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Let $\Omega$ be a finitely connected closed point set in the complex plane with a piecewise smooth boundary $\partial \Omega$. The approximation of functions analytic on $\Omega$ by rational functions determined by interpolation or least squares approximation at preselected nodes is discussed. Attention is focused on simple methods for selecting an appropriate rational space and obtaining a fairly well-conditioned rational basis. Applications include the determination of conformal mappings. Numerical examples illustrate the approximation method.

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SIGNIFICANCE AND EXPLANATION

Complex rational approximation by interpolation has a long history in the theory of approximation. The following numerical questions, however, do not appear to have received much attention:

1) Given a region on which an analytic function shall be approximated by rational functions, and given a set of interpolation points on the boundary of this region, how should one numerically determine a suitable rational space?

2) How does a well-conditioned basis of this rational space look?

3) If one is free to select interpolation points on the boundary, how should they be chosen?

4) Can the selection of the rational space be simplified if one allows least squares approximation instead of interpolation?

The present paper discusses these questions.
ON COMPLEX RATIONAL APPROXIMATION
BY INTERPOLATION AT PRESELECTED NODES

Lothar Reichel

1. Introduction

Let \( \Omega \) be a closed region of finite connectivity in the complex plane, and assume that the boundary \( \partial \Omega \) is piecewise smooth. Let \( f(z) \) be a function analytic on \( \Omega \) and assume that \( f(z) \) is explicitly known on \( \partial \Omega \) or on a finite point set on \( \partial \Omega \). The purpose of the present paper is to describe a numerical method for determining a rational approximant \( r(z) \) to \( f(z) \) on \( \Omega \). The method consists of the following steps

1) select finitely many nodes \( z_k \) in the point set of \( \partial \Omega \) on which \( f(z) \) is known.

2) choose a rational space from which the approximant \( r(z) \) is to be selected. The choice will depend on the distribution of nodes \( z_k \).

3) select a well-conditioned basis of the rational space.

4) compute \( r(z) \) by interpolation or least squares approximation at the nodes \( z_k \).

Complex approximation by interpolation has a long history in the theory of approximation. Our scheme differs from previously described methods, see [1], [3], [7], [9], in that we use the selection of nodes as starting point. This allows us to treat cases when \( f(z) \) is known on a finite point set only, and it also allows us to let the allocation of nodes depend on properties of \( f(z) \). In turn we discuss necessary and sufficient conditions on the selection of rational space (section 2), a simple method for choosing rational space (section 3), the selection of nodes (section 4), choice of rational basis (section 5), and approximation on multiply connected regions (section 6). We

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present applications to conformal mapping, and indicate generalizations to the
numerical solution of Dirichlet problems for the Laplace equation on multiply
connected regions (section 7). Comments on approximation by the discrete
least squares method conclude the paper (section 8).
2. Convergence results

Throughout this paper $\{z_{k,m}\}_{k=1}^m$ denotes a set of interpolation or least squares nodes on $\partial \Omega$. The set $\{w_{k,n}\}_{k=1}^{n-1}$ denotes a set of not necessarily distinct poles in $\Omega_c$, the complement of $\Omega$ with respect to the extended complex plane, and defines the rational space

$$Q_n := \text{span}\{1, (z-w_{1,n})^{-1}, (z-w_{2,n})^{-1}, \ldots, (z-w_{k,n})^{-1} \}.$$

The approximation error we measure in the maximum norm

$$\|f\|_{\partial \Omega} := \max_{z \in \partial \Omega} |f(z)|.$$

The next definitions follow [9].

Definitions.

Let the real valued function $\sigma$ be positive a.e. on a Jordan arc $\gamma$, and assume $\int_{\gamma} \sigma(z)|dz| = 1$. The direction of integration defines an orientation on $\gamma$, and we let $z^*$ be the first point of $\gamma$. The mapping $F : \gamma \times [0,1], F(\zeta) := \int_{\gamma} \sigma(\zeta)|d\zeta|$ defined by integration along in the positive direction has an inverse a.e. For a sequence of sets $\{\zeta_{k,n}\}_{k=1}^n$, $n = 1, 2, 3, \ldots$ of points $\zeta_{k,n}$ on $\gamma$, let for constants $0 < d_1 < d_2 < 1,$ $N_n(d_1, d_2)$ denote the number of points of $\{\zeta_{k,n}\}_{k=1}^n$ on the subarc of $\gamma$ with end points $F^{-1}(d_1)$ and $F^{-1}(d_2)$. If $\lim_{n \to \infty} \frac{1}{n} N_n(d_1, d_2) = d_2 - d_1,$ and $d_1, d_2, 0 < d_1 < d_2 < 1,$ then the point sets are said to be uniformly distributed on $\gamma$ with respect to $\sigma$ as $n \to \infty$. A set $\{\zeta_{j,n}\}_{j=1}^n$ of points $\zeta_{j,n}$ on $\gamma$ is said to be equidistributed with respect to $\gamma$ if

$$F^{-1}(\zeta_{j,n}) - F^{-1}(\zeta_{j-1,n}) = \frac{1}{n}, \quad k = 2(1)n.$$ Sequences of equidistributed sets are uniformly distributed. The definitions carry over to Jordan curves if we let $\zeta^*$ be a point on the curve, and identify the curve with an arc with $\zeta^*$ both as first and last point.
The following theorem covers approximation on bounded simply connected regions. Extensions are provided in the remark below.

**Theorem 2.1**

Let \( \Gamma_{\text{nodes}} \) and \( \Gamma_{\text{poles}} \) be piecewise smooth Jordan curves, \( \Gamma_{\text{nodes}} \) containing \( \Gamma_{\text{nodes}} \) in its interior, and \( \Gamma_{\text{poles}} \cap \Gamma_{\text{nodes}} = \emptyset \). Let \( S \) denote the open region between \( \Gamma_{\text{nodes}} \) and \( \Gamma_{\text{poles}} \). Let \( U(z) \) solve the Dirichlet problem

\[
\begin{aligned}
U(z) &\text{ is harmonic in } S \text{ as a function of } x, y, z = x + iy, x, y \text{ real} \\
U(z) &\text{ is continuous on } S \cup \Gamma_{\text{poles}} \cup \Gamma_{\text{nodes}} \\
U(z) &= 1 \text{ on } \Gamma_{\text{nodes}} \\
U(z) &= 0 \text{ on } \Gamma_{\text{poles}}.
\end{aligned}
\]

(2.2)

Let \( \frac{\partial}{\partial n} \) denote the outward normal derivative with respect to \( S \). Then

\[
(2.3) \quad c := \int_{\Gamma_{\text{nodes}}} \frac{\partial U(z)}{\partial n} |dz| = \int_{\Gamma_{\text{poles}}} \frac{\partial U(z)}{\partial n} |dz| > 0.
\]

Let \( f(z) \) be a function analytic on and interior to \( \Gamma_{\mu_1} := \{z, U(z) = \mu_1\} \), where \( \mu_1 \) is a constant \( 0 < \mu_1 < 1 \). Let \( \{z_{k,n}\}^n_{k=1}, n = 1, 2, 3, \ldots \) be a sequence of points sets on \( \Gamma_{\text{nodes}} \) uniformly distributed with respect to \( -1 \frac{\partial U}{\partial n} \) as \( n \to \infty \), and let \( \{w_{k,n}\}^{n-1}_{k=1}, n = 2, 3, 4, \ldots \) be a sequence of point sets on \( \Gamma_{\text{poles}} \) uniformly distributed with respect to \( -c \frac{\partial U}{\partial n} \). Let the sets \( \{w_{k,n}\}^{n-1}_{k=1} \) define a sequence of rational spaces \( Q_n, n = 2, 3, 4, \ldots \), see (2.1). Then \( r_n \in Q_n \), uniquely determined by interpolating \( f(z) \) at points in the set \( \{z_{k,n}\}^n_{k=1} \), converges to \( f(z) \) on and interior to \( \Gamma_{\mu_2} \) for all \( 0 < \mu_2 < \mu_1 \), as \( n \to \infty \).

The rate of convergence is given by

\[
(2.3) \quad \lim_{n \to \infty} \max_{z \in \Gamma_{\mu_2}} |f(z) - r_n(z)|^{1/n} \leq e^{-2\pi(\mu_1 - \mu_2)}.
\]
If the sequences \( (z_k^n, w_k^n)_{k=1}^{n-1} \) are uniformly distributed with respect to another density function such that

\[
V_n(z) := \frac{1}{n} \sum_{k=1}^{n} \ln|z-z_k^n| - \frac{1}{n} \sum_{k=1}^{n-1} \ln|z-w_k^n|, \tag{2.4a}
\]

\[
\lim_{n \to \infty} V_n(z) =: V(z) \text{ is nonconstant on } \Gamma_{\text{nodes}},
\]

then there is a function \( g(z) \) analytic on and interior to \( \Gamma_{\text{nodes}} \) such that \( r_n(z) \in Q_n, r_n(z_k^n) = f(z_k^n), k = 1(1)n, \) and \( |g(z) - r_n(z)| \to 0 \) as \( n \to \infty. \)

**Proof.** Equations (2.2) have a unique solution \( \hat{U}(z) \) with \( \frac{\partial U}{\partial n} > 0 \) a.e. on \( \Gamma_{\text{nodes}} \) and \( \frac{\partial U}{\partial n} < 0 \) a.e. on \( \Gamma_{\text{poles}}. \) Green's formula yields (2.3). By studying potentials (2.4a) Walsh has established the connection between the level curves of \( \hat{U} \) and the rate of convergence, see the proof of Theorem 9 in Walsh [10], ch. 8. If the limit potential \( V(z) \) in (2.4) is nonconstant on \( \Gamma_{\text{nodes}} \) then there are points \( \zeta_1 \in \Gamma_{\text{nodes}}, \zeta_2 \in \text{exterior of } \Gamma_{\text{nodes}} \) such that \( V(\zeta_1) > V(\zeta_2). \) Let \( g(z) := (z-\zeta_2)^{-1} \), and let \( r_n \in Q_n \) interpolate \( g(z) \) at \( z = z_k^n, k = 1(1)n. \) Then by [10], ch. 8,

\[
g(\zeta_1) - r_n(\zeta_1) = \frac{1}{n} \sum_{k=1}^{n} \frac{\zeta_1 - z_{nk}}{z_{nk} - \zeta_2} + \frac{1}{n} \sum_{k=1}^{n-1} \frac{\zeta_2 - w_{nk}}{w_{nk} - \zeta_1}
\]

and

\[
\ln|g(\zeta_1) - r_n(\zeta_1)| = n(V_n(\zeta_1) - V_n(\zeta_2)).
\]

Since

\[
V_n(\zeta_1) - V_n(\zeta_2) + V(\zeta_1) - V(\zeta_2) =: \delta > 0, \ n \to \infty,
\]

\[
|g(\zeta_1) - r_n(\zeta_1)| = e^{\delta n} + o(n^{1/2}), \ n \to \infty,
\]

which shows divergence and completes the proof.
Remark 2.1

The distribution of nodes and poles described in the theorem is invariant under conformal mapping, see [10], section 9.12: Let $\phi$ map $S$ conformally and 1 to 1 onto $\phi(S)$ and be continuous and 1 to 1 on $S$. If the node sets $\{z_{kn}\}^n_{k=1}$ are uniformly distributed on $\Gamma_{\text{nodes}}$ with respect to $c^{-1} \frac{\partial U}{\partial n}$ as $n \to \infty$, where $U$ solves (2.2), then the node sets $\{\phi(z_{kn})\}^n_{k=1}$ are uniformly distributed on $\phi(\Gamma_{\text{nodes}})$ with respect to $c^{-1} \frac{\partial \phi}{\partial n}$ as $n \to \infty$, where $U_\phi$ solves the Dirichlet problem analogous to (2.2) on the mapped region $\phi(S \cup \Gamma_{\text{nodes}} \cup \Gamma_{\text{poles}})$, and $c_\phi = \int \phi(\Gamma_{\text{nodes}}) \frac{\partial \phi}{\partial n} |dz|$. Similarly, if $\{w_{kn}\}^{n-1}_{k=1}$ are uniformly distributed on $\Gamma_{\text{poles}}$ with respect to $-c^{-1} \frac{\partial U}{\partial n}$ as $n \to \infty$, then the sets $\{\phi(w_{kn})\}^{n-1}_{k=1}$ are uniformly distributed on $\phi(\Gamma_{\text{poles}})$ with respect to $-c^{-1} \frac{\partial \phi}{\partial n}$ as $n \to \infty$.

Especially theorem 2.1 holds also if $\Gamma_{\text{nodes}}$ is exterior to $\Gamma_{\text{poles}}$, or if $\Gamma_{\text{nodes}}$ is a piecewise smooth Jordan arc.

Remark 2.2

The configuration of curves in theorem 2.1, may consist of several mutually exterior curve pairs $\{\Gamma_{\text{nodes}}, \Gamma_{\text{poles}}\}$, and the allocation of nodes and poles on each pair can be made independent of the allocation on the other pairs. This follows from the fact that for each curve pair a solution of (2.2) if extended to the exterior of the curve pair would be constant there. Its normal derivative on any other curve would vanish. This remark follows again from [10], ch. 8, theorem 9 and its proof, which covers a more general situation than theorem 2.1 above.
We close this section by indicating how theorem 2.1 can be used for computing analytic continuations. Let \( f(z) \) be known on a curve \( \Gamma_{\text{nodes}} \). In many physical problems one may know that \( f(z) \) is analytic in a specific simply connected region \( B \) containing \( \Gamma_{\text{nodes}} \) in its interior. Then let \( \Gamma_{\text{poles}} \) be the boundary curve of \( B \). The rational approximants \( r_n \) computed as described in the theorem converge to \( f(z) \) in the interior of \( \Gamma_{\text{poles}} \) as \( n \to \infty \). The next section discusses how the nodes \( z_{kn} \) and poles \( w_{kn} \) can be allocated without explicitly solving the Dirichlet problem (2.2).
3. **Selection of rational space**

We discuss approximation of analytic functions $f(z)$ on simply connected regions. Our starting point is the assumption that a density function $\sigma$ for the interpolation nodes $\{z_{k,n}\}_{k=1}^n$ on $\Gamma_{\text{nodes}}$ is known, and that the nodes are equidistributed w.r.t. $\sigma$. We assume $\sigma > 0$ a.e. on $\Gamma_{\text{nodes}}$. If a set of nodes $\{z_{k,n}\}_{k=1}^n$ is given on $\Gamma_{\text{nodes}}$, then we construct a piecewise linear density function such that the nodes are equidistributed w.r.t. the constructed density function, which we also denote by $\sigma$. A set of poles $\{w_{k,n}\}_{k=1}^{n-1}$ defining the space $Q_n$ are obtained by solving (2.2) as an initial value problem: $U$ and $\frac{\partial U}{\partial n} = \sigma$ are known on $\Gamma_{\text{nodes}}$, and we want to determine other level curves of $U$, on one of which we allocate the poles $w_{k,n}$.

**An initial value problem**

Assume that $\Gamma_{\text{nodes}}$ is a smooth Jordan curve. If $\Gamma_{\text{nodes}}$ has corners, we round them for the present computations. If $\Gamma_{\text{nodes}}$ is an arc, we replace the arc by a smooth circumscribing curve, or we could proceed as illustrated in section 8. First determine a set of $n-1$ points $\{\zeta_{k,n}\}_{k=1}^{n-1}$ equidistributed w.r.t. $\sigma$. Let $W$ denote the conjugate harmonic function to $U$ such that $W(C_{\text{in}}) = 0$. The conformal mapping

$$z + z = \phi(z) := \exp(U(z) + iW(z))$$

maps $S,$ see theorem 2.1 for a definition, conformally on an annulus with

$$\phi(\zeta_{k,n}) = \exp(1 + 2\pi i \frac{k-1}{n}), \ k = 1(1)n-1.$$ 

Now assume that $S$ is exterior to $\Gamma_{\text{nodes}}$. By the conformal invariance noted in remark 2.1, the poles $\{w_{k,n}\}_{k=1}^{n-1}$ should be allocated so that the points $\phi(w_{k,n})$, $k = 1(1)n-1$, are equidistant on a circle concentric with the unit circle. Let $\phi^{-1}$ be the inverse of $\phi$, and let $z = e^{t+is}, s,t \in R$. Then
For a fixed \( t = t_0 \), the curve \( z = z(t_0, s), 0 < s < 2\pi \), is a level curve of \( U \). We determine such level curves by solving an initial value problem for the Cauchy-Riemann equations for \( z = z(t, s) \),

\[
\frac{\partial z}{\partial t} = -i \frac{\partial z}{\partial s}.
\]

Initial value problems for (3.3a) are ill-posed, but a low accuracy solution suffices for our purpose, and generally we integrate few steps only. The ill-posedness has not caused any difficulty in the present application. On nodes (3.1) yields, with \( s_k = 2\pi \frac{k-1}{n}, k = 1(1)n \),

\[
\frac{\partial z}{\partial s} (1, s_k) \approx \frac{z_{k+1,n} - z_{k-1,n}}{2\Delta s}.
\]

Substituting (3.4) into (3.3a), integrating in the positive \( t \)-direction with Eulers method with \( \Delta t = \Delta s \), and denoting the computed approximation of \( z(1 + \Delta t, s_k) \) by \( w_{k,n} \), yields the scheme

\[
w_{k,n} := \xi_{k,n} - \frac{1}{2} (\xi_{k+1,n} - \xi_{k-1,n}), k = 1(1)n-1, \xi_n := \xi_1, \xi_0 := \xi_{n-1}.
\]

The \( w_{k,n} \) lie on an approximate level curve of \( U \), and \( w_{k,n} \) lies approximately on the same streamline as \( \xi_{k,n} \). Hence, the \( w_{k,n} \) are approximately equidistributed w.r.t. \( \frac{\partial U}{\partial n} \) on a level curve of \( U \).

Ex. 3.1. Let

\[
\Gamma_{\text{nodes}} := (z(t) := x(t) + iy(t), x(t) := 1.75 \cdot \cos(t) + 2.625 \cdot \cos(2t) - 2.625, y(t) := 3.0625 \cdot \sin(t - 0.2) + 1.225 \cdot \sin(2t) - 0.6125 \cdot \sin(4t) + 0.875, 0 \leq t \leq 2\pi).
\]

Allocate 32 points \( \xi_{k,33} \) equidistantly w.r.t. arc length on \( \Gamma_{\text{nodes}} \), and integrate according to (3.5a). Figure 3.1 is obtained.
The curve (3.6) has been used by Meiss-Markowitz [5] in a quite different example.

The integration (3.5a) can be repeated by first letting $t_k := w_k, k = 1(1)n-1$, and then performing (3.5a). The integration should be repeated until the computed level curve intersects itself, and the $w_k$ should be allocated on the last non-intersecting or near-non-intersecting computed level curve. We motivate this by considering the case where $\Gamma_{\text{nodes}}$ is the unit circle. Analogous results can be established for more general curves.

Ex. 3.2. Let $\Gamma_{\text{nodes}} = \{ z : |z| = 1 \}$, and let the interpolation nodes be equidistant on $\Gamma_{\text{nodes}}$. $S$ is exterior to $\Gamma_{\text{nodes}}$, and level curves of $U$
are circles $|z| = r > 1$. The poles $w_{k,n}$ will lie equidistantly on a circle $|z| = r_0 > 1$, and

$$U(z) = 1 - \ln |z| \cdot (\ln |z_0|)^{-1}.$$  

At any fixed point $\tilde{z}$, $1 < |\tilde{z}| < r_0$, we have that $U(\tilde{z})$ increases with $|z_0|$. Hence, approximation of analytic functions on the unit disk by interpolation at the roots of unity is by theorem 2.1, best done by the family of rational functions which correspond to $r_0 = \infty$, i.e. polynomials.

When approximation of functions on the region exterior to $\Gamma_{\text{nodes}}$ is considered, then $S$ is in the interior of $\Gamma_{\text{nodes}}$, and (3.3a) is replaced by (3.3b)

$$\frac{\partial z}{\partial t} = i \frac{\partial z}{\partial s}.$$  

The corresponding difference equation is

(3.5b)  

$$w_{k,n} := \zeta_{k,n} + \frac{1}{2} (\zeta_{k+1,n} - \zeta_{k-1,n}), \quad k = 1(1)n-1, \quad \zeta_n := \zeta_1, \quad \zeta_0 := \zeta_{n-1}.$$  

Method (3.5) as well as other integration methods for (3.3) have been studied in [6] for the case when the function to be continued analytically is analytic in a simply connected region. The analysis carries without difficulty over to the present situation.

We conclude this section with some computed examples. All computations in this paper were carried out on a VAX/780 in double precision arithmetic, i.e. with 12 significant digits.

Ex. 3.3. Approximate $f(z) := \sqrt{z-a}$ on and interior to the curve $\Gamma_{\text{nodes}}$ of example 3.1., with $a := -2.6325 + i 1.425$, see figure 3.2
The branch of the square root is chosen to make \( f(z) \) single valued and analytic in the plane cut along \( z := a + t, t < 0 \). For \( n = 1 + 32\ell, \ell = 1(1)4, \) we allocate \( n \) interpolation nodes \( z_{k,n} \) equidistantly w.r.t. arc length on \( \Gamma_{\text{nodes}} \). \( Q_{32} \) is defined by the 32 poles \( w_{j,33} \) on figure 3.1. \( Q_{33+32\ell}, \ell = 1,2,3 \) are defined by letting \( w_{j,33+32\ell} := w_{(j \mod 32)+1,33} \). Let \( r_n(z) \) denote the element in \( Q_n \) such that \( r_n(z_{k,n}) = f(z_{k,n}) \), \( k = 1(1)n \). In figure 3.3 the computed errors are marked with dots.

We note, in passing, that the nodes of course do not have to be allocated exactly equidistantly w.r.t. arc lengths. Nodes could be marked sufficiently accurately with a light pen, and also equations (3.3a,b) are sufficiently simple to allow an approximate graphic solution.
Ex. 3.4. In theory, it is also possible to approximate the function \( f(z) \) of example 3.3 by polynomials. Let \( \{z_{k,n}\}_{k=1}^{n} \) denote a set of \( n \) Fejér points for \( \Gamma_{\text{nodes}} \). For their definition, see [3] or [9]. Figure 3.4 shows the points \( \{z_{k,120}\}_{k=1}^{120} \) marked with crosses on \( \Gamma_{\text{nodes}} \). Interpolation of \( f(z) \) in \( n \) points defines a polynomial \( p_{n}(z) \) of degree \( < n \), and the polynomial sequence \( \{p_{n}(z)\}_{n=1}^{\infty} \) converges maximally to \( f(z) \), i.e. \( p_{n}(z) \) converges exponentially to \( f(z) \), \( n \to \infty \), and there is no polynomial sequence with a higher exponential rate of convergence to \( f(z) \).
Computation of $p_n$ for some $n$ gave the following table

| $n$ | $|f-p_n|_{\Gamma_{\text{nodes}}}$ |
|-----|---------------------------------
| 40  | 0.60                            |
| 80  | 0.42                            |
| 120 | 0.42                            |
| 160 | impossible to evaluate.         |

A slow rate of convergence is combined with difficulties of accurately evaluating $p_n(z)$ for large $n$. A Lagrange polynomial basis was used, and the Fejér points $z_{k,n}$ were determined with 4 significant digits.

Ex. 3.5. Approximate $f(z) := \frac{1}{z} \sqrt{(z-z_1)(z-z_2)}$ on and exterior to $\Gamma_{\text{nodes}}$ defined by (3.6), where $z_1 := -4-2i$, $z_2 = i$, see figure 3.5.
The branch of the square root is chosen so that $zf(z)$ is analytic in the finite plane cut between $z_1$ and $z_2$. We wish to approximate $f(z)$ by interpolation in 33 nodes equidistant w.r.t. arc length. Figure 3.6 shows 32 points $\zeta_j,33$ allocated equidistantly w.r.t. arc length on $\Gamma_{\text{nodes}}$, and also 32 poles $w_j,33$ obtained by applying (3.5b) once. $r_{33}$ denotes the rational interpolant to $f(z)$ in $Q_{33}$, see below.

Allocate 65 interpolation nodes on $\Gamma_{\text{nodes}}$ equidistantly w.r.t. arc length. Let $w_j,65 := w_j,32,65 := w_j,33$, $j = 1(1)32$. This defines $Q_{65}$. Let $r_{65}$ be the interpolant to $f(z)$ in $Q_{65}$. We obtain

$$n \frac{|f-r|}{n \Gamma_{\text{nodes}}}$$

$$33 \quad 1 \cdot 10^{-2}$$

$$65 \quad 1 \cdot 10^{-4}$$
4. Selection of nodes

If \( f(z) \), the function to approximate, is known on a discrete point set only, there may be no choice to make. In this section we assume that \( f(z) \) is known everywhere on \( \Gamma_{\text{nodes}} \). If no knowledge of the location of the singularities of \( f(z) \) is available, we want an allocation of nodes such that the orthogonal distance from a point of \( \Gamma_{\text{nodes}} \) to level curves \( \{z : u(z) = p\}, p > 0, \) is approximately constant for all points of \( \Gamma_{\text{nodes}} \). Then the rate of convergence will depend on the distance from \( \Gamma_{\text{nodes}} \) to a singularity of \( f(z) \) closest to \( \Gamma_{\text{nodes}} \). This is, to a first approximation, achieved by allocating the nodes equidistantly w.r.t. arc length.

Knowledge about the location of the singularities of \( f(z) \) can be used by allocating more nodes on parts of \( \Gamma_{\text{nodes}} \) close to a singularity. The level curves of \( U(z) \) will be close to \( \Gamma_{\text{nodes}} \) where the node density is highest. Example 7.2 provides an illustration.
5. **Rational basis**

The basis implicit in the definition (2.1) of \( Q_n \) is generally ill-conditioned. If the nodes \( z_{k,n} \) and poles \( w_{k,n} \) are near-equidistributed with respect to \( |c^{-1} \frac{\partial u}{\partial n}| \) on \( \Gamma_{\text{nodes}} \) and \( \Gamma_{\text{poles}} \), respectively, then the basis \( \ell_0(z) = 1, \ell_j(z) := \prod_{k=1}^{n-1} \frac{z-z_{k,n}}{z_{k,n}-z_{j,n}} \) \( j = 1(1)n-1, \) is fairly well-conditioned. A condition number of a basis we define following Gautschi [], with the map \( F_n : R^n + Q_n : a \rightarrow \sum_{k=0}^{n-1} a_k f_k(z), \) where \( a = (a_0, a_1, \ldots, a_{n-1}) \). Equip \( R^n \) with the maximum norm \( |a|_\infty := \max|a_k| \) and \( Q_n \) with the norm \( |\ell_k(z)|_{\Gamma_n} \). The induced operator norms are

\[
|F_n| := \max_{1 \leq a_n} \sum_{k=0}^{n-1} |a_k| |\ell_k(z)|_{\Gamma_n} = \max_{1 \leq a_n} \sum_{k=0}^{n-1} |\ell_k(z)|_{\Gamma_n} \leq 2n \max_{1 \leq a_n} \sum_{k=0}^{n-1} |\ell_k(z)|_{\Gamma_n}
\]

and the condition number of the basis \( \ell_j(z) \) is

\[
\text{cond } F_n := \frac{|F_n|}{|F_n^{-1}|} \leq 2 \max_{1 \leq a_n} \sum_{k=0}^{n-1} |\ell_k(z)|_{\Gamma_n} \leq 2n \max_{1 \leq a_n} \sum_{k=0}^{n-1} |\ell_k(z)|_{\Gamma_n}.
\]

Assuming that the \( z_k, w_k \) are uniformly distributed with respect to \( |c^{-1} \frac{\partial u}{\partial n}| \), we have for \( k \neq 0 \),

\[
\ln|\ell_k(z)| = (n-1) \left( \frac{1}{n-1} \sum_{j=1}^{n-1} \ln|z-z_j,n| - \ln|z_k,n-z_j,n| \right) - \frac{1}{n-1} \left( \sum_{j=1}^{n-1} \ln|z-w_j,n-1| - \ln|z_k,n-w_j,n-1| \right)
\]

\[
= (n-1) \left( \int_{\Gamma_{\text{nodes}}} \ln|z-\zeta| c^{-1} \frac{\partial u}{\partial n}(\zeta) \, |d\zeta| - \int_{\Gamma_{\text{nodes}}} \ln|z_k,n-\zeta| c^{-1} \frac{\partial u}{\partial n}(\zeta) \, |d\zeta| + \ldots \right)
\]
\begin{align*}
- \int_{\text{poles}} \ln|z-\xi|c^{-1} \frac{\partial u}{\partial n}(\xi) \mathrm{d}\xi + \int_{\text{poles}} \ln|z_k, n-\xi|c^{-1} \frac{\partial u}{\partial n}(\xi) \mathrm{d}z_k &= O(\frac{\ln(n)}{n})
\end{align*}

for any fixed $z \in \Gamma_{\text{nodes}}$, $z \neq z_j, n_1$ as $n \to \infty$. Hence, $|\mathbf{f}_k(z)| = O(n)$, $k = O(1)n^{-1}$, for any fixed $z \in \Gamma_{\text{nodes}}$, $n \to \infty$. This shows that the basis $\{\mathbf{f}_j\}_{j=0}^{n-1}$ is reasonably well-conditioned.
6. Multiply connected regions

We consider the approximation of a function \( f(z) \) on an exterior doubly connected region. Generalization to more complicated regions are immediate. Let \( f(z) \) be analytic on and exterior to the piecewise smooth curves \( \gamma_1 \) and \( \gamma_2 \). \( \gamma_1 \) bounds the region \( \Omega_1 \), see figure 6.1.

![Figure 6.1](image)

We wish to approximate \( f(z) \) by a rational function which interpolates \( f(z) \) in nodes on \( \gamma_1 \) and \( \gamma_2 \). During the allocation of poles we can regard the problem as if it were composed of the two simpler subproblems:

Approximate \( f_i(z) \), analytic on and exterior to \( \gamma_i \), \( i = 1,2 \), and \( f_2(\omega) = 0 \). Consider the case \( i = 1 \). Assume \( n \) nodes \( \{z^{(1)}_k\}_{k=1}^n \) and a density function \( \sigma^{(1)} \) are known on \( \gamma_1 \). The method of section 3, yields poles \( w_k^{(1)} \), \( k = 1(1)n-1 \), in \( \Omega_1 \), and this defines the space

\[
Q_n^{(1)} := \text{span}\{1, (z-w_1^{(1)})^{-1}, \ldots, (z-w_k^{(1)})^{-1}\}
\]

and basis

\[
\begin{align*}
\ell_k^{(1)}(z) := & \prod_{j=1}^{n-1} \frac{z-z_j^{(1)}}{z-w_j^{(1)}} \prod_{j=1}^{n-1} \frac{z-w_j^{(1)} - w_k^{(1)}}{z-w_j^{(1)}} , \quad k = 1(1)n-1 \\
\ell_0^{(1)}(z) := & 1.
\end{align*}
\]

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Repeat for problem $i = 2$. Nodes $z_k^{(2)}$, $k = 1 \ldots m$ are assumed to be known on $\mathcal{Y}_2$. Since $f_2(\omega) = 0$, the rational space we are to construct does not have to contain constants. We can identify the nodes $z_k^{(2)}$ with the points $\zeta_k$ of section 3 and obtain poles $w_k^{(2)}$, $k = 1 \ldots m$, defining the space

$$(6.3) \quad \mathcal{Q}_m^{(2)} := \text{span}((z-w_1^{(2)})^{-1}, \ldots, \prod_{k=1}^{m} (z-w_k^{(2)})^{-1})$$

and basis

$$l_k^{(2)}(z) := \prod_{j=1 \atop j \neq k}^{m} \frac{z-z_j^{(2)}}{z_k^{(2)}-z_j^{(2)}} \frac{z_k-w_j^{(2)}}{z-w_j^{(2)}}, \quad j = 1(1)m.$$  

To solve the original problems, we select the function $r_{n,m} \in \mathcal{Q}_n^{(1)} \cap \mathcal{Q}_m^{(2)}$ which interpolates $f(z)$ in $\{z_k^{(1)}\}_{k=1}^{n} \cup \{z_k^{(2)}\}_{k=1}^{m}$. Convergence results analogous to theorem 2.1 can be shown, see remark 2.2, provided that both $n,m \to \infty$. The basis $\{l_0^{(1)}, \ldots, l_{n-1}^{(1)}, l_1^{(2)}, \ldots, l_m^{(2)}\}$ is fairly well-conditioned on $\mathcal{Y}_1 \cup \mathcal{Y}_2$, under the assumption that $\|l_k^{(1)}\|_{\mathcal{Y}_2}^{(1)}$, $k = 1(1)n-1$, and $\|l_k^{(2)}\|_{\mathcal{Y}_1}^{(2)}$, $k = 1(1)m$ are small. The method of proof is similar to that used in section 5. The assumption is reasonable due to the relation between nodes and poles.

**Ex. 6.1.** Let $f(z) = \frac{\sqrt{(z+\frac{5}{2})(z+\frac{3}{2})}}{\sqrt{(z-\frac{1}{2})(z+\frac{1}{2})}} \cdot (z+\frac{1}{2})^{-1} + \frac{\sqrt{(z-2-\frac{1}{2})(z-2+\frac{1}{2})}}{\sqrt{(z+\frac{1}{2})(z+\frac{3}{2})}} \cdot (z-\frac{1}{2})^{-1}$

where the branches are selected so that the first term is analytic in the complex plane cut along the line segment between $z = -\frac{5}{3}$ and $z = -\frac{3}{2}$, and the second term is analytic in the plane cut along the line segment between $z = 2 + \frac{1}{2}$ and $z = 2 - \frac{1}{2}$. Approximate $f(z)$ in the region exterior to both curves

$$\mathcal{Y}_1 = \{z = 2 + \cos(t) + i\sin(t)(2+\cos(t))^2, \quad 0 < t < 2\pi\}$$
$$\mathcal{Y}_2 = \{z = -2 - \sin(t)(2+\cos(t))^2 + i\cos(t), \quad 0 < t < 2\pi\}.$$  

Figure 6.1 shows $\mathcal{Y}_1$, $\mathcal{Y}_2$ and the branch points of $f(z)$ marked with crosses inside $\mathcal{Y}_1$ and $\mathcal{Y}_2$. 

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We wish to interpolate $f(z)$ in 31 nodes $z_k^{(1)}$ equidistant w.r.t. arc length on $Y_1$, and in 30 nodes $z_k^{(2)}$ equidistant w.r.t. arc length on $Y_2$. Figure 6.2 shows 30 points $z_k^{(2)}$ equidistant w.r.t. arc length on $Y_2$, the 30 points $z_k^{(2)} = z_k^{(2)}$ on $Y_2$, and the poles $w_j^{(1)}$ and $w_j^{(2)}$ obtained by applying (3.5b) twice for each curve $Y_i$. Denote the interpolating rational approximant by $r(z)$. Then

$$\|f - r\|_{Y_1} = 4 \cdot 10^{-7}, \quad \|f - r\|_{Y_2} = 4 \cdot 10^{-7}.$$
7. Applications to Dirichlet problems and conformal mapping

The close connection between rational approximation and approximation by rational harmonics, see [11], suggests applications to the numerical solution of Dirichlet problems for the Laplace equation on simply and multiply connected regions. We discuss in some detail the special Dirichlet problems whose solution yields a conformal mapping from a simply connected region to \(|w| > 1\) or to \(|w| < 1\).

Let \( \Omega \) be simply connected with boundary \( \partial \Omega = \Gamma_{\text{nodes}} \) and complement \( \Omega_c \). Allocate nodes \( z_j, j = 1(1)2n-1 \), on \( \Gamma_{\text{nodes}} \) for some \( n \). Determine \( n-1 \) points \( \zeta_j \) on \( \Gamma_{\text{nodes}} \), such that the distribution functions for \( z_j \) and \( \zeta_j \) agree. Allocate \( n-1 \) poles \( w_j \) in \( \Omega_c \) by application of (3.5a) or (3.5b). This defines the \( 2n-1 \)-dimensional space of harmonic functions

\[
\text{span}\{l_0(z), \Re(l_1(z)), \Im(l_1(z)), \ldots, \Re(l_{n-1}(z)), \Im(l_{n-1}(z))\},
\]

where

\[
\begin{align*}
    l_0(z) &:= 1 \\
    l_j(z) &:= \prod_{k=1}^{n-1} \frac{z-z_{2k}}{z-w_{2k}} \prod_{k \neq j} \frac{z-w_{2k}}{z_{2j}-z_{2k}}, \quad j = 1(1)n-1.
\end{align*}
\]

An approximate solution to

\[
\begin{cases}
    \Delta u = 0 \quad \text{in } \Omega \\
    u = q \quad \text{on } \partial \Omega
\end{cases}
\]

is obtained by solving

\[
\Re\left( \sum_{k=0}^{n-1} a_k \ell_k(z_j) \right) = g(z_j), \quad j = 1(1)2n-1
\]

for the \( a_k \). This system may not always be uniquely solvable, which is why, in the computed examples below, we solved (7.3) by singular value
decomposition of the matrix. In none of the computed examples the matrix was near-singular.

The special choice

\begin{equation}
\tag{7.4}
g(z) = -\ln|z|
\end{equation}

leads to approximate conformal mappings for \( \Omega \) provided that \( 0 \notin \partial \Omega \).

Ex. 7.1. Let \( \Omega \) be the bounded region of figure 3.5. Compute a conformal mapping \( \phi : \Omega + |w| < 1 \), such that \( \phi(0) = 0 \). Allocate 129 nodes \( z_j \) on \( \partial \Omega \) equidistantly w.r.t. arc length. Use the poles \( \{w_j\}_{j=1}^{32} \) shown in figure 3.1, and define \( w_{j+32} := w_j \), \( j = 1(1)32 \). Thus we obtain a basis (7.2) for \( n = 65 \), and solve (7.3) with \( g(z) \) defined by (7.4). This yields

\begin{equation}
\phi_{65}(z) := z \exp\left( \sum_{k=0}^{64} a_k \ell_k(z) \right)
\end{equation}

which approximates \( \phi \). Figure 7.1 shows \( \phi_{65}(\partial \Omega) \) and a reference circle of radius 1.1. The error \( \|\phi_{65}(z) - 1\|_{\partial \Omega} = 7 \cdot 10^{-5} \) is well below the resolution of the picture.

![Figure 7.1](image-url)
Ex. 7.2. Let $\Omega$ be the exterior of the curve in figure 7.2. Compute a conformal mapping $\psi: \Omega \rightarrow |w| > 1$, such that $\psi(\infty) = \infty$. Allocate 65 nodes $z_j$ on $\partial \Omega$, equidistantly w.r.t. arc length, and use the 32 poles $\omega^j$ of figure 3.6. Solving (7.3), with $q$ defined by (7.4), one obtains the approximate map

$$
\psi_n(z) := z \exp\left( \sum_{k=0}^{n-1} a_k \ell_k(z) \right)
$$

for $n = 33$.

Figure 7.2

Figure 7.3 shows $\psi_{33}(\partial \Omega)$ and a reference circle of radius 1.1. $|\psi_{33}(z)| - 11,_{\partial \Omega} = 3 \times 10^{-2}$. $\psi_{33}(\partial \Omega)$ intersects itself at the blob, which is roughly the image of the part of $\partial \Omega$, which lies strictly interior to the convex hull of $\partial \Omega$. This suggests that in order to achieve higher accuracy more nodes should be allocated between the points $z(\frac{\pi}{2})$ and $z\left(\frac{3\pi}{2}\right)$ in figure 7.2.
7.1. The allocation of nodes to be described next is simplified by the fact that the parameter \( t \) in (3.6) satisfies \( \frac{d(\text{arc length})}{dt} \approx \text{constant} \), \( 0 < t < 2\pi \). (Figure 2.2 shows 32 points on \( \partial \Omega \) equidistant w.r.t. \( t \).) This simplifies the determination of the nodes and poles to be used, but is not essential for the discussion, why we chose not to use this fact in the first part of this example. Figure 7.1 shows 32 points marked with crosses on \( \Gamma \) and allocated equidistantly with respect to the boundary parameter \( t \), see (3.6). In figure 7.4 we have allocated 11 points \( \zeta_j \) equidistantly with respect to \( t \) for \( 0 < t < \frac{\pi}{2} \), 44 points \( \zeta_j \) equidistantly with respect to \( t \) for \( \frac{\pi}{2} < t < \frac{3\pi}{2} \), and 11 points equidistantly with respect to \( t, \frac{3\pi}{2} < t < 2\pi \). By (3.5b), we obtain 66 poles \( w_j \), and finally we allocate 133 nodes on \( \Gamma \) having the same distribution as the \( \zeta_j \). This yields the mapping \( \phi_67(z) \). Figure 7.5 shows \( \phi_67(\Gamma) \) and a circumscribed concentric reference circle. \( \| \phi_67(z) \| - 11_{\Gamma} = 3 \cdot 10^{-3} \).
The abbreviation error is near the resolution of the plotter. Double the number of poles in figure 7.4 by \( w_{j+66} := w_j, j = 1(1)66 \). This yields 
\[ 4 \times 66 + 1 = 265 \] harmonic basis function, which we determine by interpolation at 265 points \( z_j \), which we allocated with the same density function as we used for \( \phi_{67} \). This gives \( \phi_{133}(z) \) and \( \| \phi_{133}(z) \| = 6 \times 10^{-6} \). Figure 7.6 shows \( \phi_{133}(\Omega) \) and a reference circle of radius 1.1.

![Figure 7.6](image-url)
8. Least squares approximation

We consider an example, where the distribution of nodes and poles does not agree very well with the conditions of theorem 2.1, and point out that in such cases least squares approximation may give higher accuracy than interpolation.

Approximate \( f(z) := \sqrt{z^2 - 1} \) on \( \Omega = [-5i, 5i] \), by using function values at equidistant nodes \( z_{k,m} := i(10 \frac{k-1}{m} - 5), k = 1(1)m, \) on \( \Omega \). The branch of the square root is chosen to make \( f(z) \) analytic in the finite plane cut on the real axis from 1 to \( \infty \) and from \(-1\) to \(-\infty\). The poles we allocate in a simple manner: for \( n \) even, let for some \( s > 0 \)

\[
\begin{align*}
\omega_{2k,n} &= i(10 \frac{2k-2}{n-2} - 5) + s, \\
\omega_{2k-1,n} &= i(10 \frac{2k-2}{n-2} - 5) - s,
\end{align*}
\]

First, consider the selection of \( s \). For \( s \) large the distribution of nodes \( z_{k,m} \) and poles \( \omega_{k,n} \) does not agree at all with the distribution suggested in theorem 2.1. In fact, for \( s = \infty \), i.e. polynomial approximation, approximation by interpolation diverges. This follows from the similarities of our approximation problem and the classical example of Runge, see [2]. On the other hand, for \( s > 0 \) small the rate of convergence becomes unnecessarily slow.

Ex. 8.1. Let \( n = m + 1 \) and compute the rational approximant \( r_n(z) \) to \( f(z) \) for \( s = 1, 2, 3 \).
Only points with markers in figure 6.1 correspond to computed approximation errors.

Example 8.1 shows that the selection of $s$ is of some importance, c.f. also example 3.2. When it is difficult to determine an appropriate allocation of poles, a crude determination of poles combined with discrete least squares approximation at the nodes $z_{k,m}$ can be a good strategy. For a justification when the poles all are at $\infty$, see [8]. Approximation by rationals with a different but fixed distribution of poles can be treated similarly as in [8]. Given $m$ nodes $z_{k,m}$ and a distribution of poles $w_{k,n}$, one generally does not know a priori how to select the ratio $m/n$. One has to select several values of $n$ and select the best of the computed approximants, see [8]. The difficulties in choosing $m/n$ are illustrated in the next example.

Ex. 8.2. Consider the same approximation problem as in example 8.1 with $s = 2$, but let $\frac{n}{m} = 0.9$. For $m = 25, 51, 25$, we let $n$ be the even
integer closest to 0.9m. The "+" in Figure 8.2 show the approximation error. The dots correspond to interpolation \( n = m - 1 \) and are the same as in Figure 8.1.

![Graph showing approximation error](image)

**Figure 8.2**

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On Complex Rational Approximation by Interpolation at Preselected Nodes

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Let \( \Omega \) be a finitely connected closed point set in the complex plane with a piecewise smooth boundary \( \partial \Omega \). The approximation of functions analytic on \( \Omega \) by rational functions determined by interpolation or least squares approximation at preselected nodes is discussed. Attention is focussed on simple methods for selecting an appropriate rational space and obtaining a fairly well-conditioned rational basis. Applications include the determination of conformal mappings. Numerical examples illustrate the approximation method.