The Flexural Behavior of PACSAT in Orbit

William Solfrey

35th Year Rand

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PREFACE

This report presents an analysis of the libration and vibration of a passive communications satellite known as PACSAT. Previous related Rand reports include N-1780-ARPA, PACSAT: A Passive Communication Satellite for Survivable Command and Control, November 1981, and R-2920-ARPA, Electromagnetic Properties and Communication Characteristics of PACSAT, August 1982.

The research is sponsored by the Strategic Technology Office, Defense Advanced Research Projects Agency, under the supervision of Dr. Sherman Karp. It is an element of a broad range of strategic communications studies being conducted for ARPA by Rand.

SUMMARY

The passive communications satellite known as PACSAT has been under consideration for many years. The satellite could be useful for command and control of strategic forces because it can provide survivable communications. Many studies have shown that it can provide low-to-medium data rates at distances up to a few thousand miles between moderate sized mobile terminals.

PACSAT consists of a uniform linear array of electromagnetically resonant spheres which provides a very large scattering cross section in a particular direction. It is nominally vertical and straight, and is maintained in that condition by gravity-gradient stabilization. Previous studies have shown that verticality must be maintained to 1.5 deg, and straightness to 1/8 wavelength (.5 cm at the desired operating frequency). This report addresses the effects of various disturbing mechanical forces on verticality and straightness. Certain of these effects are quite strong, and it is shown in particular that micrometeoroid impacts and thermal bending effects cause motions which far exceed the permissible displacements.

The general problem is formulated, with gravity-gradient tension and elastic forces regarded as the principal forces, and other forces treated as perturbations. The motion is separated into center-of-mass motion, rigid-body libration around the local vertical, and vibration transverse to the libration. The fundamental modes of PACSAT in a circular orbit with no external forces are studied in detail. The motion is stable, undamped, and maintains the original amplitude, so the verticality and straightness requirements must be satisfied by the initial conditions, a very difficult task.

The effect of orbital ellipticity is the first perturbation studied. It induces an in-plane libration of amplitude in radians equal to the eccentricity. The out-of-plane libration is unstable, but the exponent is fourth order in the eccentricity, and the instability takes thousands of years to develop. There is no significant effect on vibration.

Earth oblateness effects are studied next. The precession of the orbital plane couples the in-plane and out-of-plane motions, producing a resonant excitation of the out-of-plane motion, with an amplitude proportional to the product of eccentricity and inclination for near-circular, near-equatorial orbits. Despite the resonance, the displacement is small. There are no discernible effects on vibration. Lunar and solar gravity and solar radiation pressure all prove to be negligible.

Micrometeoroid impacts produce important vibrations. It is shown that a particle of a few micrograms mass at meteoric velocity can cause deflections which exceed the straightness requirements, and the impact of such a particle can be expected as often as once a month. Even if the satellite is initially vertical and straight, it will not be able to maintain its condition under expected meteoric bombardment.

Thermal bending proves to be the most important cause of vibration. The solar heat input causes differential expansion, which makes PACSAT curve away from the sun. Eclipsing by the earth, or shadowing within PACSAT, causes the solar input to vary so as to excite the lowest vibrational frequency of PACSAT, which is at approximately three times orbital frequency. The resonance causes large displacements. Unless the absorptivity of the surfaces is kept below .0007, which is well beyond the current state of the art, the straightness requirements cannot be met.

It is concluded that the flexural misbehavior of PACSAT in orbit is such that it is most improbable that the present design (unsupported linear array) can perform its communications function.

ACKNOWLEDGMENTS

The author is indebted to Edward Bedrosian for many helpful suggestions. The computer programs used in the research were prepared by Marianne Lakatos. Thanks are also due to the reviewers, Roy Danchick and Richard Frick.

CONTENTS

GLOSSARY OF SYMBOLS

Symbol		Equation of First Appearance
a	semi-major axis of elliptical orbit	4.1
a	parameter in Mathieu equation	4.24
$\vec{a}$	acceleration of sphere	6.8
a	magnitude of acceleration	6.10
$a_{k,\ell}$	coefficient of Y_ℓ in expansion of X_k	3.47
b	velocity of sound	7.12
c	velocity of light	6.8
$c_{\ell,m}$	matrix element	3.49
c_p	specific heat	8.13
d_i	diameter of impacting body	7.1
e	eccentricity of elliptical orbit	4.1
e	thermal expansion coefficient	8.1
f	function describing in-plane libration in elliptical orbit	4.7
g	function describing out-of-plane libration in elliptical orbit	4.7
g	argument of perigee	5.1
g	acceleration of gravity at earth's surface	6.11
h	longitude of node	5.1
$\vec{i}_x, \vec{i}_y, \vec{i}_z$	rectangular earth-centered coordinates	3.1
$\vec{i}_R, \vec{i}_\theta, \vec{i}_z$	cylindrical earth-centered coordinates	3.1
$\vec{i}_R, \vec{i}_L, \vec{i}_\phi$	spherical earth-centered coordinates	5.4

Symbol		Equation of First Appearance
$\vec{i}_1, \vec{i}_2, \vec{i}_3$	radial, in-plane, and out-of-plane unit vectors	5.10
j	oblateness parameter	5.5
k	summation index	3.31
k	thermal conductivity	8.1
ℓ	summation index	3.47
ℓ	mean anomaly	5.2
ℓ	length of rod	7.9
ℓ_u	unshadowed length of rod	8.9
m	summation index	3.49
n	orbital angular rate	4.1
q	parameter in Mathieu equation	4.24
$\vec{r}$	vector from center of earth to point on satellite	2.2
r	radius of sphere or cylinder	6.9
s	element spacing	1.1
s	distance along satellite from center	2.2
t	time	2.1
$\vec{u}(t)$	libration unit vector	2.27
$\vec{v}(s,t)$	total displacement	2.27
v_i	velocity of impacting body	7.1
$\vec{w}(s,t)$	vibrational displacement	2.27
x	normalized length	3.22
y	distance of rod filament from center	2.14
$\dot{y}_0$	initial transverse velocity	7.1

Symbol		Equation of First Appearance
z	longitudinal displacement	7.12
$\dot{z}_0$	initial longitudinal velocity	7.13
A	projected area	6.8
A,B,C,D	constants describing initial conditions	4.10
$A_k(\theta)$	expansion coefficients of vibration	3.28
$B_k(\theta)$	expansion coefficient for vibration	4.34
D_s	diameter of sun	6.13
E	Young's modulus	2.14
E	eccentric anomaly	4.1
F	function describing in-plane vibration in elliptical orbit	4.31
$\vec{F}(\vec{r},t)$	external force per unit length	2.7
$F(\theta_s)$	amplitude of thermal vibration	8.26
G	function describing out-of-plane vibration in elliptical orbit	4.31
I	moment of inertia around bending axis	2.14
I	inclination of orbit plane to equator	5.1
I_D	inclination to equator of orbit plane of disturbing body	6.3
J	solar constant	6.8
J_2	oblateness coefficient	5.4
K	curvature of rod	8.2
$K_{k,\ell}$	matrix element	4.36
L	PACSAT length	2.2
$\mathcal{L}$	latitude	5.1

Symbol		Equation of First Appearance
M	mass of sphere	6.8
M_D	mass of disturbing body	6.3
M_i	mass of earth	6.3
M_E	mass of impacting body	7.1
M_s	mass of sun	6.11
N	number of spheres	7.17
N_k	normalization coefficient	3.41
P	twist of precessing coordinate system	5.12
P_k	Legendre polynomial	Fig. 5
Q	curvature function	8.14
Q_i	expansion coefficient of Q	8.20
$\vec{R}(t)$	vector from center of earth to center of mass of satellite	2.27
R	distance from center of earth to center of mass of the satellite	4.1
R_C	crater diameter	7.22
R_D	distance of disturbing body	6.3
R_e	radius of earth	5.4
R_K	radius of curvature	2.14
R_s	distance to sun	6.11
S	yield strength of sphere	7.22
T	kinetic energy	2.1
T	satellite period	6.17
$T(s,t)$	tension in rod	2.10

Symbol		Equation of First Appearance
$T_0(t)$	measure of tension due to gravity gradient	2.28
$T_1(s,t)$	tension due to vibration	2.28
$T(r,\theta)$	temperature of rod	8.1
$U(\vec{r})$	gravitational potential energy	2.6
V	potential energy	2.1
V	thermal displacement	8.16
V_B	strain energy due to bending	2.16
V_T	strain energy due to tension	2.10
$V_k(\theta)$	expansion coefficient of V	8.18
W	amplitude of thermal vibration	8.26
$W(r,t)$	potential energy associated with external forces	2.7
$X_k(x)$	characteristic function	3.28
Y	amplitude of transverse motion	7.5
$Y_k(x)$	characteristic function of bar	3.36
Z	amplitude of longitudinal motion	7.15
α	angle of incidence	1.1
α	absorptivity	8.1
β	angle of diffraction	1.1
β	elasticity parameter	3.23
β	angle between satellite axis and sunline	8.8
γ	angle subtended by sun	6.13
δ	variation symbol	2.1
ε	emissivity	8.1

Symbol		Equation of First Appearance
ε	inclination of ecliptic to equator	8.7
θ	angular coordinate in orbital plane	3.1
θ_s	longitude of sun	8.7
λ	wavelength	1.1
μ	earth's gravitational parameter	2.6
v_i	expansion frequency	8.20
v_k	characteristic frequency of bar	3.36
ρ	mass density per unit length	2.2
$\vec{\sigma}$	unit vector from satellite to sun	6.8
σ	Stefan's constant	8.1
σ	volume density	8.13
σ_i	volume density of impacting body	7.22
τ	thermal time constant	8.13
ϕ	longitude	5.1
ϕ	angular coordinate around rod	8.1
ϕ_i	expansion phase	8.20
χ	librational out-of-plane angle from vertical	3.4
ψ	librational in-plane angle from vertical	3.4
ω_k	characteristic frequency	3.26
Δv	velocity change	6.18
Φ	radar cross section	7.18
$\Phi(\theta_s)$	phase of thermal vibration	8.26

FIGURES

TABLES

I. INTRODUCTION

The survivability and endurance of the communications systems used for the command and control of strategic forces are vital issues. Concern that existing systems may be vulnerable, despite their diversity and specialized design, has led to numerous suggestions for their improvement. Among these is a proposal by Yater[1] for an advanced passive communications satellite concept. If it proves practical, this design, now called PACSAT, may be useful for the command and control of strategic forces because it can provide survivable point-to-point communication at low-to-medium data rates at distances up to a few thousand miles between moderate sized mobile terminals.

The PACSAT concept was investigated at the Stanford Research Institute[2], and in a recent treatment by Bedrosian[3]. PACSAT consists of a uniform linear array of scatterers whose axis is nominally pointed toward the center of the earth. Gravity-gradient stabilization is used to keep the array erect. When illuminated by a transmitter on the earth, the array scatters energy much like a diffraction grating. If the angle of incidence (the angle between the line of sight from the transmitter to the array and the array axis) is denoted by α, the first-order grating lobe is directed back toward the earth at a half-angle, β, given by the relation:

$$s(\cos\alpha + \cos\beta) = \lambda \tag{1.1}$$

where s is the element spacing and λ is the wavelength.

The region illuminated by the grating lobe on the earth will be a narrow annulus. A communication link is established by choosing the wavelength so the annulus includes the desired receiver. The radar cross section, available bandwidth, and other communication parameters are calculated in Ref. 3, where it is also shown that coverage considerations lead to placing PACSAT in a nominally circular equatorial orbit at an altitude of 1-1.5 earth radii.

Reference 3 calculates the limits on PACSAT performance caused by departure from the ideal configuration. Effects considered are near-field effects (PACSAT, at a length of 1,500 m, a wavelength of 3.75 cm, and the indicated altitude, is in the transition zone between Fresnel and Fraunhofer diffraction), librational motion about the vertical, and distortion of the array from perfect straightness into a parabolic shape. The reduction of the radar cross section is calculated. Upon extracting the information from the extensive curves of Ref. 3, the requirements for verticality and straightness are obtained and shown in Table 1.

Thus, if the radar cross-section loss is not to exceed 1 dB, the array must maintain verticality to within 1.5 deg, and must be straight to about 0.5 cm at the wavelength of interest. For an array length of 1,500 m, if the parabola be approximated by an arc of a circle, the radius of curvature of the circle must exceed 50,000 km, about eight times the radius of the earth. This is clearly a very difficult task, since the straightness must be maintained in the presence of a variety of disturbing forces. If the distorted shape were other than parabolic, the limitation would be applied to the rms displacement. The purpose of this report is to investigate the phenomena and ascertain whether PACSAT can meet the flexural requirements.

The behavior of PACSAT in orbit is closely related to the theory of the attitude dynamics of satellites with flexible appendages. This subject has received extensive treatment in the literature. A review article by Modi[4], written in 1974, lists over 200 references, and there have been numerous publications since. Although PACSAT at 1,500 m length

Table 1

PACSAT FLEXURAL REQUIREMENTS

Reduction in Cross Section (dB)	Departure from Vertical (deg)	Maximum Parabolic Displacement (wavelengths)
1	1.5	.127
3	2.0	.217
10	2.7	.401

is longer than any satellite ever flown, it is most similar to the Radio Astronomy Explorer satellite (1968-55A), which was placed in a 5,850 km altitude circular orbit in July 1968 and carried four 228 m extensible booms. The dynamics of that satellite have been well studied[5, 6]. More recently, the motion of a gravity-gradient stabilized satellite driven by orbital eccentricity has been treated by Hablani and Shrivastava[7] (see also comments by Alfriend[8]), and the stability diagram for the Mathieu equation which governs the dynamics of a flexible body has been investigated by Kumar and Bainum[9] (see also comments by MacNeal[10]).

The dynamics of PACSAT in orbit involve the motion of the body arising from at least the following causes:

1. Earth's central gravitational field
2. Elastic forces of tension and bending
3. Effects of initial conditions
4. Ellipticity of orbit
5. Earth oblateness effects
6. Solar and lunar gravity
7. Solar radiation pressure
8. Micrometeoroid impacts
9. Thermal bending

The first two of these will be regarded as fundamental, and all of the rest will be treated as perturbations. It will be shown that the most important effects which cause libration about the vertical are initial conditions, ellipticity, and oblateness, while the most important causes of departure from straightness are initial conditions, micrometeoroid impacts, and thermal bending. In particular, thermal bending effects are so serious that it is unlikely that the present design of PACSAT can even approach the straightness requirements.

A partial study of PACSAT dynamics has been conducted by Burke[11], who considered the fundamental mode problem, but did not treat the various forces. Some of Burke's ideas are included in this report.

The several topics are investigated in the following sections. The dynamic problem is formulated in general in Section II. Several mutually consistent assumptions are employed to simplify the equations. The fundamental modes of the simplest problem--circular orbit and only gravity gradient and elastic forces--are found in Section III. This proves to be a mathematical problem involving singular perturbations and rather delicate analysis. All modes have real frequencies, so the unperturbed problem is stable for small motions (an assumption used in the analysis). The effects of initial conditions are evaluated at this point.

Ellipticity is treated in Section IV. It is shown that small ellipticity drives the in-plane motion to produce a libration at the orbital frequency with amplitude equal to the eccentricity. The frequencies of the librational modes are changed slightly. The out-of-plane equation is proved to be unstable, but the time for instability effects to appear is over 4,500 years. There is no significant effect on vibration.

Earth oblateness effects are studied in Section V. Coupling between equations excites a librational resonance in the out-of-plane motion for near-circular, near-equatorial orbits, but the amplitude is small, being proportional to the product of eccentricity and inclination. In Section VI, solar and lunar gravity and solar radiation pressure are investigated, and all prove to have small effects.

Micrometeoroid impacts are studied in Section VII. Assuming the particle is small enough that it does not destroy the integrity of PACSAT, the effect of micrometeoroid impact is closely related to the effect of initial conditions, since the impact is equivalent to a point discontinuity in velocity. It is shown that the effect on straightness may be quite significant, since a meteoric particle with a few micrograms mass can produce vibratory motion comparable to the requirements, and such particles may be encountered as often as once a month.

Thermal bending effects are considered in Section VIII. The solar heat input causes differential expansion, which makes PACSAT curve away from the sun. Eclipsing by the earth, or shadowing within PACSAT, causes the solar input to vary in time to excite the lowest vibrational mode of

PACSAT as a resonance, and the resulting displacement is major. Unless the absorptivity of the material can be reduced below .0007, a very difficult task, the straightness requirements cannot be met.

Finally, conclusions appear in Section IX.

II. FORMULATION OF THE GENERAL PROBLEM

The geometrical configuration is shown in Fig. 1. PACSAT is in a near-circular, near-equatorial orbit. It is nominally vertical and straight, but is actually somewhat tilted and disturbed into a sinuous shape. The figure is not to scale, since PACSAT is short compared to the orbit radius and the distortions are small. The vector $\vec{r}$ is the radius vector from the center of the earth to a point P on PACSAT, and the directed length s is the distance from the center of mass C to the point on the undistorted axis of PACSAT corresponding to P. With L the length of PACSAT (1,500 m), the range of s is from -L/2 to L/2.

A series of assumptions is made to permit analytic treatment of the problem:

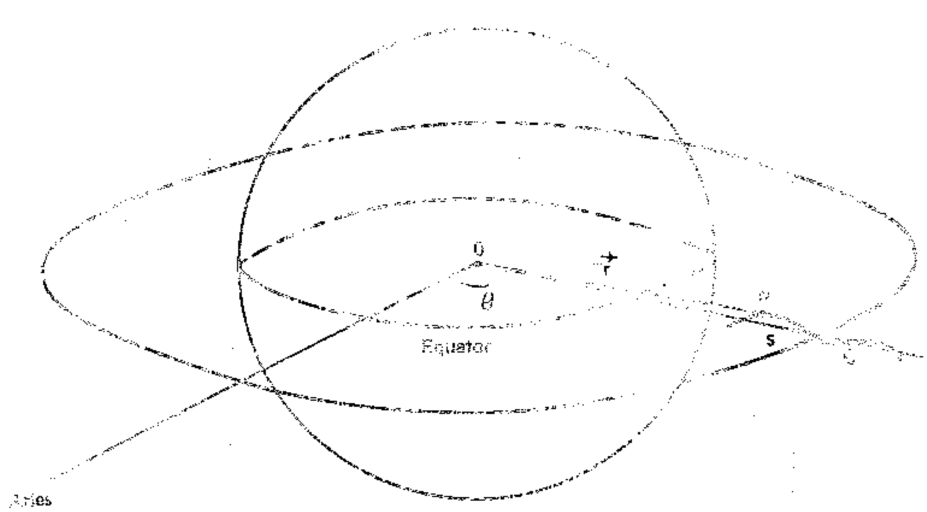

Fig. 1 — PACSAT in circular equatorial orbit

1. Since the ratio of the length of PACSAT (1,500 m) to the radius of the orbit (~13,000 km) is on the order of 10^{-4}, all expressions will be expanded in powers of this ratio, and only first-order terms kept.
2. All slopes and curvatures are small, so in forming the Lagrangian, only second-order terms will be kept. This is consistent with the requirement of small displacements established in Section I, and leads to linear differential equations.
3. All forces other than gravity and elastic forces may be treated as perturbations. This assumption will be tested as each force is considered.
4. The various effects will be considered mutually independent, unless they can interact to produce resonances.
5. The PACSAT structure is inextensible. This is consistent with the properties of steel. A consequence of this assumption is that there are no longitudinal vibrations.
6. The PACSAT rod structure is very long compared with its diameter. Hence, the axial moments of inertia may be neglected. Further, there will be no torsional vibrations.
7. The PACSAT structure is uniform, and the motion is continuous.
8. There is no damping present. Although PACSAT contains "sticky hinges," the gravity-gradient tension will cause them to lock. It is easily shown that the internal friction in steel is not sufficient to provide appreciable damping.

Under these assumptions, the equations of the system may be formulated and simplified. The total motion may be separated into three parts, the motion of the center of mass, a rigid-body libration about the center of mass, and vibrations transverse with respect to the libration. The formulation will use the Lagrangian technique. Although other methods could have been employed, the Lagrangian is the most convenient when thermal bending is treated.

The structure of PACSAT in the undistorted state is depicted in Fig. 2. The central steel rod, with a diameter of .132 cm, carries 80,001 spheres. These are spaced 1.875 cm between centers (1/2 wavelength at 3.75 cm), and have radius .595 cm ($3.75/2\pi$), the radius at which the radar scattering of the sphere is a maximum. The spheres may be solid aluminum, through-drilled, or they may be plastic, to save weight, coated with aluminum to provide radar reflectivity.

Following Burke[11], we assume that the massive solid spheres prevent bending of the rod within the sphere. Hence, the effective length of the rod bearing solid spheres is taken to be $.685 \times 1{,}500/1.875 = 548$ m. The plastic spheres are sufficiently light that they do not inhibit the bending, and the effective length is the full 1,500 m. The effective length will be further modified when thermal bending is considered. This use of effective length permits us to treat the motion as continuous, as per assumption 7, instead of inserting immobile sections and having to investigate a very complicated boundary value problem.

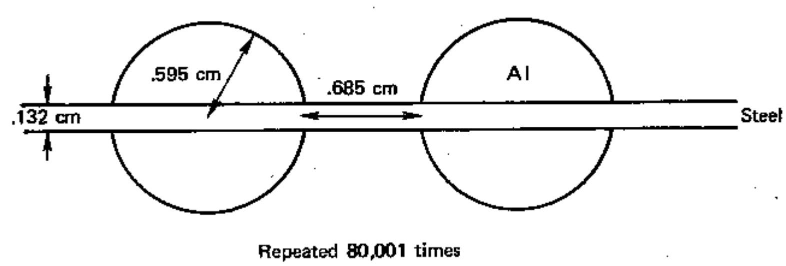

Fig. 2 — PACSAT structure

The differential equations and boundary conditions which govern the motion will be derived from the basic Hamilton's principle[12],

$$\delta\int_{t_1}^{t_2} dt[T - V] = 0 \qquad 2.1$$

where δ denotes the operation of variation, T is the kinetic energy, and V is the potential and strain energy. The energies must be expressed in terms of the coordinates. As shown in Fig. 1, the position of a point P is described by a vector $\vec{r}$ which determines the distance of P from the center of the earth, and by a distance s which identifies the point P. Under the assumption that the rod is long compared with its radius, the internal distortions of the cross section of the rod will be neglected, and it will be assumed that the position of each point on the cross section is characterized by the distance s. This also corresponds to the assumption of small displacements, slopes, and curvatures. Under these assumptions, the motion of the rod may be described by writing the position vector $\vec{r}$ as a function of s and t, the time, and ignoring the internal coordinates.

Let ρ denote the mass density per unit length of the structure. For the steel rod with aluminum spheres, the total mass per 1.875 cm section is the mass of the rod part plus the mass of the sphere. The volume density of steel is 7.86 g/cm^3, that of aluminum is 2.69.[1] The mass of the steel rod, radius .066 cm, length 1.875 cm, is .202 g. The volume of the entire sphere (radius .595 cm) is .882 cm^3, that of the section of rod within the sphere is .0163 cm, and hence the mass of the drilled aluminum sphere is 2.330 g, the total mass per 1.875 cm section is 2.531 g, and the linear mass density ρ is 1.35 g/cm. For the plastic spheres loaded rod, we assume that the spheres are sufficiently light that

[1]These and all other numerical values pertaining to materials are taken from International Critical Tables, McGraw Hill Book Co., New York, 1927.

the mass density is that of the rod alone, which from the above figures is .107 g/cm. These values were first derived by Burke[11].

From here on, PACSAT will be referred to as "the rod," except where the spheres are to be specifically discussed. The total kinetic energy of the rod is

$$T = \frac{1}{2}\rho\int_{-\frac{L}{2}}^{\frac{L}{2}} ds\left(\frac{\partial\vec{r}}{\partial t}\right)^2 . \tag{2.2}$$

All integrations over time in the ensuing derivation have the limits t_1, t_2; all integrations over s have the limits -L/2, L/2. The limits will be omitted and implicitly understood. According to the principle of variation as discussed in Ref. 12, the time integral of the Lagrangian is stationary with respect to variations of the coordinates about the "trajectory" which connects the state of the system at the initial time t_1 to the state at the final time t_2, provided these variations vanish at the endpoints. Then the variation of the kinetic action is

$$\delta T = \delta\int dt\int ds\frac{1}{2}\rho\left(\frac{\partial\vec{r}}{\partial t}\right)^2 = \rho\int dt\int ds\frac{\partial\vec{r}}{\partial t}\cdot\delta\frac{\partial\vec{r}}{\partial t} . \tag{2.3}$$

The definition of variations permits the commutation of the operations of variation and time differentiation. Integrating by parts,

$$\delta T = \rho\int ds\left[\frac{\partial\vec{r}}{\partial t}\cdot\delta\vec{r}\Big|_{t_1}^{t_2} - \int dt\frac{\partial^2\vec{r}}{\partial t^2}\cdot\delta\vec{r}\right] . \tag{2.4}$$

The integrated term vanishes, since the variations must vanish at the endpoints. The variation of the kinetic action is thus

$$\delta T = -\rho \int dt \int ds \frac{\partial^2 \vec{r}}{\partial t^2} \cdot \delta \vec{r} \qquad 2.5$$

and is immediately recognizable as the acceleration term.

The potential energy is composed of the gravitational potential and the work done by the external forces. In most cases the external forces are derivable from a potential, but the more general case can be treated just as easily. Let $U(\vec{r})$ be the gravitational potential per unit length:

$$U(\vec{r}) = -\rho\mu/|\vec{r}| \qquad 2.6$$

where μ is the product of Newton's constant and the mass of the earth, and let $\vec{F}(\vec{r},t)$ denote the external forces per unit length. The potential energy associated with these external forces is

$$W(\vec{r},t) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,',t) \cdot (\vec{r}\,' - \vec{r}_0) d\vec{r}\,' \qquad 2.7$$

where $\vec{r}_0$ is an initial state of the system. The variation of the gravitational action is

$$\delta U = -\delta \int dt \int ds \rho\mu/|\vec{r}(s,t)| = +\int dt \int ds \rho\mu(\vec{r} \cdot \delta\vec{r})/|r|^3 \qquad 2.8$$

and the variation of the external potential action is

$$\delta W = -\int dt \int ds \vec{F}(\vec{r},t) \cdot \delta\vec{r} \; . \qquad 2.9$$

The strain energy is found by the methods of the theory of elasticity. The strain in a rod element may be resolved into a tension along the rod and a bending couple around the rod axis. For the tension, see Fig. 3a. The action of the tension is to stretch the rod element, originally of length ds, into a section of length $d\ell$. The work done is the force times the elongation. The change in rod length is approximately $1/2(\partial\vec{r}/\partial s)^2 ds$,[2] and the strain energy of the entire rod due to tension is

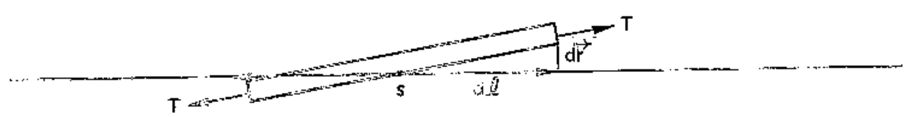

Fig. 3a — Tension in a rod element

[2]Here and hereafter the square of a vector quantity means the square of the magnitude of the vector.

$$V_T = \frac{1}{2}\int dsT(s,t)\left(\frac{\partial\vec{r}}{\partial s}\right)^2 . \qquad 2.10$$

Varying the time integral of this quantity,

$$\delta\int dtV_T = \int dt\int dsT(s,t)\frac{\partial\vec{r}}{\partial s} \cdot \delta\frac{\partial\vec{r}}{\partial s} . \qquad 2.11$$

The variation operation and length differentiation can be commuted. Integrating by parts,

$$\delta\int dtV_T = \int dt\left[T(s,t)\frac{\partial\vec{r}}{\partial s} \cdot \delta\vec{r}\Big|_{-\frac{L}{2}}^{\frac{L}{2}} - \int ds\frac{\partial}{\partial s}\left(T(s,t)\frac{\partial\vec{r}}{\partial s}\right) \cdot \delta\vec{r}\right] . \qquad 2.12$$

Since the variations need not be constrained at the ends, this imposes the boundary condition:

$$T(s,t) = 0 \qquad s = \pm L/2 . \qquad 2.13$$

This condition, that the rod cannot sustain tension at its ends, is natural to the physics of the problem.

For the bending, see Fig. 3b. Suppose that the initially undistorted rod is bent corresponding to a radius of curvature R_K. Consider the extension of a filament of the rod at a distance y from the center of the section. The length of the filament is altered by the

Fig. 3b — Bending of a rod element

bending in the ratio $1/(1 + y/R_K)$. Thus, on the side of the axis for which y is positive, the outward side, the filament is extended, while on the inward side it is compressed. By the definition[13] of Young's modulus, E, the force necessary to produce the extension y/R over the element of area dA is $EdAy/R_K$. The couple by which the bending is resisted amounts to

$$E\int dAy \cdot y/R_K = EI/R_K \qquad 2.14$$

where I is the moment of inertia of the area (not the mass) around the bending axis. The angle of bending corresponding to a length of axis ds is ds/R_K. Since the mean value is half the final value of the couple, the work required to bend ds to curvature $1/R_K$ is

$$\frac{1}{2}EIds/R_K^2 \ . \tag{2.15}$$

This argument is substantially due to Lord Rayleigh[14].

The radius of curvature is the reciprocal of the curvature itself. By the assumption of small slopes, the curvature can be replaced by $\partial^2\vec{r}/\partial s^2$. Hence, the strain energy of the rod due to bending is

$$V_B = \frac{1}{2}EI\int ds\left(\frac{\partial^2\vec{r}}{\partial s^2}\right)^2 \ . \tag{2.16}$$

Forming the variation of the time integral of V_B,

$$\delta\int dtV_B = EI\int dt\int ds\left(\frac{\partial^2\vec{r}}{\partial s^2} \cdot \delta\frac{\partial^2\vec{r}}{\partial s^2}\right) . \tag{2.17}$$

Again the variation and s derivative can be commuted. Integrating by parts twice,

$$\delta\int dtV_B = EI\int dt\left[\left(\frac{\partial^2\vec{r}}{\partial s^2} \cdot \frac{\partial}{\partial s}\delta\vec{r} - \frac{\partial^3\vec{r}}{\partial s^3} \cdot \delta\vec{r}\right)\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}} + \int ds\frac{\partial^4\vec{r}}{\partial s^4} \cdot \delta\vec{r}\right] . \tag{2.18}$$

Since the variation $\delta\vec{r}$ and its s derivative are independent arbitrary quantities, this yields two further boundary conditions. Finally, collect all the terms proportional to $\delta\vec{r}$ under the integral sign in the expressions 2.5, 2.8, 2.9, 2.12, and 2.18. The arbitrariness of $\delta\vec{r}$ along the rod requires that its coefficient vanish, yielding the governing differential equation. Thus, the problem to be solved is formulated as the differential equation:

$$\rho\frac{\partial^2\vec{r}}{\partial t^2} = \frac{-\rho\mu\vec{r}}{r^3} + \frac{\partial}{\partial s}\left(T(s,t)\frac{\partial\vec{r}}{\partial s}\right) - EI\frac{\partial^4\vec{r}}{\partial s^4} + \vec{F}(\vec{r},t) \; . \qquad 2.19$$

Inertia **Gravity** **Tension** **Bending** **External**

The boundary conditions are

At $s = \pm L/2$ $\quad T(s,t) = 0$ 2.20a

$\partial^2\vec{r}/\partial s^2 = 0$ 2.20b

$\partial^3\vec{r}/\partial s^3 = 0$. 2.20c

These mean that at the ends the rod cannot sustain tension, bending, or shearing, all of which are clearly required conditions.

In deriving 2.19 and 2.20, we have assumed that the kinetic energy is unaffected by the bending. Actually, there is kinetic energy associated with the rotatory motion of the rod, the so-called "rotary inertia" term[14], but this contributes a term to the differential equation which is of the order of magnitude of the square of the ratio of the rod radius to its length, relative to the potential energy term derived from 2.18, and this can safely be ignored. This neglect of the "rotary inertia" is customary in vibration problems.

The condition of inextensibility leads to a restriction on the possible motions. Consider two points on the rod, which in undistorted motion are at positions s_1, s_2, separated by a distance ds. As the rod bends, they will be at positions $\vec{r}(s_1,t)$, $\vec{r}(s_2,t)$. The distance between the points must remain unchanged, so,

$$|\vec{r}(s_1 + ds,t) - \vec{r}(s_1,t)| = ds \ . \qquad 2.21$$

Dividing by ds and taking the limit as ds approaches zero yields

$$\left|\frac{\partial\vec{r}(s,t)}{\partial s}\right| = 1 \ . \qquad 2.22$$

This relation will be used for simplification of the equations.

We first show that equations 2.19 and 2.20 imply the conservation of angular momentum. This is in contradistinction to the analysis of Burke[11], who imposes angular momentum conservation as an additional constraint and deduces that the librational motion cannot exist, a result clearly in disagreement with observation. To show conservation of angular momentum, take the vector cross product of the vector $\vec{r}$ and the vector differential equation 2.19, then integrate over the length of the rod. For the left side, we have (limits of integration -L/2, L/2 implied)

$$\rho\int ds\vec{r} \times \frac{\partial^2\vec{r}}{\partial t^2} = \frac{\partial}{\partial t}\rho\int ds\vec{r} \times \frac{\partial\vec{r}}{\partial t} \ . \qquad 2.23$$

This expression is immediately recognized as the time derivative of the total angular momentum of the rod. The first term on the right is zero ($\vec{r} \times \vec{r} = 0$). For the second term, integrate by parts

$$\int ds\vec{r} \times \frac{\partial}{\partial s}T(s,t)\frac{\partial\vec{r}}{\partial s} = \vec{r} \times T(s,t)\frac{\partial\vec{r}}{\partial s}\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}} - \int dsT(s,t)\frac{\partial\vec{r}}{\partial s} \times \frac{\partial\vec{r}}{\partial s} \ . \qquad 2.24$$

The first term on the right side of 2.24 vanishes by the boundary condition 2.20a, the second from the cross product of identical vectors. For the third term on the right, integrate by parts twice:

$$\int ds\vec{r} \times \frac{\partial^4\vec{r}}{\partial s^4} = \left(\vec{r} \times \frac{\partial^3\vec{r}}{\partial s^3} - \frac{\partial\vec{r}}{\partial s} \times \frac{\partial^2\vec{r}}{\partial s^2}\right)\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}} + \int ds\frac{\partial^2\vec{r}}{\partial s^2} \times \frac{\partial^2\vec{r}}{\partial s^2} \; . \qquad 2.25$$

The integrated terms vanish by the boundary conditions 2.20b and 2.20c, the integral from the cross product of identical vectors. Hence, the result of integration is

$$\frac{\partial}{\partial t}\rho\int ds\vec{r} \times \frac{\partial\vec{r}}{\partial t} = \int ds\vec{r} \times \vec{F}(\vec{r},t) \; . \qquad 2.26$$

Equation 2.26 asserts that the rate of change of angular momentum equals the torque applied by the external forces. If there are no external forces, the angular momentum is conserved. Hence, no additional constraints need be applied to 2.19 and 2.20, and Burke's result[11] is erroneous.

The differential equation 2.19 and boundary conditions 2.20 describe the motion of a point on the rod, relative to the center of the earth. However, PACSAT is known to be in orbit, executing small motions around the local vertical. Hence, we separate the motion into the orbital, librational, and vibrational motions, and treat the several motions separately. To this purpose, write

$$\vec{r}(s,t) = \underset{\substack{\text{Center}\\ \text{of mass}}}{\vec{R}(t)} + \underset{\text{Libration}}{s\vec{u}(t)} + \underset{\text{Vibration}}{\vec{w}(s,t)} = \underset{\substack{\text{Center}\\ \text{of mass}}}{\vec{R}(t)} + \underset{\text{Displacement}}{\vec{v}(s,t)} \; . \qquad 2.27$$

The vector $\vec{R}(t)$ represents the motion of the center of mass of PACSAT. It is of the order of magnitude of the distance of PACSAT from the center of the earth ($1.2 - 1.5 \times 10^4$ km). The expression $s\vec{u}(t)$ is the rigid-body libration about the vertical (the displacement is proportional to the distance from the center of mass), and is of the order of the length of PACSAT (1.5 km). The vector $\vec{w}(s,t)$ measures the distortion of PACSAT from straightness. This distortion executes free and forced vibrations about the librating rigid body. It is of the order of magnitude of the permissible displacement (.5 cm). Hence, the terms of 2.27 are such that expansion of the differential equation 2.19, boundary conditions 2.20, and subsidiary condition 2.22 in ratios of the libration to center of mass, or vibration to libration, are all rapidly convergent, and only first-order terms need be kept. We further assume that the scale length of variation of the perturbing force $\vec{F}$ is at least on the order of magnitude of $\vec{R}$. This is satisfied for the oblateness (order R), lunar gravity (order distance earth to moon = 4×10^5 km), and solar gravity and radiation pressure (order distance earth to sun = 1.5×10^8 km), all the forces we will consider.

The tension $T(s,t)$ may also be expanded. As is well known[9], a rigid, vertical, straight rod in circular orbit experiences a tension which is parabolic along its length. Hence, we take for the tension the form:

$$T(s,t) = T_0(t)\left(1 - 4\frac{s^2}{L^2}\right) + T_1(s,t) \qquad 2.28$$

where the first term on the right represents the tension associated with the center-of-mass and librational motion, displaying a parabolic variation which vanishes at the ends (2.20a); the second term is the additional tension associated with vibration.

The gravity term expands as follows:

$$\frac{\vec{r}}{r^3} = \frac{\vec{R} + \vec{v}}{(R^2 + 2\vec{R} \cdot \vec{v} + v^2)^{3/2}} \cong \frac{(\vec{R} + \vec{v})}{R^3}\left(1 - 3\frac{\vec{R} \cdot \vec{v}}{R^2}\right) \qquad 2.29a$$

$$\rightarrow \frac{\vec{R}}{R^3} + \frac{1}{R^3}\left(\vec{v} - 3\frac{\vec{R} \cdot \vec{v}}{R^2}\vec{R}\right) \qquad 2.29b$$

where the neglected terms are on the order of 10^{-8} relative to the leading term. The external force expands in the form:

$$\vec{F}(\vec{r},t) \cong \vec{F}(\vec{R},t) + [\nabla\vec{F}(\vec{R},t)] \cdot \vec{v} \qquad 2.30$$

where the expression $\nabla\vec{F}(\vec{R},t)$ denotes the 3 × 3 dyadic which in rectangular coordinates has the representation:

$$\nabla\vec{F}(\vec{R},t) = \begin{Bmatrix} \frac{\partial F_x}{\partial X} & \frac{\partial F_x}{\partial Y} & \frac{\partial F_x}{\partial Z} \\ \frac{\partial F_y}{\partial X} & \frac{\partial F_y}{\partial Y} & \frac{\partial F_y}{\partial Z} \\ \frac{\partial F_z}{\partial X} & \frac{\partial F_z}{\partial Y} & \frac{\partial F_z}{\partial Z} \end{Bmatrix} \qquad 2.31$$

and which transforms under coordinate transformations like a tensor of second rank. All components of 2.31 are evaluated at the point R. All other terms in 2.19 and 2.20 are linear and have obvious expansions. We obtain for the center of mass motion:

$$\ddot{\vec{R}} = -\frac{\mu\vec{R}}{R^3} + \frac{1}{\rho}\vec{F}(\vec{R},t) \quad , \tag{2.32}$$

for the libration:

$$\ddot{\vec{u}} = -\frac{\mu}{R^3}\left(\vec{u} - 3\frac{\vec{R}\cdot\vec{u}}{R^2}\vec{R}\right) - \frac{8T_0(t)}{\rho L^2}\vec{u} + \frac{1}{\rho}(\nabla F)\cdot\vec{u} \quad , \tag{2.33}$$

and for the vibrational motion:

$$\ddot{\vec{w}} = -\frac{\mu}{R^3}\left(\vec{w} - 3\frac{\vec{R}\cdot\vec{w}}{R^2}\vec{R}\right) + \frac{T_0(t)}{\rho}\frac{\partial}{\partial s}\left[\left(1 - \frac{4s^2}{L^2}\right)\frac{\partial\vec{w}}{\partial s}\right] + \frac{1}{\rho}\left(\frac{\partial}{\partial s}T_1(s,t)\right)\vec{u}$$
$$- \frac{EI}{\rho}\frac{\partial^4\vec{w}}{\partial s^4} + \frac{1}{\rho}(\nabla\vec{F})\cdot\vec{w} \; . \tag{2.34}$$

The boundary conditions become

$$\text{At } s = \pm\frac{L}{2} \qquad \frac{\partial^2\vec{w}}{\partial s^2} = \frac{\partial^3\vec{w}}{\partial s^3} = T_1(s,t) = 0 \tag{2.35}$$

The expansion of the condition 2.2 yields

$$|\vec{u}(t)|^2 = 1 \qquad 2.36a$$

$$\vec{u} \cdot \vec{w} = 0 \qquad 2.36b$$

$$|w|^2 \text{ negligible .} \qquad 2.36c$$

The condition 2.36a, that the vector $\vec{u}$ have unit magnitude, demonstrates the rigid-body character of the libration. The condition 2.36b shows that the vibrational motion is always transverse to the libration, and 2.36c demonstrates that higher order terms in the vibration are to be neglected. Thus, we have the motion of the center of mass under central gravity and external forces, as described by 2.32, the rigid libration about the center of mass under gravity gradient, tension, and external forces, described by 2.33 and 2.36, and the linearized transverse vibration about the libration under gravity gradient, tension, elastic bending, and external forces, with no bending or shearing at the ends, described by 2.34, 2.35, and 2.36b. These equations, under the appropriate restrictions, will be considered in the subsequent sections. The equations will be modified when thermal bending is considered.

The assumption of linearity may be relaxed if we are willing to accept a complicated computer program and implement numerical methods for treating the nonlinear equations. A recent analysis by Kane and Levinson[15, 16] describes such a treatment, but the limitations on the motion required to meet the requirements on radar cross section make such complications unnecessary.

III. THE BASIC PROBLEM

In this section, the simplest problem will be considered. The orbit of PACSAT will be taken as circular and equatorial. All perturbing forces will be neglected. We shall find the modes of libration and vibration of PACSAT under these circumstances, and determine the effects of initial conditions.

The motion is conveniently described in cylindrical coordinates. Referring to Fig. 1, the distance R to the center of mass of the satellite will be constant. The angle θ, which locates the position of the center of mass in the orbital plane, measured from a reference direction (the first point of Aries is convenient), will increase linearly with time at the orbital rate $\dot{\theta}$, which is also constant. The displacements $\vec{u}$ and $\vec{w}$ will be expressed in terms of $\vec{i}_R$, $\vec{i}_\theta$, and $\vec{i}_z$, the unit vectors in the radial, velocity, and normal to the orbit plane directions. If we also use rectangular coordinates, with the z axis in the direction of the North Pole and the x direction toward Aries, we have the time dependences

$$\vec{i}_R = \vec{i}_x \cos\dot{\theta}t + \vec{i}_y \sin\dot{\theta}t \qquad 3.1a$$

$$\vec{i}_\theta = -\vec{i}_x \sin\dot{\theta}t + \vec{i}_y \cos\dot{\theta}t \ . \qquad 3.1b$$

The vector $\vec{R}$ of equation 2.32 is simply $R\vec{i}_R$. The vectors $\vec{u}$ and $\vec{w}$ are expressed as

$$\vec{u} = u_1\vec{i}_R + u_2\vec{i}_\theta + u_3\vec{i}_z \qquad 3.2a$$

$$\vec{w} = w_1\vec{i}_R + w_2\vec{i}_\theta + w_3\vec{i}_z \ . \qquad 3.2b$$

The conditions 2.36 yield

$$u_1^2 + u_2^2 + u_3^2 = 1 \tag{3.3a}$$

$$u_1 w_1 + u_2 w_2 + u_3 w_3 = 0 \ . \tag{3.3b}$$

If the rod were vertical and straight, we would have $u_1 = 1$, all other terms zero. The condition 3.3a then permits the representation

$$u_1 = \cos\chi\cos\psi \tag{3.4a}$$

$$u_2 = \cos\chi\sin\psi \tag{3.4b}$$

$$u_3 = \sin\chi \ . \tag{3.4c}$$

The angle ψ is the in-plane angular displacement from vertical, the angle χ is the out-of-plane angular displacement from vertical. These will be treated as first-order quantities. The displacements w_2 and w_3 are the transverse displacements from straightness, also to be treated as first order. Equation 3.3b shows that the radial displacement w_1 is a second-order quantity.

The angular velocity $\dot{\theta}$ and orbit radius R satisfy the relation

$$R^3\dot{\theta}^2 = \mu = 3.986 \times 10^5 \ \mathrm{km}^3/\mathrm{sec}^2 \ . \tag{3.5}$$

A corresponding relation (4.1e) holds in elliptic orbits. For PACSAT at an altitude of 1 earth radius (6,378 km), the orbital period is 3 hours, 59 minutes; at 1.5 earth radius, the period is 5 hours, 34 minutes.

The velocities and accelerations must be represented in the cylindrical coordinates. Differentiating 3.1 yields

$$\dot{\vec{i}}_R = \dot{\theta}\vec{i}_\theta \tag{3.6a}$$

$$\dot{\vec{i}}_\theta = -\dot{\theta}\vec{i}_R \tag{3.6b}$$

$$\ddot{\vec{i}}_R = -\dot{\theta}^2\vec{i}_R \tag{3.6c}$$

$$\ddot{\vec{i}}_\theta = -\dot{\theta}^2\vec{i}_\theta \ . \tag{3.6d}$$

The acceleration $\ddot{\vec{u}}$ is given by

$$\ddot{\vec{u}} = \ddot{u}_1\vec{i}_R + 2\dot{u}_1\dot{\vec{i}}_R + u_1\ddot{\vec{i}}_R + \ldots \tag{3.7a}$$

$$= (\ddot{u}_1 - 2\dot{\theta}\dot{u}_2 - \dot{\theta}_2 u_1)\vec{i}_R + (\ddot{u}_2 + 2\dot{\theta}\dot{u}_1 - \dot{\theta}^2 u_2)\vec{i}_\theta + \ddot{u}_3\vec{i}_z \ . \tag{3.7b}$$

The gravity-gradient term on the right side of 2.33 becomes

$$-\dot{\theta}^2(-2u_1\vec{i}_R + u_2\vec{i}_\theta + u_3\vec{i}_z) \ . \tag{3.8}$$

There are corresponding terms for $\vec{w}$.

We first consider the libration. With the perturbing force omitted and some simplification, 2.33 becomes

$$\ddot{u}_1 - 3\dot{\theta}^2 u_1 - 2\dot{\theta}\dot{u}_2 = -\frac{8T_0}{\rho L^2}u_1 \tag{3.9a}$$

$$\ddot{u}_2 + 2\dot{\theta}\dot{u}_1 = -\frac{8T_0}{\rho L^2}u_2 \qquad 3.9b$$

$$\ddot{u}_3 + \dot{\theta}^2 u_3 = -\frac{8T_0}{\rho L^2}u_3 \ . \qquad 3.9c$$

Multiply the first equation by u_1, the second by u_2, the third by u_3, and add. This provides an equation for the tension T_0. Equations 3.4 may be differentiated to express the derivatives of u_1, u_2, and u_3 in terms of ψ, χ, and their derivatives. It is convenient to introduce the angle θ as the independent variable. Derivatives with respect to θ will be denoted by a prime. Also, multiply 3.9a by u_2, 3.9b by u_1, and subtract. This will yield an equation which is appropriate for determining ψ. Finally, substitute the expression for the tension into the right side of 3.9c and simplify. This equation is suitable for χ. After the manipulations, there results:

$$T_0 = \frac{1}{8}\rho L^2\dot{\theta}^2[\chi'^2 + \cos^2\chi(1 + \psi')^2 + 3\cos^2\chi\cos^2\psi - 1] \qquad 3.10a$$

$$\psi'' + 3\cos\psi\sin\psi = 2\tan\chi(1 + \psi')\chi' \qquad 3.10b$$

$$\chi'' + [(1 + \psi')^2 + 3\cos^2\psi]\sin\chi\cos\chi = 0 \ . \qquad 3.10c$$

To this point, there have been no simplifications produced by the small values of ψ and χ. Now we wish to restrict the libration amplitude to 1.5 deg = .025 rad. Keeping terms of orders 0 and 1 in equation 3.10a and of orders 1 and 2 in equation 3.10b and c should be more than sufficient accuracy. Thus, we expand

$$\psi = \psi_1 + \psi_2 \qquad 3.11a$$

$$\chi = \chi_1 + \chi_2 \qquad 3.11b$$

and obtain the set of equations:

$$T_0 = \frac{3}{8}\rho L^2\dot{\theta}^2[1 + \frac{2}{3}\psi'] \qquad 3.12a$$

$$\psi_1'' + 3\psi_1 = 0 \qquad 3.12b$$

$$\chi_1'' + 4\chi_1 = 0 \qquad 3.12c$$

$$\psi_2'' + 3\psi_2 = 2\chi_1\chi_1' \qquad 3.12d$$

$$\chi_2'' + 4\chi_2 = -2\chi_1\psi_1' \quad . \qquad 3.12e$$

The first equation, 3.12a, represents the gravity-gradient tension, which has been derived many times previously[9, 10, and other papers], as modified by libration. For PACSAT at 1 earth radius, $\rho = 1.35$ g/cm, $L = 1{,}500$ m, $\dot{\theta} = 2\pi/14{,}340$ rad/sec, the tension is

$$T_0 = 2{,}187(1 + \frac{2}{3}\psi') \text{ dynes} \quad . \qquad 3.13$$

The full length is used, since the entire body partakes of the rigid motion. This value was also obtained by Burke[11].

The equations 3.12b and 3.12c solve immediately as

$$\psi_1 = \psi_0 \cos\sqrt{3}\theta + \frac{\psi_0'}{\sqrt{3}}\sin\sqrt{3}\theta \qquad 3.14a$$

$$\chi_1 = \chi_0 \cos 2\theta + \frac{1}{2}\chi_0' \sin 2\theta \qquad 3.14b$$

where the subscript 0 denotes initial value and θ is measured from the initial time. Thus, the first-order in-plane libration oscillates at a frequency equal to $\sqrt{3}$ times the orbital frequency, the first-order out-of-plane libration oscillates at twice the orbital frequency. These results also are well known[9, 10]. If we assume the initial libration velocities are zero, which should be at least approximately satisfied for any reasonable deployment scheme, the sin terms may be dropped.

The variation of the tension, as shown by equations 3.12a or 3.13, has a maximum equal to $2\psi_0/\sqrt{3}$, if the initial displacement is entirely in the in-plane direction. For an initial value 1.5^0, the second term in 3.12a or 3.13 is at most 2.9 percent, so the tension is substantially constant.

The equations 3.12d and 3.12e may be solved easily, and yield:

$$\psi_2 = \frac{2}{13}\chi_0^2[\sin 4\theta - \frac{4}{\sqrt{3}}\sin\sqrt{3}\theta] \qquad 3.15a$$

$$\chi_2 = -\frac{1}{13}\psi_0\chi_0[(4 - \sqrt{3})\sin(2 + \sqrt{3})\theta + (4 + \sqrt{3})\sin(2 - \sqrt{3})\theta - 5\sin 2\theta] \qquad 3.15b$$

where the second-order terms have been fitted to zero initial conditions. If the motion is limited to 1.5 deg, the maximum values of these expressions are respectively .00032 (χ_0 = .0262), and .00033 ($\psi_0 = \chi_0$ = .0185). Thus, the second-order corrections to the libration are on the order of 1 percent and may be neglected.

The first-order libration, 3.14, represents a complicated time dependence for the orientation of the libration. The motions in the two planes are at different frequencies, so the attitude orientation of the body follows a Lissajous figure. If we assume the initial displacement is of unit amplitude and displaced 45 deg out of plane, the resulting motion for the first four orbits is as plotted in Fig. 4. The curve may be thought of as the motion of the tip, plotted in the local east (in-plane) and north (out-of-plane) coordinate system (a plane perpendicular

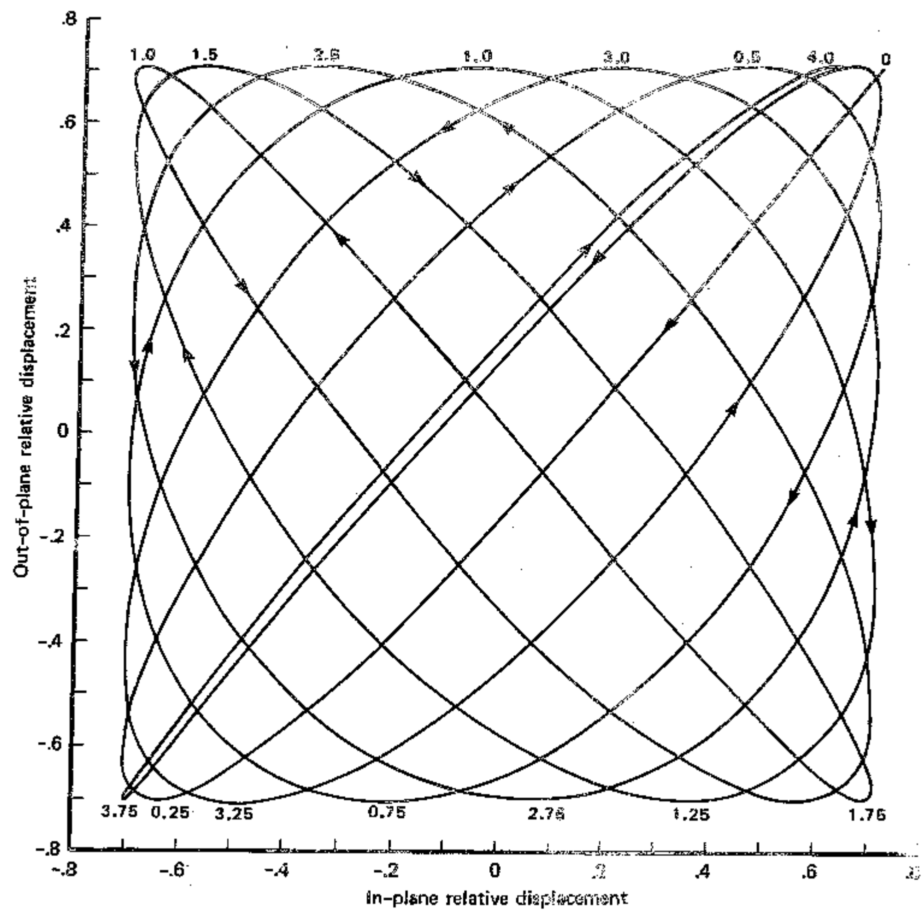

Fig. 4 — Libration Lissajous pattern

to the radius vector). The numbers at the top and bottom of the curve represent the time in fractions of an orbit. The motion begins at $t = 0$ in the upper right corner of Fig. 4, and is at the top each half and whole orbit, at the bottom each quarter orbit, as can be seen from 3.14b. Since the frequencies in the two planes are incommensurable, the curve will never repeat, and will eventually cover the entire square. However, it can come very close to repeating. Table 2 shows some near repetitions of the initial value. The first column is the orbit number, the second the elapsed time, and the third the leftward displacement from the upper right corner in units of the initial displacement $(\psi_0^2 + \chi_0^2) = 1$.

The closeness to repetition at such relatively low values of orbit number is remarkable. The 34.69 day value corresponds to the approximation

$$\sqrt{3} = 1.7320508 \sim 362/209 = 1.7320574 \quad . \qquad 3.16$$

Closeness to other corners of the square may be even better, but does not represent pattern repetition.

Table 2

LISSAJOUS REPETITIONS

Orbit Number	Elapsed Time (days)	Displacement
4	.664	.070747
11	1.83	.038208
15	2.49	.005159
41	6.80	.002766
56	9.29	.000371
153	25.39	.000199
209	34.69	.000027
571	94.77	.000014
780	129.46	.000002

It is shown in Ref. 3 that the mutual viewing time of PACSAT in equatorial orbit of an altitude of one earth radius is about 45 minutes for two observers at the NE and NW corners of the continental United States, the minimum time, and is about 110 minutes for adjacent observers at 30°N, the maximum time. These correspond to .188 and .460 part of an orbit. In the first case, about three-quarters of an arc of Fig. 4 will be covered, in the second case nearly two arcs will be traversed. Thus, practically the full range of libration orientations will be presented to the observers, and the attendant reductions in cross section will have to be overcome.

This completes our discussion of free libration, driven only by initial conditions. Before discussing the vibrational motion, we briefly digress to consider the accuracy of the assumption that the rod is inextensible. With the possible exception of forces induced by micrometeoroid impacts, which we will treat at the appropriate occasion, the gravity-gradient tension is the strongest force acting to produce extension of the rod.

The total elongation of the rod, obtained by integrating the tension over the length, then dividing by the product of the cross-sectional area and by Young's modulus (2×10^{12} dynes/cm^2 for steel), is given by

$$\Delta s = \frac{2}{3} \frac{T_o L}{AE} \tag{3.17}$$

For PACSAT, this has the value .03 cm. Considering the total manufacturing tolerances and possible deformations during deployment, this is certainly not a significant quantity. The stretching of a section of length 1.875 cm (sphere-to-sphere spacing) at the center is 6×10^{-7} cm, or 60 Angstrom units, far below any manufacturing tolerance. The assumption of inextensibility is certainly verified.

We now take up the subject of free vibratory modes. This involves solving equation 2.34 with the boundary condition 2.35 and subsidiary condition 2.36b. Only first-order quantities will be kept, which implies that the effect of the libration on the vibratory motion will be neglected. We have carried out a more detailed analysis, and shown that there are no first-order modifications to the frequencies, but there may be second-order effects. Since the libration is limited to .025, these are changes of less than 0.1 percent and will not be considered further.

When the representation 3.2b for $\vec{w}$ is substituted into equation 2.34, the only surviving first-order terms in the radial component determine the correction to the tension as

$$\frac{\partial T_1}{\partial s} = -2\rho\dot{\theta}^2 w_2' \quad . \tag{3.18}$$

Anticipating later results, we have the order-of-magnitude relation

$$T_1/T_0 \sim w/L = 5 \times 10^{-3}/1{,}500 \sim 3.3 \times 10^{-6} \tag{3.19}$$

so the change in tension is very small.

The in-plane and out-of-plane components of the vibratory motion yield the differential equations

$$\ddot{w}_2 = \frac{3}{8}L^2\dot{\theta}^2\frac{\partial}{\partial s}\left[\left(1 - 4\frac{s^2}{L^2}\right)\frac{\partial w_2}{\partial s}\right] - \frac{EI}{\rho}\,\frac{\partial^4 w_2}{\partial s^4} \tag{3.20a}$$

$$\ddot{w}_3 + \dot{\theta}^2 w_3 = \frac{3}{8}L^2\dot{\theta}^2\frac{\partial}{\partial s}\left[\left(1 - 4\frac{s^2}{L^2}\right)\frac{\partial w_3}{\partial s}\right] - \frac{EI}{\rho}\,\frac{\partial^4 w_3}{\partial s^4} \tag{3.20b}$$

with boundary conditions

$$\text{At } s = \pm\frac{L}{2} \qquad \frac{\partial^2 w_2}{\partial s^2} = \frac{\partial^3 w_2}{\partial s^3} = 0 \tag{3.21a}$$

$$\frac{\partial^2 w_3}{\partial s^2} = \frac{\partial^3 w_3}{\partial s^3} = 0 \ . \tag{3.21b}$$

We shall use θ as one independent variable, and introduce the normalized length x as the other where

$$s = \frac{1}{2}Lx \qquad -1 \le x \le 1 \ . \tag{3.22}$$

In addition, we define the elasticity parameter β as

$$\beta = 32EI/3\rho L^4\dot{\theta}^2 \ . \tag{3.23}$$

The parameter β measures the relative effects of the elasticity and gravity-gradient tension on the motion. For PACSAT, at one earth radius altitude, we have the values

$$E = 2 \times 10^{12} \text{ dynes/cm}^2 \tag{3.24a}$$

$$I = \frac{\pi}{4}(.066)^4 = 1.49 \times 10^{-5} \text{ cm}^4 \tag{3.24b}$$

$$\dot{\theta} = 2\pi/14{,}340 \text{ sec}^{-1} \ . \tag{3.24c}$$

For the two cases of bare rod and rod loaded with aluminum spheres, we have respectively $\rho = .107$ g/cm, $L = 1{,}500$ m, and $\rho = 1.35$ g/cm, $L = 548$ m. These yield

$$\beta = 3 \times 10^{-5} \qquad \text{bare rod} \tag{3.25a}$$

$$\beta = 1.36 \times 10^{-4} \quad \text{rod with spheres .} \tag{3.25b}$$

Thus, β is a small quantity.

We see that the differential equations and boundary conditions are uncoupled, so in-plane and out-of-plane motions vibrate independently. Suppose the characteristic frequencies of the in-plane (out-of-plane) motion are denoted by $\omega_{IP,k}(\omega_{OP,k})$. Then it is apparent from the left sides of 3.20 that

$$\omega_{OP,k}^2 = \omega_{IP,k}^2 + \dot{\theta}^2 \ . \tag{3.26}$$

Hence, only the in-plane motion need be treated in detail. The out-of-plane motion will be deduced by comparison of coefficients and the use of 3.26. We note that the libration frequencies ($\omega_{IP,0} = \sqrt{3}\dot{\theta}$, $\omega_{OP,0} = 2\dot{\theta}$) also satisfy 3.26, from which we are justified in calling the libration the lowest odd (in z) vibration mode.

With the new independent variables, we have

$$\frac{2}{3}w_2'' = \frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial w_2}{\partial x}\right] - \beta\frac{\partial^4 w_2}{\partial x^4} \tag{3.27a}$$

$$\text{At } x = \pm 1 \qquad \frac{\partial^2 w_2}{\partial x^2} = \frac{\partial^3 w_2}{\partial x^3} = 0 \ . \tag{3.27b}$$

These may be solved by separation of variables, thus

$$w_2 = \sum_1^\infty A_k(\theta)X_k(x) \tag{3.28}$$

which in turn yield the equations:

$$A_k'' + \omega_k^2 A_k = 0 \qquad 3.29$$

$$-\frac{2}{3}\omega_k^2 X_k = \frac{d}{dx}\left[(1 - x^2)\frac{dX_k}{dx}\right] - \beta\frac{d^4X_k}{dx^4} \qquad 3.30a$$

$$\text{At } x = \pm 1 \qquad \frac{d^2X_k}{dx^2} = \frac{d^3X_k}{dx^3} = 0 \ . \qquad 3.30b$$

The solution of 3.29 is

$$A_k = A_k(0)\cos\omega_k\theta + \frac{A_k'(0)}{\omega_k}\sin\omega_k\theta \ . \qquad 3.31$$

The principal task of this section is to investigate the eigenvalue problems described by 3.30.

With some changes of notation, this is the same as the eigenvalue problem studied by Burke[11]. He actually investigated a set of problems which approximate 3.30, but his treatment of the full problem did not consider the fact that, although the elastic term has a small coefficient, the problem is of the character of a singular perturbation, and limit procedures cannot be applied. Therefore, Burke's results, which are very limited because he considered only the lowest eigenvalue, must be regarded at best as suspect.

The singular nature of the problem arises from the presence of the small coefficient in the highest derivative term in 3.30a. Suppose β approaches zero. For $\beta = 0$, the differential equation 3.30a reduces to Legendre's equation. The eigenvalues are given by

$$\omega_k(0) = \left[\frac{3}{2}k(k + 1)\right]^{1/2}$$

and the eigenfunctions are the Legendre polynomials $P_k(x)$. But this solution does not satisfy the boundary conditions 3.30b. Hence, the limit of the solution of 3.30 as β approaches zero is not the solution for $\beta = 0$, displaying the singular character. We expect that any attempt to apply a perturbation theory corresponding to an expansion in powers of β is doomed to fail.

This failure of an expansion in powers of β was demonstrated by an abortive treatment. The eigenfunction X_k could be expanded as a series of Legendre polynomials, and it would be expected that the coefficient of $P_k(x)$ would be the largest. The fourth derivative of a Legendre polynomial P_k can be expressed as a sum of Legendre polynomials of orders up to $k - 4$. Hence, equation 3.30a can be written as a set of equations for the coefficients in the expansion, and the boundary conditions yield two further relations for these coefficients. The condition that this set of linear homogeneous equations possess a solution other than zero requires that the determinant of the set vanish, which provides an equation for the eigenvalue ω_k. The coefficients of the expansion then are found from the cofactors of the determinant.

This is a reasonable procedure, but it is found that the terms of the determinant increase rapidly away from the diagonal. Suppose the determinant is truncated at some value N, and we wish to find the eigenvalues to first order in β. This can indeed be done, with a result depending on N. The limit as N approaches infinity can be taken, and the eigenvalue does approach a limit. However, if we attempt to calculate the eigenfunctions, also to first order in β, the expansion coefficients are all of order β except those corresponding to $N - 1$ and N, which are of order unity. Hence, the representation clearly has no well-defined convergence process, since the coefficients may be made to change arbitrarily by changing the number of rows kept in the truncated determinant.

We have developed a convergent algorithm for computing the eigenvalues and eigenfunctions of the differential equation 3.30a with boundary conditions 3.30b. Before embarking on this analysis, we will prove that the eigenvalues of 3.30 are all real and positive. This demonstrates that the free motion is stable, that is, there are no growing oscillations. The proof is limited to small displacements, the range of validity of 3.30, but that is the only range of interest.

To prove the eigenvalues are real, suppose ω is a complex eigenvalue, with eigenfunction X. Let a bar above a symbol denote the complex conjugate. Then, since 3.30 is a linear system with real coefficients, other than ω^2, we see that $\bar{\omega}$ is also an eigenvalue, and $\bar{X}$ is its eigenfunction. Multiply the differential equation for X by $\bar{X}$, the differential equation for $\bar{X}$ by X, subtract, and integrate over x from -1 to 1. There results

$$-\frac{2}{3}(\omega^2 - \bar{\omega}^2)\int dx X\bar{X} = \int dx \begin{bmatrix} \bar{X}\left\{\frac{d}{dx}(1 - x^2)\frac{d}{dx}X - \beta\frac{d^4X}{dx^4}\right\} \\ \\ -X\left\{\frac{d}{dx}(1 - x^2)\frac{d}{dx}\bar{X} - \beta\frac{d^4\bar{X}}{dx^4}\right\} \end{bmatrix} \qquad 3.32$$

where the limits are implied. Integrate the right sides by parts. For the terms independent of β, a single integration by parts yields boundary terms which vanish because of $1 - x^2$, and integral terms which cancel. For the β terms, two integrations yield boundary terms which vanish by 3.30b and its conjugate, and also integral terms which cancel. Therefore, the right side of 3.32 vanishes. The integral on the left has a positive integrand and cannot vanish, so the only possibility is $\omega^2 = \bar{\omega}^2$, requiring the imaginary part to vanish and ω^2 be real.

To prove ω_k^2 is positive, multiply 3.30a by X_k and integrate from -1 to 1. Integrate by parts as above. The boundary terms will vanish, and there will result

$$\frac{2}{3}\omega_k^2\int dx X_k^2 = \int dx \left[(1 - x^2)\left(\frac{dX_k}{dx}\right)^2 + \beta\left(\frac{d^2X_k}{dx^2}\right)^2\right] . \qquad 3.33$$

Since all integrals are positive, ω_k^2 is positive. The same procedure can be used to prove the solutions are orthogonal, i.e.,

$$\int dx X_k X_\ell = 0 \qquad k \neq \ell \; . \tag{3.34}$$

We shall normalize the eigenfunction so that

$$\int dx X_k^2 = 1 \; . \tag{3.35}$$

These results are all derived without finding the explicit eigenvalues and eigenfunctions. To determine the solutions, we consider the differential equation and boundary conditions with the gravity-gradient term omitted. We call this the bar problem. This seems perverse, since we are dropping a term with unit coefficient and keeping a term with a small coefficient. However, this simplified problem contains all the singular properties of the general problem. The eigenfunctions of the bar problem form a basis for expansion of the complete problem, and, whereas we may expect slow convergence, the solution should actually converge, unlike the previous situation.

The bar problem has been considered by many writers[13, 14, and others]. We will separate the eigenfunctions into the two classes of even and odd functions. The eigenvalues of the bar problem will be denoted by ν_k, the eigenfunctions by $Y_k(x)$.

The bar problem is defined by the fourth-order differential equation and boundary conditions

$$\frac{d^4 Y_k}{dx^4} = \nu_k^4 Y_k \tag{3.36a}$$

$$\text{At } x = \pm 1 \qquad \frac{d^2Y_k}{dx^2} = \frac{d^3Y_k}{dx^3} = 0 \ . \tag{3.36b}$$

The solution of 3.36a for even eigenfunctions is

$$Y_k = B_k \cos\nu_k x + C_k \cosh\nu_k x \ . \tag{3.37}$$

To satisfy the boundary conditions, we require

$$-B_k \cos\nu_k + C_k \cosh\nu_k = 0 \tag{3.38a}$$

$$+B_k \sin\nu_k + C_k \sinh\nu_k = 0 \ . \tag{3.38b}$$

These equations must have a vanishing determinant, which yields the eigenvalue equation:

$$\tan\nu_k = -\tanh\nu_k \ . \tag{3.39}$$

For ν_k at all large, the right side tends rapidly to -1, so ν_k will tend to $(k - 1/4)\pi$. We have the explicit sequence

$$\nu_1/\pi = .7528094 \tag{3.40a}$$

$$\nu_2/\pi = 1.7500053 \tag{3.40b}$$

$$\nu_k/\pi = k - 1/4 \qquad \text{to 8 decimals} \qquad k > 2 \ . \tag{3.40c}$$

The eigenfunctions take the form

$$Y_k = N_k[\cosh\nu_k \cos\nu_k x + \cos\nu_k \cosh\nu_k x] \qquad 3.41$$

where N_k denotes a normalizing factor. The bar eigenvalues will also be normalized to unit mean square:

$$\int dx Y_k^2 = 1 \quad . \qquad 3.42$$

The integration is straightforward, and yields

$$N_k = 1/(\sqrt{2}\cosh\nu_k |\cos\nu_k|) \quad . \qquad 3.43$$

For the odd eigenvalues, we obtain the equation analogous to 3.39,

$$\tan\nu_k = \tanh\nu_k \qquad 3.44$$

with the solutions

$$\nu_1/\pi = 1.2498763 \qquad 3.45a$$

$$\nu_2/\pi = 2.2499998 \qquad 3.45b$$

$$\nu_k/\pi = k + 1/4 \qquad \text{to 9 decimals} \qquad k > 2 \ . \qquad 3.45c$$

The normalized odd eigenfunctions are

$$Y_k = N_k[\sinh\nu_k \sin\nu_k x + \sin\nu_k \sinh\nu_k x] \qquad 3.46a$$

$$N_k = 1/(\sqrt{2}\sinh\nu_k|\sin\nu_k|) \ . \qquad 3.46b$$

Let us expand the solutions of the full problem 3.30 in terms of the solutions of the bar problem, namely,

$$X_k(x) = \sum_1^\infty a_{k,\ell} Y_\ell(x) \ . \qquad 3.47$$

The boundary conditions are automatically satisfied. The differential equation becomes

$$\sum_1^\infty a_{k,\ell}\left(-\frac{2}{3}\omega_k^2 + \beta\nu_\ell^4\right)Y_\ell(x) = \sum_1^\infty a_{k,\ell}\frac{d}{dx}(1 - x^2)\frac{d}{dx}Y_\ell(x) \ . \qquad 3.48$$

Multiply by $Y_m(x)$ and integrate. The left side reduces to the term in $a_{k,m}$. Define a matrix element coefficient $c_{\ell,m}$ by

$$c_{\ell,m} = \int dx(1 - x^2)\frac{dY_\ell}{dx}\frac{dY_m}{dx} \ . \qquad 3.49$$

Then the differential equation 3.48 becomes the set of algebraic equations:

$$\left[c_{\ell,\ell} + \beta\nu_\ell^4 - \frac{2}{3}\omega_k^2\right]a_{k,\ell} + \sum_{m=1}^{\infty}{}' c_{\ell,m}a_{k,m} = 0 \qquad \ell = 1,\ldots \qquad 3.50$$

where the prime on the sum means the term $m = \ell$ is to be omitted.

The evaluation of the coefficient $c_{\ell,m}$ is a tedious task in integration of elementary functions. Omitting the details, we have for both even and odd functions

$$c_{\ell,m} = -32\nu_\ell^4\nu_m^4/(\nu_\ell^4 - \nu_m^4)^2 \qquad \ell \neq m \ . \qquad 3.51$$

The diagonal elements are given by

$$c_{\ell,\ell} = \frac{2}{3}\nu_\ell^2\tan^2\nu_\ell + \frac{5}{2} \qquad \text{even} \qquad 3.52a$$

$$c_{\ell,\ell} = \frac{2}{3}\nu_\ell^2\cot^2\nu_\ell + \frac{5}{2} \qquad \text{odd} \ . \qquad 3.52b$$

We see from 3.51 that if m is large compared to ℓ, that is, far above the diagonal, the coefficient is approximately $-32(\ell/m)^4$ and the terms decrease rapidly. The diagonal elements, which are the largest, increase as ℓ^2, so the equations may be normalized by dividing by the diagonal elements and the resulting determinants are all convergent.

With the procedures proven legitimate, we did not attempt to treat the eigenvalues and eigenfunctions by solving the determinants analytically, but resorted to a computer program available for such purposes at Rand. The basic program, EIGRS, was prepared and is supported by IMSL, Inc., Houston, Texas (International Mathematical Statistical Libraries). The program to calculate the coefficients $c_{\ell,m}$ and use EIGRS was written by M. D. Lakatos.

The fundamental modes were calculated for the three values $\beta = 0$, 3×10^{-5}, 10^{-4}. The infinite set of equations 3.50 were truncated by multiples of 5 up to 40 equations, and the eigenvalues ω_k and expansion coefficients $a_{k,\ell}$ were determined. The first 20 even and odd eigenvalues and eigenfunctions were found accurately to four significant figures (no changes of 10^{-4} in the increase from 35 to 40 equations), and the lower ones were much more accurate than that. The first four odd and even eigenvalues for ω_k, and the 20th, are listed in Table 3 for the three values of β.

We may immediately draw some important conclusions from Table 3. First, in spite of the small values of β, even the fourth eigenvalues experience appreciable changes (5 percent), and the 20th eigenvalues experience 40 percent increases. Second, and more important, the lowest eigenvalue, corresponding to the first even mode, is very nearly an integer, that is, the lowest vibrational mode is at nearly three times the orbital frequency. If the perturbations contain terms at thrice orbital, and it

Table 3

CHARACTERISTIC FREQUENCIES

	β		
k	0	3×10^{-5}	10^{-4}
1E	3.0000	3.00032	3.00107
1O	4.2426	4.2454	4.2506
2E	5.4772	5.4875	5.5079
2O	6.7083	6.7380	6.7916
3E	7.9376	8.0057	8.1157
3O	9.1661	9.2980	9.4882
4E	10.3950	10.6191	10.9105
4O	11.6248	11.9709	12.3808
20E	54.88	64.18	76.82
20O	56.37	66.16	79.57

will be demonstrated later that thermal bending effects do involve such frequencies, then strong resonant responses can be expected.

A computer program, also written by M. D. Lakatos, has been used to calculate the eigenfunctions $X_k(x)$ from the expansion coefficients $a_{k,\ell}$ and the bar eigenfunctions $Y_\ell(x)$. The first and second even eigenfunctions are plotted in Fig. 5, the fifth even eigenfunction in Fig. 6. Only the portion for positive x is shown. All curves are for $\beta = 10^{-4}$. Also graphed on Figs. 5 and 6 are the suitably normalized Legendre polynomials P_2, P_y, and P_{10}. In Fig. 5, the departure of the true eigenfunction from the Legendre polynomial is only discernible at the extreme right end of the curve, that is, near the end of the rod. The Legendre polynomials have strong curvature near the ends, whereas the true eigenfunctions are straight to fourth-order terms, so the closeness

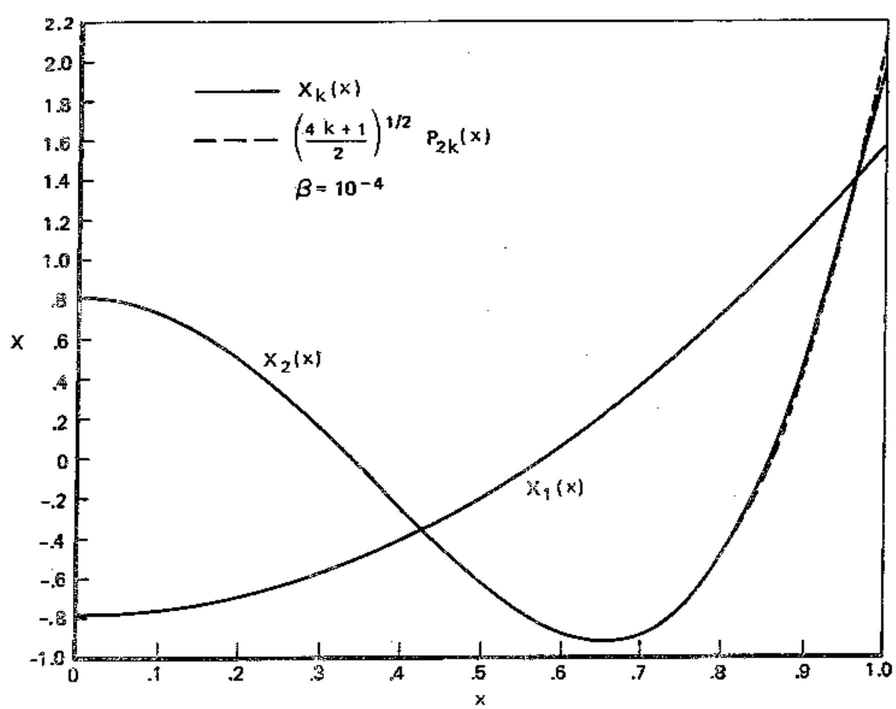

Fig. 5 — First two even fundamental modes

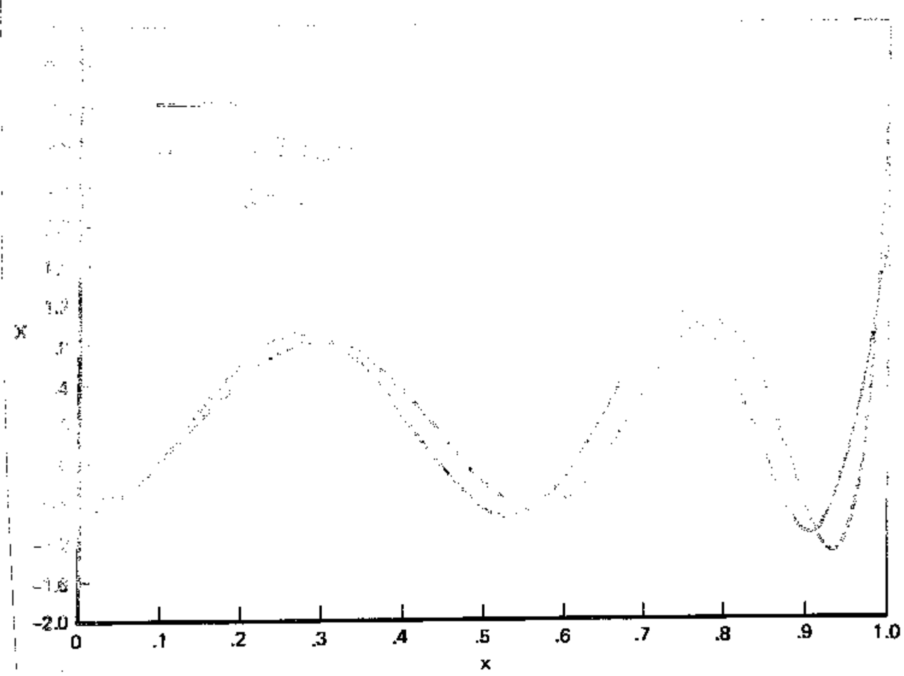

Fig. 6 — Fifth even fundamental mode

of the approximation is pleasing. The fifth modes show significant departures throughout, reaching a difference of 1.4 units at the end. As is plausible, more of the energy density per unit length of the modes is toward the center of the rod for the true eigenfunctions than for the gravity-gradient modes, since the elasticity suppresses bending at the ends. The Nth eigenfunction has N zeros, just like the Legendre polynomial.

With the eigenvalues and eigenfunctions constructed, we are now ready to determine the effects of initial conditions on vibratory motion. The initial values of the coefficients A_k are given by

$$A_k(0) = \int dx X_k(x) w_2(x, 0) \tag{3.53a}$$

$$A_k'(0) = \int dx X_k(x)\dot{w}_2(x,0)/\dot{\theta} \ . \tag{3.53b}$$

If the rod is started from rest, $A_k'(0) = 0$. We will return to this point when we discuss micrometeoroid impacts, which impart an initial velocity to a short section of rod, but limit ourselves to $A_k'(0) = 0$ at present, and look for suitable initial conditions.

Previous investigations[2, 3, 11] have assumed the rod is distorted into a parabolic shape. However, this does not meet the boundary conditions, and the ends of the rod would jump discontinuously if the constraints establishing the initial distortion were released. We choose as our first initial condition the simplest even polynomial which meets the boundary conditions, has vanishing average value, and is normalized to unit deflection between center and ends. This proves to be the sixth-degree expression:

$$w_6 = (-29 + 105x^2 - 35x^4 + 7x^6)/77 \ . \tag{3.54}$$

This expression and the comparison parabola $(3x^2 - 1)/3$ are plotted in Fig. 7. The two curves are very close, the maximum difference being .043 at the center and ends (x = 0 and 1), and .036 near x = .65. Hence, expression 3.54 is a proper representation for a near-parabolic displacement.

Insert 3.54 into 3.53a, and replace $X_k(x)$ by the expansion 3.47. The differential equation 3.36a may be used to replace Y_ℓ by

$$\frac{d^8 Y_\ell}{dx^8} \Big/ \nu_\ell^8$$

and the resulting expression integrated by parts seven times. The boundary conditions make the integrated terms vanish for the first six integrations. Since $d^6 w_6/dx^6$ is constant, the last integration evaluates

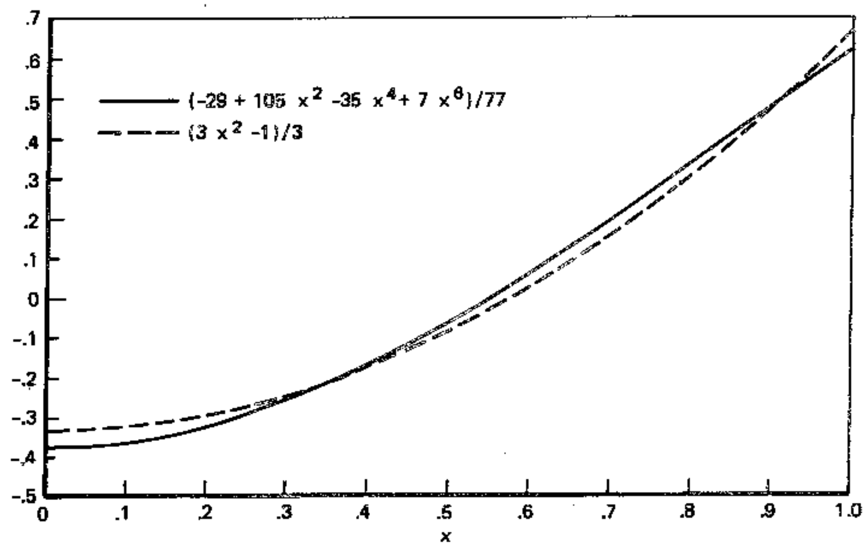

Fig. 7 — Smooth initial conditions

the integral as

$$\int dx w_6(x) Y_\ell(x) = \frac{1,440}{11\nu_k^8} \left.\frac{dY_\ell}{dx}\right|_{x=1} . \qquad 3.55$$

Evaluating the derivative from 3.41 and using the closeness of ν_ℓ to $(\ell - 1/4)\pi$ yields for 3.55:

$$\int dx w_6(x) Y_\ell(x) = \frac{1,440\sqrt{2}}{11} \frac{(-1)^{\ell-1}}{\nu_\ell^7} . \qquad 3.56$$

The first three coefficients have the numerical values .43955, -.00122, .00005. Thus, almost all the contribution to A_k will come from the term $a_{k,1}Y_1$ in the expansion. From the coefficients as determined by the computer program, we have $a_{1,1} = .9966$, $a_{2,1} = -.0827$, $a_{3,1} = .0055$. The response to the initial displacement 3.54 is therefore given to better than 1 percent accuracy by

$$w_2(x,\theta) = .4381X_1(x)\cos\omega_1\theta - .0364X_2(x)\cos\omega_2\theta \ . \tag{3.57}$$

The motion will be a near-parabolic shape at frequency near thrice orbital, with an 8 percent admixture of double-noded shape (see Fig. 5), at frequency near 5.5 times orbital. The "throw" of $X_1(x)$, that is, the difference between maximum and minimum, is 2.341, so the throw of the first term is 1.0256. We observe that the response to a near parabolic disturbance is primarily an oscillation at the initial amplitude, thrice orbital frequency, and still near-parabolic shape.

To obtain a response involving many modes, we use a "broken-line" initial displacement. This is equivalent to a plucked string. The simplest form puts the break at the center, exciting only even modes, so we choose

$$w_0(x) = .5 - |x| \tag{3.58}$$

which has the required properties (even, break at center, unit amplitude, satisfies the boundary conditions). The integral involving w_0 and $Y_\ell(x)$ may be evaluated easily, yielding

$$\int dx w_0(x)Y_\ell(x) = \frac{2}{\nu_\ell^2}\frac{1}{\sqrt{2}|\cos\nu_\ell|}\left[1 - \frac{\cos\nu_\ell}{\cosh\nu_\ell}\right] . \tag{3.59}$$

The factor multiplying $2/\nu_\ell^2$ is 1.1329 for $\ell = 1$, .9942 for $\ell = 2$, and essentially unity thereafter. We see that the terms in the expansion with respect to bar eigenfunctions of $A_k(0)$, the initial value of the coefficient of the true eigenfunction, only fall as $2/\nu_\ell^2$. To obtain 1 percent accuracy, nine terms have to be kept. Thus, the coefficient $A_k(0)$ is approximately given by

$$A_k(0) = \frac{2.2658}{\nu_1^2} a_{k,1} + \frac{1.9884}{\nu_2^2} a_{k,2} + 2\sum_3^9 a_{k,\ell} \Big/ \nu_\ell^2 \; . \qquad 3.60$$

With the coefficients from the computer program, these expressions, which we call excitation coefficients, have been worked out using the author's HP-34C, and are listed in Table 4.

Again, the first term is dominant, but the higher frequency terms are now much more important than they were for the near-parabolic excitation 3.54. The shape of the rod at the initial time and after 1 and 10 orbits has been calculated, and is plotted in Fig. 8. Because the most important term, that at nearly thrice orbital frequency, is back at

Table 4

EXCITATION COEFFICIENTS FOR PLUCKED ROD

k	$A_k(0)$
1	.3954
2	-.0886
3	.0400
4	-.0220
5	.0147
6	-.0096
7	.0057
8	-.0036
9	.0031

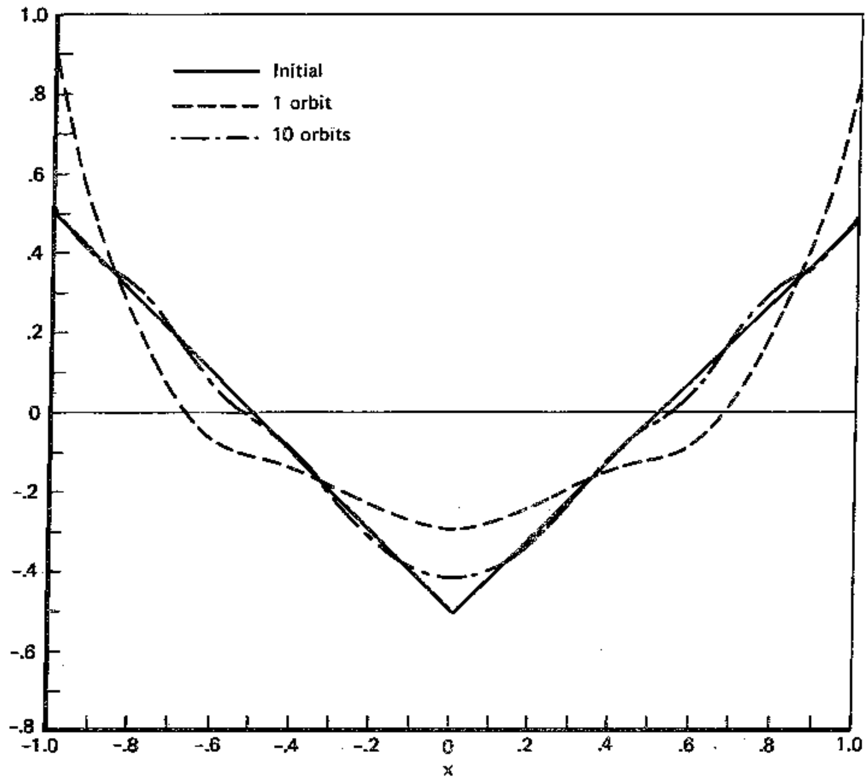

Fig. 8 — Response to initial displacement in-plane

its initial value after an integral number of orbits, the distorted shape is quite similar to the original broken line, but the changes are apparent. Thus, the vibration continues at the original amplitude, but the shape will now be a complicated function of time.

The analysis has all been for in-plane motion, but the same techniques adapt directly to out-of-plane motion, with equation 3.26 providing the frequency relation and everything else being identical. The investigation shows that initial tilting or distortion will yield libration or vibration with amplitude substantially equal to the initial amplitude. The verticality and straightness requirements must therefore be satisfied by the initial deployment of the satellite. The verticality should not be difficult, but an rms lateral displacement of .5 cm over a length of 1,500 m promises to be a very difficult deployment task, since the satellite must be installed in a dispenser of radius limited to 4 m by the payload bay dimensions of the space shuttle, and then unrolled from that dispenser, without acquiring any permanent bends which would lead to such distortions.

The libration and vibration would decay with time if there were any damping. It has been shown by Burke[11] that the "sticky hinges" of the Stanford design will be locked by the gravity-gradient tension and will not provide the required damping. Internal friction in steel is also insufficient. (The damping of metal strings in musical instruments is primarily due to interaction with the air.) Viscous damper booms like those used on the Radio Astronomy Explorer[5, 6] cannot be attached to PACSAT. If damping is to be provided, some other procedure must be found.

This completes our discussion of the basic problem. We next consider the effects of various perturbations, beginning with orbital ellipticity.

IV. ELLIPTICITY EFFECTS

There have been previous studies[7, 8] of the effect of ellipticity on the libration of a satellite. We shall employ a different technique, which yields the same driven solution and stability analysis, and is valid for both in-plane and out-of-plane motions. We shall also investigate the effect on vibration, which will prove to consist of small variations of frequencies and amplitudes.

First, let us consider the center-of-mass motion. This will be an ellipse in the equatorial plane with the earth at one focus. The semi-major axis of the ellipse is denoted by a, the eccentricity by e, and the angular coordinate by θ, measured from perigee. The eccentric anomaly (see any book on celestial mechanics) is denoted by E, and the mean orbital rate by n. Then we have the equations:

$$\vec{R} = R\vec{i}_R = R(\vec{i}_x\cos\theta + \vec{i}_y\sin\theta) \tag{4.1a}$$

$$R = a(1 - e^2)/(1 + e\cos\theta) \tag{4.1b}$$

$$\cos\theta = (\cos E - e)/(1 - e\cos E) \tag{4.1c}$$

$$nt = E - e\sin E \tag{4.1d}$$

$$n^2a^3 = \mu \ . \tag{4.1e}$$

Equation 4.1a is the proper generalization of 3.1a; 4.1b is the equation of an ellipse of the indicated dimensions. The angular coordinate and eccentric anomaly are related through 4.1c. Kepler's equation 4.1d expresses the time in terms of the eccentric anomaly, and can be inverted to express the anomaly, and hence the angular coordinate θ, in terms of the time. Finally, 4.1e is the generalization of 3.5 to elliptic motion.

The libration and vibration will again be resolved in cylindrical coordinates. While equations 3.6a and 3.6b, which give the first time derivatives of the unit vectors $\vec{i}_R$ and $\vec{i}_\theta$, remain valid, equations 3.6c and 3.6d must be generalized to include the variable angular velocity, hence,

$$\ddot{\vec{i}}_R = -\dot{\theta}^2\vec{i}_R + \ddot{\theta}\vec{i}_\theta \qquad 4.2a$$

$$\ddot{\vec{i}}_\theta = -\ddot{\theta}\vec{i}_R - \dot{\theta}^2\vec{i}_\theta \; . \qquad 4.2b$$

The libration equation 2.33 becomes

$$\ddot{u}_1 - \left(\dot{\theta}^2 + \frac{2\mu}{R^3}\right)u_1 - 2\dot{\theta}\dot{u}_2 - \ddot{\theta}u_2 = -\frac{8T_0}{\rho L^2}u_1 \qquad 4.3a$$

$$\ddot{u}_2 - \left(\dot{\theta}^2 - \frac{\mu}{R^3}\right)u_2 + 2\dot{\theta}\dot{u}_1 + \ddot{\theta}u_1 = -\frac{8T_0}{\rho L^2}u_2 \qquad 4.3b$$

$$\ddot{u}_3 + \frac{\mu}{R^3}\,u_3 = -\frac{8T_0}{\rho L^2}u_3 \; . \qquad 4.3c$$

The angles ψ (in-plane motion) and χ (out-of-plane motion) are introduced by 3.4, and the same manipulation used to separate the equations. The angle θ is to be used as the independent variable, but the transformation is somewhat more complicated. It is found that

$$\dot{\theta} = na^2\sqrt{1 - e^2}/R^2 \qquad 4.4a$$

$$\frac{\mu}{R^3\dot{\theta}^2} = \frac{1}{1 + e\cos\theta} \qquad 4.4b$$

$$\frac{\ddot{\theta}}{\dot{\theta}^2} = -\frac{2e\sin\theta}{1 + e\cos\theta} . \qquad 4.4c$$

The equations analogous to 3.10 are

$$T_0 = \frac{1}{8}\rho L^2\dot{\theta}^2\left[\chi'^2 + \cos^2\chi(1 + \psi')^2 + \frac{1}{1 + e\cos\theta}(3\cos^2\chi\cos^2\psi - 1)\right] \qquad 4.5a$$

$$\psi'' - \frac{2e\sin\theta}{1 + e\cos\theta}\psi' + \frac{3}{1 + e\cos\theta}\sin\psi\cos\psi = \frac{2e\sin\theta}{1 + e\cos\theta} + 2\tan\chi\chi'(1 + \psi') \qquad 4.5b$$

$$\chi'' - \frac{2e\sin\theta}{1 + e\cos\theta}\chi' + \left[(1 + \psi')^2 + \frac{3\cos^2\psi}{1 + e\cos\theta}\right]\sin\chi\cos\chi = 0 . \qquad 4.5c$$

In 4.5a $\dot{\theta}$ is regarded as variable. The most important difference between 4.5 and 3.10 is the inhomogeneous term on the right of 4.5b. Although equations 3.10 have the possible solution $\psi = \chi = 0$, a permanently vertical rod, this is not possible for 4.5. This driven solution is physically very reasonable. The velocity vector of the center of mass is not perpendicular to the vertical, so an in-plane attitude oscillation can be expected.

The intention is to launch PACSAT into a circular orbit, so it can be expected that e will be small. The previously cited Radio Astronomy Explorer satellite (1968-55A), whose orbit and physical character is most similar to PACSAT, had an orbital eccentricity of .0004. An eccentricity of .01 at an altitude of one earth radius corresponds to a velocity injection error of 183 fps, far above any reasonable value. We will calculate the amplitude and frequency of the motion to second order in e, which should be more accurate than necessary. The amplitude can be found by expanding 4.5b and 4.5c to second order, but the frequency requires some third-order terms. The initial conditions will be left unspecified, since PACSAT may be injected at an arbitrary point along the orbit. To second order, equations 4.5b and 4.5c become

$$\psi'' - \frac{2e\sin\theta}{1 + e\cos\theta}\psi' + \frac{3}{1 + e\cos\theta}\psi = \frac{2e\sin\theta}{1 + e\cos\theta} + 2\chi\chi' \qquad 4.6a$$

$$\chi'' - \frac{2e\sin\theta}{1 + e\cos\theta}\chi' + \left[1 + \frac{3}{1 + e\cos\theta}\right]\chi = -2\chi\psi' \ . \qquad 4.6b$$

The first-derivative terms may be eliminated by the transformation

$$\psi = f/(1 + e\cos\theta) \qquad 4.7a$$

$$\chi = g/(1 + e\cos\theta) \qquad 4.7b$$

which yields the simpler equations, correct to second order,

$$f'' + (3 - 2e\cos\theta)f = 2e\sin\theta + 2gg' \qquad 4.8a$$

$$g'' + (4 - 2e\cos\theta)g = \qquad - 2gf' \ . \qquad 4.8b$$

Separating the first-order and second-order parts yields

$$f = f_1 + f_2 \qquad 4.9a$$

$$g = g_1 + g_2 \qquad 4.9b$$

$$f_1'' + 3f_1 = 2e\sin\theta \qquad 4.9c$$

$$g_1'' + 4g_1 = 0 \qquad 4.9d$$

$$f_2'' + 3f_2 = +2e\cos\theta f_1 + 2g_1g_1' \qquad 4.9e$$

$$g_2'' + 4g_2 = +2e\cos\theta g_1 - 2g_1 f_1' \ . \qquad 4.9f$$

The solutions of 4.9c and 4.9d, subject to the initial conditions, are

$$f_1 = e\sin\theta + A_1\cos\sqrt{3}\theta + B_1\sin\sqrt{3}\theta \qquad 4.10a$$

$$g_1 = \qquad C_1\cos 2\theta + D_1\sin 2\theta \ . \qquad 4.10b$$

Thus, the driven part of the in-plane motion has amplitude e and oscillates at the orbital frequency, whereas the initial conditions provide an oscillation at the natural frequency $\sqrt{3}$ times orbital with the amplitude $(A_1^2 + B_1^2)^{1/2}$. There is no driven out-of-plane motion, only the natural double-frequency oscillation.

The right side of 4.9e involves terms in $(\sqrt{3} - 1)\theta$, $(\sqrt{3} + 1)\theta$, 2θ, and 4θ. The solution is

$$\begin{aligned} f_2 = {} & \frac{1}{11}eA_1[(2\sqrt{3} + 1)\cos\sqrt{3} - 1)\theta - (2\sqrt{3} - 1)\cos(\sqrt{3} + 1)\theta] \\ & + \frac{1}{11}eB_1[(2\sqrt{3} + 1)\sin(\sqrt{3} - 1)\theta - (2\sqrt{3} - 1)\sin(\sqrt{3} + 1)\theta] \\ & + \frac{2}{13}\left(C_1^2 - D_1^2\right)\sin 4\theta - \frac{4}{13}C_1D_1\cos 4\theta \\ & - e^2\sin 2\theta + A_2\cos\sqrt{3}\theta + B_2\sin\sqrt{3}\theta \ . \qquad 4.11 \end{aligned}$$

The right side of 4.9f involves terms in $(2 + \sqrt{3})\theta$ and $(2 - \sqrt{3})\theta$, with solution:

$$g_2 = \frac{1}{13}\Big[(4 - \sqrt{3})(A_1D_1 + B_1C_1)\cos(2 + \sqrt{3})$$

$$+ (4 + \sqrt{3})(A_1D_1 - B_1C_1)\cos(2 - \sqrt{3})\theta$$

$$- (4 - \sqrt{3})(A_1C_1 - B_1D_1)\sin(2 + \sqrt{3})\theta$$

$$- (4 + \sqrt{3})(A_1C_1 + B_1D_1)\sin(2 - \sqrt{3})\theta + C_2\cos 2\theta + D_2\sin 2\theta\Big] . \quad 4.12$$

To second order, the out-of-plane motion is independent of the eccentricity except as e appears in the constants A_1, B_1, C_1, D_1 which involve the initial conditions.

Thus, the amplitudes of the in-plane and out-of-plane motions have been obtained to second order. To determine the frequencies, it is necessary to expand 4.5 to third order. The expressions for f_1, f_2, g_1, and g_2 are substituted, and the terms of apparent frequency $\sqrt{3}$ and 2 are collected. The combined amplitude of these terms provide the correction to the frequency. Thus, indicating only the third-order parts on the right, we have

$$f'' + (3 + e^2)f = 2e\cos\theta f_2 - e^2\cos 2\theta f_1$$

$$+ 2[g_1g_2' + g_2g_1' - e\cos\theta g_1g_1' + e\sin\theta g_1^2 + g_1g_1'f_1' + f_1^3] \quad 4.13a$$

$$g'' + (4 + e^2)g = 2e\cos\theta g_2 - e^2\cos 2\theta g_1$$

$$- 2[g_1 f_2' + g_2 f_1' - e\cos\theta g_1 f_1' + e\sin\theta f_1 g_1$$

$$+ \frac{1}{2}f_1'^2 g_1 - \frac{3}{2}f_1^2 g_1 - \frac{4}{3}g_1^3] \; . \qquad 4.13b$$

Terms in $A_1\cos\sqrt{3}\theta$ collect on the right side of 4.13a as

$$\frac{35}{11}e^2 + \frac{3}{2}(A_1^2 + B_1^2) + \frac{6}{13}(C_1^2 + D_1^2) \qquad 4.14$$

from which we deduce the modified in-plane libration frequency

$$\omega_{IP}^2 = 3 - \frac{24}{11}e^2 - \frac{3}{2}(A_1^2 + B_1^2) - \frac{6}{13}(C_1^2 + D_1^2) \qquad 4.15a$$

$$\omega_{IP} = \sqrt{3}\left[1 - \frac{4}{11}e^2 - \frac{1}{4}(A_1^2 + B_1^2) - \frac{1}{13}(C_1^2 + D_1^2)\right] \; . \qquad 4.15b$$

Similarly, collecting terms in $\cos 2\theta$ on the right side of 4.13b yields

$$e^2 + \frac{6}{13}(A_1^2 + B_1^2) + \frac{18}{13}(C_1^2 + D_1^2) \qquad 4.16$$

from which we deduce the modified out-of-plane libration frequency

$$\omega_{OP}^2 = 4 - \frac{6}{13}(A_1^2 + B_1^2) - \frac{18}{13}(C_1^2 + D_1^2) \qquad 4.17a$$

$$\omega_{OP} = 2\left[1 - \frac{3}{52}(A_1^2 + B_1^2) - \frac{9}{52}(C_1^2 + D_1^2)\right] \; . \qquad 4.17b$$

The expressions 4.15b and 4.17b are generalizations of the results of Refs. 7 and 8, which treat only the in-plane motion. We see that the frequency of the out-of-plane motion is independent of e to second order, except through the implicit dependence via the initial conditions.

The simplest case which retains full physical meaning is for the rod to start at rest at perigee with an infinitesimal displacement in an arbitrary direction. Then the constants become

$$A_1 = \psi_0 \qquad B_1 = -e/\sqrt{3} \tag{4.18a}$$

$$C_1 = \chi_0 \qquad D_1 = 0 \ . \tag{4.18b}$$

The frequencies become

$$\omega_{IP} = \sqrt{3}\left[1 - \frac{59}{132}e^2 - \frac{1}{4}\psi_0^2 - \frac{6}{13}\chi_0^2\right] \tag{4.19a}$$

$$\omega_{OP} = 2\left[1 - \frac{1}{52}e^2 - \frac{3}{52}\psi_0^2 - \frac{9}{52}\chi_0^2\right] \ . \tag{4.19b}$$

The solutions for the displacements are, to second order,

$$\begin{aligned}\psi = {} & e\sin\theta + \psi_0\cos\omega_{IP}\theta - \frac{e}{\sqrt{3}}\sin\omega_{IP}\theta - \frac{3}{2}e^2\sin2\theta + \frac{2}{13}\chi_0^2\sin4\theta \\ & + \frac{1}{22}e\psi_0[(4\sqrt{3} - 9)\cos(\sqrt{3} - 1)\theta - (4\sqrt{3} + 13)\cos(\sqrt{3} + 1)\theta] \\ & - \frac{1}{\sqrt{3}}e^2[(4\sqrt{3} - 9)\sin(\sqrt{3} - 1)\theta - (4\sqrt{3} + 13)\sin(\sqrt{3} + 1)\theta] \\ & + \frac{9}{11}e\psi_0\cos\omega_{IP}\theta + \frac{1}{\sqrt{3}}\left[\left(\frac{35}{11} - \frac{1}{\sqrt{3}}\right)e^2 - \frac{8}{13}\chi_0^2\right]\sin\omega_{IP}\theta\end{aligned} \tag{4.20a}$$

$$\chi = \chi_0\cos\omega_{OP}\theta + \frac{1}{13}\chi_0\Bigg\{-(4 - \sqrt{3})\left[\frac{e}{\sqrt{3}}\cos(2 + \sqrt{3})\theta + \psi_0\sin(2 + \sqrt{3})\theta\right]$$
$$+ (4 + \sqrt{3})\left[\frac{e}{\sqrt{3}}\cos(2 - \sqrt{3})\theta - \psi_0\sin(2 - \sqrt{3})\theta\right] + 11e\cos\omega_{OP}\theta$$
$$+ 5\psi_0\sin\omega_{OP}\theta\Bigg\} - \frac{1}{2}e\chi_0(\cos\theta + \cos 3\theta) \ . \qquad 4.20b$$

For most reasonable values of eccentricity and initial displacement, the second-order terms are very small. The tension is given to first order by

$$T_0 = \frac{3}{8}\rho L^2\dot{\theta}^2[1 + 2\sqrt{3}(-A_1\sin\sqrt{3}\theta + B_1\cos\sqrt{3}\theta)] \qquad 4.21$$

where the e dependence is in $\dot{\theta}^2$.

There is a question concerning the stability of the out-of-plane libration. Let equation 4.5b be linearized by dropping the χ^3 terms. An initial deflection of 1 meter at the tip of the rod corresponds to $\chi = .00133$, so expressions of the order of 10^{-6} are being neglected. Further, to simplify the analysis while retaining the significant effects, let the initial conditions be chosen so $A_1 = B_1 = 0$ (start the in-plane motion from rest, 90 deg from perigee, with initial deflection e). Then we have the simplified form for ψ:

$$\psi = e\sin\theta - \frac{3}{2}e^2\sin 2\theta \qquad 4.22$$

and the differential equation for g becomes, to second order in e,

$$g'' + [4 - 3e^2\cos 2\theta]g = 0 \ . \qquad 4.23$$

Equation 4.23 is in the standard form of the Mathieu equation[17]:

$$y'' + (a - 2q\cos 2\theta)y = 0 \quad . \qquad 4.24$$

The stability theory of such equations is developed in Ref. 17, Chapter 4. It may be summarized as follows. When $q = 0$, equation 4.24 will have periodic solutions, with period π, if a is the square of an even integer. If q is not zero, there will be two specific values of a, regarded as a function of q, such that the equation has a solution of period π. The two values correspond to even and odd solutions. These values of a, call them a_E and a_O, then describe two curves in the a,q plane, which issue from the a axis at $a = 4k^2$, k integral. If the value of a which actually appears in the equation 4.24 lies between the pair of curves, the solution of 4.24 is unstable; if a lies outside the region between the curves the solution is stable. According to Ref. 17, p. 17, the curves issuing from $a = 4$ are

$$a_E = 4 + \frac{5}{12}q^2 + 0(q^4) \qquad 4.25a$$

$$a_O = 4 - \frac{1}{12}q^2 + 0(q^4) \quad . \qquad 4.25b$$

We see that for 4.23 the value of a, 4, is at the junction of the curves, and the problem of stability is not resolved to the second order and should be carried to the fourth order.

If we assume χ is infinitesimal, the solution of 4.5a is found to fourth order as

$$\psi = e\sin\theta - \frac{3}{2}e^2\sin 2\theta + e^3\left(\frac{13}{12}\sin 3\theta + \frac{3}{4}\sin\theta\right)$$

$$+ e^4\left(\frac{13}{4}\sin 2\theta - \frac{83}{104}\sin 4\theta\right) \qquad 4.26$$

and when this is substituted into 4.5b and the lengthy manipulations are carried out, the fourth-order generalization of 4.23 is

$$g'' + \left[4 - \frac{15}{8}e^4 - 3e^2\cos 2\theta + \frac{9}{4}e^3(\cos\theta - \cos 3\theta)\right.$$

$$\left. + e^4\left(\frac{33}{2}\cos 2\theta + \frac{1{,}143}{104}\cos 4\theta\right)\right]g = 0 \ . \qquad 4.27$$

This equation is in the form of a generalization of Mathieu's equation known as Hill's equation (Ref. 17, Chapter 6). The stability theory proceeds in the same manner. The presence of the odd harmonics of θ requires that the periodic solutions have period 2π. After extensive calculations, the curves a_E and a_0 are found to be

$$a_E = 4 - \frac{237}{52}e^4 \qquad 4.28a$$

$$a_0 = 4 + \frac{69}{13}e^4 \qquad 4.28b$$

and the solution of 4.27 is unstable. However, if e = .01, it will take about 10^7 orbits for the instability to produce observable effects, and this is over 4,500 years. Hence, although the out-of-plane motion is nominally unstable, there will be no effects during the operating lifetime of PACSAT.

We next consider the effect of ellipticity on vibration. The equations 3.20 must be generalized. We will keep only first-order changes produced by the ellipticity. The differential equation 2.34, expressed in cylindrical coordinates, becomes

$$\ddot{w}_1 - \left(\dot{\theta}^2 + \frac{2\mu}{R^3}\right)w_1 - 2\dot{\theta}\dot{w}_2 - \ddot{\theta}w_2$$
$$= \left(\frac{\partial}{\partial s}\frac{T_1}{\rho}\right)u_1 + \frac{T_0}{\rho}\frac{\partial}{\partial s}\left(1 - 4\frac{s^2}{L^2}\right)\frac{\partial w_1}{\partial s} - \frac{EI}{\rho}\frac{\partial^4 w_1}{\partial s^4} \qquad 4.29a$$

$$\ddot{w}_2 - \left(\dot{\theta}^2 - \frac{\mu}{R^3}\right)w_2 + 2\dot{\theta}\dot{w}_1 + \ddot{\theta}w_1$$
$$= \left(\frac{\partial}{\partial s}\frac{T_1}{\rho}\right)u_2 + \frac{T_0}{\rho}\frac{\partial}{\partial s}\left(1 - \frac{4s^2}{L^2}\right)\frac{\partial w_2}{\partial s} - \frac{EI}{\rho}\frac{\partial^4 w_2}{\partial s^4} \qquad 4.29b$$

$$\ddot{w}_3 \qquad + \frac{\mu}{R^3}w_3$$
$$= \left(\frac{\partial}{\partial s}\frac{T_1}{\rho}\right)u_3 + \frac{T_0}{\rho}\frac{\partial}{\partial s}\left(1 - 4\frac{s^2}{L^2}\right)\frac{\partial w_3}{\partial s} - \frac{EI}{\rho}\frac{\partial^4 w_3}{\partial s^4} \; . \qquad 4.29c$$

If we multiply the first equation by u_1, the second by u_2, the third by u_3, and add, we obtain an expression for the perturbation in the tension. After extensive simplification, it becomes to second order

$$\frac{\partial}{\partial s}\frac{T_1}{\rho} = -2(\dot{\theta} + \dot{\psi})\dot{w}_2 + 6\frac{\mu}{R^3}(\psi w_2 + \chi w_3) - 2\dot{\chi}\dot{w}_3 + 2\dot{\theta}^2\chi w_3 \; . \qquad 4.30$$

The important difference between 4.3 and 4.29 is that in 4.3b, the term $\ddot{\theta}u_1$ is first order and contributes the driving term, but in 4.29b the term $\ddot{\theta}w_1$ is third order, since w_1 is second order. Hence, there is no driving term in 4.29. In 4.29b and 4.29c, the expression 4.30 can be substituted in the first term on the right, and we see that it contributes only second-order quantities. (In all this discussion, e, ψ, χ, w_2, w_3 are regarded as first order.) We define the convenient quantities:

$$F = w_2/(1 + e\cos\theta) \tag{4.31a}$$

$$G = w_3/(1 + e\cos\theta) \ . \tag{4.31b}$$

We are not interested in the interrelation between the ellipticity and the initial conditions on the librations as they affect the vibration. Hence, we will choose the initial conditions so that $A_1 = B_1 = C_1 = D_1 = 0$ (see equations 4.10). Then the equations for the vibration, complete to first-order corrections due to ellipticity, become analogous to 3.27:

$$\frac{2}{3}F'' = \frac{\partial}{\partial x}(1 - x^2)\frac{\partial F}{\partial x} - \beta(1 - 4e\cos\theta)\frac{\partial^4 F}{\partial x^4} \tag{4.32a}$$

$$\frac{2}{3}(G'' + G) = \frac{\partial}{\partial x}(1 - x^2)\frac{\partial G}{\partial x} - \beta(1 - 4e\cos\theta)\frac{\partial^4 G}{\partial x^4} \tag{4.32b}$$

with the same boundary conditions as in 3.27. If the first-order and second-order parts of 4.32a are separated (analysis of 4.32b is superfluous for the same reason as in 3.20), the equations for the first-order part reduce to 3.27. The second-order part is the forced oscillation governed by

$$\frac{2}{3}F_2'' - \frac{\partial}{\partial x}(1 - x^2)\frac{\partial F_2}{\partial x} + \beta\frac{\partial^4 F_2}{\partial x^4} = 4e\beta\cos\theta\frac{\partial^4 F_1}{\partial x^4} . \qquad 4.33$$

Substituting for F_1 the expansion 3.28 and for F_2 a corresponding expansion with B_k replacing A_k, there results:

$$\sum_1^\infty \frac{2}{3}(B_k'' + \omega_k^2 B_k)X_k(x) = 4e\beta\cos\theta\sum_1^\infty A_k(\theta)\frac{\partial^4 X_k}{\partial x^4} \qquad 4.34$$

$$B_k'' + \omega_k^2 B_k = 6e\beta\cos\theta\sum_1^\infty A_\ell(0)\cos\omega_\ell\theta\int_{-1}^{1} dx X_k(x)\frac{\partial^4 X_\ell}{\partial x^4} . \qquad 4.35$$

The integral on the right side of 4.35 may be expressed in terms of the expansion coefficients of the true eigenfunctions with respect to the bar eigenfunctions (see equation 3.47) as

$$K_{k,\ell} = \int_{-1}^{1} dx X_k(x)\frac{\partial^4 X_\ell}{\partial x^4} = \sum_1^\infty a_{k,m}a_{\ell,m}\nu_m^4 \qquad 4.36$$

from which we obtain the solution for B_k,

$$B_k = 3e\beta\sum_1^\infty A_\ell(0)K_{k,\ell}\left[\frac{\cos(\omega_\ell - 1)\theta}{\omega_k^2 - (\omega_\ell - 1)^2} + \frac{\cos(\omega_\ell + 1)\theta}{\omega_k^2 - (\omega_\ell + 1)^2}\right] . \qquad 4.37$$

There are no possible vanishing denominators in 4.37, so there are no resonances, and the perturbations remain small. We see that the effect of ellipticity on vibration is to provide only small corrections.

The changes in the vibration frequencies are of order e^2 and may be neglected completely.

This completes the analysis of ellipticity effects. We next consider effects of external forces, beginning with the oblateness of the earth.

V. EARTH OBLATENESS EFFECTS

The analysis of the earth oblateness is quite complicated. Oblateness affects the libration and vibration equations in two ways: the direct effect caused by the ∇F term in equations 2.33 and 2.34, and the indirect effect caused by the change in the orbit of the center of mass. Furthermore, it develops that there is a resonance in the out-of-plane libration, so many terms must be kept in coefficients to make sure that all significant terms are included.

The investigation begins by finding the modified orbit of the center of mass. As is well known, oblateness causes the orbit plane to precess around the pole, and causes the perigee to precess around the orbit. The equations of libration and vibration must be expressed in terms of axes moving with the precessing plane if the motion is to be understood. They are expressed thusly, and the resonance treated carefully. The vibration analysis will be very simple, since there are no resonances to the order treated.

The analysis of the center-of-mass motion will follow Brouwer and Clemence[18]. It would have been much easier if we could just write down the results. However, although the precession frequencies are matters of general knowledge, we have not found a reference that states the amplitudes of the oscillatory terms to the desired accuracy. We have had to derive them, and do not mean to imply that we have used the simplest procedure.

Consider a satellite moving in a near-circular, near-equatorial orbit of eccentricity e and inclination I. Let the latitude of the satellite be $\mathcal{L}$, the longitude ϕ. We express the coordinates in terms of Delaunay variables (Ref. 18, p. 541). The mean anomaly (nt in equation 4.1d) is denoted by ℓ, the argument of perigee by g, and the longitude of the node by h. Then the latitude and longitude are given by

$$\sin\mathcal{L} = \sin I \sin(\theta + g) \qquad 5.1a$$

$$\phi = h + \tan^{-1}[\cos I \tan(\theta + g)] \ . \qquad 5.1b$$

The radius R and angular coordinate θ are still given by 4.1b-e.

The coordinates will be expanded to third-order terms in the eccentricity and inclination. The result is

$$R = a\left[1 + \frac{1}{2}e^2 - \left(e - \frac{3}{8}e^3\right)\cos\ell - \frac{1}{2}e^2\cos 2\ell - \frac{3}{8}e^3\cos 3\ell\right] \qquad 5.2a$$

$$\sin\mathcal{L} = I\left[\left(1 - e^2 - \frac{1}{6}I^2\right)\sin(\ell + g) + e[\sin(2\ell + g) - \sin g]\right.$$

$$\left. + \frac{9}{8}e^2\sin(3\ell + g) + \frac{1}{8}e^2\sin(\ell - g)\right] \qquad 5.2b$$

$$\phi = \ell + g + h + \left(2e - \frac{1}{4}e^3\right)\sin\ell + \frac{5}{4}e^2\sin 2\ell + \frac{13}{12}e^3\sin 3\ell$$

$$- \frac{1}{4}I^2[\sin(2\ell + 2g) + 2e\sin(3\ell + 2g) - 2e\sin(\ell + 2g)] \ . \qquad 5.2c$$

If there were no perturbations, ℓ would vary linearly with time, and all the other elements would be constant. The disturbing force induces element variations, which will be treated as first-order quantities because they are proportional to at least the first power of the strength of the disturbing force. Denoting the variations by δa, δe, etc., we have for the variations of the coordinates:

$$\frac{1}{a}\delta R = \frac{\delta a}{a}\frac{R}{a} + \left[e - \left(1 - \frac{9}{8}e^2\right)\cos\ell - e\cos 2\ell - \frac{9}{8}e^2\cos 3\ell\right]\delta e$$

$$+ \left[\left(1 - \frac{3}{8}e^2\right)\sin\ell + e\sin 2\ell + \frac{9}{8}e^2\sin 3\ell\right]e\delta\ell \qquad 5.3a$$

$$\delta(\sin\mathcal{L}) = \frac{\delta I}{I}\sin\mathcal{L} + I\Big[-2e\sin(\ell + g) + \sin(2\ell + g) - \sin g$$

$$+ \frac{9}{4}e\sin(3\ell + g) + \frac{1}{4}e\sin(\ell - g)\Big]\delta e$$

$$+ I\Big\{\cos(\ell + g) + e[\cos(2\ell + g) - \cos g]\Big\}\delta(\ell + g)$$

$$+ I\Big[\cos(2\ell + g) + \cos g + \frac{9}{4}e\cos(3\ell + g) + \frac{1}{4}e\cos(\ell - g)\Big]e\delta\ell$$

$$- \frac{1}{2}I^2\sin(\ell + g)\delta I \qquad 5.3b$$

$$\delta\phi = \delta(\ell + g + h) + \left[2\left(1 - \frac{3}{8}e^2\right)\sin\ell + \frac{5}{2}e\sin 2\ell + \frac{13}{4}e^2\sin 3\ell\right]\delta e$$

$$+ \left[2\left(1 - \frac{1}{8}e^2\right)\cos\ell + \frac{5}{2}e\cos 2\ell + \frac{13}{4}e^2\cos 3\ell\right]e\delta\ell - \frac{1}{2}I\delta I\sin(2\ell + g)$$

$$- \frac{1}{2}I^2\Big\{[\sin(3\ell + 2g) - \sin(\ell + 2g)]\delta e + \cos(2\ell + 2g)\delta(\ell + g)$$

$$+ [\cos(3\ell + 2g) + \cos(\ell + 2g)]e\delta\ell\Big\} \qquad 5.3c$$

The terms of 5.3 have been arranged to eliminate apparent singularities in the perturbations of the elements when e and I are small.

The disturbing force is the gradient of a potential:

$$U = \frac{\mu J_2 R_e^2}{2R^3}\left(1 - 3\sin^2\mathcal{L}\right) \qquad 5.4a$$

$$\vec{F} = \nabla U = -\frac{3\mu J_2 R_e^2}{2R^4}[(1 - 3\sin^2\mathcal{L})\vec{i}_R + 2\sin\mathcal{L}\cos\mathcal{L}\vec{i}_{\mathcal{L}})] \quad . \qquad 5.4b$$

The oblateness coefficient J_2 has the numerical value 1.0826×10^{-3}, and R_e is the radius of the earth. The theory as developed in Ref. 18 uses a Hamiltonian formalism. Coordinates L, G, H, canonically conjugate to ℓ, g, h, are introduced, and the Hamiltonian is represented in terms of them. A canonical transformation is performed, which introduces a new set of elements such that the dependence of the Hamiltonian on ℓ, g, h disappears to first order in J_2. The variations of the elements are given in a closed but very clumsy form on p. 568 of Ref. 18. Expanding to second order in e and I yields the variations in the elements as required by equation 5.3, as follows:

$$\frac{\delta a}{a} = \frac{3}{4}j[4e\cos\ell + 6e^2\cos 2\ell + 2I^2\cos(2\ell + 2g)] \qquad 5.5a$$

$$\delta e = \frac{3}{4}j\left\{\begin{aligned}&2\left(1 + \frac{1}{8}e^2 - \frac{3}{2}I^2\right)\cos\ell + 3e\cos 2\ell + \frac{53}{12}e^2\cos 3\ell\\ &+ \frac{1}{2}I^2\left[\cos(\ell + 2g) + \frac{7}{3}\cos(3\ell + 2g)\right]\end{aligned}\right\} \qquad 5.5b$$

$$\delta I = \frac{3}{4}jI\left[\cos(2\ell + 2g) - e\cos(\ell + 2g) + \frac{7}{3}e\cos(3\ell + 2g)\right] \qquad 5.5c$$

$$e\delta\ell = -\frac{3}{4}j\left\{\begin{aligned}&2\left(1 - \frac{5}{8}e^2 - \frac{3}{2}I^2\right)\sin\ell + 3e\sin 2\ell + \frac{53}{12}e^2\sin 3\ell\\ &- \frac{1}{2}I^2\left[\sin(\ell + 2g) - \frac{7}{3}\sin(3\ell + 2g)\right]\end{aligned}\right\} \qquad 5.5d$$

$$I(\delta\ell + \delta g) = \frac{3}{4}jI\left[13e\sin\ell - \sin(2\ell + 2g) + e\sin(\ell + 2g) - \frac{7}{3}e\sin(3\ell + 2g)\right] \qquad 5.5e$$

$$\delta(\ell + g + h) = \frac{3}{4}j\,[7e\sin\ell + 6e^2\sin2\ell + I^2\sin(2\ell + 2g)] \qquad 5.5f$$

$$j = J_2R_e^2/a^2 = 2.7 \times 10^{-4} \qquad \text{[PACSAT at 1 earth radius]} \ . \qquad 5.5g$$

Where these are substituted into 5.3, there is very extensive simplification. The expressions which finally result for the coordinates themselves are given to second order by

$$R = a\left\{\begin{matrix} 1 - \frac{3}{2}j + \frac{1}{2}e^2\left(1 - \frac{9}{2}j\right) + \frac{9}{4}jI^2 - e\left(1 - \frac{3}{4}j\right)\cos\ell \\ - \frac{1}{2}e^2\left(1 - \frac{1}{4}j\right)\cos2\ell + \frac{1}{4}jI^2\cos(2\ell + 2g) \end{matrix}\right\} \qquad 5.6a$$

$$\sin\mathcal{L} = I\left[\left(1 - \frac{3}{4}j\right)\sin(\ell + g) + e\left(1 + \frac{7}{2}j\right)\sin(2\ell + g) - e\left(1 + \frac{9}{2}j\right)\sin g\right] \qquad 5.6b$$

$$\phi = \ell + g + h + 2e\left(1 + \frac{9}{4}j\right)\sin\ell + e^2\left(\frac{5}{4} + \frac{7}{2}j\right)\sin2\ell$$

$$- \frac{1}{4}(1 + j)I^2\sin(2\ell + 2g) \ . \qquad 5.6c$$

The perturbed frequencies are given on p. 572 of Ref. 18 as

$$\dot{\ell} = n\left[1 + \frac{3}{2}j\left(1 + \frac{3}{2}e^2 - \frac{3}{2}I^2\right)\right] \qquad 5.7a$$

$$\dot{g} = 3nj\left(1 + 2e^2 - \frac{5}{4}I^2\right) \qquad 5.7b$$

$$\dot{h} = -\frac{3}{2}nj\left(1 + 2e^2 - \frac{1}{2}I^2\right) \tag{5.7c}$$

$$n^2a^3 = \mu \; . \tag{5.7d}$$

The frequencies may be interpreted as: $\dot{\ell}$ is the perigee to perigee or anomalistic frequency, $\dot{\ell} + \dot{g}$ is the nodal frequency, and $\dot{\ell} + \dot{g} + \dot{h}$ is the longitudinal recurrence frequency. The changes indicated by 5.7 show that the perigee advances and the node regresses. It is convenient to redefine the orbital elements a, e, and I, which are arbitrary constants, to simplify 5.6. We shall choose a new set of parameters such that the constant term and the term in $\cos\ell$ in 5.6a, and the term in $\sin(\ell + g)$ in 5.6b, have the same coefficients as in 5.2a and 5.2b. With the new parameters, we obtain

$$R = a\left[1 + \frac{1}{2}e^2 - e\cos\ell - \frac{1}{2}e^2\left(1 - \frac{1}{4}j\right)\cos2\ell + \frac{1}{4}jI^2\cos(2\ell + 2g)\right] \tag{5.8a}$$

$$\sin\mathcal{L} = I\left[\sin(\ell + g) + e\left(1 + \frac{7}{2}j\right)\sin(2\ell + g) - e\left(1 + \frac{9}{2}j\right)\sin g\right] \tag{5.8b}$$

$$\begin{aligned}\phi = \ell + g + h + 2e\left(1 + \frac{3}{2}j\right)\sin\ell + e^2\left(\frac{5}{4} + \frac{13}{8}j\right)\sin2\ell \\ - \frac{1}{4}I^2\left(1 + \frac{5}{2}j\right)\sin(2\ell + g)\end{aligned} \tag{5.8c}$$

$$\dot{\ell} = n\left[1 - \frac{3}{4}j\left(1 + 3e^2 - \frac{3}{2}I^2\right)\right] \tag{5.8d}$$

$$\dot{\ell} + \dot{g} = n\left[1 + \frac{3}{4}j\left(3 + 5e^2 - \frac{7}{2}I^2\right)\right] \tag{5.8e}$$

$$\dot{\ell} + \dot{g} + \dot{h} = n\left[1 + \frac{3}{4}j\left(1 + e^2 - \frac{5}{2}I^2\right)\right] \qquad 5.8f$$

The angular rate in orbit $\dot{\theta}$ is given by

$$\dot{\theta} = n\left[1 + \frac{3}{4}j\left(1 + e^2 - \frac{3}{2}I^2\right) + 2e\left(1 + \frac{3}{4}j\right)\cos\ell + e^2\left(\frac{5}{2} + \frac{11}{4}j\right)\cos 2\ell - \frac{7}{8}jI^2\cos(2\ell + 2g)\right] . \qquad 5.9$$

With the results 5.8 and 5.9, which describe the motion of the center of mass in the presence of oblateness, we can proceed to determine the libration and vibration equations. We first must introduce a coordinate system to represent the in-plane and out-of-plane motions. This will be chosen as the radial unit vector, the local unit horizontal velocity vector, and the cross product of the first two. Expressing these coordinates in terms of the latitude and longitude, we obtain

$$\vec{i}_1 = \vec{i}_R \qquad 5.10a$$

$$\vec{i}_2 = (\dot{\mathcal{L}}\vec{i}_{\mathcal{L}} + \cos\mathcal{L}\dot{\phi}\vec{i}_{\phi})/\dot{\theta} \qquad 5.10b$$

$$\vec{i}_3 = (\cos\mathcal{L}\dot{\phi}\vec{i}_{\mathcal{L}} - \dot{\mathcal{L}}\vec{i}_{\phi})/\dot{\theta} . \qquad 5.10c$$

The displacements $\vec{u}$, $\vec{w}$ will be resolved into components in the directions defined by 5.10, thus

$$\vec{u} = u_1\vec{i}_1 + u_2\vec{i}_2 + u_3\vec{i}_3 \qquad 5.11a$$

$$\vec{w} = w_1\vec{i}_1 + w_2\vec{i}_2 + w_3\vec{i}_3 \ . \qquad 5.11b$$

The terms u_1, w_1 are the radial components, u_2, w_2 are the in-plane components, and u_3, w_3 are the out-of-plane components. As before, u_1 is close to unity, w_1 is second order, and the others are first order.

The time derivatives of the unit vectors are needed to express the acceleration in components. We introduce a quantity P, which we will call the twist, by the relation

$$P = \left[\ddot{\mathcal{L}} + \cos\mathcal{L}\sin\mathcal{L}\dot{\phi}^2 - \frac{\ddot{\theta}}{\dot{\theta}}\right]/\dot{\phi}\cos\mathcal{L} \ . \qquad 5.12$$

The time derivatives of the unit vectors are given by

$$\dot{\vec{i}}_1 = \dot{\theta}\vec{i}_2 \qquad 5.13a$$

$$\dot{\vec{i}}_2 = -\dot{\theta}\vec{i}_1 + P\vec{i}_3 \qquad 5.13b$$

$$\dot{\vec{i}}_3 = \qquad - P\vec{i}_2 \ . \qquad 5.13c$$

We see that the twist P measures the internal rotation of the coordinate system. It is easily shown from the equations of motion that P vanishes for a purely elliptic orbit. For motion under oblateness forces, the various derivatives in 5.12 may be reduced using 5.8b and 5.8c, and the result to second order is

$$P = -3njI\left[\sin(\ell + g) + \frac{5}{2}e\sin(2\ell + g) + \frac{1}{2}e\sin g\right] \ . \qquad 5.14$$

The acceleration and gravity-gradient terms combine to form the three expressions as left sides of equations analogous to 4.3,

$$\ddot{u}_1 - \left(\dot{\theta}^2 + \frac{2\mu}{R^3}\right)u_1 - 2\dot{\theta}\dot{u}_2 - \ddot{\theta}u_2 + P\dot{\theta}u_3 \qquad 5.15a$$

$$\ddot{u}_2 - \left(\dot{\theta}^2 - \frac{\mu}{R^3} + P^2\right)u_2 + 2\dot{\theta}\dot{u}_1 + \ddot{\theta}u_1 - 2P\dot{u}_3 - \dot{P}u_3 \qquad 5.15b$$

$$\ddot{u}_3 + \left(\frac{\mu}{R^3} - P^2\right)u_3 + P\dot{\theta}u_1 + 2P\dot{u}_2 + \dot{P}u_2 \qquad 5.15c$$

and similar expressions for w. Since P is proportional to jI, which is on the scale of 10^{-5}, the terms in P^2 will be neglected. The term $P\dot{\theta}u_1$ in 5.15c shows that the twist induces a driving term in the out-of-plane libration.

We next have to transform the dyadic $\nabla\vec{F} \cdot \vec{u}$ on the right side of 2.33. We observe from 5.4b that the force vector $\vec{F}$, evaluated at an arbitrary point, r, $\mathcal{L}$, ϕ, can be written in the form:

$$\vec{F}(\vec{r}) = F_1(r,\mathcal{L})\vec{i}_r + F_2(\vec{r},\vec{\mathcal{L}})\vec{i}_{\mathcal{L}} \; . \qquad 5.16$$

The displacement from the center of mass can be written in terms of the spherical coordinates, and also in the form 5.11. Using $\vec{V}$ as the generic term for the displacement, we have

$$V_r = V_1 \qquad 5.17a$$

$$V_{\mathcal{L}} = (\dot{\mathcal{L}}V_2 + \cos\mathcal{L}\dot{\phi}V_3)/\dot{\theta} \qquad 5.17b$$

$$V_{\phi} = (\cos\mathcal{L}\dot{\phi}V_2 - \dot{\mathcal{L}}V_3)/\dot{\theta} \; . \qquad 5.17c$$

The changes in latitude and longitude are given by

$$\delta\mathcal{L} = V_{\mathcal{L}}/R \tag{5.18a}$$

$$\delta\phi = V_{\phi}/R\cos\mathcal{L} \tag{5.18b}$$

and the variations in the unit vectors $\vec{i}_r$, $\vec{i}_{\mathcal{L}}$ by

$$\delta\vec{i}_r = (V_{\mathcal{L}}\vec{i}_{\mathcal{L}} + V_{\phi}\vec{i}_{\phi})/R \tag{5.19a}$$

$$\delta\vec{i}_{\mathcal{L}} = -(V_{\mathcal{L}}\vec{i}_1 + \tan\mathcal{L}V_{\phi}\vec{i}_{\phi})/R \tag{5.19b}$$

$$\delta\vec{i}_{\phi} = -(V_{\phi}\vec{i}_1 - \tan\mathcal{L}V_{\mathcal{L}}\vec{i}_{\mathcal{L}})/R\ . \tag{5.19c}$$

In equations 5.17 - 5.19, the quantities $\mathcal{L}$, R, and the time derivatives are evaluated at the center of mass. We now have for the variation of $\vec{F}$,

$$\delta\vec{F} = \left[\frac{\partial F_1}{\partial r}\delta r + \frac{\partial F_1}{\partial\mathcal{L}}\delta\mathcal{L}\right]\vec{i}_r + F_1\delta\vec{i}_r + \ldots . \tag{5.20}$$

After extensive reduction, this leads to the form:

$$\delta\vec{F} = \left[\frac{\partial F_1}{\partial R}V_1 + \frac{1}{R}\left(\frac{\partial F_1}{\partial \mathcal{L}} - F_2\right)[\dot{\mathcal{L}}V_2 + \cos\mathcal{L}\dot{\phi}V_3]/\dot{\theta}\right]\vec{i}_1$$
$$+ \left[\frac{\dot{\mathcal{L}}}{\dot{\theta}}\frac{\partial F_1}{\partial R}V_1 + \frac{1}{R}\left(F_1 + \frac{\dot{\mathcal{L}}^2}{\dot{\theta}^2}\frac{\partial F_1}{\partial \mathcal{L}} - \frac{\sin\mathcal{L}\cos\mathcal{L}\dot{\phi}^2}{\dot{\theta}^2}F_2\right)V_2\right.$$
$$\left. + \frac{\cos\mathcal{L}\dot{\mathcal{L}}\dot{\phi}}{R\dot{\theta}^2}\left(\frac{\partial F_2}{\partial \mathcal{L}} + \tan\mathcal{L}F_2\right)V_3\right]\vec{i}_2$$
$$+ \left[\frac{\cos\mathcal{L}\dot{\phi}}{\dot{\theta}}\frac{\partial F_2}{\partial R}V_1 + \frac{\cos\mathcal{L}\dot{\mathcal{L}}\dot{\phi}}{R\dot{\theta}^2}\left(\frac{\partial F_2}{\partial \mathcal{L}} + \tan\mathcal{L}F_2\right)V_2\right.$$
$$\left. + \frac{1}{R}\left(F_1 + \frac{\cos\mathcal{L}\dot{\phi}^2}{\dot{\theta}^2}\frac{\partial F_2}{\partial \mathcal{L}} - \tan\mathcal{L}\frac{\dot{\mathcal{L}}^2}{\dot{\theta}^2}F_2\right)V_3\right]\vec{i}_3 \quad . \qquad 5.21$$

We will keep only first-order terms in e and I. On substituting from 5.4b and 5.8, we have

$$\delta\vec{F} = \frac{3}{2}\frac{\mu J_2 R_e^2}{R^5}\left\{\begin{array}{l}[4V_1 + 8I\sin(\ell + g)V_3]\vec{i}_1 - [V_2 + 2I\cos(\ell + g)V_3]\vec{i}_2 \\ - [3V_3 - 8I\sin(\ell + g)V_1 + 2I\cos(\ell + g)V_2]\vec{i}_3\end{array}\right\} . \qquad 5.22$$

If we keep the same order of accuracy in the expressions 5.15, we obtain the equations of libration:

$$\ddot{u}_1 - n^2\left[3 + \frac{15}{2}j + 10e(1 + 3j)\cos\ell\right]u_1 - 2n\left(1 + \frac{3}{4}j + e\cos\ell\right)\dot{u}_2$$
$$+ 2en^2\left(1 - \frac{3}{4}j\right)\sin\ell u_2 - 15n^2 jI\sin(\ell + g)u_3 = -\frac{8T_0}{\rho L^2}u_1 \qquad 5.23a$$

$$\ddot{u}_2 - n^2\left(1 - \frac{15}{2}j\right)e\cos\ell u_2 + 2n\left(1 + \frac{3}{4}j + 2e\cos\ell\right)\dot{u}_1 - 2en^2\left(1 - \frac{3}{4}j\right)\sin\ell u_1$$
$$+ 6njI\sin(\ell + g)\dot{u}_3 + 6n^2 jI\cos(\ell + g)u_3 = -\frac{8T_0}{\rho L^2}u_2 \qquad 5.23b$$

$$\ddot{u}_3 + n^2\left[1 + \frac{9}{2}j + 3\left(1 + \frac{15}{2}j\right)e\cos\ell\right]u_3 - 15n^2jI\sin(\ell + g)u_1$$

$$- 6njI\sin(\ell + g)\dot{u}_2 - 3n^2jI\cos(\ell + g)u_2 = -\frac{8T_0}{\rho L^2}u_3 \ . \qquad 5.23c$$

We apply the same transformations to these equations as used in 3.10, except that the time is retained as the independent variable. Keeping only first-order terms, we find:

$$T_0 = \frac{1}{8}\rho L^2\left\{n^2\left[3 + \frac{15}{2}j + 10e(1 + 3j)\cos\ell\right] + 2n\left(1 + \frac{3}{4}j + e\cos\ell\right)\dot{\psi}\right.$$

$$\left. - 2en^2\left(1 - \frac{3}{4}j\right)\sin\ell\psi + 15n^2jI\sin(\ell + g)\chi\right\} \qquad 5.24a$$

$$\ddot{\psi} + n^2\left[3\left(1 + \frac{5}{2}j\right) + (11 + 54j)e\cos\ell\right]\psi$$

$$= 2en^2\left(1 - \frac{3}{4}j\right)\sin\ell - 6njI\sin(\ell + g)\dot{\chi} - 6n^2jI\cos(\ell + g)\chi \qquad 5.24b$$

$$\ddot{\chi} + n^2[4(1 + 3j) + (15 + 69j)e\cos\ell]\chi = 15n^2jI\sin(\ell + g)$$

$$+ 15n^2jeI\sin(2\ell + g) + 6njI\sin(\ell + g)\dot{\psi} + 3n^2jI\cos(\ell + g)\psi \ . \qquad 5.24c$$

We see that the in-plane and out-of-plane frequencies and the tension are slightly modified by the perturbations. The term in $\dot{\psi}$ in the tension has been approximated to obtain 5.24b and c. If we regard χ and ψ as infinitesimal and omit the initial conditions, the solution of 5.24b and 5.24c, with only the driving terms kept, is

$$\psi_1 = e\left(1 - \frac{21}{4}j\right)\sin\ell \qquad 5.25a$$

$$\chi_1 = 5jI\sin(\ell + g) \ . \qquad 5.25b$$

Substituting χ_1 for χ in the last term on the left of 5.24c, ψ_1 for ψ on the right, keeping only the resonant terms yields the critical second-order equation

$$\ddot{\chi}_2 + 4(1 + 3j)n^2\chi_2 = -18n^2 jeI\sin(2\ell + g) \ . \qquad 5.26$$

The right side is the driving resonant term, with frequency $2(1 + 3/4j)n$. This differs from the natural frequency on the left by a term of order j, so the factor j will disappear from the solution. This resonant displacement has the value

$$\chi_2 = -3eI\sin(2\ell + g) \ . \qquad 5.27$$

If $e = .01$, $I = 1.5$ deg, the amplitude of this resonant term is .045 deg. Hence, although the resonance exists, its effect is very slight.

The same type of analysis can be applied to the vibration equations. However, the lack of a driving term in the equation analogous to 5.24b, and the fact that the lowest vibration frequency is thrice rather than twice orbital, allows no resonances to exist to at least second order in the parameters. Hence, the effect of oblateness on vibration is at most to change the frequencies of the fundamental modes by amounts of order j, which is about .00027 for PACSAT at an altitude of one earth radius. We deduce that there are no significant effects of oblateness on vibration, and the only effect is the resonant libration 5.27.

We next consider other orbital perturbations.

VI. OTHER ORBITAL PERTURBATIONS

In this section we shall consider the effects of solar and lunar gravity and of solar radiation pressure. The theory has been developed by previous investigators. Just as oblateness causes a precession of the orbit plane around the equator, solar effects produce precession around the ecliptic, and lunar gravity produces precession around the lunar orbit plane. We will show that the magnitude of the precession rates produced by the sun and moon is very small compared to the oblateness precession rate, and infer that the effects on libration and vibration will be correspondingly small.

The parameter we will use for comparison is the precession rate of the node. This is given from equation 5.7c as

$$\dot{h} = -\frac{3}{2} n J_2 \frac{R_e^2}{R^2} . \qquad 6.1$$

For PACSAT at an altitude of one earth radius, this has the numerical value

$$\dot{h} = -.875 \text{ deg/day} . \qquad 6.2$$

The corresponding expressions for the solar and lunar precession rates are[19]

$$\dot{h} = -\frac{3}{4} n \frac{M_D}{M_E} \left(\frac{R}{R_D}\right)^3 \left(1 - \frac{3}{2}\sin^2 I_D\right) \qquad 6.3$$

where D denotes the disturbing body, of mass M_D, distance R_D, and inclination to the equator I_D. For the sun, we have

$$\frac{M_D}{M_E} = 3 \times 10^5 \qquad 6.4a$$

$$\frac{R}{R_D} = \frac{2 \times 6.378 \times 10^3}{1.495 \times 10^8} = 8.53 \times 10^{-5} \qquad 6.4b$$

$$I_D = 23.45 \text{ deg} . \qquad 6.4c$$

These yield

$$\dot{h} = -\frac{3}{4} \times 2.16 \times 10^3 \times 3 \times 10^5 \times (8.53 \times 10^{-5})^3 \times .762$$

$$= -.00023 \text{ deg/day} . \qquad 6.5$$

Similarly, for the moon, we have

$$\frac{M_D}{M_E} = \frac{1}{81.3} \qquad 6.6a$$

$$\frac{R}{R_D} = \frac{2 \times 6.378 \times 10^5}{3.844 \times 10^5} = .0332 \qquad 6.6b$$

$$18.3 < I_D < 28.5 \qquad 6.6c$$

where the third of equations 6.6 describes the precession of the lunar orbit plane around the ecliptic. Using 18.3 deg, the value which produces the greatest precession, yields

$$\dot{h} = -\frac{3}{4} \times 2.16 \times 10^3 \times \frac{1}{81.3} \times (.0332)^3 \times .852 = -.00062 \text{ deg/day} . \qquad 6.7$$

Thus, the ratio of the lunar precession rate to the oblateness precision rate is about .0007, and we infer that the effects on libration and vibration will be reduced by about the same relative value. Hence, luni-solar gravity effects will be negligible.

We next consider the effect of solar radiation pressure. This effect has caused strange behavior on the part of past satellites. It produced the large changes observed in the perigee of the ECHO balloon, and an interaction between solar radiation pressure and bending produced by solar heating caused the rapid reduction in spin rate of the boom-carrying satellites Alouette and Explorer XX[20]. However, we shall show that for PACSAT the solar radiation pressure effects are negligible.

The solar radiation pressure on a specularly reflecting sphere produces the acceleration[20],

$$\vec{a} = -\frac{JA}{cM}\vec{\sigma} \qquad 6.8$$

where J is the solar incident flux, A the projected area, c the velocity of light, M the mass, and $\vec{\sigma}$ the unit vector directed from the sphere toward the sun. If the satellite is not specularly reflecting, the magnitude of the acceleration is reduced. This acceleration is in the direction opposite to the solar gravity, and may be thought of as a reduction in the effective gravitational constant. For PACSAT, we have the numerical values

$$J = 1.37 \times 10^3 \text{w/m}^2 \qquad 6.9a$$

$$c = 3 \times 10^8 \text{m/sec} \qquad 6.9b$$

$$\frac{A}{M} = \frac{3}{4}\,\frac{1}{r\rho} = \frac{3}{4} \times \frac{1}{.595 \times 10^{-2} \times 2.69 \times 10^{3}} = .0468\ \mathrm{m}^2/\mathrm{kg} \qquad 6.9c$$

from which we deduce the acceleration:

$$a = 2.14 \times 10^{-7}\mathrm{m/sec}^2\ . \qquad 6.10$$

The acceleration due to solar gravity is

$$a = g\frac{M_s}{M_E}\left(\frac{R_e}{R_s}\right)^2 \qquad 6.11$$

where g is the acceleration of gravity at the earth's surface. This has the numerical value

$$a = 9.8 \times 3 \times 10^5 \times (4.266 \times 10^{-5})^2 = 5.35 \times 10^{-3}\ . \qquad 6.12$$

We see that the relative reduction in the effective solar gravity caused by solar radiation pressure is about 4×10^{-5}. Hence, solar radiation pressure causes a very small change in an already small effect and can safely be ignored.

There is another solar radiation pressure effect which should be considered. The satellite will pass into and out of the earth's shadow. During its passage into the penumbra, the only effect is a steady decrease in the solar flux. The decrease is uniform along the satellite. However, during the time interval which begins when the lower end of the satellite enters the umbra and ends when the upper end enters, a "wave of darkening" passes up the satellite, and the solar radiation pressure is extinguished. This could potentially cause both libration and vibration. The effect, however, is negligible. We will calculate the change in velocity of the upper end of the satellite relative to the lower, and show that this change is far below detectability.

We will calculate the effect at the time when the sun is on the equator, that is, the equinox, since that is clearly when the effect is greatest. The configuration is depicted in Fig. 9. The sun, of diameter D_S, is at distance R_S. The satellite enters the penumbra at P, the umbra at Q. The figure is not to scale, and the lines connecting the sun to P and Q should be nearly vertical. The angle γ which measures the sun's angular diameter is

$$\gamma = \frac{D_s}{R_s} = \frac{1.398 \times 10^6}{1.495 \times 10^8} = .00933 \text{ rad} = .53 \text{ deg.} \qquad 6.13$$

For PACSAT at one earth radius altitude, the angle θ_0 is 60 deg. It is easily demonstrated that the angle POQ, the width of the penumbra as seen from the center of the earth, is also equal to γ. The arc length PQ is then $R\gamma$ = 119 km. The velocity of the satellite in orbit is 5.59 km/sec, so the time to cross the penumbra is 21.3 sec.

If we draw the plane from the satellite instantaneous position, measured by the angle θ, tangent to the earth's surface and extended to the sun, and neglect atmospheric refraction, the portion of the sun's area between the intersection of that plane and the left edge of the sun (Fig. 9) is observed. It is easily shown by the law of cosines and a simple calculation that the solar flux is reduced from its "full sun" value J to the effective value

$$J' = J \times \frac{2}{\pi}\sin^{-1}[(\theta_0 - \theta)/\gamma]^{1/2} \ . \qquad 6.14$$

At the time when the lower end of the satellite just enters the umbra, the upper end is at the angular position $\theta_0 - L/\sqrt{3}R$. The flux is reduced to the amount:

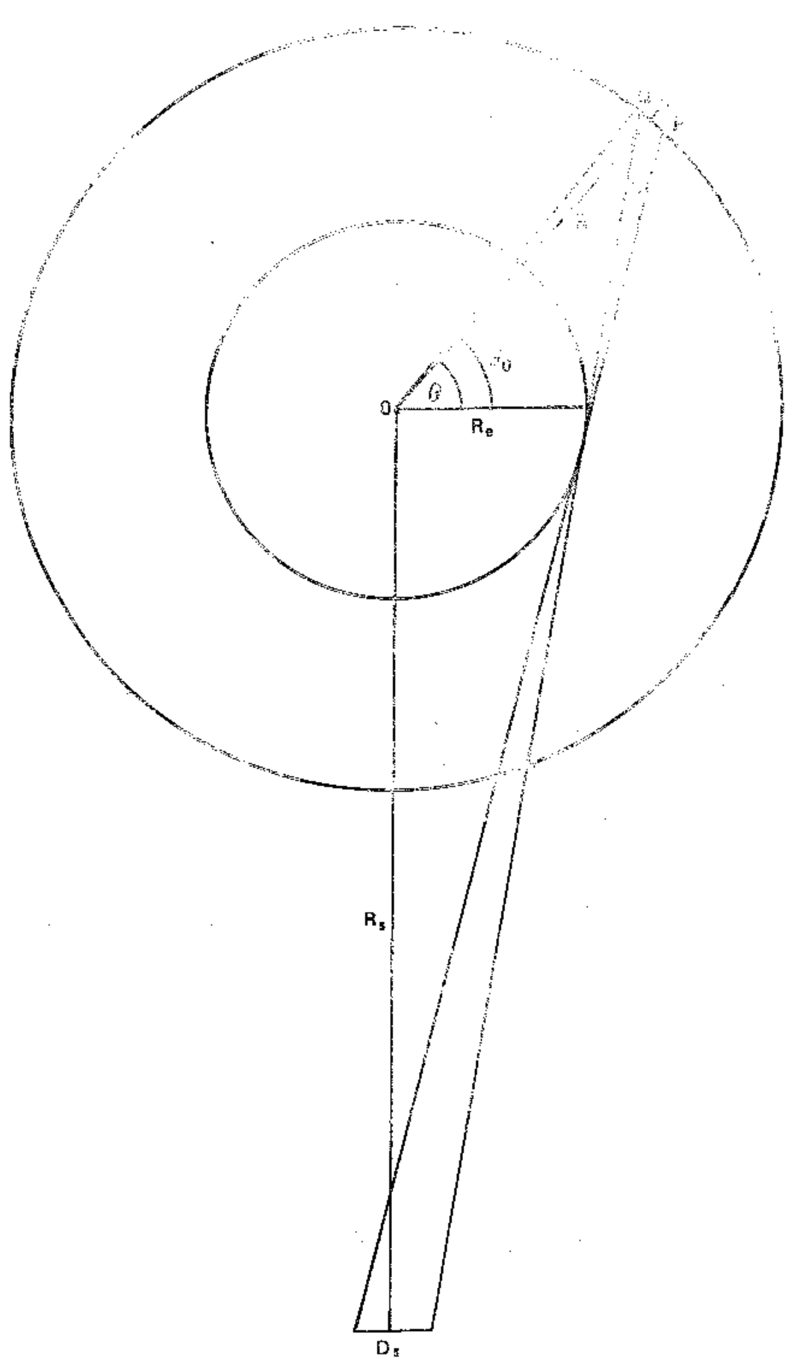

Fig. 9 — Passage through penumbra

$$J' = 1.37 \times 10^{+3} \times \frac{2}{\pi} \times \left(\frac{L}{\sqrt{3}R\gamma}\right)^{1/2} = 74.4 \text{ w/m}^2 \qquad 6.15$$

and the acceleration to the amount:

$$a = 1.16 \times 10^{-8} \text{m/sec}^2 \ . \qquad 6.16$$

The time to pass into the full umbra is

$$\frac{L}{\sqrt{3}R} \times \frac{T}{2\pi} = .3 \text{ sec} \qquad 6.17$$

where T is the satellite period, so the change in velocity of the upper end with respect to the lower is

$$\Delta v = 1.16 \times 10^{-8} \times .3 = 3.5 \times 10^{-9} \text{m/sec} \ . \qquad 6.18$$

This is far below any detectable amount, and also much below any error caused by emplacement, so the effect of passing into or out of the earth's shadow, and thereby losing solar radiation pressure, is completely negligible.

VII. MICROMETEOROID IMPACT EFFECTS

All the external forces considered to this point have very large scale factors. The gravity-gradient and oblateness terms have scales for variation on the order of the orbit radius, the solar and lunar effects scale like the distance to the sun or moon. By contrast, micrometeoroid impacts have as a geometrical scale the radius of the impacting body, which is small compared to the size of PACSAT. Hence, we can expect the excitation of many vibrational modes by an impact. We shall assume that the impacting body is sufficiently small that it does not sever or produce other gross effects on PACSAT.

The momentum of the incoming body will be absorbed in a region of length about equal to the meteoroid diameter. There will be cratering and spallation, which cause ejection of mass from the impacted body. The theory of this effect can be found in Ref. 21. Approximately half the kinetic energy of the meteoroid is transferred to the ejecta, so we shall take the momentum transfer to PACSAT as $M_i v_i/\sqrt{2}$, where M_i is the mass of the meteoroid and v_i is its velocity. Furthermore, the direction of the impacting body is arbitrary. The component of velocity transverse to PACSAT has the rms value $v_i/\sqrt{3}$, so the transverse component of momentum transfer will be $M_i v_i/\sqrt{6}$.

The theory of the transverse vibrations of struck bars was developed by Rayleigh (Ref. 14, p. 270), and the theory of bars in orbit is almost the same. We ignore the ellipticity and perturbing forces, and let the meteoroid, with effective momentum $M_i v_i/\sqrt{6}$ and diameter d_i, strike at a point s_0. The section of rod about s_0 of length d_i will absorb the momentum, and will be set into motion with an initial velocity $\dot{y}_0$, which is determined from conservation of momentum:

$$\dot{y}_0 \rho d_i = M_i v_i/\sqrt{6} \qquad 7.1a$$

$$\dot{y}_0 = M_i v_i/\sqrt{6}\rho d_i \qquad s_0 - .5d_i < s < s_0 + .5d_i \ . \qquad 7.1b$$

Suppose the direction of impact is in-plane. The theory is the same for out-of-plane motion. Let the initial displacement be zero. Then the solution of the vibration equation is given by 3.31, where $A_k(0) = 0$, and $A_k'(0)$ is

$$A_k'(0) = \frac{2}{nL}\int_{-1}^{1} \dot{y}_0 X_k(x)dx \qquad 7.2a$$

$$= \frac{2}{nL}\frac{M_i v_i}{\sqrt{6}\rho} X_k(x_0) \qquad 7.2b$$

and thus the complete motion is

$$y(x,t) = \frac{2}{nL}\frac{M_i v_i}{\sqrt{6}\rho}\sum_0^{\infty} X_k(x_0)X_k(x)\sin\omega_k\tau/\omega_k \qquad 7.3$$

where the libration is described by $k = 0$, $X_0(x) = x$, $\omega_0 = \sqrt{3}$. Even and odd vibratory modes are to be included in 7.3.

We calculate the motion of the point of impact. If we then average over possible impact locations, we obtain the simpler expression:

$$\bar{y}(t) = \frac{2}{nL}\frac{M_i v_i}{\sqrt{6}\rho}\sum_0^{\infty}\sin\omega_k\tau/\omega_k \ . \qquad 7.4$$

The series which appears is an oscillatory function of time, whose peak value is approximately 1.5. Hence, the amplitude of the transverse motion is

$$Y = \sqrt{\frac{3}{2}}\,\frac{M_i v_i}{nL\rho} \qquad 7.5$$

and the libration is about 3/8 of this.

Data for the meteoroid population are given in Ref. 21. The incoming velocity is taken to be 16 km/sec. This appears rather low, since meteoroids are observed with incoming velocities up to 70 km/sec. We will use the referenced value, but observe that the momentum transfer scales linearly with the incoming velocity. With the appropriate constants for PACSAT, the vibration and libration amplitudes are given by

$$Y_V = 6.05 \times 10^4 M_i(g) \qquad 7.6a$$

$$Y_L = 2.27 \times 10^4 M_i(g) \qquad 7.6b$$

in centimeters. If we had taken the impact at the tip, these values are approximately doubled.

The permissible vibration amplitude is .5 cm, the permissible tip libration amplitude is 20 m (1.5 deg). The impacting mass which will produce these deflections is

$$M_i = 8.3 \times 10^{-6} g \qquad \text{(vibration)} \qquad 7.7a$$

$$M_i = .088g \qquad \text{(libration)}\ . \qquad 7.7b$$

A particle of mass .088g and a velocity of 16 km/sec has a kinetic energy of about 11,000 joules. Such an impact would undoubtedly smash PACSAT to the proverbial smithereens.

We now must consider the probability of impact of particles of the indicated size. The flux of meteoroid particles per square meter per second is given in Table I of Ref. 21, which we reproduce in graphical form in Fig. 10. We see that the flux at the indicated levels is

$$\text{Flux } (8.3 \times 10^{-6}) = 5.3 \times 10^{-8} \quad m^{-2}s^{-1} \qquad 7.8a$$

$$\text{Flux } (.088) = 10^{-12} \quad m^{-2}s^{-1} . \qquad 7.8b$$

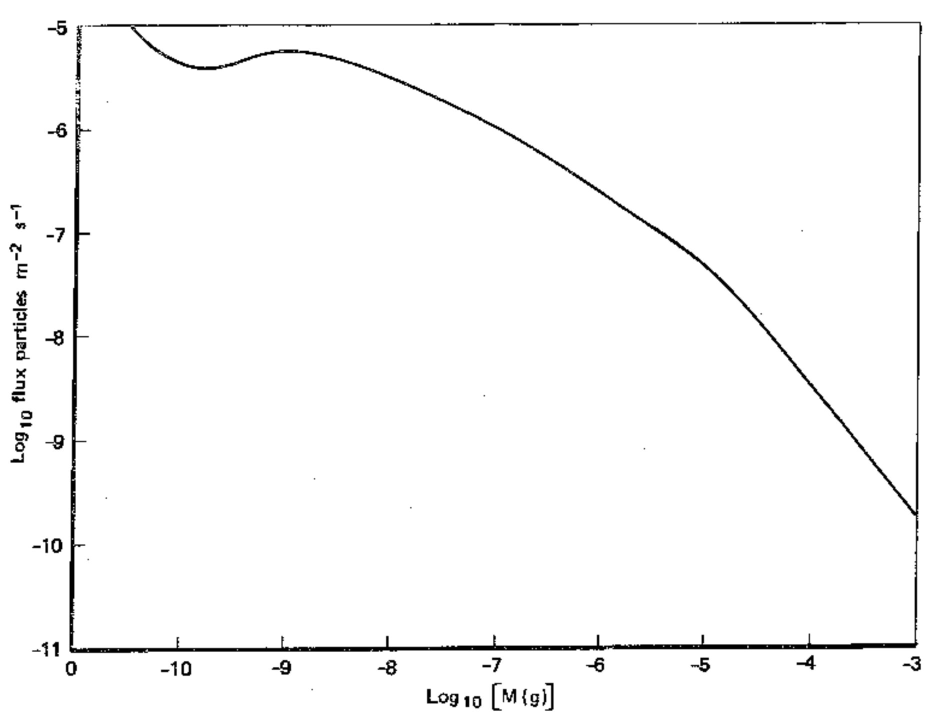

Fig. 10 — Meteoroid flux (from Barengoltz)

The projected frontal area is

$$A = N(\pi r^2 + \ell d) \tag{7.9}$$

where r is the radius of the sphere (.595 cm), ℓ the length (.685 cm) and d the diameter (.132 cm) of the rod, and N is the number of sections. This works out to a frontal area of 9.62 m^2. The product of this and the flux gives the probability of impact per day as

$$\text{Probability (vibration)} = .044 \tag{7.10a}$$

$$\text{Probability (libration)} = .83 \times 10^{-6} \tag{7.10b}$$

from which the expected time between impacts is

$$\text{Interval (vibration)} = 22 \text{ days} \tag{7.11a}$$

$$\text{Interval (libration)} = 1.2 \times 10^6 \text{ days} = 3300 \text{ years} . \tag{7.11b}$$

If the representative velocity were 50 km/sec instead of 15, the interval for vibratory excitation exceeding the permissible would be only 8 days. Since the intervals should have a Poisson distribution, the standard deviation should be about the same as the mean.

We should check the consistency of our assumptions. If the incoming particle is a small grain, its density is about .6 g/cm^3 (Ref. 21). The diameter of a spherical particle of mass 8.3×10^{-6}g and density .6g/cm^3 is .025 cm, small compared to the size of a PACSAT sphere. Hence, it is reasonable to treat the momentum absorption as concentrated. The kinetic energy of such a particle is about 1 joule, and the initial velocity is given by 7.1b as about 2.5 m/sec. The PACSAT structure should be able to sustain such an impulse without rupturing.

We see that the effect of micrometeoroid impact on PACSAT vibrations is very important. Even if the initial alignment is perfect, we can expect excessive vibration to be induced by meteoroid impact within about a 20-day interval. This is much too frequent for satisfactory operation. Some method of damping must be provided.

Before leaving the subject of meteoroid impacts, we shall consider the excitation of longitudinal vibrations. Analogous to the calculation of the transverse momentum, the effective longitudinal component of momentum transfer is $M_i v_i/\sqrt{6}$. The theory of longitudinal vibrations of bars has also been worked out by Rayleigh (Ref. 14, p. 245). The longitudinal motion $z(s,t)$ is governed by the differential equation and boundary conditions:

$$\frac{\partial^2 z}{\partial t^2} = b^2 \frac{\partial^2 z}{\partial s^2} \qquad 7.12a$$

$$\text{at} \quad s = \pm\frac{L}{2} \qquad \frac{\partial z}{\partial s} = 0 \qquad 7.12b$$

$$b = \left(\frac{E}{\sigma}\right)^{\frac{1}{2}} = 5.04 \times 10^5 \text{ cm/sec (steel)} . \qquad 7.12c$$

Here σ is the volume density of steel and E is Young's modulus. The parameter b is the velocity of sound waves in steel. We choose the initial deflection equal to zero, and the initial velocity, analogous to 7.1b, with impact at s_0,

$$\dot{z}_0 = M_i v_i/\sqrt{6}\rho d_i \qquad s_0 - .5d_i < s < s_0 + .5d_i . \qquad 7.13$$

The solution of 7.12a, subject to the boundary conditions 7.12b and initial condition 7.13, is

$$z(s,t) = \frac{M_i v_i}{\sqrt{6}\pi\rho b} \sum_1^\infty \cos\frac{2\pi ks}{L}\cos\frac{2\pi ks_0}{L}\sin 2\pi kbt/k \,. \qquad 7.14$$

The series has the peak value $\pi/2$, so the amplitude for longitudinal vibration is

$$Z = \frac{M_i v_i}{2\sqrt{6}\rho b} \;. \qquad 7.15$$

For a 16 km/sec particle, this becomes

$$Z = .46 M_i(g) \qquad \text{cm} \;. \qquad 7.16$$

The change in the radar cross section of PACSAT produced by longitudinal vibrations is not given by the analysis of Ref. 3 which led to Table 1, and we will calculate it here. The position of the m^1th sphere is given by ms + z(ms), where s is the spacing between elements. Assuming z is small, if we define the mean and mean square values of z by

$$\bar{z} = \frac{1}{N}\sum_{-N/2}^{N/2} z_m \qquad 7.17a$$

$$\overline{z^2} = \frac{1}{N}\sum_{-N/2}^{N/2} z_m^2 \qquad 7.17b$$

the change in radar cross section is

$$\frac{\Delta\Phi}{\Phi} = \left(\frac{2\pi}{s}\right)^2\left(\overline{z^2} - \bar{z}^2\right) . \qquad 7.18$$

The moments $\bar{z}$ and $\bar{z}^2$ will be functions of time. If we assume the impact is at the center, $s_0 \equiv 0$, then the moments $\bar{z}$ and $\bar{z}^2$ are calculated to be

$$\bar{z} = \frac{M_i v_i}{\sqrt{6}\pi\rho b} \cdot \frac{1}{N}\sum_1^\infty \frac{(-1)^k}{k} \sin 2\pi kat \qquad 7.19a$$

$$\overline{z^2} = N\bar{z}^2 . \qquad 7.19b$$

The series in 7.19a has the amplitude $\pi/2$, so we obtain for the change in cross section

$$\Delta\Phi/\Phi = \left(\frac{2\pi Z}{s}\right)^2 / N . \qquad 7.20$$

With $s = 1.875$ cm, $N = 8 \times 10^4$, and z given by 7.16, this gives

$$\Delta\Phi/\Phi = 2.97 \times 10^{-5} M_i^2 . \qquad 7.21$$

For a 1 percent change in cross section, the required mass is 18 grams. A particle sufficient to produce a transverse vibration of .5 cm would produce a longitudinal vibration of 2.7×10^{-6} cm, clearly a negligible effect. The lowest frequency associated with longitudinal vibrations is $b/L = 3.3\ \text{sec}^{-1}$, so the longitudinal vibrations are very small and very rapid compared to the transverse vibrations.

The meteoroid impact can also produce torsional vibrations. These will not have any effect on the radar cross section, so we need not analyze them. The effects on the motion should be comparable to the displacements associated with longitudinal vibrations.

We can make a crude estimate of the force involved in the impact of a particle which produces .5 cm vibration amplitude. According to Ref. 21, a meteoroid impact produces a hemispherical crater, with a radius R_c given by

$$\frac{R_c}{d_i} = \frac{1}{2}(\sigma_i v_i^2/S)^{1/3} \quad . \tag{7.22}$$

Here σ_i, v_i, and d_i are the volume density, velocity, and diameter of the meteoroid, and S is the yield strength of the material. Assuming the impact is on a sphere, we use for S the tensile strength of aluminum, which depends on the forming, annealing, and other properties of the spheres and should be between $.4 \times 10^8$ and 2×10^8 Newtons/m^2. With the previous numbers ($\sigma_i = 600$ kg/m^3, $v_i = 1.6 \times 10^4$ m/sec, $d_i = 2.5 \times 10^{-4}$ m), 7.22 yields a crater radius between .1 cm and .17 cm. We use .14 cm as representative. The crater is formed by a fracturing and yielding process, which we will estimate to propagate at the speed of sound in bulk aluminum (6.4×10^5 cm/sec). Since the impact is hypersonic, we should actually use the velocity of propagation of shock waves, but this differs from the sonic velocity by a factor of 2 or 3 and we are only interested in the order of magnitude of the force. The time for the meteoroid to be brought to rest we estimate by dividing the crater radius by the speed of sound, and obtain

$$\Delta t = R_c/b = .14/6.4 \times 10^5 = 2.2 \times 10^{-7} \text{ sec} . \qquad 7.23$$

The average force is the momentum change $M_i v_i/\sqrt{2}$ divided by the time interval Δt, or

$$F = M_i v_i/\sqrt{2}\Delta t = 8.3 \times 10^{-6} \times 1.6 \times 10^6/\sqrt{2} \times 2.2 \times 10^{-7} \qquad 7.24a$$

$$= 4 \times 10^7 \text{ dynes} . \qquad 7.24b$$

We see that this force is very large compared with other forces that may act, but it has a very short duration.

The calculations of meteoroid effects have been based on the assumption that the solid sphere stops the meteoroid. If PACSAT were built of hollow spheres, the meteoroid might punch its way through and exit carrying off some of the momentum. We take as a reasonable approximation that the meteoroid should have sufficient velocity to punch two craters of depth, given by 7.22, equal to the thickness of the sphere wall. The tensile strength of plastic should be used, which we expect to cover the same range as does aluminum (phenolic plastics, the most up-to-date type in the critical tables, cover the range .7 to 1.7 $\times 10^8$ Newtons/m^2). According to the immediately preceding calculation, an 8.3 $\times 10^{-6}$g particle will just penetrate a shell with wall thickness .1 cm, and will certainly not have enough energy remaining to penetrate the shell a second time. Hence, if the hollow spherical shell has a thickness of one millimeter, which is certainly reasonable, the analysis for the solid sphere should still be correct.

We conclude that meteoroid impacts can have a very important effect on the usage of PACSAT, since impacts sufficient to cause a radar cross-section loss of 1 dB can be expected to occur about every 20 days. A loss of 10 dB can be expected about every 50 days. The oscillations associated with these impacts must be damped if PACSAT is to operate satisfactorily.

The next and final topic for investigation is the effect of bending caused by thermal stresses.

VIII. THERMAL BENDING EFFECTS

There have been numerous investigations of the bending of long booms under solar input. Most of these studies have been applied to the Radio Astronomy Explorer satellite (1968-55A), which carries four 750-foot booms in a cruciform arrangement around the satellite body, plus a damper boom. The booms are extendible beryllium-copper thin-walled tubing with overlapping portions. A study by Yu[22], with comments by Augusti[23], Jordan[24], and Yu[25], concerning the stability of a boom making a fixed angle with the sun-line, showed after much disagreement that a boom is stable when the angle between it and the sun-line is less than 90 deg, but is unstable if the angle exceeds 90 deg. Sikka et al.[26] determined the temperature distribution and curvature in long solid cylinders in space. We shall use their results directly. Thermal curvatures in long booms have been studied by Eby and Karam[27], Vigneron[28], and Graham[29], with particular attention paid to the effects of thermal time constants on the motion.

The problem to be treated may be visualized as follows. PACSAT, in orbit around the earth in a nominally vertical position, receives direct heat from the sun and reflected heat from the earth. The reflected heat causes temperature variations along the rod which would primarily lead to longitudinal displacements and vibrations, which we may neglect per the investigation on micrometeoroids. The direct heat causes temperature variations across the rod. The rod expands more on the heated side than on the unheated, which causes it to curve away from the sun-line. This thermal curvature will couple to the mechanical curvature used in all preceding sections of this report, and we can expect substantial excitation of vibratory modes. In particular, if the solar input contains third harmonics of the orbital frequency, there will be resonant excitation of the lowest vibrational mode, and large deflections may result. This proves to be the case, and thermal bending effects will apparently render the present design of PACSAT unfeasible for satisfactory communications.

We begin by summarizing the investigation of Sikka et al.[26]. They treat a long solid cylinder of homogeneous isotropic material. The flux from the sun is incident normally, and a fraction α is absorbed. Thermal variations along the cylinder are neglected. The steady-state temperature is found. This temperature distribution satisfies Laplace's equation, with the boundary condition which matches the thermal input with the losses by conduction to the interior and radiation to space. In terms of the radial coordinate r from the center of the cylinder and the angular coordinate ϕ around the cylinder, measured from the sun-line, the temperature distribution is approximately given by

$$T(r,\theta) = \left(\frac{1}{\pi}\frac{\alpha}{\varepsilon}\frac{J}{\sigma}\right)^{\frac{1}{4}} + \frac{1}{2}\frac{\alpha J}{k} r\cos\phi \qquad 8.1$$

where α is the absorptivity, ε the emissivity, J the solar input [1.37 $\times 10^3 w/m^2$], σ is Stefan's constant [$5.67 \times 10^{-8} w/m^2/(°K)^4$], and k is the thermal conductivity. Terms of higher order angular dependence do not contribute to the curvature. For a gray body whose absorptivity and emissivity are equal, the first term in 8.1 is 296°K. For a steel rod, k = 16 w/m/°K, r = .066 cm, the second term is $.028\alpha\cos\phi$, so even for a perfectly absorbing rod the variation in temperature around the rod is much less than 0.1°K.

The solution of the thermoelastic problem of the bending of a cylinder under thermal expansion is well known[30]. The curvature K produced in the cylinder due to the temperature distribution T is the same as the curvature induced in a long cylinder due to application of stress eET at the ends, where e is the coefficient of thermal expansion and E is Young's modulus. When the stress is calculated with the temperature distribution 1.1, the simple result from Ref. 26 for the curvature is

$$K = \frac{1}{2}\frac{\alpha e J}{k} \quad . \qquad 8.2$$

For a steel rod, the coefficient of expansion e is $1.8 \times 10^{-5}/°K$, so the curvature is

$$K = 7.7 \times 10^{-4}\alpha \qquad m^{-1} \tag{8.3}$$

and the radius of curvature R_K, the reciprocal of the curvature, is

$$R_K = 1.30/\alpha \qquad km \ . \tag{8.4}$$

It is apparent at this stage of the analysis that PACSAT is in trouble. The requirements on straightness indicate a radius of curvature of 50,000 km, which would necessitate an absorptivity of 2.6×10^{-5} (.0026 percent) if the thermal radius of curvature 8.4 represents the actual curvature. This absorptivity appears to be beyond the current state of the art. We shall find the actual radius of curvature, and will show that it leads to an absorptivity requirement of .07 percent, which is also beyond the state of the art.

Equation 8.2 gives the curvature when the rod is straight and the sunlight is perpendicular to the axis. When the rod is bent, and the sun is in some other direction, the curvature should be perpendicular to the slope, be in the plane containing the slope and the sun-line vector, be directed away from the sun, and be proportional to the cosine of the angle between the normal and the sun-line. These conditions are satisfied by the expression[6]

$$\vec{K} = K\frac{\partial\vec{r}}{\partial s} \times \left(\frac{\partial\vec{r}}{\partial s} \times \vec{\sigma}\right) = K\left[\left(\frac{\partial\vec{r}}{\partial s} \cdot \vec{\sigma}\right)\frac{\partial\vec{r}}{\partial s} - \vec{\sigma}\right] \tag{8.5}$$

where K is expression 8.2, $\partial\vec{r}/\partial s$ is the slope, and $\vec{\sigma}$ is the unit sun-line vector. We have used the unit magnitude of $\partial\vec{r}/\partial s$.

The shadows cast by the spheres will shield portions of the rods from solar input. The configuration is shown in Fig. 11. The satellite is in equatorial circular orbit and is approximately radial. In equatorial coordinates, the slope is approximately

$$\frac{\partial \vec{r}}{\partial s} \sim \vec{i}_R = \vec{i}_x \cos\theta + \vec{i}_y \sin\theta \qquad 8.6$$

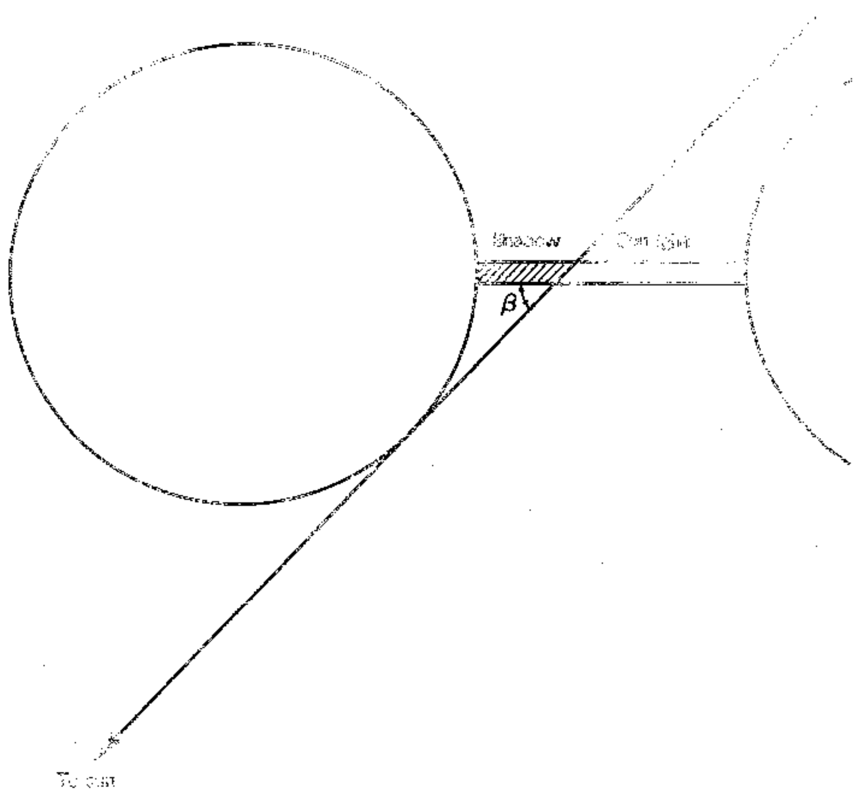

Fig. 11 — Shadowing of rods by spheres

while the sun-line vector $\vec{\sigma}$ is

$$\vec{\sigma} = \vec{i}_x \cos\theta_s + \vec{i}_y \cos\varepsilon \sin\theta_s + \vec{i}_z \sin\varepsilon \sin\theta_s \qquad 8.7$$

where ε is the inclination to the ecliptic (23.45 deg) and θ_s is the solar longitude measured from Aries. Ellipticity of the earth's orbit around the sun has been neglected. The angle β of Fig. 11 is given by

$$\cos\beta = \frac{\partial\vec{r}}{\partial s} \cdot \vec{\sigma} = \cos\theta_s \cos\theta + \cos\varepsilon \sin\theta_s \sin\theta \, . \qquad 8.8$$

With the length of the rod equal to ℓ, and the radius of the sphere equal to r, it is easily seen that the length of the unshadowed part of the rod is given by

$$\ell_u = \ell - r(\csc\beta - 1) \; .$$

The radius of the rod has been ignored, so the shadow is regarded as going straight across the rod. For PACSAT, $\ell = .685$ cm, $r = .595$ cm, and for ℓ_u to be positive β must be greater than 27.7 deg. Thus, during the portions of the orbit when PACSAT is sufficiently close to being directly between the earth and the sun, or for those when PACSAT is nearly opposite the sun, there will be no thermal bending effects. The opposition condition will put PACSAT into eclipse. If the altitude exceeds 1.15 earth radii, PACSAT will be in full internal shadow before it goes into eclipse. We shall consider this condition only.

The thermal curvature K is to be added to the mechanical curvature to form the strain energy. Thus, we have the new form for the strain energy 2.16:

$$V_B = \frac{1}{2}EI\int ds\left[\frac{\partial^2\vec{r}}{\partial s^2} + K\frac{\partial\vec{r}}{\partial s} \times \left(\frac{\partial\vec{r}}{\partial s} \times \vec{\sigma}\right)\right]^2 . \qquad 8.9$$

The vector $\vec{\sigma}$ has a maximum variation L/R_S, the ratio of the satellite length to the distance to the sun. This is numerically 10^{-8}, so $\vec{\sigma}$ will be treated as independent of s.

When the expression in brackets in 8.9 is squared (squared means the square of the magnitude), the square of the first term gives the previous results. The square of the second term may be shown to be equivalent to a 10^{-8} relative change in the tension. For the cross-term, we may simplify, since the curvature is perpendicular to the slope, and obtain

$$V'_B = -EIK\int ds\frac{\partial^2\vec{r}}{\partial s^2} \cdot \vec{\sigma} \qquad 8.10$$

forming the variation and integrating by parts,

$$\delta V'_B = -EIK\int ds\vec{\sigma} \cdot \frac{\partial^2}{\partial s^2}\delta\vec{r} \qquad 8.11a$$

$$= -EIK\left(\vec{\sigma} \cdot \frac{\partial}{\partial s}\delta\vec{r}\Big|_{-\frac{L}{2}}^{\frac{L}{2}} - \int ds\frac{\partial\vec{\sigma}}{\partial s} \cdot \frac{\partial}{\partial s}\delta\vec{r}\right) . \qquad 8.11b$$

The second term is zero, since $\vec{\sigma}$ is independent of s. Hence, the thermal curvature does not affect the differential equation 2.19. The first term in 8.11b must be combined with the corresponding term in 2.18. Consequently, we obtain the new boundary conditions:

$$\text{At } s = \pm\frac{L}{2} \qquad \frac{\partial^2 \vec{r}}{\partial s^2} = K\vec{\sigma} \tag{8.12a}$$

$$\frac{\partial^3 \vec{r}}{\partial s^3} = 0 \, . \tag{8.12b}$$

These equations still correspond to no bending or shearing at the ends. We see that there are no thermal bending effects on the libration. This is obvious anyway, since an induced curvature cannot to first order affect a rigid-body motion.

Before proceeding to solve the vibration equations, we need an estimate of the validity of the use of the steady-state thermal theory. The thermal time constant of a sphere or rod is given by

$$\tau = \frac{c_p \sigma r^2}{k} \tag{8.13}$$

where c_p is the specific heat, σ the volume density, r the radius of the sphere or rod, and k the thermal conductivity. This expression gives for the aluminum sphere and the steel rod the values .3 sec and .1 sec. The vibratory motions and the rate of variation of the solar input both have time scales of many minutes, so the condition of steady-state temperature is well satisfied.

Assume a circular equatorial orbit and form the equations of vibration. The complete rod participates in the bending, but only the unshadowed part experiences thermal input. Thus, when we introduce normalized variables, a factor involving ℓ_u should appear in the boundary conditions, but not in the differential equation. The result for the in-plane vibration, analogous to 3.27, is

$$\frac{2}{3} w_2'' = \frac{\partial}{\partial x}(1 - x^2)\frac{\partial w_2}{\partial x} - \beta\frac{\partial^4 w_2}{\partial x^4} \tag{8.14a}$$

$$\text{at } x = \pm 1 \qquad \frac{\partial^2 w_2}{\partial x^2} = L^2(\ell_U/\ell)^2 K\sigma_2/4 = Q \tag{8.14b}$$

$$\frac{\partial^3 w_2}{\partial x^3} = 0 \tag{8.14c}$$

where σ_2 is the component of the sun-line vector in the in-plane direction. We obtain readily

$$\sigma_2 = -\cos\theta_s \sin\theta + \cos\varepsilon \sin\theta_s \cos\theta \ . \tag{8.15}$$

Noting the definition of Q in 8.14b, we solve the system 8.14 by the trial form

$$w_2 = \frac{1}{6}(3x^2 - 1)Q + V \ . \tag{8.16}$$

The first factor has been chosen to satisfy the boundary conditions 8.14b and 8.14c and to have vanishing average value. Hence, V will satisfy the homogeneous form of the boundary conditions 8.14. The differential equation for V becomes

$$\frac{2}{3}V'' = \frac{\partial}{\partial x}(1 - x^2)\frac{\partial V}{\partial x} - \beta\frac{\partial^4 V}{\partial x^4} - (3x^2 - 1)\left(\frac{1}{9}Q'' + Q\right) \ . \tag{8.17}$$

The solution may be expanded in the form

$$V = \sum_1^\infty V_k(\theta)X_k(x) \tag{8.18}$$

which satisfies the boundary conditions, and leads to the differential equation for V_k:

$$V_k'' + \omega_k^2 V_k = -\frac{1}{6}(Q'' + 9Q)\int_{-1}^{1}(3x^2 - 1)X_k(x)dx \ . \qquad 8.19$$

Let the effective solar input be expanded into the as yet unspecified set of frequencies ν_i by

$$Q = \Sigma Q_i \sin(\nu_i\theta + \phi_i) \ . \qquad 8.20$$

Then the solution of 8.19 is

$$V_k = -\frac{1}{6}\left[\int_{-1}^{1}(3x^2 - 1)X_k(x)dx\right]\Sigma Q_i \frac{9 - \nu_i^2}{\omega_k^2 - \nu_i^2}\sin(\nu_i\theta + \phi_i) \qquad 8.21a$$

$$= -\frac{1}{6}\left[\int_{-1}^{1}(3x^2 - 1)X_k(x)dx\right]\left[Q + \Sigma Q_i \frac{9 - \omega_k^2}{\omega_k^2 - \nu_i^2}\sin(\nu_i\theta + \phi_i)\right] . \qquad 8.21b$$

When this solution is substituted into 8.18, and the result into 8.16, the term in Q in the second bracket in 8.21b leads to the cancellation of the first term in 8.16. The -1 in the integral does not contribute, since all X_k have vanishing average value. The numerical values of the integrals are found to be

$$\int_{-1}^{1} x^2 X_1(x)dx = .4216 \qquad 8.22a$$

$$\int_{-1}^{1} x^2 X_k(x)dx < .0005 \qquad k > 1 \ . \qquad 8.22b$$

Hence, the sum over k in 8.18 reduces to a single term. We have

$$w_2 = .2108X_1(x)\Sigma Q_i \frac{\omega_1^2 - 9}{\omega_1^2 - \nu_i^2} \sin(\nu_i\theta + \phi_i) \ . \qquad 8.23$$

Since $\omega_1^2 = 9.00645$, we see that the deflection is small unless ν_i is near 3. Hence, the input Q, which depends on the orbital and solar frequencies, should be expanded in Fourier series, and those terms near thrice orbital frequency retained. Thus, we represent the resonant part of Q in the form

$$Q_R = \sum_{-\infty}^{\infty} Q_i \sin(3\theta + i\theta_s + \phi_i) \ . \qquad 8.24$$

The calculation of the coefficients Q_i is very laborious. They are represented as double integrals, which must be expanded as slowly convergent series in the ratio $\sin^2\varepsilon/\sin^2\beta_{max} = .733$. Rather than present the details, we simply give the result. Using the numerical value of K from 8.3 and L = 548 m, the deflection w_2 is

$$w_2 = \alpha X_1(x)\begin{bmatrix} \sin(3\theta - 3\theta_s)(2.5078 + .6058\cos 2\theta_s + .0374\cos 4\theta_s) \\ + \cos(3\theta - 3\theta_s)(\qquad .5509\sin 2\theta_s + .0469\sin 4\theta_s) \end{bmatrix} .$$

8.25

As mentioned at the end of Section III, the "throw" of $X_1(x)$, that is, the difference between its maximum and minimum values, is 2.341. This enables us to define the deflection amplitude as

$$W = \alpha F(\theta_s)\sin[3\theta - 3\theta_s + \Phi(\theta_s)] \tag{8.26}$$

where F and Φ are the amplitude and phase of 2.341 times the bracketed expression in 8.25. We plot $F(\theta_s)$ and $\Phi(\theta_s)$ in Fig. 12. They have a period of one-half year. As can be seen from the figure, Φ differs from zero by only about 14 deg, but F varies from about 7.5 m at the equinox to 4.5 m at the solstice.

We see that if W is to be less than .5 cm, α must be less than .0007. The excitation of the resonant mode is caused by shadowing, which induces the third harmonic in σ_2. If there were no shadowing, the third harmonic would be produced by eclipsing. Incidentally, there is no resonant excitation in the out-of-plane motion.

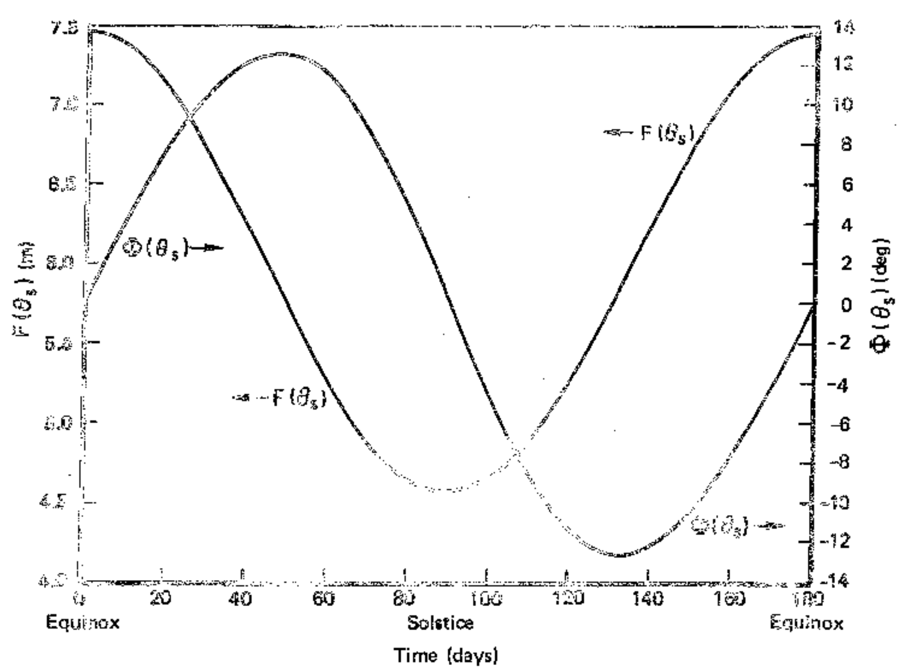

Fig. 12 — Annual variation of thermal deflection
$W = \alpha F(\theta_s)\sin\left[3(\theta-\theta_s) + \Phi(\theta_s)\right]$

The requirement that α be less than .0007 is probably impossible with the current state of plating and coating materials. Silver plating achieves an α of only .1 (Refs. 27, 29). A fresh mirror surface, produced by evaporating silver on glass, has absorptivity of .01 - .02 over the solar spectrum[31]. It is highly unlikely that this surface could be formed on a metal substrate, even with heroic polishing.

It is possible to improve the situation somewhat by making the rod of Invar (34 percent Ni steel), the standard low-expansion material. Unfortunately, the curvature depends on the quotient of the expansion coefficient and the thermal conductivity, and these tend to vary in the same manner. For steel, $e = 18 \times 10^{-6}$, $k = 16$, and the quotient is 1.1×10^{-6}. For Invar, $e = 1.5 \times 10^{-6}$, $k = 11$, and the quotient is 1.4×10^{-7}, an improvement of about a factor of 9. This leaves a required α of .006, still beyond the state of the art, and says nothing about the difficulty of working Invar.

The use of plastic or ceramic materials would make matters worse. Their expansion coefficients are somewhat lower than metals, but their thermal conductivities are much lower. Data[32], with the necessary changes of units, show that for vinyl acetate rigid copolymer, $e = 7 - 18 \times 10^{-6}$, $k = .08$, $e/k = 88 - 226 \times 10^{-6}$, for phenolic plastics, $e = 30 - 40 \times 10^{-6}$, $k = .09 - .16$, $e/k = 190 - 440 \times 10^{-6}$, and for glass $e = .8 - 14 \times 10^{-6}$, $k = .05$, $e/k = 16 - 280 \times 10^{-6}$. All of these are considerably worse than steel. It appears that little can be done by choice of materials to improve the thermal bending situation.

It appears from equation 8.26 and the numerical data that, under the best of circumstances, thermal bending will cause PACSAT to oscillate with a period of about 80 minutes and an amplitude of several centimeters, far too great for satisfactory communications. There may be some possible redesigns. If a thermal shield could be put around the rods, it might be possible to apply coatings in cascade to reduce the absorptivity. Also, if the satellite is built with flexible sections, so that the curvature is not maintained along the length, it would present a scalloped appearance and have a much smaller amplitude of oscillation. As was shown by Burke[11], the "sticky hinges" proposed by Stanford will not work for this purpose, since they will lock and allow the curvature to propagate. The flexible portions must transmit the gravity-gradient tension, or the whole stabilization system will not function. The engineering of such concepts is beyond the scope of the present investigation (a euphemism for saying, "I don't know how").

IX. CONCLUSIONS

Although the analysis in this report has been quite lengthy, since there were many topics to cover, the conclusions are quite brief:

1. The verticality and straightness conditions must be satisfied initially, which will be at least an extremely difficult deployment task.
2. The major causes of flexure are micrometeoroid impacts and thermal bending. Micrometeoroid impact may produce excessive vibration at tens of days occurrence times. Thermal bending effects will excite resonant oscillations of large amplitude and are so severe that it is highly unlikely that they can be overcome.
3. The flexural misbehavior of PACSAT in orbit is such that it is most improbable that the existing design of PACSAT can perform its communications function.

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