-C-0088 and the National Science Foundation under Grant No. MEA-8115340.

APPENDIX

In this Appendix we derive the expression in Eq. (49) with Eq. (50) for the local orthogonal transformation containing terms up to quadratic in the mechanical displacement gradients. To this end we first note that Green's deformation tensor is defined by

$$ C_{KL} = y_{,KL} y_{,LL}, $$

and that the polar decomposition theorem takes the form

$$ y_{,LL} = R_{,KL} U_{KL}, $$

where

$$ U_{KM} U_{ML} = C_{KL}. $$

and, of course, if $U_{KL}$ is the reciprocal of $U_{KL}$, we have

$$ R_{IK} = y_{,iL} U_{LX}^{-1}. $$

We now substitute from Eqs. (4) and (37a) into Eq. (A1) to obtain

$$ C_{KL} = \delta_{KL} + 2S_{KL} + u_{,KL} u_{,LL}. $$

Since the magnitude of $u_{,KL}$ is small, on account of Eqs. (A3) and (A5) $U_{KM}$ must differ from the identity $\delta_{KL}$ by a small (but nonlinear) quantity $\gamma_{KM}$, thus

$$ U_{KM} = \delta_{KM} + \gamma_{KM}. $$

Substituting from Eq. (A6) into Eq. (A3) and employing Eq. (A5) we obtain

$$ \gamma_{KL} = S_{KL} + \frac{1}{2} u_{,KL} u_{,LL} - \frac{1}{2} \gamma_{KM} \gamma_{ML}. $$

Also, since $u_{,KL}$ is small, $U_{ML}^{-1}$ must differ from $\delta_{ML}$ by a small (but nonlinear) quantity $\alpha_{ML}$, thus

$$ U_{ML}^{-1} = \delta_{ML} + \alpha_{ML}. $$

Since

$$ U_{KM} U_{ML}^{-1} = \delta_{KL}, $$

the substitution of Eqs. (A6) and (A8) in Eq. (A9) yields

$$ \alpha_{KL} = - \gamma_{KL} - \gamma_{KM} \gamma_{ML}. $$

Furthermore, from Eq. (A4), with Eqs. (4) and (A8), we have

$$ R_{IK} = \delta_{IK} \delta_{MK}, $$

where

$$ R_{MK} = \delta_{MK} + u_{,MK} + \alpha_{MK} + u_{,ML} \alpha_{LK}. $$

and we have employed the translation operator $\delta_{MK}$ discussed in Sec. 1 in obtaining Eqs. (A11) with (A12). Equation (A12) shows that if we can find $\alpha_{KM}$ to any order in the small quantities $u_{,KL}$, we can obtain $R_{MK}$ and, of course, $R_{KL}$ to that order in $u_{,KL}$.

To lowest order, from Eqs. (A7) and (A10), successively,

$$ o_f^{KL} = S_{KL}, $$

$$ \alpha_{KL} = - S_{KL} $$

the substitution of the latter of which in Eq. (A12) yields the usual linear relation given in Eq. (52). To second order, using (A13) in the nonlinear terms in Eqs. (A7) and (A10), we obtain

$$ \begin{align*}
\gamma_{KL} &= S_{KL} + \frac{1}{2} u_{,MK} u_{,ML} - \frac{1}{2} S_{KM} S_{ML}, \\
\alpha_{KL} &= - S_{KL} - \frac{1}{2} u_{,MK} u_{,ML} + \frac{1}{2} S_{KM} S_{ML}. 
\end{align*} $$

Substituting from Eq. (A13b) into the nonlinear terms and (A14b) into the linear terms in (A12) and employing Eqs. (37a) and (51), we obtain

$$ R_{MK} = \delta_{MK} + \Omega_{KM}, $$

where

$$ \Omega_{KM} = w_{KM} + \frac{1}{2} w_{KL} w_{LM} + \frac{1}{2} S_{ML} w_{LK} - S_{KL} w_{LM}. $$

which, respectively, correspond to Eqs. (49) and (50). It is a simple matter to show that the orthogonality relations

$$ R_{KL} R_{LM} = \delta_{KM}, \quad R_{LK} R_{MK} = \delta_{LM} $$

are satisfied to second order in the $u_{,KL}$.


5. H. F. Tiersten, Linear Piezoelectric Plate Vibrations (Plenum, New York, 1969), Chap. 4, Sec. 4.


8. Since only linear electric terms are retained, this is the consistent approach.

9. In this work we are interested in obtaining the extensional equation for the rod essentially in the limit as the cross-section collapses to a line.

10. Reference 5, Chap. 7, Sec. 1.


13. Reference 5, Chap. 15, Sec. 1.

14. Only terms with resonant denominators for driving frequencies in the vicinity of the resonant frequencies are large and need be retained in the nonlinear products.

15. Reference 2, Sec. II. These are the only frequency combinations occurring at the lowest orders of nonlinearity that are in the vicinity of $\omega_n$.

16. Only one term is retained because only the largest term is considered to be of any potential importance. If more are thought to be required, they may be included without difficulty.

17. The procedure is a straightforward iterative procedure except when the functional form resulting from the nonlinear terms is identical with that satisfying the basic linear differential equation, at which point the coefficients associated with the functional form due to the nonlinearity are treated as unknown and identical with the coefficients of the linear solution.
This is the reason Eq. (104) is a homogeneous equation whereas Eq. (99) is an inhomogeneous equation. This procedure is appropriate because the term on the rhs of (104) actually is a homogeneous term and appears to be inhomogeneous (known) only because of the nature of the formal procedure employed which tends to be misleading whenever this type of coincidence of functions occurs.

A similar effect was inadvertently neglected in Ref. 3 and should have been included. However, the quadratic coefficient in Ref. 3 for the AT-cut turns out to be so small that the effect has a negligible influence on the numerical results.

It should be recalled that the last term in Eq. (116) is the large one for the case \( N = 1 \) only. For other values of \( N \), values of \( M \neq N \) have to be considered, perhaps more than one. However, if for a particular orientation \( \kappa \) turns out to be very small, the last term in Eq. (116) is negligible and the solution holds for any odd \( N \).


Reference 20, Eq. (37.8). The Appendix does no more than carry out to second order what the reference states can be done.